EXIT CHARTS FOR TURBO RECEIVERS IN MIMO SYSTEMS

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ABSTRACT
Iterative exchange of information between a decoder and a front end receiver was shown to improve the performance when comparing to non-iterative approach. This so-called turbo processing may be described using Extrinsic Information Transfer (EXIT) charts which were already used in case of turbo decoders and turbo-equalizers. This work extends the EXIT charts technique to describe the behavior of MIMO narrowband turbo receivers and explains the main difficulties in the EXIT analysis for Rayleigh fading channels.

1. INTRODUCTION
The growth of interest in the wireless access have lead researchers to develop new transmission techniques that can accommodate higher data rates in the limited bandwidth. MIMO (Multiple Input, Multiple Output) systems are able to transmit multiple data streams increasing significantly the spectral efficiency, but the theoretical capacity of MIMO systems [1] can be approached only if signal processing is introduced at the transmitter or the receiver.

In particular, so called turbo processing has been proposed for the MIMO receivers [2]. A Turbo receiver is composed of a front-end (FE) receiver and a channel decoder which iteratively exchange information. Turbo receivers improve significantly the overall performance measured by Bit- or Block Error Rates (BER, BLER) when comparing to non-iterative solutions.

Extensive simulations are commonly used for design and for evaluation of communication systems but simple analytical or quasi-analytical methods are always useful to reduce the simulations time and to yield optimal solutions. In case of turbo processing, Extrinsic Information Transfer (EXIT) charts have been already applied for evaluation and design of Turbo decoders [3] and Turbo Receivers [4, 5]. The idea behind the EXIT chart is to reduce the description of the whole iterative (turbo) process to the exchange of one parameter, called mutual information (MI) defined between bits and their Logarithmic Likelihood Ratios (LLR) at the output and the input of the SISO receiver/decoder [3].

In this paper we describe how to use EXIT charts for analysis of turbo receivers in MIMO systems. The main contributions when comparing to previous work on this subject [3, 5] are: description of linear FE MIMO receivers using EXIT charts and definition of limitations of EXIT charts for analysis of fading channels.

2. SYSTEM MODEL
Consider a multiantenna transmitter, using Vertical Encoding (VE) [6, 7], meaning that a single data stream is encoded and demultiplexed to M transmitter antennas (Fig 1.a) while N antennas are used for reception (cf. Fig 1.b).

A narrow-band channel is considered which leads to the following transmission model [1]

$$r(n) = H s(n) + \eta(n)$$

where \(r(n)\) is the received vector at sample time \(n\), \(H\) is a \(N \times M\) matrix whose entries are unitary norm complex gaussian, \(\eta(n)\) is a vector of complex normally distributed noise with variance \(\sigma_\eta^2\) and \(s(n) = [s_0(n), s_2(n), \ldots, s_{M-1}(n)]^T\) is composed of sub-stream symbols which belong to 4QAM constellation \(s_k(n) = \frac{1}{\sqrt{2}}(c_{k,0} + j c_{k,1})\), where \(c_{k,0}\) and \(c_{k,1}\) are coded bits. The resulting SNR is then \(M/\sigma_\eta^2\). With narrow-band assumption, time index \(n\) is dropped in the subsequent analysis. The simulations presented in this work are done for \(N=M=4\) antennas and for a 1/2 rate convolutional encoder with octal generator polynomial \([5 7]\).

3. RECEIVER ALGORITHMS
Turbo Receiver is composed of a front-end (FE) receiver and a decoder. The FE receiver should produce soft outputs under the form of Logarithmic Likelihood Ratios (LLRs) and the immediate candidate is a Maximum A Posteriori
(MAP) receiver (called here T-MAP). It produces the extrinsic LLRs of the $j$-th bit of the $k$-th substream [4, 5]:

$$\Lambda_{k,j}^{ex} = \ln \frac{P(c_{k,j} = 1|x)}{P(c_{k,j} = 0|x)} - \Lambda_{k,j}^{a}$$  \hspace{1cm} (2)

using a priori LLR $\Lambda_{k,j}^{a}$ obtained from the decoder in the previous iteration.

The complexity of the algorithm T-MAP grows exponentially with the number of antennas and the modulation constellation size, so less complex receivers using linear data combining were proposed in the literature. Such linear FE receivers consist of a soft interference canceller followed by a linear combiner of resulting signal [2, 4, 7]

$$y_{k} = w_{k}^{H}(r - b_{k}) \quad b_{k} = HE[s] - h_{k}E[s_{k}]$$  \hspace{1cm} (3)

where $b_{k}$ is the $k$-th column of matrix $H$. The symbol expectations are computed from a priori LLRs delivered by the decoder [7, 8].

The computation of the vectors $w_{k}$ depends on the criterion used to design the FE receiver. We considered the following receivers:

- **T-MRC.** The FE receiver is Maximum Ratio Combiner where $w_{k} = h_{k}$. Errors introduced by imperfect interference cancellation are ignored.

- **T-MMSE.** The FE receiver minimizes MMSE between $y_{k}$ and $s_{k}$ [2, 7, 8]. Following the ideas presented in [8], we implement its low-complexity version.

A detailed description of the algorithms may be found in [2, 7, 8].

### 4. EXIT CHARTS

The EXIT charts technique explained in details in [3, 4] allows us to analyze the exchange of information between two devices (e.g. a FE receiver and a decoder) in the two-parameters space. Mutual information $I(\Lambda; x)$ (MI) between LLRs and the corresponding bits proved to be suitable for such description. MI is obtained supposing that the input and output LLRs of each device are i.i.d. random variables defined by their pdfs $p_{\Lambda|x}(\xi|x = b)$ conditioned on the bits sent $b = -1, 1$ [3],

$$I(\Lambda; x) = \frac{1}{2} \sum_{b = -1, 1} \int_{-\infty}^{\infty} p_{\Lambda}(\xi|x = b) \log_{2} \left( \frac{2p_{\Lambda}(\xi|x = b)}{p_{\Lambda}(\xi|x = 1) + p_{\Lambda}(\xi|x = -1)} \right) d\xi$$  \hspace{1cm} (4)

Since LLR’s pdf is not known it has to be estimated from the data. Here we used a simple method presented in [9]

$$I(\Lambda; x) \approx 1 - \frac{1}{N} \sum_{n=0}^{N-1} \log_{2}(1 + \exp(-x_{n}\Lambda_{n}))$$  \hspace{1cm} (5)

where $N$ is the number of bits transmitted in the simulation and $\Lambda_{n}$ is the LLR associated with the bit $x_{n}$. Each device transforming LLRs (i.e. the FE receiver or the decoder) is characterized by the transfer function relating MI defined for the input and the output. Let $I_{o}^{R} = I(\Lambda_{o}^{c}; c)$ and $I_{o}^{D} = I(\Lambda_{o}^{D,ex}; c)$ are input and output MI of the receiver associated with respective LLRs. Then the receiver and the decoder are described by their EXIT functions $I_{o}^{R} = f_{MI}(I_{o}^{R})$ and $I_{o}^{D} = f_{MI}(I_{o}^{D})$ where $I_{o}^{R} = I_{o}^{R}$ and $I_{o}^{D} = I(\Lambda_{o}^{D,ex}; c)$ are the input and output MI defined for the decoder’s LLRs.

The numerical procedure for obtaining the EXIT transfer function can be described as follows [4]. Randomly generate input bits $x_{n} \epsilon \{-1, 1\}$ and next LLRs $\Lambda_{n}$ according to gaussian distribution with variance $\sigma_{1}^{2}$ and mean $x_{n} \cdot \sigma_{1}^{2}/2$. Apply a SISO device (decoder or FE receiver) to the sequence $\Lambda_{n}$. From the obtained output LLRs calculate MI using Eq.(5). The input MI may be obtained via numerical integration of Eq.(4) (because pdf of LLR is known to be gaussian).

Repeating the above-defined procedure for different values of $\sigma_{1}^{2}$ results in pairs $(I_{o}^{R}, I_{o}^{D})$, $(I_{o}^{P}, I_{o}^{D})$ which are used as knots to interpolate functions $f_{MI}^{R}(\cdot)$ and $f_{MI}^{P}(\cdot)$. Note
that for the FE receiver, received data for given channel state and SNR, has to be simulated as well.

The output of one device is the input to the another, so both transfer function are shown in the same EXIT plane with axes \((I_R^o, I_R^i)\). The lines connecting MI points evaluated during iterations are called a trajectory. The great advantage of using EXIT charts comes from the fact that the mean trajectory (averaged over different realization of data and noise) measured for both devices actually exchanging information falls on their EXIT functions obtained independently \([3, 4]\) i.e.

\[
I_{o,m}^D = f_{M^D}(I_{o,m}^R), \quad I_{R,m}^R = f_{R^D}(I_{R,m}^D)
\]

(6) where \(I_{o,m}^R\) and \(I_{R,m}^D\) are output MIIs produced in \(m\)-th iteration. The relation \(\text{Eq.}(6)\) is usually well satisfied for a particular channel realization as will be shown further on.

The convergence occurs at the crossing of EXIT functions of the decoder and the FE receiver. This graphical representation gives us the immediate insight into the number of iterations required to attain the best performance. Also, because MI at the decoder’s output may be mapped into the final BER/BLER, the EXIT chart allows us to compare turbo receivers in terms of their performance \([3, 4]\) (i.e. higher the final MI is, lower the corresponding BER/BLER will be).

### 4.1. EXIT for fading channels

The EXIT charts were applied for fading channels in context of turbo decoders \([3]\) but we will show that such application requires particular attention in case of turbo-receivers.

To describe the process for fading channels let us analyze the transition from the mean MI at the FE receiver’s output \(I_R^o = E[I_R^o]\) to the mean MI at the decoder’s output \(I_R^D = E[I_R^D] = E[f_{M^D}(I_R^o)]\); in both equations, expectations are taken over different realization of the channel.

The mean EXIT functions will define the trajectory if \(E[f_{M^D}(E[I_R^D])] = E[f_{M^D}^{R^D}(E[I_R^R])]\) which may be satisfied only when the function \(f_{M^D}^{R^D}(\cdot)\) is linear over the interval of variations of \(I_R^R\). Therefore the EXIT analysis for fading channels will be accurate only if the distribution of \(I_R^R\) is very “narrow” function with mean close to 0.5 where \(f_{M^D}^{R^D}(\cdot)\) is almost linear with \(I_R^D\). When MI approaches 1.0, we should always expect the discrepancies because the function \(f_{M^D}^{R^D}(\cdot)\) become highly non-linear. This analysis is illustrated in the next section.

### 5. NUMERICAL RESULTS

Figure 2 shows the EXIT charts obtained by means of turbo receivers defined in Section 3 (T-MAP, T-MRC, T-MMSE), for a fixed channel matrix \(H\) with entries:

\[
[0.926 -1.187i \quad -0.24 -0.535i \quad 1.305 +0.184i \quad 0.483 -0.852i; \quad -0.432 -0.235i \quad 0.448 +0.122i \quad -0.32 -0.007i \quad 0.507 +0.417i; \quad -0.211 -0.877i \quad 0.649 +0.294i \quad 0.316 -0.209i \quad -0.969 -0.312i; \quad -0.198 -0.688i \quad -1.054 +0.14i \quad 0.44 +0.371i \quad 0.948 -0.304i]
\]

Solid lines represent the EXIT function obtained for each device using the method presented in section 4. Dashed lines are trajectories whose vertices are computed when running actual simulations of the Turbo Receiver.

We observe that they all fall accurately on the EXIT functions of the corresponding devices; this illustrates the usefulness of EXIT chart analysis.

The EXIT charts obtained for studied turbo receivers in fading channels are shown in Fig.3. In this case the EXIT trajectory and the FE receiver EXIT function are obtained through averaging over different channel realizations. We may observe that EXIT chart looses its useful interpretation, especially for T-MRC, because some of the vertices in the trajectory do not coincide with the decoder’s EXIT function. To illustrate our explanation of this very fact (cf. Section...
4.1) we show in Fig.4 the pdf of $I_o^D$ (obtained by means of histograms) for the first and the second iteration of the algorithm T-MRC; the decoder’s EXIT function $f_{MI}^D(\cdot)$ is also show the indicate non-linearity regions.

From the Fig.4 we may see that the pdf obtained in the first iteration falls in a linear range of the decoder EXIT function (i.e. $f_{MI}^D((0.3, 0.8))$). Then, as explained in Section 4.1, $f_{MI}^D(E[I_o^D]) \approx E[f_{MI}^D(I_o^D)]$. In the second iteration however, the pdf is shifted to the right (as expected, since the tendency is to move to higher values of MI and enters nonlinear region of $f_{MI}^D(\cdot)$; then of course $I_o^D \neq f_{MI}^D(I_o^D)$.

We observe in Fig.3 that the error between the trajectories and the EXIT function is lower for receivers T-MMSE and T-MAP, so in Fig.5 we show pdfs obtained for T-MMSE (similar to those presented in Fig.4 for T-MRC). It may be seen that variance of output MI is smaller than in case of the receiver T-MRC. Then, the linear approximation of the function $f_{MI}^D(\cdot)$ holds reasonably well at least for visual analysis; the discrepancy will be noticeable only when zooming-in the exit chart close to $I_o^D \approx 1$ (i.e. in highly nonlinear region of $f_{MI}^D(\cdot)$).

6. CONCLUSIONS

In this paper, EXIT charts were applied to analyze Turbo Receivers in MIMO systems. It has been shown that in static channels, the EXIT chart is a useful tool for prediction of turbo iterations in linear and non-linear FE receivers.

It is also explained what are the main obstacles in applying EXIT chart in fading channels. We conclude that the EXIT analysis is accurate if the FE receiver produces mutual information relatively insensitive to the channel state. This assumption is well satisfied by the T-MMSE and T-MAP receivers, but not in the T-MRC case.

As a promising research venue we indicate the application of EXIT charts for different types of receivers (e.g. V-BLAST [6]) and transmitters.

7. REFERENCES


