A numerical view of Spectral Decomposition Theorem for SFT

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Abstract
In this paper algorithmizable conditions of Spectral Decomposition Theorem for SFT are presented.

1. Introduction
Finite type subshifts with finite number of symbols are important sort of discrete dynamical systems (often called symbolic dynamics) and are intensively studied [2, 3, 4, 5, 6]. Mañé proved a theorem (for finite type subshifts generated by 0, 1-square matrices), stating that topological transitivity and topological mixing can be reduced to algebraic properties of associated matrices [5]. In the paper [3] algorithmical conditions for topological transitivity and mixing were considered. The Spectral Decomposition Theorem is an important result in the theory of dynamical systems, giving a separation of a nonwandering set into basic sets with topological transitivity and elementary sets with topological mixing [1]. In this paper an algorithmizable method of Spectral Decomposition Theorem [1] for subshifts of finite type (SFT) is formulated. The classical Smale’s Spectral Decomposition Theorem [4, 5, 6] and algorithmic results contained in papers [2, 3] has been utilised.

2. Preliminaries
Let us recall the relevant material from [3, 6].

Let $\mathcal{M}_n(0, 1)$ be the set of all $n \times n$ matrices whose entries are either 0’s or 1’s.

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Definition 2.1 Let $G_A$ be an oriented graph with $n$ vertices associated with a matrix $A \in \mathcal{M}_n(0,1)$. There is an edge in $G_A$ from the $i$-th to the $j$-th vertex iff $a_{ij} = 1$, where $A = [a_{ij}]_{i,j=1,...,n}$. A cycle is a path in the graph having the same initial and terminal vertex.

Definition 2.2 The full shift on $n$ symbols is the space
$$\Sigma_n := \{x = \{x_i\}_{i \in \mathbb{Z}} : x \in \{1,\ldots,n\}^\mathbb{Z}\}$$
with the shift mapping $\sigma : \Sigma_n \to \Sigma_n$ defined by
$$\sigma(x)_i = x_{i+1}.$$

Definition 2.3 Let $A \in \mathcal{M}_n(0,1)$. Define
$$\Sigma_A := \{x \in \Sigma_n : \forall i \in \mathbb{Z} \ a_{x_ix_{i+1}} = 1\}.$$
The pair $(\Sigma_A, \sigma|_{\Sigma_A})$ is called a subshift of finite type (SFT).

Definition 2.4 A matrix $A \in \mathcal{M}_n(0,1)$ is said to be a transition matrix iff there is an entry 1 in every column and every row of $A$.

For $A \in \mathcal{M}_n(0,1)$ and $m \in \mathbb{N}$ the entries of the matrix $A^m$ will be denoted by $a^m_{ij}$, $i,j \in \{1,\ldots,n\}$.

Definition 2.5 A matrix $A$ is irreducible iff for every $i,j \in \{1,\ldots,n\}$ there is a positive integer $m$ such that $a^m_{ij} > 0$.

Remark 2.6 If $A$ is irreducible, then for every $i,j \in \{1,\ldots,n\}$ there exists $m \in \{1,\ldots,n\}$ such that $a^m_{ij} > 0$.

Proof. "This is elementary, since every path of the length greater than $n$ in the graph associated with $A$ contains a cycle that can be cut out to yield a shorter path" [3].

Definition 2.7 A matrix $A$ is mixing iff for every $i,j \in \{1,\ldots,n\}$ there is a positive integer $m_0$ such that $a^m_{ij} > 0$ for every integer $m > m_0$. 
Definition 2.8 A matrix $A$ is primitive iff there exists $M \in \mathbb{N}$ such that $A^M > 0$.

Corollary 2.9 A matrix $A$ is mixing iff it is primitive.

Let $X$ be a compact metric space with a metric $d$ and let $f : X \to X$ be a homeomorphism.

Definition 2.10 A point $x \in X$ is said to be a nonwandering point for $f$ iff for every open neighbourhood $U$ of $x$ there is a positive integer $m$ such that $f^{-m}(U) \cap U \neq \emptyset$.

The set of all nonwandering points of $f$ will be denoted by $\Omega(f)$.

Definition 2.11 We call $f$ topologically transitive iff for all open nonempty sets $U, V \subset X$ there is a positive integer $m$ such that $f^{-m}(U) \cap V \neq \emptyset$.

Definition 2.12 We call $f$ topologically mixing iff for all open nonempty sets $U, V \subset X$ there is a positive integer $m_0$ such that $f^{-m}(U) \cap V \neq \emptyset$ for every integer $m > m_0$.

Definition 2.13 A homeomorphism $f$ is said to be expansive with an expansive constant $\varepsilon$ if for all $x, y \in X$ such that $x \neq y$ the condition

$$d(f^m(x), f^m(y)) > \varepsilon$$

is satisfied for some $m \in \mathbb{Z}$.

Definition 2.14 A sequence $\{x_i\}_{i \in \mathbb{Z}}$ is called a $\delta$-pseudo orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For given $\varepsilon > 0$ we say that a $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ is $\varepsilon$-traced by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. A homeomorphism $f$ has the pseudo orbit tracing property (POTP) - shadowing property if every $\delta$-pseudo orbit of $f$ is $\varepsilon$-traced by some point of $X$.

Theorem 2.15 (see [4]) For a given transition matrix $A$ the following are true:

1. $\sigma|_{\Sigma_A}$ is topologically transitive iff $A$ is irreducible
2. $\sigma|_{\Sigma_A}$ is topologically mixing iff $A$ is mixing
We need some algorithmical conditions for topological transitivity and mixing for SFT.

**Theorem 2.16** (see [3]) For a given transition matrix $A$ the following are true:

1. $\sigma |_{\Sigma^A}$ is topologically transitive $\iff A + A^2 + \ldots + A^n > 0$,
2. $\sigma |_{\Sigma^A}$ is topologically mixing $\iff A$ is irreducible and

$$1 \in \sum_{i=1}^n \sum_{j=1}^n \mathbb{Z} \cdot j \cdot \text{sgn}(a_{ij}^j),$$

where $\mathbb{Z}$ is the set of integers. The equivalent statement of the above condition is

$$\exists \{c_{ij}\}_{i,j\in\{1,\ldots,n\}} \subset \mathbb{Z} : 1 = \sum_{i,j=1}^n c_{ij} \cdot j \cdot \text{sgn}(a_{ij}^j).$$

Let $\text{GCD}(m_1, \ldots, m_k)$ be the greatest common divisor of positive integers $m_1, \ldots, m_k$. Since there exist integers $t_1, \ldots, t_k$ such that

$$\text{GCD}(m_1, \ldots, m_k) = t_1m_1 + t_2m_2 + \ldots + t_km_k,$$

the following corollary is satisfied.

**Corollary 2.17** Let $A$ be a transition matrix. Then:

$\sigma |_{\Sigma^A}$ is topologically mixing $\iff A$ is irreducible and there exist cycles $\pi_1, \ldots, \pi_k$ of lengths $m_1, \ldots, m_k \leq n$ in the graph $G^A$ such that

$$\text{GCD}(m_1, \ldots, m_k) = 1.$$

**Theorem 2.18** Let $A \in \mathcal{M}_n(0,1)$. The columns of $A$ can be reordered in such a manner that $A$ takes the form of a block matrix:

$$A = \begin{bmatrix}
A_1 & * & \ldots & * \\
0 & A_2 & * & \\
& \ldots & \ldots & \ldots \\
0 & 0 & \ldots & A_k
\end{bmatrix}$$

where each $A_i$ is either irreducible or a zero matrix, the terms * denote arbitrary numbers and all the terms below the blocks $A_i$ are equal to 0.
For the completeness we present a proof - see [3].

**Proof.** Let \( V = \{1, \ldots, n\} \) be the set of columns. There is a simple correspondence between \( V \) and the set of all vertices of a graph associated with \( A \). Define the following equivalence relation on \( V \):

\[
x R y \iff (x = y \lor \exists m, l > 0 : (a^m_{xy} > 0) \land (a^l_{yx} > 0)).
\]

Then \( V/R \) is partially ordered by relation \( S \) defined as

\[
[x] S [y] \iff ([x] = [y] \lor \text{there is a path from } x \text{ to } y).
\]

It is obvious that any reordering of the classes of abstraction of the columns nondecreasing with respect to \( S \) is a solution of our problem.

Matrices \( A_i \) correspond to the classes of abstraction of \( R \). \( \square \)

### 3. The main result

In this section an algorithmization of the Spectral Decomposition Theorem is introduced.

**Theorem 3.1** (see [1]) Let \( f : X \to X \) be an expansive homeomorphism with the shadowing property. Then:

1. there exists a finite sequence \( B_1, \ldots, B_l \) (basic sets) of \( f \)-invariant, closed, pairwise disjoint subsets such that
   - \( \Omega(f) = \bigcup_{i=1}^l B_i \),
   - \( f|_{B_i} : B_i \to B_i \) is topologically transitive for all \( i \in \{1, \ldots, l\} \),

2. for each basic set \( B_j \) there exists \( a_j \in \mathbb{N} \) and a finite sequence
   \[
   (C_{j1}, \ldots, C_{ja_j-1})
   \]
   (elementary sets) of closed, pairwise disjoint subsets such that for every \( i \in \{1, \ldots, a_j-1\} \)
   - \( f((C_{ji})_i) = (C_{ji})_{i+1}, \ f^a((C_{ji})_i) = (C_{ji})_i \),
   - \( B_j = \bigcup_{i=1}^{a_j-1} (C_{ji})_i \),

\( \square \)
Let \( A \in \mathcal{M}_n(0,1) \). The mapping \( \sigma : \Sigma_A \to \Sigma_A \) is expansive and has the shadowing property (see for example [7]), thus Theorem 3.1 is true for SFT. We analyze the first part of Theorem 3.1. According to Theorem 2.18 the matrix \( A \) can be reordered to matrix \( \bar{A} \) of the form

\[
A = \begin{bmatrix}
A_1 & * & \ldots & * \\
0 & A_2 & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_m
\end{bmatrix}
\]

where each \( A_i \) is either irreducible or a zero matrix. We choose only irreducible matrices \( A_1, \ldots, A_k \), because for zero matrix the associated SFT is empty set. Matrices \( A_1, \ldots, A_k \) are irreducible so, as a consequence, the maps

\[
\sigma|_{\Sigma_{A_1}}, \ldots, \sigma|_{\Sigma_{A_k}}
\]

are topologically transitive, sets \( \Sigma_{A_1}, \ldots, \Sigma_{A_k} \) are invariant under \( \sigma \) and

\[
\Omega(\sigma) = \Sigma_{A_1} \cup \ldots \cup \Sigma_{A_k}.
\]

So \( \Sigma_{A_1}, \ldots, \Sigma_{A_k} \) are the basic sets.

**Algorithmization of the period.**

If \( A \) is irreducible we define the sequence \( \{s_i\}_{i \in \mathbb{N}} \) in the following way

\[
s_i = \begin{cases} 
  i, & \text{if } \text{tr}(A^i) > 0, \\
  0, & \text{if } \text{tr}(A^i) = 0. 
\end{cases} \quad i \in \mathbb{N}.
\]

We choose only nonzero elements

\[
s_{i_1}, \ldots, s_{i_p}, \ldots
\]

of the sequence \( \{s_i\}_{i \in \mathbb{N}} \). Consider

\[
d_A = \text{GCD}\{s_{i_1}, \ldots, s_{i_p}, \ldots\} = \text{GCD}\{i_1, \ldots, i_p, \ldots\}.
\]

It is easy to see, that \( d_A \) is the period of the matrix \( A \) [4].

**Remark 3.2** \( d_A \leq n \).
Proof. Assume that
\[d_A = \gcd\{s_{i_1}, \ldots, s_{i_p}, \ldots\} > n.\]

By the definition of \(s_{ij}\) there exists \(k > n\) such that
\[\text{tr}(A^k) > 0.\]

From this we deduce that there exist: a cycle with length \(k\) and a cycle with
length \(k_0 \leq n\) in the graph \(G_A\) ("every path of the length greater than \(n\) in
the graph associated with \(A\) contains a cycle that can be cut out to yield a
shorter path" [3]). According to this remark, we have
\[\text{tr}(A^{k_0}) > 0.\]

Thus
\[d_A = \gcd\{k_0, \gcd\{s_{i_1}, \ldots, s_{i_p}, \ldots\}\} \leq n.\]

This contradicts the assumption. \(\square\)

Corollary 3.3 \(d_A = \gcd\{s_{i_1}, \ldots, s_{i_n}\} \).

We can formulate a known result.

Corollary 3.4 The following conditions are equivalent:

1. \(A\) is mixing,
2. \(A\) is primitive,
3. \(d_A = 1.\)

According to the above results we can rewrite Theorem 2.16 in the following
form:

Theorem 3.5 For a given transition matrix \(A\) the following are true:

1. \(\sigma|_{\Sigma_A}\) is topologically transitive \(\iff\) \(A + A^2 + \ldots + A^n > 0,\)
2. \(\sigma|_{\Sigma_A}\) is topologically mixing \(\iff\) \(A + A^2 + \ldots + A^n > 0 \land d_A = 1.\)
Proof. The proof is simple and therefore it is omitted here. □

The construction of elementary sets for basic set.

First, we have to compute periods of matrices \( A_1, \ldots, A_s \) (for simplicity of notation, we write \( d_1, \ldots, d_s \)). If \( d_i = 1 \) for some \( i \), then the basic set is elementary set (matrix \( A_i \) is primitive and \( \sigma|_{\Sigma_{A_i}} \) is topologically mixing). Assume that \( d_i > 1 \). Let \( \psi^1, \ldots, \psi^s \) be sets of symbols associated matrices \( A_1, \ldots, A_s \). For every \( i \in \{1, \ldots, s\} \) we can write \( \psi^i \) in the following form

\[
\psi^i = \psi^i_0 \cup \ldots \cup \psi^i_{d_i - 1},
\]

where \( \psi^i_j \) are nonempty, pairwise disjoint sets and there is an edge connecting \( \psi^i_j \) with \( \psi^i_{j+1} \) for \( j \in \{0, \ldots, d_i - 2\} \) and \( \psi^i_{d_i - 1} \) with \( \psi^i_0 \) (a consequence of classical results \([?]\)). We can write the basic set \( \Sigma_{A_i} \) in the form

\[
\Sigma_{A_i} = \Sigma^0_{A_i} \cup \ldots \cup \Sigma^{d_i - 1}_{A_i},
\]

for fixed \( i \in \{1, \ldots, s\} \), where

\[
\Sigma^j_{A_i} = \left\{ \{x_i\}_{i \in \mathbb{Z} \in \Sigma_{A_i}} : x_0 \in \psi^i_j \right\}, \quad j = 0, \ldots, d_i - 1.
\]

**Corollary 3.6** The condition

\[
w \in \psi^i_j
\]

is satisfied iff in the graph \( G_{A_i} \) there exists a cycle with length \( d_i \) contained \( w \).

**Corollary 3.7** If \( A_i \) is an irreducible matrix with period \( d_i \), then \( \sigma^{d_i}|_{\Sigma_{A_i}} : \Sigma_{A_i} \rightarrow \Sigma_{A_i} \) is topologically mixing.

Since \( A_i \) are irreducible matrices with period \( d_i \), we conclude from the Corollary 3.7 that the mappings

\[
\sigma^{d_i}|_{\Sigma^j_{A_i}} : \Sigma^j_{A_i} \rightarrow \Sigma^j_{A_i}
\]

are topologically mixing for fixed \( i \). Moreover, the condition

\[
\sigma(\Sigma^j_{A_i}) = \Sigma^{j+1}_{A_i}, \quad \sigma(\Sigma^{d_i - 1}_{A_i}) = \Sigma^0_{A_i}
\]

is satisfied for \( j \in \{0, \ldots, d_i - 1\} \). This clearly forces \( \Sigma^j_{A_i} \) are elementary sets for \( \Sigma_{A_i} \) with constant \( a_i = d_i \).
Example 3.8 Let $\Sigma_A$ be the subshift induced by

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

We can reorder $A$ to the following matrix

$$\tilde{A} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

The matrix $\tilde{A}$ is of the form

$$\tilde{A} = \begin{bmatrix} A_1 & * & * \\
0 & A_2 & * \\
0 & 0 & A_3 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.$$
The condition
\[ \sum_{i=1}^{k_j} A_{ij} > 0, \] where \( k_j = 6 - j \) for \( j \in \{1, 2, 3\} \)
implies irreducibility of matrices \( A_1, A_2, A_3 \) (see Theorem 2.16). As a consequence topological transitivity for maps
\[ \sigma|_{\Sigma A_1}, \sigma|_{\Sigma A_2}, \sigma|_{\Sigma A_3} \]
is obtained. So, \( \Sigma A_1, \Sigma A_2, \Sigma A_3 \) are basic sets. The periods of the matrices \( A_1, A_2, A_3 \) are
\[ d_1 = 5, \ d_2 = 4, \ d_3 = 3. \]
According to the above construction we have:
\[ \Sigma A_1 = \Sigma^0 A_1 \cup \Sigma^1 A_1 \cup \Sigma^2 A_1 \cup \Sigma^3 A_1 \cup \Sigma^4 A_1, \]
where
\[ \Sigma^i A_1 = \{ x \in \Sigma A_1 : x_0 = i + 1 \}, \ i = 0, \ldots, 4. \]
Moreover
\[ \Sigma A_2 = \Sigma^0 A_2 \cup \Sigma^1 A_2 \cup \Sigma^2 A_2 \cup \Sigma^3 A_2, \]
where
\[ \Sigma^i A_2 = \{ x \in \Sigma A_2 : x_0 = i + 6 \}, \ i = 0, \ldots, 3. \]
Furthermore
\[ \Sigma A_3 = \Sigma^0 A_3 \cup \Sigma^1 A_3 \cup \Sigma^2 A_3, \]
where
\[ \Sigma^i A_3 = \{ x \in \Sigma A_3 : x_0 = i + 10 \}, \ i = 0, \ldots, 2. \]
Sets \( \Sigma^i A_j \) are elementary sets.

**Example 3.9** For any \( k \) we can give an example of SFT with nonwandering set consisting of exactly \( k \) basic and elementary sets. We present an example for \( k = 4 \). Consider the following matrix:
\[ A = \begin{bmatrix} C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{bmatrix}, \quad \text{where} \ C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

The oriented graph associated with \( A \) can look like this:
Let $\sigma = \sigma|_{\Sigma_A}$. It is easy to see, that

$$\Sigma_A = \Omega(\sigma) = B_1 \cup B_2 \cup B_3 \cup B_4$$

where $B_1, B_2, B_3, B_4$ are basic and elementary sets with $a = 2$.

References


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