Projection Methods for Solving Urysohn Integral Equations with Multiwavelet Bases

1* Nasser Aghazadeh and 2 Medya Siadat
1, 2 Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz 53751 71379, Iran
E-mail: aghazadeh@azaruniv.ac.ir

*Corresponding author

ABSTRACT

Urysohn integral equations appear in many applications, for example it occurs in solving problems arising in economics, engineering and physics. Equations of this type have been used to model many thermostatic devices. In this paper the Galerkin and the Petrov-Galerkin methods have been used to solve the nonlinear integral equation of the Urysohn type. Alpert (1993) constructed a class of wavele bases and applied it to approximate solutions of the Fredholm second kind integral equations by the Galerkin method. We use Alpert multiwavelet bases with orthonormal Legendre polynomials to approximate the solution of nonlinear integral equation of the Urysohn type. The numerical examples show the good accuracy of the method.

Keywords: Urysohn integral equation, Fredholm, Petrov-Galerkin method, Galerkin method, Multiwavelet bases.

1. INTRODUCTION

In this paper, we study the nonlinear Urysohn integral equation:

\[ x(t) = \int_0^1 k(t, s, x(s))ds + f(t), \quad 0 \leq t \leq 1 \] (1)

for known functions \( k \) and \( f \), an unknown solution \( x \) is to be approximated.

Equation (1) was discussed by Urysohn (1924). Equations of this type appear in many applications, for example it occurs in solving problems arising in economics, engineering and physics (Zabreiko et al (1975)). Also equations
of this type have been used to model many thermostatic devices; see Glashoff and Sprekels (1981) and Glashoff and Sprekels (1981). Several authors have considered the problem of establishing the existence of solutions for (1) using different techniques. For example see (El-Sayed et al. 2003) and O’Regan (1998). In Atkinson and Rotra (1987), equation (1) is considered in a nonlinear operator equation

\[ x = \mathcal{K}(x) \]  

with \( \mathcal{K} \) a completely continuous mapping of a domain in the Banach space \( X \) into \( X \).

Let \( X_n, \ n \geq 1 \) denote a sequence of finite dimensional approximating subspaces and let \( P_n \) be a projection of \( X \) onto \( X_n \). The projection method for solving \( x = \mathcal{K}(x) \) consists of solving \( x_n = P_n \mathcal{K}(x_n) \).

This method was analyzed by (Krasnoselskii et al. 1972) and the rate of convergence of \( \{x_n\} \) to the exact solution were obtained. Detailed convergence results for both Galerkin and collocation projection methods for nonlinear case have been studied extensively in Atkinson and Rotra (1987).

2. MULTIWAVELET BASES

We provide below a brief review of Alpert’s wavelets (see Alpert (1993)). The wavelet bases for \( L^2[0,1] \) is comprised of dilates and translates of a set of functions \( h_1, h_2, ..., h_n \). In particular, for \( k \) a positive integer, and for \( m = 0,1,... \), we define a space \( S^k_m \) of piecewise polynomial functions:

\[ S^k_m = \{ f : \text{there striction off to the interval } (2^{-m}n, 2^{-m}(n + 1)) \text{ is a polynomial of degree less than } k, \text{ for } n = 0, ..., 2^m - 1 \text{ and } f \text{ vanish elsewhere} \} \]

(3)

Note that

\[ \dim S^k_m = 2^m k, \]

and

\[ S^k_0 \subset S^k_1 \subset \cdots \subset S^k_m \subset \cdots \]

The orthogonal complement of \( S^k_m \) in \( S^k_{m+1} \) is denoted by \( R^k_m \) so that \( \dim R^k_m = 2^m k \) and
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\[ S_m^k \oplus R_m^k = S_{m+1}^k, \quad R_m^k \perp S_m^k \]

Also note that

\[ S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k \quad (4) \]

The set of functions \( h_1, h_2, \ldots, h_k \) mentioned above is taken as an orthonormal bases for \( R_0^k \). Since \( R_0^k \) is orthogonal to \( S_0^k \), the first \( k \) moments of \( h_1, h_2, \ldots, h_k \) vanish,

\[
\int_0^1 h_j(x)x^i \, dx = 0, \quad i = 0, 1, \ldots, k - 1 \quad (5)
\]

The wavelet bases of Alpert is constructed by defining orthogonal systems

\[
h_{j,m}^n(s) = 2^m h_j(2^m s - n), \quad j = 1, \ldots, k, \quad m, n \in \mathbb{Z}. \quad (6)
\]

We refer the reader to Alpert (1993) for detailed constructions of \( h_1, h_2, \ldots, h_k \). The function \( h_{j,m}^n \) generated in (6) becomes an orthonormal bases for \( R_m^k \).

\[
R_m^k = \text{linear span} \{ h_{j,m}^n; \quad j = 1, \ldots, k; \quad n = 0, \ldots, 2^m - 1 \}.
\]

If \( \{ u_1, u_2, \ldots, u_k \} \) denote an orthonormal bases for \( R_0^k \), then the orthonormal system

\[
B_k = \{ u_j, j = 1, \ldots, k \} \cup \{ h_{j,m}^n; \quad j = 1, \ldots, k; \quad m = 0, \ldots; n = 0, \ldots, 2^m - 1 \},
\]

becomes the multiwavelet bases of order \( k \) for \( L^2[0,1] \). In practice we take an arbitrarily large but a fixed value for \( m \) and use

\[
B_k = \{ u_j, j = 1, \ldots, k \} \cup \{ h_{j,m}^n, \quad j = 1, \ldots, k, \quad n = 0, 1, \ldots, 2^m - 1 \} = \{ b_j \}_{j=1}^{k(2^m+1)} \quad (7)
\]

for an orthonormal bases for \( S_m^k \). For example the \( S_1^4 \) space has the basis elements as the following form:
\[ b_1 = 1 \]
\[ b_2 = \sqrt{3}(2t - 1) \]
\[ b_3 = \sqrt{5}(6t^2 - 6t + 1) \]
\[ b_4 = \sqrt{7}(20t^3 - 30t^2 + 12t - 1) \]
\[ b_5 = \begin{cases} 
\sqrt{15} 17 (3 - 56t + 216t^2 - 224t^3), & 0 \leq t \leq \frac{1}{2}; \\
\sqrt{15} 17 (-61 + 296t - 456t^2 + 224t^3), & \frac{1}{2} < t \leq 1. 
\end{cases} \]
\[ b_6 = \begin{cases} 
\sqrt{21} 21 (-11 + 270t - 1320t^2 + 1680t^3), & 0 \leq t \leq \frac{1}{2}; \\
\sqrt{21} 21 (-619 + 2670t - 3720t^2 + 1680t^3), & \frac{1}{2} < t \leq 1. 
\end{cases} \]
\[ b_7 = \begin{cases} 
\sqrt{35} 68 (2 - 60t + 348t^2 - 512t^3), & 0 \leq t \leq \frac{1}{2}; \\
\sqrt{35} 68 (-222 + 900t - 1188t^2 + 512t^3), & \frac{1}{2} < t \leq 1. 
\end{cases} \]
\[ b_8 = \begin{cases} 
\sqrt{5} 84 (-2 + 72t - 492t^2 + 840t^3), & 0 \leq t \leq \frac{1}{2}; \\
\sqrt{5} 84 (-418 + 1608t - 2028t^2 + 840t^3), & \frac{1}{2} < t \leq 1. 
\end{cases} \]

The approximating power of the wavelets is given as follows (see Alpert (1993)).

**Lemma 1.** Let \( Q^k_m \) be the orthogonal projection of \( L^2[0,1] \) onto \( S^k_m \). If \( f \in C^k [0,1] \), then

\[ \|Q^k_m f - f\| \leq 2^{-mk} \frac{2}{4^kk!} \sup_{x \in [0,1]} |f^{(k)}(x)|. \]

**2. THE GALERKIN METHOD**

Consider the following Urysohn integral equation:
where $k$ and $f$ are known functions and $x$ is the unknown function to be approximated.

Here we use the Galerkin method with multiwavelet bases to give numerical solutions of Urysohn integral equation.

Let $X = L^2[0,1]$. Alpert (1993) proved that $L^2[0,1] = \bigcup_{m=0}^{\infty} S_m^k$, so we choose

$$X_n = S_m^k, \quad m, k \in \mathbb{Z}, \quad m \geq 0, \quad k \geq 2.$$  

Now we define the orthogonal projection operator as the following form:

$$P_n : X \rightarrow X_n$$

so

$$x_n(t) = \sum_{i=1}^{n} c_i b_i(t)$$  

where $\{b_i(t)\}_{i=1}^{n}$ are orthonormal bases elements of $S_m^k$ and $n = 2^m k$.

In order to solve equation (8) we write it in operator form:

$$x = \mathcal{K}(x) + f$$  

where $\mathcal{K}$ is a completely continuous operator defined on $X$ as follows:

$$\mathcal{K}(x) = \int_0^1 k(t,s,x(s))ds, \quad 0 \leq t \leq 1, \quad x \in \mathbb{R}.$$  

In order to obtain the approximate solution $x_n(t)$, it needs to satisfy equation (10):

$$x_n(t) = \mathcal{K}(x_n) + f$$

$$x_n = \int_0^1 k(t,s,x_n(s))ds + f(t)$$
The expansion of $k$ in this bases is given by the formula

$$k(t, s, x_n(s)) = \sum_{i,j=1}^{n} k_{ij} b_i(s) b_j(t)$$

(12)

where the coefficients $k_{ij}$ are given by the expansion

$$k_{ij} = \int_0^1 \int_0^1 k(t, s, x_n(s)) b_i(s) b_j(t) ds dt.$$

Using (12) and (9), integral equation (11) is thereby approximated by the equation:

$$\sum_{i=1}^{n} c_i b_i(t) - \int_0^1 \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} b_i(s) b_j(t) ds = f(t), \quad 0 \leq t \leq 1$$

multiplying both sides of the above equations with bases elements $\{b_j(t)\}$ and then integrating from 0 to 1 with respect to $t$ and using the orthonormality of the bases $\{b_j(t)\}$, we have:

$$c_j - \sum_{i=1}^{n} k_{ij} \left( \int_0^1 b_i(s) ds \right) = f_j, \quad j = 1, \ldots, n$$

(13)

where

$$f_j = \int_0^1 f(t) b_j(t) dt, \quad j = 1, \ldots, n$$

Now we have $n$ equations with $n$ unknowns which can be solved easily.

4. THE PETROV-GALERKIN METHOD

We begin this section with a brief review of the Petrov-Galerkin method. We follow closely the paper by Chen and Xu (1998). Let $X$ be a Banach space and $X^*$ its dual space of continuous linear functionals. For each positive integer $n$, we assume that $X_n \subset X$, $Y_n \subset X^*$, and $X_n$ and $Y_n$ are finite dimensional vector spaces with

$$\dim X_n = \dim Y_n.$$ 

We further assume the following approximation property.
(H) If \( x \in X \) and \( y \in Y \), then there are sequences \( \{x_n\} \), \( \{y_n\} \) with \( x_n \in X_n \), \( y_n \in Y_n \) for all \( n \) such that \( x_n \to x \), and \( y_n \to y \).

Define, for \( x \in X \), an element \( P_n x \in X_n \) called the generalized best approximation from \( X_n \) to \( x \) with respect to \( Y_n \) by the equation

\[
(x - P_n x, y_n) = 0, \quad \forall y_n \in Y_n.
\]  

(14)

It is proved by Chen and Xu (1998), that for each \( x \in X \), the generalized best approximation from \( X_n \) to \( x \) with respect to \( Y_n \) exists uniquely if and only if

\[
Y_n \cap X_n^\perp = \{0\}
\]  

(15)

where \( X_n^\perp = \{x^* \in X^*: \langle x, x^* \rangle = 0 \mbox{ for all } x \in X_n\} \). When condition (15) is satisfied, \( P_n \) defines a projection; \( P_n^2 = P_n \). Throughout the remainder of this paper, condition (15) is assumed.

In order to formulate the Petrov-Galerkin method as part of the general projection scheme and render an appropriate error analysis accordingly, it is important to establish that \( P_n \) converges pointwise to the identity operator \( I \). To this end, the notion of regular pair is introduced. Assume that for each \( n \), there is a linear operator \( \Pi_n : X_n \to Y_n \) with \( \Pi_n X_n = Y_n \) and satisfying the following two conditions.

\[
(H - 1) \quad \|x_n\| \leq C_1(x_n, \Pi_n x_n)^{\frac{1}{2}} \text{ for all } x_n \in X_n.
\]

\[
(H - 2) \quad \|\Pi_n x_n\| \leq C_2\|x_n\| \text{ for all } x_n \in X_n.
\]

Here \( C_1 \) and \( C_2 \) are constants independent of \( n \).

If a pair of space sequences \( \{X_n\} \) and \( \{Y_n\} \) satisfy (H-1) and (H-2), we call \( \{X_n, Y_n\} \) a regular pair. It is shown by Chen and Xu (1998), that, if a regular pair \( \{X_n, Y_n\} \) satisfies \( \dim X_n = \dim Y_n \) and condition (H), then the corresponding generalized projection \( P_n \) satisfies:

\[
(P1) \quad \|P_n x - x\| \to 0 \text{ as } n \to 0 \text{ for all } x \in X,
\]

\[
(P2) \quad \|P_n\| \leq C, \quad n = 1, 2, \ldots, \text{ for some constant } C, \text{ and}
\]

\[
(P3) \quad \|P_n x - x\| \leq c\|Q_n x - x\|, \quad n = 1, 2, \ldots, \text{ for some constant } c,
\]

where \( Q_n x \) is the best approximation from \( X_n \) to \( x \).
5. ESTABLISHMENT OF THE PETROV-GALERKIN CONDITIONS

In this section we show that all necessary conditions shown in previous section hold with Alpert Multiwavelet basis. At first we should choose the appropriate spaces for $\mathcal{X}$ and $\mathcal{X}^*$. Let put $\mathcal{X} = L^2[0,1]$, so the dual space will be the same: $\mathcal{X}^* = L^2[0,1]$. (The dual of $L^p$ space is $L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$).

We saw in section 2 that $\mathcal{X} = L^2[0,1] = \bigcup_{m=0}^{\infty} S_m^k$, which $S_m^k$, $m = 0, 1, \ldots$ are the spaces constructed from Alpert Multiwavelets. Now we introduce the space sequences $X_n$ and $Y_n$:

$$X_n = S_m^k \subseteq \bigcup_{m=0}^{\infty} S_m^k = L^2[0,1] = \mathcal{X}$$

$$Y_n = S_m^{k'} \subseteq \bigcup_{m=0}^{\infty} S_m^{k'} = L^2[0,1] = \mathcal{X}^*$$

In order to satisfy in condition $\dim X_n = \dim Y_n < \infty$ we should have $2^mk = 2^mk' < \infty$ and this equation implies that $2^m - m' = \frac{k}{k'}$.

In this article, we consider $m' = m + 1$ and $k' = \frac{k}{2}$.

Condition (H):

$$\forall x_n(t) \in X = L^2[0,1] = \bigcup_{m=0}^{\infty} S_m^k \exists x_n(t) \in X_n = S_m^k; \quad Px_n(t) - x_P \to 0, \quad asn \to \infty.$$ 

$$\forall y_n(t) \in \mathcal{X}^* = L^2[0,1] = \bigcup_{m=0}^{\infty} S_m^{k'} \exists y_n(t) \in Y_n = S_m^{k'}; \quad Py_n(t) - y_P \to 0, \quad asn \to \infty.$$ 

In above equations, $\| \cdot \|_2$ is defined in the space $L^2[0,1]$. We introduce $x_n(t)$ and $y_n(t)$ as follows. If $\{b_i(t)\}_{i=1}^{\infty}$ be a basis for the space $X_n = S_m^k$ and $\{b'_i(t)\}_{i=1}^{\infty}$ be another one for the space $Y_n = S_m^{k'}$, with the condition $n = 2^mk = 2^mk'$, then we define the projection operator $P_n$ as the same as we did in Galerkin method:
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\[ P_n : X = \bigcup_{m=0}^{\infty} S^k_m \rightarrow X_n = S^k_m \]

\[ P_n(x(t)) = x_n(t) = \sum_{i=1}^{n} c_i b_i(t). \]

Also we can define the operator \( P'_n \) from \( X^* \) to \( Y_n \) in a same way. Now we should define the operator \( \prod_n \) which used in definition of regular pair:

\[ \prod_n : X_n = S^k_m \rightarrow Y_n = S^{k'}_m, \]

\[ \prod_n (x_n(t)) = \prod_n \left( \sum_{i=1}^{n} \alpha_i b_i(t) \right) = \sum_{i=1}^{n} \alpha b_i^*(t). \]

Whit this definition \( \prod_n \) is clearly linear and one to one, so with respect to that \( \dim X_n = \dim Y_n \) we can conclude that \( \prod_n X_n = Y_n \).

The issue we are concerned with is to show that conditions (H-1) and (H-2) are established.

\textbf{H-1}: \( \forall x_n(t) \in X_n = S^k_m \| x_n \| < c_1 \langle x_n, \prod_n x_n \rangle^{\frac{1}{2}}. \)

\textbf{Proof}.

\[ \langle x_n, \prod_n x_n \rangle = \int_0^1 x_n(t) \left( \prod_n x_n(t) \right) dt \]

\[ = \sum_{n=0}^{2^{m-1}} \left[ \sum_{i=1}^{2^{m_k}} \alpha_i b_i(t) \right] \left[ \sum_{j=1}^{2^{m_k}} \alpha_j b_j^*(t) \right] dt \]

\[ = \sum_{n=0}^{2^{m-1}} \sum_{i,j=1}^{2^{m_k}} \int_0^{n+1} b_i(t) b_j^*(t) \alpha_i \alpha_j dt \]

\[ = \sum_{i,j} \alpha_i \alpha_j \int_0^1 b_i(t) b_j^*(t) dt \]
Let \( \alpha^T = (\alpha_1, \cdots, \alpha_n) \) and \( B = [b_{i,j}]_{i,j=1}^n \), then:

\[
\langle x_n, \prod_n x_n \rangle = \alpha^T B \alpha
\]  \hspace{1cm} (16)

Now we choose spaces \( X_n \) and \( Y_n \) in such a way that the condition (H-1) holds. If \( k \) be a positive integer, by putting \( X_n = S_0^k \) and \( Y_n = S_1^k \), the matrix \( B \) will be diagonal with positive members. So we have:

\[
\langle x_n, \prod_n x_n \rangle = \sum_{i=1}^n \alpha_i^2 b_{ii} \geq b_{qq} \|x_n\|_2^2
\]

Where \( b_{qq} = \min_{1 \leq i \leq n} b_{ii} \), so

\[
\langle x_n, \prod_n x_n \rangle \geq b_{qq} \|x_n\|
\]

and

\[
\|x_n\| \leq \frac{1}{\sqrt{b_{qq}}} \langle x_n, \prod_n x_n \rangle^{\frac{1}{2}}
\]

Considering \( c_1 = \frac{1}{\sqrt{b_{qq}}} \), condition (H-1) holds.

(H-2): \( \forall x_n(t) \in X_n = S_0^k, \quad \|\prod_n x_n\| \leq c_2 \|x_n\| \)

**Proof.**

\[
\left\| \prod_n x_n \right\|_2^2 = \int_0^1 \left( \prod_n x_n(t) \right)^2 dt = \int_0^1 \left( \sum_{i=1}^{2^m k'} \alpha_i b_i^*(t) \right)^2 dt
\]

By dividing the interval \([0,1]\) into subintervals \( \left[ \frac{n}{2^m}, \frac{n+1}{2^m} \right], \quad n = 0, 1, \ldots, 2^m - 1 \), we can continue the previous equation like this:
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\[ \|\Pi_n x_n\|_2 = \sum_{n=0}^{2^m-1} \int_{2^n}^{2^{n+1}} \left( \sum_{i=1}^{2^m k} a_i b_i^*(t) \right)^2 \, dt \]

\[ = \sum_{n=0}^{2^m-1} \int_{2^n}^{2^{n+1}} \left[ \sum_{i=1}^{2^m k} a_i^2 b_i^2(t) + 2 \sum_{i=1}^{2^m k} \sum_{j=1, j \neq i}^{2^m k} a_i a_j b_i^*(t) b_j^*(t) \right] dt \]

\[ = \sum_{i=1}^{2^m k} a_i^2 \int_0^1 b_i^2(t) dt + \sum_{i=1}^{2^m k} (2 \sum_{j=1, j \neq i}^{2^m k} a_i a_j) \int_0^1 b_i^*(t) b_j^*(t) dt \]

Orthonormality of \( Y_n \) basis elements implies that the second integral vanishes, so:

\[ \left\| \prod_n x_n \right\|_2^2 = \sum_{i=1}^{2^m k} a_i^2 \int_0^1 b_i^2(t) dt \]

\[ \Rightarrow \left\| \prod_n x_n \right\| = \| x_n \| \leq c_2 \| x_n \| \]

Now it is sufficient to take \( c_2 \geq 1 \).

6. APPLYING THE METHOD

The Petrov-Galerkin approximation to the Urysohn integral equation (2) is obtained by solving the following equation for \( x_n \)

\[ \langle x_n - \mathcal{K}x_n, b_j^* \rangle = \langle f, b_j^* \rangle \]

where \( \{b_j^*(t)\} \) are the bases elements of subspace \( Y_n \). Using (9) and the fact that \( \mathcal{K}(x_n) \in X_n \), it follows that

\[ \sum_{i=1}^{n} c_i b_{ij} - \sum_{i=1}^{n} \hat{c}_i b_{ij} = f_j \quad (16) \]

where \( b_{ij} = \langle b_i(t), b_j^*(t) \rangle \) and \( f_j = \langle f(t), b_j^*(t) \rangle \). Note that since \( \mathcal{K}x_n \in X_n \), so

\[ \mathcal{K}x_n = \sum_{i=1}^{n} \hat{c}_i b_i(t), \]
To modify the coefficients $\hat{c}_j$, by multiplying both sides of the above equation by $\{b_j(t)\}$ and integrate over the interval $[0,1]$ with respect to $t$, we have

$$\hat{c}_j = \langle \mathcal{K}x_n(t), b_j(t) \rangle = \int_0^1 \int_0^1 k(t,s, \sum_{i=1}^n c_i b_i(s)) b_j(t) dsdt.$$  

By substituting in Eq. (16)

$$\sum_{i=1}^n c_i b_{ij} - \sum_{i=1}^n b_{ij} \left[ \int_0^1 \int_0^1 k(t,s, \sum_{i=1}^n c_i b_i(s)) b_i(t) dsdt \right] = f_j, \quad j = 1,2,...,n$$

Let

$$K_i = \int_0^1 \int_0^1 k(t,s, \sum_{i=1}^n c_i b_i(s)) b_i(t) dsdt,$$

so

$$\sum_{i=1}^n c_i b_{ij} = f_j + \sum_{i=1}^n K_i b_{ij}, \quad j = 1,...,n.$$  

The iterated projection method solution is defined by

$$\tilde{x}_n = \mathcal{K}(x_n).$$

7. NUMERICAL EXAMPLES

In this section we illustrate the method discussed in section 3 with some examples.

Example 1. In this example we solve the following equation

$$x(t) = \int_0^1 \left( e^{t+s} + \frac{1}{2} x(s) \right) ds + 2e^t + \frac{1}{2} - \frac{1}{2} e - e^{(t+1)}$$

with the exact solution $e^t$. Numerical results are presented in Table 1.
Example 2. *In this example we solve the following equation*

\[ x(t) = \int_0^1 \sin(t + s + x(s))ds + (t - \sin1\sin(1 + t)) \]

*with the exact solution* \( t \). *Numerical results are presented in Table 2.*

<table>
<thead>
<tr>
<th>iteration</th>
<th>( S_0^6 )</th>
<th>( S_0^8 )</th>
<th>( S_1^4 )</th>
<th>( S_2^4 )</th>
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<td>10</td>
<td>7.757 E--4</td>
<td>7.014 E--4</td>
<td>5.512 E--5</td>
<td>4.231 E--6</td>
</tr>
<tr>
<td>20</td>
<td>3.313 E--4</td>
<td>6.853 E--7</td>
<td>2.316 E--7</td>
<td>3.251 E--8</td>
</tr>
<tr>
<td>30</td>
<td>3.312 E--4</td>
<td>0.854 E--8</td>
<td>1.015 E--9</td>
<td>0.168 E--10</td>
</tr>
<tr>
<td>40</td>
<td>3.312 E--4</td>
<td>1.853 E--8</td>
<td>0.158 E--10</td>
<td>3.012 E--11</td>
</tr>
</tbody>
</table>

**TABLE 2: Absolute errors for example 1(Petrov-Galerkin method).**

<table>
<thead>
<tr>
<th>iteration</th>
<th>( X_n = S_0^4 )</th>
<th>( Y_n = S_0^4 )</th>
<th>( X_n = S_0^8 )</th>
<th>( Y_n = S_0^8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>8.810 E--3</td>
<td>8.844 E--3</td>
<td>8.803 E--3</td>
<td>7.213 E--3</td>
</tr>
</tbody>
</table>

**TABLE 3: Absolute errors for example 2(Galerkin method).**

<table>
<thead>
<tr>
<th>iteration</th>
<th>( S_0^4 )</th>
<th>( S_0^8 )</th>
<th>( S_1^4 )</th>
<th>( S_2^4 )</th>
</tr>
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<tbody>
<tr>
<td>20</td>
<td>3.359 E--4</td>
<td>5.268 E--6</td>
<td>2.158 E--7</td>
<td>6.268 E--7</td>
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<tr>
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<td>3.759 E--5</td>
<td>4.268 E--8</td>
<td>3.124 E--9</td>
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<td>2.168 E--5</td>
<td>3.684 E--9</td>
<td>1.589 E--10</td>
<td>2.154 E--12</td>
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</table>

**CONCLUSION**

The Galerkin and the Petrov-Galerkin methods have been used to solve the nonlinear integral equation of the Urysohn type by using a class of Alpert multiwavelets with orthonormal Legendre polynomials. The numerical examples show that the presented methods are effective and trustworthy.

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REFERENCES


