LIGHT CLASSES OF GENERALIZED STARS
IN POLYHEDRAL MAPS ON SURFACES

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Dedicated to Professor Hanjo Walther on the occasion of his 60th birthday

Abstract

A generalized $s$-star, $s \geq 1$, is a tree with a root $Z$ of degree $s$; all other vertices have degree $\leq 2$. $S_i$ denotes a generalized 3-star, all three maximal paths starting in $Z$ have exactly $i + 1$ vertices (including $Z$). Let $M$ be a surface of Euler characteristic $\chi(M) \leq 0$, and $m(M) := \lceil \frac{5 + \sqrt{24\chi(M)}}{2} \rceil$. We prove:

(1) Let $k \geq 1, d \geq m(M)$ be integers. Each polyhedral map $G$ on $M$ with a $k$-path (on $k$ vertices) contains a $k$-path of maximum degree $\leq d$ in $G$ or a generalized $s$-star $T, s \leq m(M)$, on $d + 2 - m(M)$ vertices with root $Z$, where $Z$ has degree $\leq k \cdot m(M)$ and the maximum degree of $T \setminus \{Z\}$ is $\leq d$ in $G$. Similar results are obtained for the plane and for large polyhedral maps on $M$. 
(2) Let $k$ and $i$ be integers with $k \geq 3, 1 \leq i \leq \frac{k}{2}$. If a polyhedral map $G$ on $\mathbb{M}$ with a large enough number of vertices contains a $k$-path then $G$ contains a $k$-path or a 3-star $S_i$ of maximum degree $\leq 4(k + i)$ in $G$. This bound is tight. Similar results hold for plane graphs.

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1. Introduction

In this paper all manifolds are compact 2-dimensional manifolds. We shall consider graphs without loops and multiple edges. Multigraphs can have multiple edges and loops. If a multigraph $G$ is embedded in a manifold $\mathbb{M}$ then the connected components of $\mathbb{M} - G$ are called the faces of $G$. If each face is an open disc then the embedding is called a 2-cell embedding. If each vertex of a 2-cell embedding has degree $\geq 3$ and each vertex of degree $h$ is incident with $h$ different faces then $G$ is called a map in $\mathbb{M}$. If, in addition, $G$ is 3-connected and the embedding has representativity at least three, then $G$ is called a polyhedral map in $\mathbb{M}$, see e.g. Roberton and Vitray [19] or Mohar [17]. Let us recall that the representativity $\text{rep}(G, \mathbb{M})$ (or the face width) of a (2-cell) embedded graph $G$ into a compact 2-manifold $\mathbb{M}$ is equal to the smallest number $k$ such that $\mathbb{M}$ contains a noncontractible closed curve that intersects the graph $G$ in $k$ points.

Let $S_g$ ($N_q$) be an orientable (a non-orientable) compact 2-dimensional manifold (called also a surface, see [18]) of genus $g$ ($q$, respectively). Let us recall that the relationship between Euler characteristic and the genus of a surface is the following

$$\chi(S_g) = 2 - 2g \quad \text{and} \quad \chi(N_q) = 2 - q.$$ 

We say that $H$ is a subgraph of a polyhedral map $G$ if $H$ is a subgraph of the underlying graph of the map $G$.

The boundary of a face $\alpha$ of an embedded graph consists of all vertices and edges incident with $\alpha$. Note that the boundary of $\alpha$ can be disconnected. Let $D_1, D_2, \cdots, D_s$ be the components of the boundary of $\alpha$. Let $W_i$ be the shortest closed walk induced by all edges of $D_i$, and let $\partial(W_i)$ be its length, i.e., the number of edges met at the walk $W_i$ (edges met twice are
counted twice). The degree of a face $\alpha$ is

$$\deg_G(\alpha) = \sum_{i=1}^{s} \partial(W_i).$$

Hence the *degree* $\deg_G(\alpha)$ of a face $\alpha$ of a 2-cell embedding is the length of its facial walk. Vertices and faces of degree $i$ are called $i$-vertices and $i$-faces, respectively. The number of $i$-vertices and $j$-faces in a map is denoted by $v_i$ and $f_j$, respectively. For a map $G$ let $V(G)$, $E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of $G$, respectively. The degree of a vertex $A$ in $G$ is denoted by $\deg_G(A)$ or $\deg(A)$ if $G$ is known from the context. A path and a cycle on $k$ distinct vertices is defined to be the $k$-path and the $k$-cycle, respectively. $P_k$ will denote a $k$-path. The *length* of a path or a cycle is the number of its edges.

A generalized $s$-star, $s \geq 1$, is a tree with a root $Z$ of degree $s$; all other vertices have degree 2. The maximal paths starting in $Z$ are called beams. The symbol $S_i$, $i \geq 0$, denotes a generalized 3-star, all three beams of it are paths with $i + 1$ vertices (including the root). Obviously, $S_0 = K_1$, and $S_1 = K_{1,3}$.

It is a consequence of Euler’s formula that each planar graph contains a vertex of degree at most 5. It is well known that any graph embedded in a surface $M$ with Euler characteristic $\chi(M)$ has minimum degree

(1) $\delta(G) \leq \left\lceil \frac{5 + \sqrt{49 - 24\chi(M)}}{2} \right\rceil =: m(M)$, if $M \neq S_0$, and

$$\delta(G) \leq 5 =: m(S_0),$$

where $S_0$ is the sphere.

(For a proof see e.g. Sachs [20], p. 227).

A further consequence of Euler’s formula is

$$\sum_{A \in V(G)} (\deg(A) - 6) + 2 \sum_{\alpha \in F(G)} (\deg(\alpha) - 3) = 6(-\chi(M)).$$

For any graph $G$ embedded in a surface $M$ of Euler characteristic $\chi(M) \leq 0$ this implies

(2) if $\sum_{\deg(A) > 6} (\deg(A) - 6) > 6|\chi(M)|$ then $\delta(G) \leq 5$, and

(3) if $G$ has more than $6|\chi(M)|$ vertices then $\delta(G) \leq 6$.

A theorem of Kotzig [15] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. This result
was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example Ivančo [6] has proved that every polyhedral map on $S_g$ contains an edge with degree sum of their end vertices being at most $2g+13$ if $0 \leq g \leq 3$ and at most $4g+7$, if $g \geq 4$. The bounds are best possible. For other results in this topic see e.g. [1, 4, 14, 21].

2. The General Problem

In the past subgraphs have been investigated which are light in a family of graphs (see our survey article [14]). There we have generalized this concept to a light class $L$ of subgraphs in a family $H$ of graphs.

**Problem.** Let $H$ be a family of graphs and $L$ be a finite class of connected graphs having the property that every member of $L$ is isomorphic to a proper subgraph of at least one member of $H$. Let $\varphi(L, H)$ be the smallest integer with the property that every graph $G \in H$, which has a subgraph isomorphic with a member of $L$, also contains a subgraph $K, K \simeq H, H \in L$, such that for every vertex $A \in V(K)$

$$\deg_G(A) \leq \varphi(L, H).$$

If such a $\varphi(L, H)$ does not exist we write $\varphi(L, H) = +\infty$. If $\varphi(L, H) < +\infty$ we call the class $L$ light in the family $H$. Obviously, if $L' \subseteq L$ then $\varphi(L, H) \leq \varphi(L', H)$. The corresponding problem of a light subgraph $H$ is again obtained if $L = \{H\}$ is chosen. In this case let $\varphi(\{H\}, H) = \varphi(H, H)$.

3. Results

A. Polyhedral maps

Let $G(\delta, \rho; M)$ denote the set of all polyhedral maps on the surface $M$ of Euler characteristic $\chi(M)$ having minimum vertex degree at least $\delta$ and minimum face degree at least $\rho$. The following theorem has been proved for the planes $S_0$ and $N_1$ by Fabrici and Jendrol’ [1] and for 2-dimensional manifolds $M$ other than the planes by Jendrol’ and Voss [8].

**Theorem 1** ([1], [8]). Let $k$ be an integer, $k \geq 1$, and $M$ a surface with Euler characteristic $\chi(M)$. Then
(i) \( \varphi(P_k, G(3, 3; S_0)) = \varphi(P_k, G(3, 3; N_1)) = 5k \),
(ii) \( 2 \left( \frac{k}{2} \right) \cdot m(M) \leq \varphi(P_k, G(3, 3; M)) \leq k \cdot m(M), \) if \( M \notin \{S_0, N_1\} \),
(iii) \( \varphi(H, G(3, 3; M)) = \infty \) for any connected graph \( H \neq P_k \).

By the same arguments used in the proof of (iii) for the sphere \( S_0 \) and the projective plane \( N_1 \) by Fabrici and Jendrol’ [1] it can be proved that a class \( L \) of plane graphs is light in \( G(3, 3; M) \) if and only if \( L \) contains a path. So, if \( L \) contains \( P_k \) then \( \varphi(L, G(3, 3; M)) = \varphi(P_k, G(3, 3; M)) \). We will study how small can \( \varphi(L, G(3, 3; M)) \) be if besides \( P_k \) the class \( L \) contains some trees different from \( P_k \).

The class \( T_k \) of all trees of order \( k \) contains a \( k \)-path. Obviously, \( \varphi(T_k, G(3, 3; M)) = \varphi(P_k, G(3, 3; M)) \) for \( k \in \{1, 2, 3\} \). For the sphere \( S_0 \) Fabrici and Jendrol’ [2] and for each surface \( M, M \neq S_0 \), Jendrol’ and Voss [13] proved

Theorem 2 ([2], [13]). Let \( k \) be an integer, \( k \geq 4 \), and \( M \) a surface with Euler characteristic \( \chi(M) \). Then

\[
\begin{align*}
(i) \quad & \varphi(T_k, G(3, 3; S_0)) = \varphi(T_k, G(3, 3; N_1)) = 4k + 3, \\
(ii) \quad & \frac{2k+2}{3} \left( \frac{5 + \sqrt{49 - 24\chi(M)}}{2} - \frac{3}{2} \right) \leq \varphi(T_k, G(3, 3; M)) \leq \frac{\left( k + 1 \right) \left( 5 + \sqrt{49 - 24\chi(M)} \right)}{4} \quad \text{if} \quad M \notin \{S_0, N_1\}.
\end{align*}
\]

In the Theorem 1(i) not all vertices of a \( P_k \) must have the degree \( 5k \). Really, Madaras [16] improved Theorem 1(i) by showing

Theorem 3 ([16]). Let \( k \) be an integer, \( k \geq 2 \). Then each map of \( G(3, 3; S_0) \) containing a path \( P_k \) has also a path \( P_k \) such that one vertex has a degree \( \leq 5k \) and all other \( k - 1 \) vertices have a degree \( \leq \frac{5k}{2} \).

Let \( M \) be a surface of Euler characteristic \( \chi(M) \) and \( m(M) \) as defined in (1). Using the arguments of Madaras [16] we can show that \( G \) contains at least one tree from a family of specified trees with given degree constraints.

Theorem 4. Let \( M \) be a surface of Euler characteristic \( \chi(M) \). Let \( k \geq 1 \) and \( d \geq m(M) \) be integers. Let \( G \in G(3, 3; M) \) contain a \( k \)-path. Then \( G \) contains at least one of the following subgraphs:

(i) a \( k \)-path of maximum degree \( \leq d \) in \( G \), or
(ii) a generalized s-star \( T, s \leq m(\mathbb{M}) \), on \( d + 2 - m(\mathbb{M}) \) vertices with root \( Z \), where \( Z \) has a degree \( \leq k \cdot m(\mathbb{M}) \) in \( G \) and the maximum degree of \( T \setminus \{Z\} \) is \( \leq d \) in \( G \).

The generalized star \( T \) contains a path with \( \frac{2d+1-m(\mathbb{M})}{m(\mathbb{M})} + 1 \) vertices.

If \( d = \lceil \frac{1}{2} m(\mathbb{M}) \rceil \) then the generalized star \( T \) contains a \( P_k \).

Hence Theorem 4 implies the validity of the following result.

**Theorem 5.** Let \( \mathbb{M} \) be a surface of Euler characteristic \( \chi(\mathbb{M}) \) and \( k \geq 1 \) an integer. Then each map \( G \in \mathcal{G}(3, 3; \mathbb{M}) \) containing a \( k \)-path, has a \( k \)-path \( P_k \) with the property: besides one vertex \( Z \) all vertices have a degree \( \leq \frac{1}{2} \cdot m(\mathbb{M}) \) and the vertex \( Z \) has a degree \( \leq k \cdot m(\mathbb{M}) \) in \( G \).

For the sphere \( m(S_0) = 5 \) holds and Theorem 5 implies the validity of Theorem 3. If \( d = k \cdot m(\mathbb{M}) \) then the generalized star \( T \) contains a \( P_k \) not meeting the root of \( T \). Hence Theorem 4 implies the validity of the upper bound in Theorem 1. Interesting special variants of Theorem 4 are also obtained for \( d = k \) and \( d = k + 4 \).

### B. Large polyhedral maps

Let \( \chi(\mathbb{M}) \leq 0 \) throughout section \( B \).

For large maps of \( \mathcal{G}(3, 3; \mathbb{M}) \) we await a smaller bound for the maximum degree of light paths. A large polyhedral map is one with a large number of vertices or a large positive charge. A positive \( k \)-charge \( \text{ch}_k(G) \) is defined \( \text{ch}_k(G) := \sum_{\deg_G(A) \geq 6k} (\deg_G(A) - 6k) \). Let \( \mathcal{G}(3, 3; \mathbb{M}, n(a)) \) and \( \mathcal{G}(3, 3; \mathbb{M}, c_k(b)) \) denote the sets of the graphs \( G \) of \( \mathcal{G}(3, 3; \mathbb{M}) \) with \( > a \) vertices or a \( k \)-charge \( \text{ch}_k(G) > b \), respectively. Let \( b_k \) denote the largest number of vertices in a connected graph with maximum degree \( \leq 6k \) containing no path of \( k \) vertices. Obviously, \( b_k \leq (6k)^{k/2+2} \).

Let \( l_k(\mathbb{M}) := 3 \cdot 10^4 (|\chi(\mathbb{M})| + 1)^3 (b_k + 3(|\chi(\mathbb{M})| + 1)) \). We have proved

**Theorem 6** ([9], [10]). For any surface \( \mathbb{M} \) with Euler characteristic \( \chi(\mathbb{M}) \leq 0 \), any integer \( k \geq 1 \), any integer \( a > l_k(\mathbb{M}) \) and any integer \( b > 6k|\chi(\mathbb{M})| \),

(i) \( \varphi(P_k, \mathcal{G}(3, 3; \mathbb{M}, n(a))) = \begin{cases} 6k, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 6k - 2, & \text{if } k \geq 3 \text{ is odd,} \end{cases} \)

(ii) \( \varphi(P_k, \mathcal{G}(3, 3; \mathbb{M}, c_k(b))) = 5k, \)
(iii) $\varphi(H, G(3, 3; M, n(a))) = \varphi(H, G(3, 3; M, c_k(a))) = \infty$ for any $H \not\cong P_k$ and any $a$.

In [9] we could show that $\varphi(P_k, G(3, 3; M, n(a))) \leq 6k$ even for the smaller bound $a > (14(k - 1)b_k + 6)|\chi(M)|$. For the class $T_k$ of all trees of order $k$ we could prove [11]

**Theorem 7** ([11]). For any surface $M$ with Euler characteristic $\chi(M) \leq 0$, any integer $k \geq 3$, and any integer $a > (8k^2 + 6k - 6)|\chi(M)|$,

(i) $\varphi(T_k, G(3, 3; M, n(a))) = 4k + 4$, and 
(ii) $\varphi(T_k, G(3, 3; M, c_k(a))) = 4k + 3$.

With the arguments of Madaras [16] we will prove: for the graphs of $G(3, 3; M, n(a))$ and $G(3, 3; M, c_k(b))$ with large $a$ and $b$ the Theorem 4 is again valid, if $m(M)$ is replaced by 6 or 5, respectively.

**Theorem 8.** Let $M$ be a surface of Euler characteristic $\chi(M) \leq 0$. Let $k, d$ and $a, b$ be integers with $k \geq 1, d \geq 6, a > (14(k - 1)b_k + 6)|\chi(M)|$, and $b > 6k|\chi(M)|$. Let $G_1 \in G(3, 3; M, n(a))$ and $G_2 \in G(3, 3; M, c_k(b))$ contain a $k$-path. Let $m_1 := 6$ and $m_2 := 5$.

Then for $i = 1, 2$ the map $G_i$ contains at least one of the following subgraphs:

(i) a $k$-path of maximum degree $\leq d$ in $G_i$, or 
(ii) a generalized $s$-star $T, s \leq m_i$ on $d + 2 - m_i$ vertices with root $Z$, where $Z$ has a degree $\leq k \cdot m_i$ in $G_i$ and the maximum degree of $T \setminus \{Z\}$ is $\leq d$ in $G_i$.

Finally we deal with light classes $H \not\cong T_k, k \geq 1$.

Since by Theorem 7 each polyhedral map $G$ on $M$ of large order contains a tree of order $k$ such that each vertex has a degree at most $4k + 4$, if $k \geq 3$, the map $G$ also contains a $P_k$ or a $K_{1,3}$ with the same bound. Examples in [11] show that the bound is best possible.

**Theorem 9.** For any surface $M$ of Euler characteristic $\chi(M) \leq 0$ and any integer $k \geq 3$ let $a > (8k^2 + 6k - 6)|\chi(M)|$. Then

(i) $\varphi\{P_k, K_{1,3}\}, G(3, 3; M, n(a))) = 4k + 4$, and 
(ii) $\varphi\{P_k, K_{1,3}\}, G(3, 3; M, c_k(a))) = 4k + 3$. 


Next the class \( \{P_k, S_i\}, i \geq 2 \), will be considered. If \( \chi(M) \leq 0 \), \( a > 6k(2b_k + 1)|\chi(M)| \) and \( i \geq \frac{k}{2} \) then \( P_k \subseteq S_i \) and \( \varphi(\{P_k, S_i\}, G(3, 3; M, n(a))) = \varphi(\{P_k\}, G(3, 3; M, n(a))) \). For different \( i \) we will prove the following theorem.

**Theorem 10.** Let \( M \) be a surface with Euler characteristic \( \chi(M) \), and let \( k \geq 3, i \geq 1 \) be integers.

(i) If \( M \) is the sphere \( S_0 \) or the projective plane \( \mathbb{N}_1 \), then

\[
\varphi(\{P_k, S_i\}, G(3, 3; S_0)) = \varphi(\{P_k, S_i\}, G(3, 3; N_1)) \leq 4(k + i) - 1.
\]

(ii) If \( \chi(M) \leq 0 \), then for each integer \( b > 4(k + i)|\chi(M)| \) it holds

\[
\varphi(\{P_k, S_i\}, G(3, 3; M, c_k(b))) \leq 4(k + i) - 1.
\]

(iii) If \( \chi(M) \leq 0 \), then for each integer \( a > (6k + 1)(2b_k + 1)|\chi(M)| \) it holds

\[
\varphi(\{P_k, S_i\}, G(3, 3; M, n(a))) \leq 4(k + i).
\]

Taking into the consideration the Theorems 1 and 6 we obtain tight bounds in some subclasses of \( G(3, 3; M) \).

**Theorem 11.** Let \( k \) and \( i \) be integers with \( k \geq 3 \) and \( i \geq 1 \). If \( M \) is the sphere \( S_0 \) or the projective plane \( \mathbb{N}_1 \), then

\[
\varphi(\{P_k, S_i\}, G(3, 3; S_0)) = \varphi(\{P_k, S_i\}, G(3, 3; N_1)) = \min\{4(k + i) - 1; 5k\}.
\]

**Theorem 12.** For any surface \( M \) with Euler characteristic \( \chi(M) \leq 0 \), any integers \( k \geq 3, i \geq 1 \), and \( b > 6k|\chi(M)| \) it holds:

\[
\varphi(\{P_k, S_i\}, G(3, 3; M, c_k(b))) = \min\{4(k + i) - 1; 5k\}.
\]

**Theorem 13.** For any surface \( M \) with Euler characteristic \( \chi(M) \leq 0 \), any integers \( k \geq 3, i \geq 1 \), and \( a > l_k(M) \) it holds:

\[
\varphi(\{P_k, S_i\}, G(3, 3; M, n(a))) = \begin{cases} 
\min\{4(k + i); 6k\} & \text{for even } k, \\
\min\{4(k + i); 6k - 2\} & \text{for odd } k.
\end{cases}
\]
4. Proof of Theorems 4 and 8

Assume there is a counterexample $G$ to Theorem 4 or 8 having $v = |V(G)|$ vertices, where in Theorem 8 the number of vertices $v > (14(k - 1)b_k + 6)|\chi(M)|$ or the positive $k$-charge
\[
ch_k(G) = \sum_{\deg_G(A) > 6k} (\deg_G(A) - 6k) > 6k|\chi(M)|.
\]

Let $G$ be a counterexample with the maximum number of edges among all counterexamples having $v$ vertices. A vertex $A$ of the graph $G$ is major (minor) if its degree is $\geq d + 1 (\leq d)$, respectively. The assertions (1) – (3) can be found in the introduction.

(4) Each path $P_k$ of $k$ vertices contains a major vertex.

Hence $G$ contains at least one major vertex.

(5) Each $r$-face $\alpha, r \geq 4$, contains at most two major vertices; if $\alpha$ has precisely two major vertices then they are adjacent.

**Proof of (5).**

Suppose $G$ has an $r$-face with two nonadjacent vertices of degree $\geq d + 1$. Since $G$ is a polyhedral map we can join these two vertices by an edge. The resulting embedding is again a counterexample but with one edge more, a contradiction.

Let $H$ denote the subgraph of $G$ induced by the major vertices, and let $v(H)$ be the number of vertices of $H$.

(6) The subgraph $H$ contains a vertex $Z$ of degree $s := \deg_H(Z)$ with

- (i) $s \leq m(M)$ if $G \in \mathcal{G}(3, 3; M)$, or
- (ii) $s \leq 6$, if $G \in \mathcal{G}(3, 3; M, n(a)), \chi(M) \leq 0$, or
- (iii) $s \leq 5$, if $G \in \mathcal{G}(3, 3; M, c_k(b)), \chi(M) \leq 0$.

**Proof of (6).**

- (i) This assertion follows from (1)(see the introduction).
- (ii) Suppose there is a $G \in \mathcal{G}(3, 3; M, n(a))$ with the subgraph $H$ of major vertices of $G$ with minimum degree $\delta(H) > 6$. In Lemma 5 of [9] we have proved that $v(H) > 6|\chi(M)|$. By (3) the subgraph $H$ has $v(H) \leq 6|\chi(M)|$ vertices. This contradiction completes the proof of (ii).
(iii) Suppose there is a $G \in \mathcal{G}(3,3;M,c_k(b))$ with the subgraph $H$ of major vertices of $G$ with minimum degree $\delta(H) > 5$. By (2) we have $\sum (\deg_H(A) - 6) \leq 6|\chi(M)|$ where the sum is taken over all vertices $A$ of $H$ with $\deg_H(A) > 6$.

Since $G$ is a polyhedral map the union of all faces incident with $Z$ forms a wheel with nave $Z$ and cycle $C_Z$; it may be that some vertices of the cycle $C_Z$ are not joined with $Z$ by an edge. By (5) each vertex of $C_Z$ not adjacent with $Z$ is a minor vertex. Hence all major vertices of $C_Z$ are neighbours of $Z$. These neighbours partition $C_Z$ into $\deg_H(Z)$ paths which have at most $k - 1$ minor vertices according (4). Therefore, $\deg_G(Z) \leq k \deg_H(Z)$. This together with $ch_k(G) > 6k|\chi(M)|$, $\delta(H) \geq 6$ and $\sum_{\deg_H(A) \geq 6} (\deg_H(A) - 6) \leq 6|\chi(M)|$ implies:

$$6k|\chi(M)| < ch_k(G) = \sum_{\deg_G(A) > 6k} (\deg_G(A) - 6k) \leq \sum_{\deg_G(A) > 6k} (k \deg_H(A) - 6k) \leq k \sum_{A \in V(H)} (\deg_H(A) - 6) \leq 6k|\chi(M)|.$$  

This contradiction completes the proof of assertion (iii).

The $s$ neighbours $Y_1,Y_2,\ldots,Y_s$ of $Z$ in $H$ are the only major vertices on the cycle $C_Z$. An upper bound for $s$ is known by (6). If $C_Z$ has no major vertices then let $s := 1$ and $Y_1$ be an arbitrary neighbour of $Z$ on $C_Z$. The cycle $C_Z$ contains altogether $\deg_G(Z) \geq d + 1$ neighbours of $Z$. Next $C_Z \setminus \{Y_1,\ldots,Y_s\}$ consists of $s$ paths $p_1,p_2,\ldots,p_s$ of minor vertices. These paths and $Z$ induce a subgraph which contains a generalized star $T$ with root $Z$ of degree $\deg_T(Z) \leq s$ and containing all $\geq \deg_G(Z) - s \geq d + 1 - s$ minor neighbours of $Z$. By (4) each path $p_i$ has at most $k - 1$ vertices. Hence the cycle $C_Z$ has at most $s \cdot k$ vertices and $\deg_G(Z) \leq s \cdot k$. Consequently, $G$ contains a generalized star $T$ of order $d + 2 - s$ with root $Z$ of degrees $\deg_T(Z) \leq s$ and $\deg_G(Z) \leq s \cdot k$, all other vertices of $T$ have a degree $\leq d$ in $G$. This contradicts our assumption that $G$ is a counterexample to Theorem 4 or 8. Thus the proof of the Theorems 4 and 8 is complete.

5. Proof of Theorems 10–13 — Upper Bounds

Theorem 1, 6, and 10 imply the validity of the upper bounds in Theorems 11–13. Hence it suffices to prove Theorem 10.
If \( i \geq \frac{k}{2} \) then \( P_k \subseteq S_i \) and \( \varphi(\{P_k, S_i\}, \mathcal{L}) = \varphi(\{P_k\}, \mathcal{L}) \) for each class \( \mathcal{L} \) of graphs. This bound is \( \leq 6k \) in each class \( \mathcal{L} \) we have considered. Hence \( \varphi(\{P_k, S_i\}, \mathcal{L}) \leq 6k \leq 4(k + i) \), and it suffices to accomplish the proof for all \( i \leq \frac{k-1}{2} \leq \frac{k}{2} \).

The proof follows the ideas of [1] and [13]. Suppose that there is a counterexample to one version of our theorem having \( v \) vertices. Let \( G \) be a counterexample with the maximum number of edges among all counterexamples having \( v \) vertices. Obviously, \( G \) contains a \( P_k \) or an \( S_i \).

(A) If \( G \) is a counterexample to Theorem 10(i) then \( M \) is the sphere \( S_0 \) or is the projective plane \( N_1 \) and each \( P_k \) and each \( S_i \) of \( G \) contains a vertex of degree \( 4(2(k + 1)) \).

(B) If \( G \) is a counterexample to Theorem 10(ii) then \( \chi(M) \leq 0 \), the map \( G \) has a positive \( k \)-charge \( ch_k(G) > 4(k + i) |\chi(M)| \), and each \( P_k \) and each \( S_i \) of \( G \) contains a vertex of degree \( \geq 4(k + i) \).

(C) If \( G \) is a counterexample to Theorem 10(iii) then \( \chi(M) \leq 0 \), the map \( G \) has an order \( v(G) > (6k + 1)(2b_k + 1) |\chi(M)| \) and each \( P_k \) and each \( S_i \) of \( G \) contains a vertex of degree \( \geq 4(k + i) + 1 \).

In the cases (A) and (B) a vertex \( A \) is a major vertex if \( \text{deg}_G(A) \leq 4(k + i) - 1 \) and is a minor vertex if \( \text{deg}_G(A) \geq 4(k + i) \). In case (C) a vertex \( A \) is a minor vertex if \( \text{deg}_G(A) \leq 4(k + i) \) and is a major vertex if \( \text{deg}_G(A) \geq 4(k + i) + 1 \). Since \( G \) is a counterexample it holds.

(1) Each \( k \)-path and each generalized star \( S_i \) in \( G \) contains a major vertex.

(2) Every \( r \)-face \( \alpha, r \geq 4 \), of \( G \) is incident only with minor vertices.

**Proof of (2).** Suppose there is a major vertex \( B \) incident with an \( r \)-face \( \alpha, r \geq 4 \). Let \( C \) be a diagonal vertex on \( \alpha \) with respect to \( B \) i.e., \( BC \) is no edge of the boundary of \( \alpha \). Because \( G \) is a polyhedral map we can insert the edge \( BC \) into the \( r \)-face \( \alpha \) The resulting embedding is again a counterexample but with one edge more, a contradiction.

Let \( H = H(G) \) and \( H' = H'(G) \) be the subgraphs of \( G \) induced on all major or minor vertices of \( G \), respectively.

(3) \( H \) is not empty.

**Proof of (3).** Since \( G \) is a counterexample it contains a \( k \)-path \( P_k \) or a 3-star \( S_i \). By (1) \( P_k \) or \( S_i \) contains a major vertex.

(2) directly implies:
(4) All faces incident with a major vertex $X$ induce a wheel with nave $X$ and a cycle $C$ of length $\geq 4(k + i)$ consisting of all neighbours of $X$. The cycle $C$ contains at least 5 major vertices.

**Proof of (4).** The first assertion is clear. If $C$ would contain at most 4 major vertices then $C$ would also contain at most 4 paths of $\leq k - 1$ minor vertices and $C$ would have a length $\leq 4 + 4(k - 1) = 4k < 4(k + i)$. This contradiction proves (4).

(5) By (4) the minimum degree of $H$ is at least 5.

Note that a triangle is always a 3-face. For the following we need Lemma 1.

**Lemma 1.** The three vertices of each triangle $D$ of $H$ are joint with the minor vertices inside $D$ by at most $2(k - 1 + i) - 1$ edges.

**Proof.** Let $D = [PQR]$ be a triangle of $H$. Let $K$ denote the subgraph of $G$ induced by the minor vertices of $G$ lying in the interior of $[PQR]$. By (2) all faces incident with $P$ induce a wheel $W_P$ with nave $P$ and a cycle containing all neighbours of $P$. Correspondingly, $Q$ and $R$ are the naves of a wheel $W_Q$ and $W_R$, respectively. Let $p, q$, and $r$ denote the path of $W_P \cap K$, $W_Q \cap K$, and $W_R \cap K$, respectively. Then $p, q$ and $q, r$ and $r, p$ have a common endvertex $Q', R', R''$, and $P'$, respectively (a sketch of the situation is depicted in Figure 1).
Case 1. Let \( p \) and \( q \) have precisely one common vertex, namely, \( Q' \). Then \( p \cup q \) is a path of \( K \) having \( \leq k - 1 \) vertices. Hence the paths \( p \cup q \) and \( r \) having at most \( k - 1 \) vertices each, and \( P, Q, R \) are joined by \( \leq 2k - 1 < 2(k - 1 + i) \) edges with \( K \).

Case 2. Let \( p, q \) and \( r \) have a second common vertex. Let \( Q'' \) and \( R'' \) denote the last common vertex of \( p, q \), and \( q, r \), and \( r, p \) by walking along \( p \) or \( q \) or \( r \) with start in \( Q' \), or \( R' \), or \( P' \), respectively. Since \( K \) does not contain a generalized star \( S_i \), w.l.o.g. the paths \( R''qR' \) and \( R'rR'' \) have at most \( i \) vertices. The paths \( P'rR'' \) and \( R''qQ' \) have precisely one common vertex, namely, \( R'' \). Hence \( P'rR''qQ' \) is a path of \( K \) with \( k - 1 \) vertices. The path \( p \) has also \( \leq k - 1 \) vertices, and \( R'rR'' - \{ R'' \} \) and \( R''qR' - \{ R'' \} \) have at most \( i - 1 \) vertices each. Therefore, \( P, Q, R \) are joined with \( K \) by \( \leq (k - 1) + (k - 1) + 1 + (i - 1) + (i - 1) = 2(k - 1 + i) - 1 \) edges with \( K \).

Let \( \alpha \) be a face of \( H \). Let \( D_1, D_2, \ldots, D_s \) be the components of the boundary of \( \alpha \). Let \( W_i \) be the shortest closed walk induced by all edges of \( D_i \), and let \( \partial(W_i) \) be its length. Then the degree of the face \( \alpha \) is \( \deg_H(\alpha) = \sum_{i=1}^{s} \partial(W_i) \). Since \( G \) is a polyhedral map any three consecutive vertices on the boundary of \( \alpha \) (i.e., in the walk \( W_i \) for some \( i \)) are pairwise different. Hence \( \partial(W_i) \geq 3 \), and

\[
(6) \quad \deg_H(\alpha) \geq 3s \geq 3.
\]

Let \( X, Y, Z \) be three consecutive vertices on the boundary of \( \alpha \). We call \( XYZ \) a corner of \( \alpha \) at the vertex \( Y \). Assertion (4) implies: In \( G \) the vertices \( X \) and \( Z \) are joined by a path \( q \) completely lying in \( \alpha \) and containing all minor neighbours of \( Y \) in this corner (\( Y \) can have some other minor neighbours at some other corners of \( \alpha \) at the vertex \( Y \), because \( Y \) can appear on the boundary of \( \alpha \) more than once).

The path \( q \setminus \{ X, Z \} \) consists of all minor neighbours of \( Y \) in this corner.

\[
(7) \quad \text{In each corner } XYZ \text{ of } \alpha \text{ at } Y \text{ the vertex } Y \text{ has at most } k - 1 \text{ minor neighbours. They form a path of } H'(G).
\]

It is obvious that

\[
(8) \quad \text{each face } \alpha \text{ of } H \text{ has precisely } \deg_H(\alpha) \text{ corners.}
\]

Let \( w(\alpha) \) denote the number of edges joining the minor vertices inside \( \alpha \) with all major vertices of \( H \) (i.e., the major vertices on the boundary of \( \alpha \)).
With (8) it follows:

(9) The minor vertices inside $\alpha$ are joined with all major vertices by $w(\alpha) \leq (k - 1) \deg_H(\alpha)$ edges.

Thus the number $w$ of all edges of $G$ joining minor vertices with major vertices is

\begin{equation}
(10) w = \sum_{\alpha \in F(H)} w(\alpha) \leq \sum_{\alpha \in F(H)} (k - 1) \deg_H(\alpha).
\end{equation}

By Lemma 1 we have a better bound if $\alpha$ is a 2-cell 3-face (triangle).

(11) If $\alpha$ is a triangle of $H$ then $w(\alpha) \leq 2(k - 1 + i) - 1$.

We proceed in three steps. First, we assign to each face $\alpha$ of $H$ the charge $w(\alpha)$. Next, we triangulate each face $\alpha$ of $H$ by introducing diagonals into the face $\alpha$ (a diagonal is an edge joining two vertices of the boundary of $\alpha$ such that no 1-face or 2-face is generated). By this method $\alpha$ is splitted into at least $t - 2$ triangles, $t = \deg_H(\alpha)$. The obtained semitriangulation $H^*$ can have loops or multiple edges (A triangulation (semitriangulation) is an embedding of a graph (multigraph) such that each face is a triangle). In the third step the charge $w(\alpha)$ is equally distributed to the triangles inside $\alpha$. The charge of a triangle $D$ of $H^*$ is denoted by $w^*(D)$. Distributing the old charges no charge has been lost. Hence,

\begin{equation}
(12) w = \sum_{\alpha \in F(H)} w(\alpha) = \sum_{D \in F(H^*)} w^*(D).
\end{equation}

(13) Each triangle $D$ of $H^*$ has a charge $w^*(D) \leq 2(k - 1 + i) - 1$.

**Proof of (13).** Let $\alpha$ be a face of $H$. We consider two cases.

**Case 1.** Let $t := \deg_H(\alpha) \geq 4$. Then with (9) each triangle $D$ inside $\alpha$ has a charge

\[
w^*(D) \leq \frac{w(\alpha)}{t} \leq \frac{t(k - 1)}{t - 2} \leq \left(1 + \frac{2}{t - 2}\right)(k - 1) \leq 2(k - 1) < 2(k - 1 + i) - 1.
\]

(Note $i \geq 1$).
Case 2. Let \( t = \deg_H(\alpha) = 3 \).

If \( \alpha \) is a triangle (2-cell 3-face) of \( H \) then \( \alpha \) is also a triangle of \( H^* \). Hence with (11) the charge \( w^*(\alpha) = w(\alpha) \leq 2(k - 1 + i) \).

Next let \( \alpha \) not be a triangle (2-cell 3-face). Then at least one diagonal \( d \) can be added so that no new face is created. The diagonal is counted twice on the boundary of \( \alpha \), i.e., \( \deg_{H+d}(\alpha) = 5 \). The charge \( w(\alpha) \leq 3(k - 1) \) is equally distributed to at least three (new) triangles of \( H^* \), each receiving a charge \( \leq \frac{w(\alpha)}{3} \leq \frac{3(k-1)}{3} = k - 1 \leq 2(k - 1 + i) - 1 \). Thus the proof of (13) is complete.

Properties (12) and (13) imply:

(14) \[ w = \sum_{\alpha \in F(H)} w(\alpha) = \sum_{D \in F(H^*)} w^*(D) \leq (2(k - 1 + i) - 1)f(H^*). \]

where \( f(H^*) = |F(H^*)| \).

The semitriangulation \( H^* \) satisfies the equation

\[ 2e(H^*) = 3f(H^*), \]

and Euler’s formula

\[ v(H^*) - e(H^*) + f(H^*) = \chi(M). \]

Hence

(15) \[ f(H^*) = 2(v(H^*) - \chi(M)), \]

and

(16) \[ e(H^*) = 3(v(H^*) - \chi(M)). \]

The number of the edges joining vertices of \( H \) and the number \( w \) of the edges joining minor vertices with major vertices in \( G \) contribute to the degree sum \( \sum_{A \in V(H)} \deg_G(A) \). Consequently, with (14) it holds

(17) \[ \sum_{A \in V(H)} \deg_G(A) = \sum_{A \in V(H)} \deg_H(A) + w \]

\[ \leq \sum_{A \in V(H)} \deg_H(A) + (2(k - 1 + i) - 1)f(H^*). \]
With (15)
\begin{equation}
\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + (4(k - 1 + i) - 2)(v(H^*) - \chi(M)).
\end{equation}

With $e(H) \leq e(H^*)$ and (16) we have
\begin{equation}
\sum_{A \in V(H)} \deg_G(A) \leq (6 + 4(k - 1 + i) - 2)(v(H^*) - \chi(M)), \text{ and}
\end{equation}
\begin{equation}
\sum_{A \in V(H)} \deg_G(A) \leq 4(k + i)(v(H^*) - \chi(M)).
\end{equation}
The inequality (18) implies with $v(H) = v(H^*)$ the existence of a major vertex $B$ of degree
\begin{equation}
\deg_G(B) \leq 4(k + i) \left(1 - \frac{\chi(M)}{v(H^*)}\right).
\end{equation}
If $M = S_0$ or $N_1$ then $\chi(M) \geq 1$ and by (5) $H$ has at least 6 vertices. Moreover (18) implies the existence of a major vertex $B$ of degree $\deg_G(B) \leq 4(k + i) - 1$. But by condition (A) each major vertex has a degree $\geq 4(k + i)$. This contradiction completes the proof of Theorem 10(i).

Next Theorem 10(ii) can be proved in the following way. By condition (B) with $6k \geq 4(k + i)$ (i.e., with $i \leq \frac{k}{2}$) and $\chi(M) \leq 0$
\begin{equation}
\sum_{\deg_G(A) \geq 4(k + i)} (\deg_G(A) - 4(k + i)) \geq \sum_{\deg_G(A) > 6k} (\deg_G(A) - 6k)
= ch_k(G) > 4(k + i)\|\chi(M)\|.
\end{equation}

With
\begin{equation}
\sum_{\deg_G(A) \geq 4(k + i)} (\deg_G(A) - 4(k + i)) = \left(\sum_{\deg_G(A) \geq 4(k + i)} \deg_G(A)\right) - 4(k + i)v(H^*)
\end{equation}
this implies
\begin{equation}
\sum_{\deg_G(A) \geq 4(k + i)} \deg_G(A) > 4(k + i)(v(H^*) + |\chi(M)|)
= 4(k + i)(v(H^*) - \chi(M)).
\end{equation}
The assertion (20) contradicts (18). Thus the proof of Theorem 10(ii) is complete.

Finally Theorem 10(iii) will be proved. For these purposes we need an upper bound for the number $v(H')$ of vertices of $H'$ in dependence on $f(H^*)$. Recall that $H'$ is the subgraph of $G$ induced by the minor vertices of $G$. Let $l$ denote the number of components of $H'$. Since $G$ is 3-connected each component $K$ of $H'$ contains the minor vertices of at least three corners of a face of $H$. The number of corners of $H$ is not greater than the number of corners of $H^*$, and $H^*$ has at most $3f(H^*)$ corners. Hence $3l \leq 3f(H^*)$, and $l \leq f(H^*)$. Since each component $K$ of $H'$ has no path with $k$ vertices and each (minor) vertex of $K$ has a degree $4(k+i)$ in $G$ the number of vertices of $K$ is $v(K) \leq b_k$, and the number of vertices of $H'$ is $v(H') \leq l \cdot b_k \leq f(H^*) \cdot b_k$. Therefore,

$$v(G) = v(H) + v(H') \leq v(H^*) + f(H^*) \cdot b_k.$$ 

Assertion (15) implies

$$v(G) \leq v(H^*) + 2(v(H^*) + |\chi(\mathcal{M})|) \cdot b_k \leq 2(v(H^*) + |\chi(\mathcal{M})|) \left(b_k + \frac{1}{2}\right).$$

With the hypothesis $v(G) > (6k+1)(2b_k+1)|\chi(\mathcal{M})|$ the number of vertices of $H^*$ is

$$v(H^*) \geq \frac{v(G)}{2b_k + 1} - |\chi(\mathcal{M})| > \frac{12k(b_k + \frac{1}{2})|\chi(\mathcal{M})|}{2b_k + 1} = 6k|\chi(\mathcal{M})|,$$

and

$$v(H^*) > 6k|\chi(\mathcal{M})| \geq 4(k+i)|\chi(\mathcal{M})|.$$ 

(19) and (21) imply: there is a vertex $B \in V(H)$ such that its degree

$$\deg_G(B) \leq 4(k+i) + \frac{4(k+i)|\chi(\mathcal{M})|}{v(H^*)}$$

$$< 4(k+i) + \frac{4(k+i)|\chi(\mathcal{M})|}{4(k+i)|\chi(\mathcal{M})|} = 4(k+i) + 1.$$ 

Therefore, the degree of the major vertex $B$ in $G$ is $\leq 4(k+i)$. But by the condition (C) each major vertex has a degree $\geq 4(k+i) + 1$. This contradiction completes the proof of Theorem 10(iii).
6. Proof of Theorem 13 for Polyhedral Maps — Lower Bound

The main goal of this part is to prove that \( \varphi(\{P_k, S_i\}, G(3, 3; \mathcal{M}, n(a))) \geq 4k + 4i, \ k \geq 3, \ \chi(\mathcal{M}) \leq 0 \), that is to construct a large polyhedral map \( G \) on surface \( \mathcal{M} \) with Euler characteristic \( \chi(\mathcal{M}) \leq 0 \) so that each path \( P_k \) with \( k \) vertices and each generalized 3-star \( S_i \) contains a vertex of degree at least \( 4k + 4i \). This construction is very similar to our construction presented in Sections 3 and 4 of [11].

Let \( P_n \times P_n \) be the cartesian product of two \( n \)-paths with vertex set \( \{(x, y) | x, y \in \mathbb{Z}, 1 \leq x \leq n, 1 \leq y \leq n\} \) and edge set \( \{(x, y), (x, y+1)| 1 \leq x \leq n, 1 \leq y \leq n - 1\} \cup \{(x, y), (x+1, y)| 1 \leq x \leq n - 1, 1 \leq y \leq n\} \). Add the edge set \( \{(x, y), (x+1, y-1)| 1 \leq x \leq n - 1, 2 \leq y \leq n\} \). The so obtained plane graph with \( 2(n-1)^2 \) triangles and an outer \( 4(n-1) \)-face is denoted by \( R_n \).

Into each triangle \( D \) of the obtained graph we insert a generalized 3-star \( S(r), 0 \leq r \leq k - 2i \), consisting of a central vertex \( Z \) and three paths \( p_1, p_2 \) and \( p_3 \) starting in \( Z \), the path \( p_1 \) of length \( k - (i + r) \), the path \( p_2 \) of length \( i + r \), and the path \( p_3 \) of length \( i \). Let the paths \( p_1, p_2, \) and \( p_3 \) be in this anticlockwise cyclic order in \( D \). If \( D_{x,y} = ((x, y), (x + 1, y), (x, y + 1)) \) then \( (x, y) \) is joined to all vertices of \( p_1 \) and \( p_2, (x + 1, y) \) is joined to all vertices of \( p_2 \) and \( p_3 \), and \( (x, y + 1) \) is joined to all vertices of \( p_3 \) and \( p_1 \) (see Figure 2). We do the same in \( D_{x,y} = ((x, y), (x - 1, y), (x, y - 1)) \). The resulting plane graph is denoted by \( R_n^{*} \).

![Figure 2](image-url)
The situation is presented in Figure 3, where in each triangle $D$ an arrow indicates which vertex of $D$ is joined with all vertices of $p_1$ and $p_2$. In this part of the proof the labels $0$ and $k - 2i, \ldots$ have no meaning. For the further proof of the lower bound $4(k + i)$ choose a fixed $r$, $0 \leq r \leq k - 2i$.

![Figure 3](image)

The inserted trees have $k - (i + r) + (i + r) + i - 2 = k + i - 2$ vertices, and the degree of each inner vertex $(x, y)$, $2 \leq x, y \leq n - 1$ is

$$
deg(x, y) = 6 + 2((k - (i + r)) + (i + r) - 1) + 2(i + r + i - 1) + 2(i + (k - (i + r) - 1) = 4k + 4i.$$

Deleting the outer face of $R_n^*$ and identifying opposite sides of the ”quadrangle” results in a toroidal map $T_n$, and reversing one side of this ”quadrangle” and then identifying opposite sides of this ”quadrangle” results in a map $Q_n$ on the Klein bottle, respectively, both satisfying the degree requirements.

The required polyhedral map on an orientable surface $S_g$ of genus $g \geq 2$ will be constructed from the toroidal triangulation $T_n^*$ with the triangulation $T_n$. We choose $2g - 2$ triangles of $T_n$ so that any two of them have a distance $\geq 2$ in $T_n$ (this is possible if $n$ is large enough). In $T_n^*$ from each of these triangles $D$ we delete the interior part so that the bounding 3-cycle of $D$
bounds now a hole of the torus. We join repeatedly two holes of \( T^n \) by a handle, and \( g-1 \) handles are added to the torus in this way.

The handles are triangulated in the following way: if \([X_1X_2X_3]\) and \([Y_1Y_2Y_3]\) are the bounding cycles of some handle which are around the handle in the same cyclic order then add the cycle \([X_1Y_1X_2Y_2X_3Y_3]\). In each of the new triangles a generalized 3-star \( S(r) \) can be placed in such a manner that the obtained polyhedral triangulation of \( S_g \) fulfills also the degree requirements.

The required polyhedral map on an unorientable surface \( \mathbb{N}_q \) of genus \( q \geq 3 \) will be constructed from the triangulation \( Q_n^* \) of the Klein bottle with triangulation \( Q_n \). We choose \( q-2 \) triangles of \( Q_n \) so that any two of them have a distance \( \geq 4 \) in \( Q_n \).

Let \( D \) be one of these triangles with bounding cycle \([X_1X_2X_3]\) and \( D_1, D_2, D_3 \) the three neighbouring triangles in \( Q_n \) with bounding cycles \([Y_1X_3X_2]\), \([Y_2X_1X_3]\), and \([Y_3X_2X_1]\) (see Figure 4). In \( Q_n^* \) we delete the inserted trees of \( D, D_1, D_2, D_3 \) and the separating edges \( X_1X_2, X_2X_3 \) and \( X_3X_1 \). A greater face \( F \) with bounding 6-cycle \( C = [X_1Y_3X_2Y_1X_3Y_2] \) is obtained (for the notation see Figure 5).

In \( F \) a crosscap is placed and the edges \( X_1X_2, X_2X_3, \) and \( X_3X_1 \) are again added so that the "interior" of \( C \) is subdivided into three quadrangles (see Figure 5). These quadrangles are subdivided by the edges \( X_iY_i, i = 1, 2, 3 \) (see Figure 6). Finally in each of the new triangles a generalized 3-star \( S(r) \) can be placed in such a manner that the obtained polyhedral triangulation of \( \mathbb{N}_q \) fulfills the degree requirements.
7. Proof of Theorems 11 and 12 for Polyhedral Maps — Lower Bounds

Let \( \mathcal{M} \) be a surface with Euler characteristic \( \chi(\mathcal{M}) \). Firstly we construct a polyhedral graph of the plane with degree sum \( \sum_{j \geq 4k+4} (j - 6k)v_j > (4k + 4i)|\chi(\mathcal{M})|, \) \( k \geq 3 \), such that each subgraph \( P_k \) and each subgraph \( S_i \) contains a vertex of degree at least \( 4(k + i) - 1, \) \( k \geq 3 \). Our method used here is very similar to that one used in [12]. We start with a plane graph \( R_{n+1} \) with \( n > (k + 1)|\chi(\mathcal{M})| + 3k \) as described in Section 6. Next the outer \( 4n \)-face is deleted and the opposite ”vertical sides” are identified, i.e., the two paths \((1, 1), (1, 2), \ldots, (1, n + 1)\) and \((n + 1, 1), (n + 1, 2), \ldots, (n + 1, n + 1)\) are identified in the given order.

The result is a triangulated cylinder \( Z_n^* \). A plane polyhedral graph \( Z_n \) is obtained by adding a bottom \( n \)-face \( F_1 \) and a top \( n \)-face \( F_2 \) which are the only \( n \)-faces of \( Z_n \), all other faces of \( Z_n \) are triangles. We use the same notation as in \( R_n \). If in all triangles of \( Z_n \) a generalized 3-star \( S(r) \) with a fixed \( r \) is inserted then all inner vertices of \( Z_n \) have the degree \( 4(k + i) \).

For instance choose \( r = 0 \). We want to increase the degrees of the vertices of the boundaries of \( Z_n \), i.e., for the vertices \((1, 1), (1, 2), \ldots, (1, n)\) and \((n + 1, 1), (n + 1, 2), \ldots, (n + 1, n)\). In order to do this we vary \( r \) so that from each inner vertex near to these boundaries a degree unit is transferred to one of these boundaries. We achieve this in the following way: According to Figure 3 insert the 3-star \( S(k - 2i) \) into \( D'_{x, 1} \), the 3-star \( S(k - 2i - 1) \) into \( D'_{x, 2} \), \ldots, the 3-star \( S(0) \) into \( D'_{x, k-2i+1} \) for all \( x, 1 \leq x \leq n \). According to Figure 3 insert the 3-star \( S(k - 2i) \) into \( D_{x, n+1} \), the 3-star \( S(k - 2i - 1) \) into \( D_{x, n} \), \ldots, the 3-star \( S(0) \) into \( D_{x, 1} \), \( n-(k-2i)M \), for all \( 2 \leq x \leq n + 1 \). Into all other triangles insert the 3-star \( S(0) \) according to Figure 3.

By the construction the vertices \((x, y), 1 \leq x \leq n, 2 + (k - 2i) \leq y \leq n - (k - 2i) \) have degree \( 4(k + i) \). The vertices \((x, y), 1 \leq x \leq n, 2 \leq y \leq k - 2i + 1 \) or \( n - (k - 2i) + 1 \leq y \leq n \) have the degree \( 4(k + i) - 1 \). The vertices on the boundaries, i.e., the vertices \((x, 1)\) and \((x, n + 1)\), \( 1 \leq x \leq n \) have the degree \( 3k + 1 \). In order to complete our construction we put into \( F_i \) a new vertex \( X_i \) and join \( X_i \) with all bounding vertices of \( F_i, i = 1, 2 \). In each new triangle \( \Delta \) a \( k \)-path \( p \) of \( F_1 \) and \( F_2 \) is inserted. One endvertex of \( p \) is joined with all three vertices of \( \Delta \), and all other vertices of \( p \) are joined with each of the two remaining vertices of \( \Delta \). In the obtained triangulation \( Z_n^{**} \) the vertices bounding \( F_i \) have degree \( 3k + 1 + 3 + 2(k - 1) - 2 = 5k \),
and $X_i$ has degree $\deg X_i \geq 2n > 2(k + i)|\chi(M)| + 6k$. Thus $\chi_k(Z_{n}^{**}) \geq (\deg X_1 - 6k) + (\deg X_2 - 6k) > 4(k + i)|\chi(M)|$.

Next the wanted polyhedral maps of $M$ will be constructed. If $M$ is an orientable 2-manifold $S_g$ of genus $g$ then $g$ handles have to be added. If $M$ is a nonorientable 2-manifold $N_q$ of genus $q$ then $q$ crosscaps have to be added. In both cases this is accomplished in the same way as in Section 6. The addition of $g$ handles or of $q$ crosscaps causes no problems according Section 6.

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