Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 149712, 15 pages
doi:10.1155/2008/149712

Research Article

Quasi-Nearly Subharmonicity and Separately Quasi-Nearly Subharmonic Functions

Juhani Riihentaus

Department of Physics and Mathematics, University of Joensuu, P.O. Box 111, 80101 Joensuu, Finland

Correspondence should be addressed to Juhani Riihentaus, juhani.riihentaus@joensuu.fi

Received 29 February 2008; Accepted 30 July 2008

Recommended by Shusen Ding

Wiegerinck has shown that a separately subharmonic function need not be subharmonic. Improving previous results of Lelong, Avanissian, Arsove, and of us, Armitage and Gardiner gave an almost sharp integrability condition which ensures a separately subharmonic function to be subharmonic. Completing now our recent counterparts to the cited results of Lelong, Avanissian and Arsove for so-called quasi-nearly subharmonic functions, we present a counterpart to the cited result of Armitage and Gardiner for separately quasinearly subharmonic function. This counterpart enables us to slightly improve Armitage's and Gardiner's original result, too. The method we use is a rather straightforward and technical, but still by no means easy, modification of Armitage's and Gardiner's argument combined with an old argument of Domar.

1. Introduction

1.1. Previous results

Solving a long standing problem, Wiegerinck [1, Theorem, page 770], see also Wiegerinck and Zeinstra [2, Theorem 1, page 246], showed that a separately subharmonic function need not be subharmonic. On the other hand, Armitage and Gardiner [3, Theorem 1, page 256] showed that a separately subharmonic function u in a domain Ω in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, is subharmonic provided $\phi(\log^+ u^+)$ is locally integrable in Ω, where $\phi : [0, +\infty) \to [0, +\infty)$ is an increasing function such that

$$\int_1^{+\infty} s^{(n-1)/(m-1)} \left(\phi(s)\right)^{-1/(m-1)} ds < +\infty. \tag{1.1}$$

Armitage's and Gardiner's result includes the previous results of Lelong [4, Theorem 1, page 315], of Avanissian [5, Theorem 9, page 140], see also [6, Proposition 3, page 24], and

[7, Theorem, page 31], of Arsove [8, Theorem 1, page 622], and of us [9, Theorem 1, page 69]. Though Armitage's and Gardiner's result is almost sharp, it is, nevertheless, based on Avanissian's result, or, alternatively, on the more general results of Arsove and us, see [10].

In [10, Proposition 3.1; Theorem 3.1, Corollary 3.1, Corollary 3.2, Corollary 3.3; pages 57–63], we have extended the cited results of Lelong, Avanissian, Arsove, and us to the so-called quasi-nearly subharmonic functions. The purpose of this paper is to extend also Armitage's and Gardiner's result to this more general setup. This is done in Theorem 4.1 below. With the aid of this extension, we will also obtain a refinement to Armitage's and Gardiner's result in their classical case, that is for separately subharmonic functions, see Corollary 4.5 below. The method of proof will be a rather straightforward and technical, but still by no means easy, modification of Domar's and Armitage's and Gardiner's argument, see [11, Lemma 1, pages 431-432 and 430] and [3, proof of Proposition 2, pages 257–259, proof of Theorem 1, pages 258-259].

Notation

Our notation is rather standard, see, for example, [7, 10, 12]. m_N is the Lebesgue measure in the Euclidean space $\mathbb{R}^N$, $N \geq 2$. We write ν_N for the Lebesgue measure of the unit ball $B^N(0,1)$ in $\mathbb{R}^N$, thus $\nu_N = m_N(B^N(0,1))$. D is a domain of $\mathbb{R}^N$. The complex space $\mathbb{C}^n$ is identified with the real space $\mathbb{R}^{2n}$, $n \geq 1$. Constants will be denoted by C and K. They will be nonnegative and may vary from line to line.

2. Quasi-nearly subharmonic functions

2.1. Nearly subharmonic functions

We recall that an upper semicontinuous function $u : D \to [-\infty,+\infty)$ is *subharmonic* if for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_N r^N}\int_{B^N(x,r)} u(y)\,dm_N(y). \tag{2.1}$$

The function $u \equiv -\infty$ is considered subharmonic.

We say that a function $u : D \to [-\infty,+\infty)$ is *nearly subharmonic*, if u is Lebesgue measurable, $u^+ \in \mathcal{L}^1_{\text{loc}}(D)$, and for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_N r^N}\int_{B^N(x,r)} u(y)\,dm_N(y). \tag{2.2}$$

Observe that in the standard definition of nearly subharmonic functions, one uses the slightly stronger assumption that $u \in \mathcal{L}^1_{\text{loc}}(D)$, see, for example, [7, page 14]. However, our above slightly more general definition seems to be more useful, see [10, Proposition 2.1(iii) and Proposition 2.2(vi) and (vii), pages 54-55].

2.2. Quasi-nearly subharmonic functions

A Lebesgue measurable function $u : D \to [-\infty,+\infty)$ is K-*quasi-nearly subharmonic*, if $u^+ \in \mathcal{L}^1_{\text{loc}}(D)$ and if there is a constant $K = K(N,u,D) \geq 1$ such that for all $\overline{B^N(x,r)} \subset D$,

$$u_M(x) \leq \frac{K}{\nu_N r^N}\int_{B^N(x,r)} u_M(y)\,dm_N(y) \tag{2.3}$$

for all $M \geq 0$, where $u_M := \sup\{u, -M\} + M$. A function $u : D \to [-\infty, +\infty)$ is *quasi-nearly subharmonic*, if u is K-quasi-nearly subharmonic for some $K \geq 1$.

A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is *K-quasi-nearly subharmonic n.s.* (*in the narrow sense*), if $u^+ \in \mathcal{L}^1_{\text{loc}}(D)$ and if there is a constant $K = K(N, u, D) \geq 1$ such that for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y)\, dm_N(y). \tag{2.4}$$

A function $u : D \to [-\infty, +\infty)$ is *quasi-nearly subharmonic n.s.*, if u is K-quasi-nearly subharmonic n.s. for some $K \geq 1$.

Quasi-nearly subharmonic functions (perhaps with a different terminology, and sometimes in certain special cases), or the corresponding generalized mean value inequality (2.4), have previously been considered at least in [9, 10, 12–24]. For properties of mean values in general, see, for example, [25]. We recall here only that this function class includes, among others, subharmonic functions, and, more generally, quasisubharmonic and nearly subharmonic functions (for the definitions of these, see above and, e.g., [4, 5, 7]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, and polyharmonic functions. Also, the class of Harnack functions is included, thus, among others, nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations. In particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations, see Vuorinen [26]. Observe that already Domar [11, page 430] has pointed out the relevance of the class of (nonnegative) quasi-nearly subharmonic functions. For, at least partly, an even more general function class, see Domar [27].

For examples and basic properties of quasi-nearly subharmonic functions, see the above references, especially Pavlović and Riihentaus [16], and Riihentaus [10]. For the sake of convenience of the reader we recall the following.

(i) A K-quasi-nearly subharmonic function n.s. is K-quasi-nearly subharmonic, but not necessarily conversely.

(ii) A nonnegative Lebesgue measurable function is K-quasi-nearly subharmonic if and only if it is K-quasi-nearly subharmonic n.s.

(iii) A Lebesgue measurable function is 1-quasi-nearly subharmonic if and only if it is 1-quasi-nearly subharmonic n.s. and if and only if it is nearly subharmonic (in the sense defined above).

(iv) If $u : D \to [-\infty, +\infty)$ is K_1-quasi-nearly subharmonic and $v : D \to [-\infty, +\infty)$ is K_2-quasi-nearly subharmonic, then $\sup\{u, v\}$ is $\sup\{K_1, K_2\}$-quasi-nearly subharmonic in D. Especially, $u^+ := \sup\{u, 0\}$ is K_1-quasi-nearly subharmonic in D.

(v) Let $\mathcal{F}$ be a family of K-quasi-nearly subharmonic (resp., K-quasi-nearly subharmonic n.s.) functions in D and let $w := \sup_{u \in \mathcal{F}} u$. If w is Lebesgue measurable and $w^+ \in \mathcal{L}^1_{\text{loc}}(D)$, then w is K-quasi-nearly subharmonic (resp., K-quasi-nearly subharmonic n.s.) in D.

(vi) If $u : D \to [-\infty, +\infty)$ is quasi-nearly subharmonic n.s., then either $u \equiv -\infty$ or u is finite almost everywhere in D, and $u \in \mathcal{L}^1_{\text{loc}}(D)$.

3. Lemmas

3.1. The first lemma

The following result and its proof are essentially due to Domar [11, Lemma 1, pages 431-432 and 430]. We state the result, however, in a more general form, at least seemingly. See also [3, page 258].

Lemma 3.1. *Let $K \geq 1$. Let $\phi : [0,+\infty) \to [0,+\infty)$ be an increasing (strictly or not) function for which there exist $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, such that $\phi(s) > 0$ and*

$$2K\phi(s - s_0) \leq \phi(s) \tag{3.1}$$

for all $s \geq s_1$. Let $u : D \to [0,+\infty)$ be a K-quasi-nearly subharmonic function. Suppose that

$$u(x_j) \geq \phi(j) \tag{3.2}$$

for some $x_j \in D$, $j \geq s_1$. If

$$R_j \geq \left(\frac{2K}{\nu_N}\right)^{1/N} \left[\frac{\phi(j+1)}{\phi(j)} m_N(A_j)\right]^{1/N}, \tag{3.3}$$

where

$$A_j := \{x \in D : \phi(j - s_0) \leq u(x) < \phi(j+1)\}, \tag{3.4}$$

then either $B^N(x_j, R_j) \cap (\mathbb{R}^N \setminus D) \neq \varnothing$ or there is $x_{j+1} \in B^N(x_j, R_j)$ such that

$$u(x_{j+1}) \geq \phi(j+1). \tag{3.5}$$

Proof. Choose

$$R_j \geq \left(\frac{2K}{\nu_N}\right)^{1/N} \left[\frac{\phi(j+1)}{\phi(j)} m_N(A_j)\right]^{1/N}, \tag{3.6}$$

and suppose that $B^N(x_j, R_j) \subset D$. Suppose on the contrary that $u(x) < \phi(j+1)$ for all $x \in B^N(x_j, R_j)$. Using theassumption (2.3) (or (2.4)) we see that

$$\begin{aligned}
\phi(j) &\leq u(x_j) \\
&\leq \frac{K}{\nu_N R_j^N} \int_{B^N(x_j,R_j)} u(x)\,dm_N(x) \\
&\leq \frac{K}{\nu_N R_j^N} \int_{B^N(x_j,R_j)\cap A_j} u(x)\,dm_N(x) + \frac{K}{\nu_N R_j^N} \int_{B^N(x_j,R_j)\setminus A_j} u(x)\,dm_N(x) \\
&< \left[\frac{K m_N(B^N(x_j,R_j)\cap A_j)}{\nu_N R_j^N} \frac{\phi(j+1)}{\phi(j)} + \frac{K m_N(B^N(x_j,R_j)\setminus A_j)}{\nu_N R_j^N} \frac{\phi(j-s_0)}{\phi(j)}\right] \phi(j) \\
&< \phi(j),
\end{aligned} \tag{3.7}$$

a contradiction. □

3.2. The second lemma

The next lemma is a slightly generalized version of Armitage's and Gardiner's result [3, Proposition 2, page 257]. The proof of our refinement is—as already pointed out—a rather straightforward modification of Armitage's and Gardiner's argument [3, proof of Proposition 2, pages 257–259].

Lemma 3.2. *Let $K \geq 1$. Let $\varphi : [0,+\infty) \to [0,+\infty)$ and $\psi : [0,+\infty) \to [0,+\infty)$ be increasing functions for which there exist $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, such that*

(i) *the inverse functions φ^{-1} and ψ^{-1} are defined on $[\inf\{\varphi(s_1 - s_0), \psi(s_1 - s_0)\}, +\infty)$,*

(ii) $2K(\psi^{-1} \circ \varphi)(s - s_0) \leq (\psi^{-1} \circ \varphi)(s)$ *for all* $s \geq s_1$,

(iii) $\sum_{j=s_1+1}^{+\infty}[((\psi^{-1} \circ \varphi)(j+1)/(\psi^{-1} \circ \varphi)(j))(1/\varphi(j - s_0))]^{1/(N-1)} < +\infty$.

Let $u : D \to [0,+\infty)$ be a K-quasi-nearly subharmonic function. Let $\widetilde{s}_1 \in \mathbb{N}$, $\widetilde{s}_1 \geq s_1$, be arbitrary. Then for each $x \in D$ and $r > 0$ such that $\overline{B^N(x,r)} \subset D$ either

$$u(x) \leq (\psi^{-1} \circ \varphi)(\widetilde{s}_1 + 1) \tag{3.8}$$

or

$$\Phi(u(x)) \leq \frac{C}{r^N}\int_{B^N(x,r)} \psi(u(y))\,dm_N(y), \tag{3.9}$$

where $C = C(N, K, s_0)$ and $\Phi : [s_2, +\infty) \to [0,+\infty)$,

$$\Phi(t) := \left(\sum_{j=j_0}^{+\infty}\left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)}\frac{1}{\varphi(j - s_0)}\right]^{1/(N-1)}\right)^{1-N}, \tag{3.10}$$

and $j_0 \in \{s_1 + 1, s_1 + 2, \ldots\}$ is such that

$$(\psi^{-1} \circ \varphi)(j_0) \leq t < (\psi^{-1} \circ \varphi)(j_0 + 1), \tag{3.11}$$

and $s_2 := (\psi^{-1} \circ \varphi)(s_1 + 1)$.

Proof. Take $x \in D$ and $r > 0$ arbitrarily such that $\overline{B^N(x,r)} \subset D$. We may suppose that $u(x) > (\psi^{-1} \circ \varphi)(\widetilde{s}_1 + 1)$. Since φ and ψ are increasing and $(\psi^{-1} \circ \varphi)(s) \to +\infty$ as $s \to +\infty$, there is an integer $j_0 \geq \widetilde{s}_1 + 1$ such that

$$(\psi^{-1} \circ \varphi)(j_0) \leq u(x) < (\psi^{-1} \circ \varphi)(j_0 + 1). \tag{3.12}$$

Write $x_{j_0} := x$, $D_0 := B^N(x_{j_0}, r)$ and for each $j \geq j_0$,

$$\begin{gathered} A_j := \{y \in D_0 : (\psi^{-1} \circ \varphi)(j - s_0) \leq u(y) < (\psi^{-1} \circ \varphi)(j+1)\}, \\ R_j := \left(\frac{2K}{\nu_N}\right)^{1/N}\left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)}m_N(A_j)\right]^{1/N}. \end{gathered} \tag{3.13}$$

If $B^N(x_{j_0}, R_{j_0}) \cap (\mathbb{R}^N \setminus D_0) \neq \varnothing$, then clearly

$$r < R_{j_0} \leq \sum_{k=j_0}^{+\infty} R_k. \tag{3.14}$$

On the other hand, if $B^N(x_{j_0}, R_{j_0}) \subset D_0$, then by Lemma 3.1 (where now

$$\phi(s) = \begin{cases} (\psi^{-1} \circ \varphi)(s), & \text{when } s \geq s_1 - s_0, \\ \dfrac{s}{s_1 - s_0}\phi(s_1 - s_0), & \text{when } 0 \leq s < s_1 - s_0, \end{cases} \tag{3.15}$$

say), there is $x_{j_0+1} \in B^N(x_{j_0}, R_{j_0})$ such that $u(x_{j_0+1}) \geq (\psi^{-1} \circ \varphi)(j_0 + 1)$.

Suppose that for $k = j_0, j_0 + 1, \ldots, j$,

$$\begin{gathered} B^N(x_k, R_k) \subset D_0, \qquad x_{k+1} \in B^N(x_k, R_k) \\ (\text{this for } k = j_0, j_0 + 1, \ldots, j - 1), \quad u(x_k) \geq (\psi^{-1} \circ \varphi)(k). \end{gathered} \tag{3.16}$$

By Lemma 3.1 there is then $x_{j+1} \in B^N(x_j, R_j)$ such that $u(x_{j+1}) \geq (\psi^{-1} \circ \varphi)(j + 1)$. Since u is locally bounded above and $(\psi^{-1} \circ \varphi)(k) \to +\infty$ as $k \to +\infty$, we may suppose that $B^N(x_{j+1}, R_{j+1}) \cap (\mathbb{R}^N \setminus D_0) \neq \varnothing$. But then,

$$r < \operatorname{dist}(x_{j_0}, x_{j_0+1}) + \operatorname{dist}(x_{j_0+1}, x_{j_0+2}) + \cdots + \operatorname{dist}(x_j, x_{j+1}) + \operatorname{dist}(x_{j+1}, \mathbb{R}^N \setminus D_0), \tag{3.17}$$

thus

$$r < R_{j_0} + R_{j_0+1} + \cdots + R_j + R_{j+1} \leq \sum_{k=j_0}^{+\infty} R_k. \tag{3.18}$$

Using, for $j = j_0 - s_0, j_0 + 1 - s_0, \ldots$, the notation

$$a_j := \{y \in D_0 : (\psi^{-1} \circ \varphi)(j) \leq u(y) < (\psi^{-1} \circ \varphi)(j + 1)\}, \tag{3.19}$$

we get from (3.18)

$$\begin{aligned} r &< \sum_{k=j_0}^{+\infty} \left(\frac{2K}{\nu_N}\right)^{1/N} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} m_N(A_k)\right]^{1/N} \\ &< \left(\frac{2K}{\nu_N}\right)^{1/N} \sum_{k=j_0}^{+\infty} \left(\left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/N} [\varphi(k - s_0) m_N(A_k)]^{1/N}\right) \\ &< \left(\frac{2K}{\nu_N}\right)^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/(N-1)}\right)^{(N-1)/N} \left[\sum_{k=j_0}^{+\infty} \varphi(k - s_0) m_N(A_k)\right]^{1/N} \end{aligned}$$

$$
\begin{aligned}
&< \left(\frac{2K}{\nu_N}\right)^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1}\circ\varphi)(k+1)}{(\psi^{-1}\circ\varphi)(k)} \frac{1}{\varphi(k-s_0)} \right]^{1/(N-1)} \right)^{(N-1)/N} \\
&\quad \times \left[\sum_{k=j_0}^{+\infty} \int_{A_k} \psi(u(y))\, dm_N(y) \right]^{1/N} \\
&< \left(\frac{2K}{\nu_N}\right)^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1}\circ\varphi)(k+1)}{(\psi^{-1}\circ\varphi)(k)} \frac{1}{\varphi(k-s_0)} \right]^{1/(N-1)} \right)^{(N-1)/N} \\
&\quad \times \left(\sum_{k=j_0}^{+\infty} \left[\sum_{j=k-s_0}^{k} \int_{a_j} \psi(u(y))\, dm_N(y) \right] \right)^{1/N} \\
&< \left[\frac{2(s_0+1)K}{\nu_N}\right]^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1}\circ\varphi)(k+1)}{(\psi^{-1}\circ\varphi)(k)} \frac{1}{\varphi(k-s_0)} \right]^{1/(N-1)} \right)^{(N-1)/N} \\
&\quad \times \left[\iint_{D_0} \psi(u(y))\, dm_N(y) \right]^{1/N}.
\end{aligned}
\tag{3.20}
$$

Thus,

$$
\Phi(u(x)) \le \frac{C}{r^N} \int_{D_0} \psi(u(y))\, dm_N(y), \tag{3.21}
$$

where $C = C(N, K, s_0)$ and $\Phi : [s_2, +\infty) \to [0, +\infty)$,

$$
\Phi(t) := \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1}\circ\varphi)(k+1)}{(\psi^{-1}\circ\varphi)(k)} \frac{1}{\varphi(k-s_0)} \right]^{1/(N-1)} \right)^{1-N}, \tag{3.22}
$$

where $j_0 \in \{s_1+1, s_1+2, \ldots\}$ is such that

$$
(\psi^{-1}\circ\varphi)(j_0) \le t < (\psi^{-1}\circ\varphi)(j_0+1), \tag{3.23}
$$

and $s_2 = (\psi^{-1}\circ\varphi)(s_1+1)$.

The function Φ may be extended to the whole interval $[0, +\infty)$, as follows:

$$
\Phi(t) := \begin{cases} \Phi(t), & \text{when } t \ge s_2, \\ \dfrac{t}{s_2}\Phi(s_2), & \text{when } 0 \le t < s_2. \end{cases} \tag{3.24}
$$

□

Remark 3.3. Write $s_3 := \sup\{s_1+3, (\psi^{-1}\circ\varphi)(s_1+3)\}$, say. (We may suppose that s_3 is an integer.) Suppose, that in addition to the assumptions (i), (ii), (iii) of Lemma 3.2, also the following assumption is satisfied:

(iv) *the function*

$$[s_1+1,+\infty) \ni s \longmapsto \frac{(\psi^{-1}\circ\varphi)(s+1)}{(\psi^{-1}\circ\varphi)(s)}\frac{1}{\varphi(s-s_0)} \in \mathbb{R} \tag{3.25}$$

is decreasing.

Then, one can replace the function $\Phi \mid [s_3,+\infty)$ by the function $\Phi_1 \mid [s_3,+\infty)$, where $\Phi_1 = \Phi_1^{\varphi,\psi} : [0,+\infty) \to [0,+\infty)$,

$$\Phi_1^{\varphi,\psi}(t) := \begin{cases} \left(\displaystyle\int_{(\psi^{-1}\circ\varphi)(t)-2}^{+\infty}\left[\frac{(\psi^{-1}\circ\varphi)(s+1)}{(\psi^{-1}\circ\varphi)(s)}\frac{1}{\varphi(s-s_0)}\right]^{1/(N-1)} ds\right)^{1-N}, & \text{when } t \geq s_3, \\ \dfrac{t}{s_3}\Phi_1^{\varphi,\psi}(s_3), & \text{when } 0 \leq t < s_3. \end{cases} \tag{3.26}$$

Similarly, if the function

$$[s_1+1,+\infty) \ni s \longmapsto \frac{(\psi^{-1}\circ\varphi)(s+1)}{(\psi^{-1}\circ\varphi)(s)} \in \mathbb{R} \tag{3.27}$$

is bounded, then in Lemma 3.2, one can replace the function $\Phi \mid [s_3,+\infty)$ by the function $\Phi_2 \mid [s_3,+\infty)$, where $\Phi_2 = \Phi_2^{\varphi,\psi} : [0,+\infty) \to [0,+\infty)$,

$$\Phi_2^{\varphi,\psi}(t) := \begin{cases} \left[\displaystyle\int_{(\psi^{-1}\circ\varphi)(t)-2}^{+\infty}\frac{ds}{\varphi(s-s_0)^{1/(N-1)}}\right]^{1-N}, & \text{when } t \geq s_3, \\ \dfrac{t}{s_3}\Phi_2^{\varphi,\psi}(s_3), & \text{when } 0 \leq t < s_3. \end{cases} \tag{3.28}$$

4. The condition

4.1. A counterpart to Armitage's and Gardiner's result

Next, we propose a counterpart to Armitage's and Gardiner's result [3, Theorem 1, page 256] for quasi-nearly subharmonic functions. The proof below follows Armitage's and Gardiner's argument [3, proof of Theorem 1, pages 258-259], but is, at least formally, more general. Compare also Corollary 4.5 below.

Theorem 4.1. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \to [-\infty,+\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied.*

(a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \longmapsto u(x,y) \in [-\infty,+\infty) \tag{4.1}$$

is K-quasi-nearly subharmonic.

(b) *For each* $x \in \mathbb{R}^m$ *the function*

$$\Omega(x) \ni y \longmapsto u(x, y) \in [-\infty, +\infty) \tag{4.2}$$

is K-quasi-nearly subharmonic.

(c) *There are increasing functions* $\varphi : [0, +\infty) \to [0, +\infty)$ *and* $\psi : [0, +\infty) \to [0, +\infty)$ *and* $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, *such that*

(c1) *the inverse functions* φ^{-1} *and* ψ^{-1} *are defined on* $[\inf\{\varphi(s_1 - s_0), \psi(s_1 - s_0)\}, +\infty)$,

(c2) $2K(\psi^{-1} \circ \varphi)(s - s_0) \le (\psi^{-1} \circ \varphi)(s)$ *for all* $s \ge s_1$,

(c3) *the function*

$$[s_1 + 1, +\infty) \ni s \longmapsto \frac{(\psi^{-1} \circ \varphi)(s + 1)}{(\psi^{-1} \circ \varphi)(s)} \in \mathbb{R} \tag{4.3}$$

is bounded,

(c4) $\int_{s_1}^{+\infty} (s^{(n-1)/(m-1)} / \varphi(s - s_0)^{1/(m-1)}) ds < +\infty$,

(c5) $\psi \circ u^+ \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then, u is quasi-nearly subharmonic in Ω.

Proof. Recall that $s_3 = \sup\{s_1 + 3, (\psi^{-1} \circ \varphi)(s_1 + 3)\}$ and write $s_4 := \sup\{s_3 + s_0, (\varphi^{-1} \circ \psi)(s_1 + 3)\}$, $s_5 := s_4 + s_0$, say. Clearly, $s_0 < s_1 < s_3 < s_4 < s_5$. (We may suppose that s_3, s_4, and s_5 are integers.) One may replace u by $\sup\{u^+, M\}$, where $M = \sup\{s_5 + 3, (\psi^{-1} \circ \varphi)(s_4 + 3), (\varphi^{-1} \circ \psi)(s_4 + 3)\}$, say. We continue to denote u_M by u.

Take $(x_0, y_0) \in \Omega$ and $r > 0$ arbitrarily such that $\overline{B^m(x_0, 2r) \times B^n(y_0, 2r)} \subset \Omega$. By [10, Proposition 3.1, page 57] (that is by a counterpart to [9, Theorem 1, page 69], say), it is sufficient to show that u is bounded above in $B^m(x_0, r) \times B^n(y_0, r)$.

Take $(\xi, \eta) \in B^m(x_0, r) \times B^n(y_0, r)$ arbitrarily. In order to apply Lemma 3.2 to the K-quasi-nearly subharmonic function $u(\cdot, \eta)$ in $B^m(\xi, r)$ check that the assumptions are satisfied. Since (i) and (ii) are satisfied, it remains to show that

$$\sum_{j=s_1+1}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(j + 1)}{(\psi^{-1} \circ \varphi)(j)} \frac{1}{\varphi(j - s_0)} \right]^{1/(m-1)} < +\infty. \tag{4.4}$$

Because of the assumption (c3), it is sufficient to show that

$$\sum_{j=s_1+1}^{+\infty} \frac{1}{\varphi(j - s_0)^{1/(m-1)}} < +\infty. \tag{4.5}$$

This is of course easy:

$$\sum_{j=s_1+1}^{+\infty} \frac{1}{\varphi(j - s_0)^{1/(m-1)}} \le \int_{s_1}^{+\infty} \frac{ds}{\varphi(s - s_0)^{1/(m-1)}} \le \int_{s_1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s - s_0)^{1/(m-1)}} ds < +\infty. \tag{4.6}$$

We know that $u(\xi,\eta) \geq s_4$. Therefore it follows from Lemma 3.2 and Remark 3.3 that

$$\Phi_2^{\varphi,\psi}(u(\xi,\eta)) = \left[\int_{(\varphi^{-1}\circ\psi)(u(\xi,\eta))-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}}\right]^{1-m} \leq \frac{C}{r^m}\int_{B^m(\xi,r)} \psi(u(x,\eta))\,dm_m(x), \tag{4.7}$$

where $\Phi_2^{\varphi,\psi}$ is defined above in (3.28).

Take then the integral mean values of both sides of (4.7) over $B^n(\eta,r)$:

$$\begin{aligned}\frac{C}{r^n}\int_{B^n(\eta,r)} \Phi_2^{\varphi,\psi}(u(\xi,y))\,dm_n(y) &\leq \frac{C}{r^n}\int_{B^n(\eta,r)}\left[\frac{C}{r^m}\int_{B^m(\xi,r)} \psi(u(x,y))\,dm_m(x)\right]dm_n(y)\\ &\leq \frac{C}{r^{m+n}}\int_{B^m(\xi,r)\times B^n(\eta,r)} \psi(u(x,y))\,dm_{m+n}(x,y)\\ &\leq \frac{C}{r^{m+n}}\int_{B^m(x_0,2r)\times B^n(y_0,2r)} \psi(u(x,y))\,dm_{m+n}(x,y).\end{aligned} \tag{4.8}$$

In order to apply Lemma 3.2 (and Remark 3.3) once more, define $\psi_1 : [0,+\infty) \to [0,+\infty)$, $\psi_1(t) := \Phi_2^{\varphi,\psi}(t)$, and $\varphi_1 : [0,+\infty) \to [0,+\infty)$,

$$\varphi_1(t) := \begin{cases} \dfrac{t}{s_3}\psi_1((\psi^{-1}\circ\varphi)(s_3)) = \dfrac{t}{s_3}\Phi_2^{\varphi,\psi}(\psi^{-1}(\varphi(s_3))), & \text{when } 0 \leq t < s_3,\\ \psi_1((\psi^{-1}\circ\varphi)(t)) = \Phi_2^{\varphi,\psi}(\psi^{-1}(\varphi(t))), & \text{when } t \geq s_3.\end{cases} \tag{4.9}$$

It is straightforward to see that both ψ_1 and φ_1 are strictly increasing and continuous. Observe also that for $t \geq s_4$, say,

$$\begin{aligned}\varphi_1(t) &= \Phi_2^{\varphi,\psi}((\psi^{-1}\circ\varphi)(t))\\ &= \left[\int_{(\varphi^{-1}\circ\psi)((\psi^{-1}\circ\varphi)(t))-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}}\right]^{1-m}\\ &= \left[\int_{t-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}}\right]^{1-m}.\end{aligned} \tag{4.10}$$

Check then that the assumptions of Lemma 3.2 (and Remark 3.3) are fullfilled for φ_1 and ψ_1. Write $\tilde{s}_0 := s_0$ and $\tilde{s}_1 := s_4$. The assumption (i) is clearly satisfied. We know that for all $s \geq s_3$,

$$\varphi_1(t) = \psi_1((\psi^{-1}\circ\varphi)(t)) \Longleftrightarrow (\psi_1^{-1}\circ\varphi_1)(t) = (\psi^{-1}\circ\varphi)(t). \tag{4.11}$$

Thus the assumption (ii) is surely satisfied for $s \geq \tilde{s}_1 = s_4$. It remains to show that

$$\sum_{j=s_4+1}^{+\infty}\left[\frac{(\psi_1^{-1}\circ\varphi_1)(j+1)}{(\psi_1^{-1}\circ\varphi_1)(j)}\,\frac{1}{\varphi_1(j-s_0)}\right]^{1/(n-1)} < +\infty, \tag{4.12}$$

say. It is surely sufficient to show that

$$\int_{s_5+s_0+2}^{+\infty} \frac{ds}{\varphi_1(s-s_0)^{1/(n-1)}} < +\infty. \tag{4.13}$$

Define $F : [s_5, +\infty) \times [s_5 + s_0 + 2, +\infty) \to [0, +\infty)$,

$$F(s,t) := \begin{cases} 0, & \text{when } s_5 \leq s < t - s_0 - 2, \\ \varphi(s-s_0)^{-1/(m-1)}, & \text{when } s_5 + s_0 + 2 \leq t - s_0 - 2 \leq s. \end{cases} \tag{4.14}$$

Suppose that $m > n$ and write $p = (m-1)/(n-1)$. Using Minkowski's inequality, see, for example, [28, page 14], one obtains, with the aid of (4.10),

$$\begin{aligned}
&\left[\int_{s_5+s_0+2}^{+\infty} \frac{dt}{\varphi_1(t-s_0)^{1/(n-1)}}\right]^{(n-1)/(m-1)} \\
&\quad = \left[\int_{s_5+s_0+2}^{+\infty} \left(\left[\int_{t-s_0-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}}\right]^{1-m}\right)^{-1/(n-1)} dt\right]^{(n-1)/(m-1)} \\
&\quad = \left(\int_{s_5+s_0+2}^{+\infty} \left[\int_{t-s_0-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}}\right]^{(m-1)/(n-1)} dt\right)^{(n-1)/(m-1)} \\
&\quad = \left(\int_{s_5+s_0+2}^{+\infty} \left[\int_{s_5}^{+\infty} F(s,t)ds\right]^{(m-1)/(n-1)} dt\right)^{(n-1)/(m-1)} \\
&\quad \leq \int_{s_5}^{+\infty} \left[\int_{s_5+s_0+2}^{+\infty} F(s,t)^{(m-1)/(n-1)} dt\right]^{(n-1)/(m-1)} ds \\
&\quad \leq \int_{s_5}^{+\infty} \left[\int_{s_5+s_0+2}^{s+s_0+2} \frac{dt}{\varphi(s-s_0)^{1/(n-1)}}\right]^{(n-1)/(m-1)} ds \\
&\quad \leq \int_{s_5}^{+\infty} \frac{[(s+s_0+2)-(s_5+s_0+2)]^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds \\
&\quad \leq \int_{s_5}^{+\infty} \frac{(s-s_5)^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds \\
&\quad \leq \int_{s_5}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty.
\end{aligned} \tag{4.15}$$

The case $m = n$ is considered similarly, just replacing Minkowski's inequality with Fubini's theorem.

Now, we can apply Lemma 3.2 (and Remark 3.3) to the left hand side of (4.8). Recall that $\tilde{s}_0 = s_0$, $\tilde{s}_1 = s_4$, $\tilde{s}_3 := \sup\{\tilde{s}_1 + 3, (\varphi_1^{-1} \circ \psi_1)(\tilde{s}_1 + 3)\}$, and $\tilde{s}_4 := \sup\{\tilde{s}_3 + \tilde{s}_0, (\varphi_1^{-1} \circ \psi_1)(\tilde{s}_1 + 3)\}$. (Here and below, in the previous definitions just replace the functions φ and ψ with the functions φ_1 and ψ_1, resp.) Write moreover $s_4^* := \sup\{\tilde{s}_4, (\psi^{-1} \circ \varphi)(s_4)\}$, say. Since $u(\xi,\eta) \geq M \geq s_4^* \geq \tilde{s}_4$ for all $(\xi,\eta) \in B^m(x_0,r) \times B^n(y_0,r)$, we obtain, using (4.8):

$$\begin{aligned}\Phi_2^{\varphi_1,\psi_1}(u(\xi,\eta)) &= \left[\int_{(\varphi_1^{-1}\circ\psi_1)(u(\xi,\eta))-2}^{+\infty} \frac{ds}{\varphi_1(s-s_0)^{1/(n-1)}}\right]^{1-n} \\ &\leq \frac{C}{r^n}\int_{B^n(\eta,r)} \Phi_2^{\varphi,\psi}(u(\xi,y))\,dm_n(y) \\ &\leq \frac{C}{r^{m+n}}\int_{B^m(x_0,2r)\times B^n(y_0,2r)} \psi(u(x,y))\,dm_{m+n}(x,y).\end{aligned} \tag{4.16}$$

From (4.16), from the facts that $(\varphi_1^{-1} \circ \psi_1)(t) = (\varphi^{-1} \circ \psi)(t) \to +\infty$ as $t \to +\infty$, from (4.13), and from the fact that

$$\int_{B^m(x_0,2r)\times B^n(y_0,2r)} \psi(u(x,y))\,dm_{m+n}(x,y) < +\infty, \tag{4.17}$$

one sees that u must be bounded above in $B^m(x_0,r) \times B^n(y_0,r)$, concluding the proof. □

Corollary 4.2. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \to [-\infty,+\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied.*

(a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \longmapsto u(x,y) \in [-\infty,+\infty) \tag{4.18}$$

is K-quasi-nearly subharmonic.

(b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \longmapsto u(x,y) \in [-\infty,+\infty) \tag{4.19}$$

is K-quasi-nearly subharmonic.

(c) *There is a strictly increasing surjection $\varphi : [0,+\infty) \to [0,+\infty)$ such that*

(c1) $\int_{s_0+1}^{+\infty}(s^{(n-1)/(m-1)}/\varphi(s-s_0)^{1/(m-1)})ds < +\infty$ *for some $s_0 \in \mathbb{N}$,*

(c2) $\varphi(\log^+ u^+) \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then, u is quasi-nearly subharmonic in Ω.

Proof. Just choose $\psi = \varphi \circ \log^+$ and apply Theorem 4.1. □

Remark 4.3. One sees easily that the condition (c1) (or (c4) above) can be replaced by the condition

(c1′) $\int_1^{+\infty}(s^{(n-1)/(m-1)}/\varphi(s)^{1/(m-1)})ds < +\infty$.

Corollary 4.4. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \to [-\infty, +\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied.*

(a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \longmapsto u(x, y) \in [-\infty, +\infty) \tag{4.20}$$

is K-quasi-nearly subharmonic.

(b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \longmapsto u(x, y) \in [-\infty, +\infty) \tag{4.21}$$

is K-quasi-nearly subharmonic.

(c) *There is a strictly increasing surjection $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

(c1) $\int_{s_0+1}^{+\infty} (s^{(n-1)/(m-1)}/\varphi(s - s_0)^{1/(m-1)})ds < +\infty$ *for some* $s_0 \in \mathbb{N}$,

(c2) $\varphi(\log(1 + (u^+)^r)) \in \mathcal{L}^1_{\text{loc}}(\Omega)$ *for some* $r > 0$.

Then, u is quasi-nearly subharmonic in Ω.

Proof. It is easy to see that the assumptions of Theorem 4.1 are satisfied. We leave the details to the reader. □

4.2. A refinement to Armitage's and Gardiner's result

Next is our slight improvement to Armitage's and Gardiner's original result.

Corollary 4.5. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$. Let $u : \Omega \to [-\infty, +\infty)$ be such that the following conditions are satisfied.*

(a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \longmapsto u(x, y) \in [-\infty, +\infty) \tag{4.22}$$

is subharmonic.

(b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \longmapsto u(x, y) \in [-\infty, +\infty) \tag{4.23}$$

is subharmonic.

(c) *There is a strictly increasing surjection $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

(c1) $\int_1^{+\infty} (s^{(n-1)/(m-1)}/\varphi(s)^{1/(m-1)})ds < +\infty$,

(c2) $\varphi(\log^+[(u^+)^r]) \in \mathcal{L}^1_{\text{loc}}(\Omega)$ *for some* $r > 0$.

Then, u is subharmonic in Ω.

Proof. By [10, Proposition 2.2(v), (vi), page 55], see also [12, Lemma 2.1, page 32] or [19, Theorem, page 188], $(u^+)^r$ satisfies the assumptions of Corollary 4.2, thus $(u^+)^r$ is quasi-nearly subharmonic in Ω, and therefore, for example, by [10, Proposition 2.2(iii), page 55] locally bounded above. Hence, also u is locally bounded above, and thus subharmonic in Ω, by [9, Theorem 1, page 69], say. □

References

[1] J. Wiegerinck, "Separately subharmonic functions need not be subharmonic," *Proceedings of the American Mathematical Society*, vol. 104, no. 3, pp. 770–771, 1988.

[2] J. Wiegerinck and R. Zeinstra, "Separately subharmonic functions: when are they subharmonic," in *Several Complex Variables and Complex Geometry, Part 1*, E. Bedford, J. P. D'Angelo, R. E. Greene, and S. G. Krantz, Eds., vol. 52 of *Proceedings of Symposia in Pure Mathematics*, pp. 245–249, American Mathematical Society, Providence, RI, USA, 1991.

[3] D. H. Armitage and S. J. Gardiner, "Conditions for separately subharmonic functions to be subharmonic," *Potential Analysis*, vol. 2, no. 3, pp. 255–261, 1993.

[4] P. Lelong, "Les fonctions plurisousharmoniques," *Annales Scientifiques de l'École Normale Supérieure*, vol. 62, pp. 301–338, 1945.

[5] V. Avanissian, "Fonctions plurisousharmoniques et fonctions doublement sousharmoniques," *Annales Scientifiques de l'École Normale Supérieure*, vol. 78, pp. 101–161, 1961.

[6] P. Lelong, *Plurisubharmonic Functions and Positive Differential Forms*, Gordon and Breach, London, UK, 1969.

[7] M. Hervé, *Analytic and Plurisubharmonic Functions in Finite and Infinite Dimensional Spaces*, vol. 198 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1971.

[8] M. G. Arsove, "On subharmonicity of doubly subharmonic functions," *Proceedings of the American Mathematical Society*, vol. 17, pp. 622–626, 1966.

[9] J. Riihentaus, "On a theorem of Avanissian-Arsove," *Expositiones Mathematicae*, vol. 7, no. 1, pp. 69–72, 1989.

[10] J. Riihentaus, "Subharmonic functions, generalizations and separately subharmonic functions," in *the 14th Conference on Analytic Functions*, Chełm, Poland, July 2007.

[11] Y. Domar, "On the existence of a largest subharmonic minorant of a given function," *Arkiv för Matematik*, vol. 3, pp. 429–440, 1957.

[12] J. Riihentaus, "A weighted boundary limit result for subharmonic functions," *Advances in Algebra and Analysis*, vol. 1, no. 1, pp. 27–38, 2006.

[13] C. Fefferman and E. M. Stein, "H^p spaces of several variables," *Acta Mathematica*, vol. 129, no. 3-4, pp. 137–193, 1972.

[14] Ü. Kuran, "Subharmonic behaviour of $|h|^p$, ($p > 0$, h harmonic)," *Journal of the London Mathematical Society*, vol. 8, pp. 529–538, 1974.

[15] M. Pavlović, "On subharmonic behaviour and oscillation of functions on balls in $\mathbb{R}^n$," *Publications de l'Institut Mathématique*, vol. 55(69), pp. 18–22, 1994.

[16] M. Pavlović and J. Riihentaus, "Classes of quasi-nearly subharmonic functions," *Potential Analysis*, vol. 29, no. 1, pp. 89–104, 2008.

[17] Y. Mizuta, *Potential Theory in Euclidean Spaces*, vol. 6 of *GAKUTO International Series. Mathematical Sciences and Applications*, Gakkōtosho, Tokyo, Japan, 1996.

[18] J. Riihentaus, "Subharmonic functions: non-tangential and tangential boundary behavior," in *Function Spaces, Differential Operators and Nonlinear Analysis*, V. Mustonen and J. Rákosnik, Eds., pp. 229–238, Academy of Sciences of the Czech Republic, Prague, Czech Republic, 2000.

[19] J. Riihentaus, "A generalized mean value inequality for subharmonic functions," *Expositiones Mathematicae*, vol. 19, no. 2, pp. 187–190, 2001.

[20] J. Riihentaus, "A generalized mean value inequality for subharmonic functions and applications," preprint, 2003, http://arxiv.org/abs/math.CA/0302261.

[21] J. Riihentaus, "Weighted boundary behavior and nonintegrability of subharmonic functions," in *Proceedings of the International Conference on Education and Information Systems: Technologies and Applications (EISTA '04)*, M. Chang, Y.-T. Hsia, F. Malpica, M. Suarez, A. Tremante, and F. Welsch, Eds., vol. 2, pp. 196–202, Orlando, Fla, USA, July 2004.

[22] J. Riihentaus, "Separately quasi-nearly subharmonic functions," in *Complex Analysis and Potential Theory*, T. A. Azeroğlu and P. M. Tamrazov, Eds., pp. 156–165, World Scientific, Hackensack, NJ, USA, 2007.

[23] V. Kojić, "Quasi-nearly subharmonic functions and conformal mappings," *Filomat*, vol. 21, no. 2, pp. 243–249, 2007.

[24] O. Djordjević and M. Pavlović, "L^p-integrability of the maximal function of a polyharmonic function," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 411–417, 2007.

[25] A. Toma, "Generating functions for the mean value of a function on a sphere and its associated ball in R^n," *Journal of Inequalities and Applications*, vol. 2008, Article ID 656329, 14 pages, 2008.

[26] M. Vuorinen, "On the Harnack constant and the boundary behavior of Harnack functions," *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica*, vol. 7, no. 2, pp. 259–277, 1982.

[27] Y. Domar, "Uniform boundedness in families related to subharmonic functions," *Journal of the London Mathematical Society*, vol. 38, no. 3, pp. 485–491, 1988.

[28] J. B. Garnett, *Bounded Analytic Functions*, vol. 236 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1st edition, 2007.