Pseudo-BL algebras and pseudo-effect algebras

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Abstract

Pseudo-BL algebras and pseudo-effect algebras arose in two rather different fields: fuzzy logics and quantum logics. In this paper, by introducing the notion of pseudo-weak MV-effect algebra which is non-commutative generalization of weak MV-effect algebra, we investigate the mutual relationship between pseudo-BL algebras and pseudo-effect algebras. We prove that dual pseudo-BL algebra under certain condition can be restricted to a pseudo-weak MV-effect algebra, and a pseudo-weak MV-effect algebra can be extended to dual pseudo-BL algebra under certain condition. Moreover, we give some examples of pseudo-BL algebra which is corresponding to some pseudo-weak MV-effect algebra. Finally, we establish the relationship between pseudo-MV algebras and pseudo-MV effect algebras.
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1. Introduction

Recently, non-commutative fuzzy logics have been actively studied. The fuzzy logic system $psBL$, $psMTL$ were introduced in [9,10] by Hájek (also see [12]); non-commutative Lukasiewicz propositional logic $PL$ based on pseudo-MV algebras was established in [13], the non-commutative t-norms (pseudo-t-norms) are discussed in [4,18]; non-commutative fuzzy logic system PL$^*$ was established and its completeness theorem was proved in [19].

On the other hand, Foulis and Bennet in 1994 introduced the notion of effect algebra to model unsharp quantum logics (see [5]). In 2001, Dvurečenskij and Vetterlein dropped the Commutative Law of effect algebras and introduced a new quantum logic structure and called it the pseudo-effect algebra (see [2]).

Are there any common properties or common structural characteristics between pseudo-effect algebras in quantum logic and non-commutative algebraic structures in fuzzy logic? Some researchers dedicate to discuss this problem. For example, the relationship between pseudo-effect algebras and pseudo-MV algebras was considered in [3]. In this paper, we intend to investigate the relationship between pseudo-BL algebras and pseudo-effect algebras. In fact, the results of this paper are non-commutative generalization (nontrivial) of the work in [15,16], and it is a development of the works in [2,21].

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We proceed as follows. In Section 2, we introduce the notion of pseudo-weak effect algebras and pseudo-weak MV-effect algebras, and give some properties of them. In Section 3, we introduce the concept of dual pseudo-BL algebra that is category equivalent to pseudo-BL algebra, and discuss the properties of partial addition operator inducing from a dual pseudo-BL algebra. In Section 4, we discuss the relation between dual pseudo-BL algebras and pseudo-weak MV-effect algebras. Under certain condition that is called condition (W), a dual pseudo-BL algebra can be restricted to a pseudo-weak MV-effect algebra; conversely, the latter can be extended to the former with condition (W). In Section 5, we establish the relationship between pseudo-MV algebras and pseudo-MV effect algebras. In Section 6, we give two figures to show the conclusions of this paper and some results by other researchers.

2. Pseudo-weak effect algebras

In [2], Dvureèenskij and Vetterlein introduced a non-commutative quantum logic structure, pseudo-effect algebras.

**Definition 2.1 (Dvureèenskij and Vetterlein [2]).** A structure \((E; +, 0, 1)\), where "+" is a partial binary operation and 0 and 1 are constants, is called pseudo-effect algebra if, for all \(a, b, c \in E\), the following hold:

- (PE1) \(a + b + (a + b) + c\) exist if and only if \(b + c\) and \(a + (b + c)\) exist, and in this case, \((a + b) + c = a + (b + c)\).
- (PE2) There is exactly one \(d \in E\) and exactly one \(e \in E\) such that \(a + d = e + a = 1\).
- (PE3) If \(a + b\) exists, there are elements \(d, e \in E\) such that \(a + b = d + a = b + e\).
- (PE4) If \(1 + a\) or \(a + 1\) exists, then \(a = 0\).

There is an equivalent definition of pseudo-effect algebra as following (the prove is omitted):

**Lemma 2.2.** An algebra structure \((L; \leq, +, 0, 1)\) is a pseudo-effect algebra if and only if it satisfies the following properties:

- (PEA1) \((L; \leq, 0, 1)\) is a poset with a smallest element 0 and a largest element 1.
- (PEA2) + is a partial binary operation such that for any \(a, b, c \in L\),
  - (a) \((a + b) + c\) is defined if and only if \(a + (b + c)\) is defined, and in this case \((a + b) + c = a + (b + c)\).
  - (b) For all \(a \in L\), \(a + 0 = 0 + a = a\).
  - (c) If \(a + b\) exists, there are elements \(d, e \in L\), such that \(a + b = d + a = b + e\).
- (PEA3) (a) If, for \(a, b, c \in L\), \(a + c\) and \(b + c\) are defined, then \(a \leq b\) if \(a + c \leq b + c\);
  - (b) If, for \(a, b, c \in L\), \(a + c\) and \(b + c\) are defined, then \(a \leq b\) if \(c + a \leq c + b\).
- (PEA4) For \(a, b \in L\), \(a \leq b\) iff \(a + x = y + a = b\) for some \(x, y \in L\).

Now, we introduce some notions that are non-commutative generalizations of respective notions in [15,16].

**Definition 2.3 (Vetterlein [15,16]).** A pseudo-weak effect algebra is a structure \((L; \leq, +, 0, 1)\) such that the axioms (PEA1), (PEA2) and (PEA3) hold as well as the following one:

- (PEA4’) For \(a, b \in L\), if \(a \leq b\), then there is a largest \(\bar{a} \leq a\) such that \(\bar{a} + x = b\) for some \(x \in L\).

**Proposition 2.4.** Let \((L; \leq, +, 0, 1)\) is a pseudo-weak effect algebra, then, for \(a, b \in L\), if \(a \leq b\), there is a largest \(a^* \leq a\) such that \(x + a^* = b\) for some \(x \in L\).

**Proof.** If \(a \leq b\), we can find a largest \(\bar{a} \leq a\) such that \(\bar{a} + y = b\) for some \(y \in L\) by (PEA4’). For any \(a' \leq a\) such that \(x + a' = b\), using (PEA2) (c), we have \(a' + t = b\) for some \(t \in L\). Since \(\bar{a}\) is the largest element, then \(a' \leq \bar{a}\).

Consequently, \(\bar{a}\) is also the largest element such that \(\bar{a} \leq a, x + \bar{a} = b\) for some \(x \in L\). \(\square\)

**Definition 2.5 (Vetterlein [15,16]).** A pseudo-weak effect algebra \((L; \leq, +, 0, 1)\) is called a pseudo-weak MV-effect algebra, if the following conditions hold:

- (PEA5) For any \(a_1, a_2, b_1, b_2 \in L\) such that \(a_1, a_2 \leq b_1, b_2\), there is a \(c \in L\) such that \(a_1, a_2 \leq c \leq b_1, b_2\).
- (PEA6) For any \(a, b, c \in L\) such that \(c \leq a + b\), there are \(a_1 \leq a\) and \(b_1 \leq b\) such that \((\alpha)c = a_1 + b_1\), and \((\beta)a_1 = a\) in case \(a \leq c\).
(PEA7) (a) For any $a, b \in L$, there are $a_1, a_2, b_1, b_2$ such that $a = a_1 + a_2$, $b = b_1 + b_2$ and $a_1 \leq b$, $b_1 \leq a$, and $a_2 \land b_2 = 0$. (b) For any $a, b \in L$, there are $a_1, a_2, b_1, b_2$ such that $a = a_2 + a_1$, $b = b_2 + b_1$ and $a_1 \leq b$, $b_1 \leq a$, and $a_2 \land b_2 = 0$.

Definition 2.6 (Vetterlein [15,16]). A pseudo-weak MV-effect algebra $(L; \leq, +, 0, 1)$ is called a weak MV-effect algebra, if $+$ is commutative.

Proposition 2.7. If $(L; \leq, +, 0, 1)$ is a pseudo-weak effect algebra, then the following statements are true:

1. (PEA6) holds if and only if (PEA6') holds, (PEA6') as follow,

   (PEA6') For any $a, b, c \in L$ such that $c \leq b + a$, there are $b_1 \leq b$ and $a_1 \leq a$ such that $(x')c = b_1 + a_1$, and $(\beta')a_1 = a$ in case $a \leq c$.

2. If the order of $L$ is a total order and for any $a \leq c \leq a + b$, there is a $b_1$ such that $c = a + b_1$, then $(L; \leq, +, 0, 1)$ is a pseudo-weak MV-effect algebra.

3. In case (PEA6) holds, for any $a, b, c, d \in L$ such that $a + b = c + d$, then $a \leq c$ if and only if $d \leq b$.

Proof. (1) Suppose (PEA6) holds. We only consider $(\beta')$, that is, $a \leq c \leq b + a$. By (PEA2) (c), there is a $d \in L$ such that $a + d = b + a$. Then, we obtain $a \leq c \leq a + d$. For (PEA6), it follows $c = a + e$ for some $e \leq d$. Therefore, $c = f + a$ for some $f \in L$ by again (PEA2) (c). Since $c \leq b + a$, and so $f + a \leq b + a$, we have $f \leq b$ by (PEA3) (a). This means that (PEA6') holds.

Similarly, we have (PEA6) from (PEA6').

(2) According to the total order of $L$, it is easily to prove (PEA5) and (PEA7). We only show that (PEA6) holds. For any $a, b, c \in L$ such that $c \leq a + b$, there are two cases:

i) $c \leq a$. Let $a_1 = c$ and $b_1 = 0$. Then $c = a_1 + b_1$, and $a_1 \leq a$ and $b_1 \leq b$.

ii) $a \leq c$. This means that $a \leq c \leq a + b$. By the condition, there is a $b_1$ such that $c = a + b_1$. We have $a + b_1 \leq a + b$, and so $b_1 \leq b$ by (PEA3) (b). Therefore, (PEA6) holds.

3) Suppose that $a, b, c, d \in L$ such that $a + b = c + d$.

If $a \leq c$, then $a \leq c \leq c + d = a + b$. By (PEA6), there exists $e \in L$ such that $c = a + e$. From $a + b = c + d$, we have $a + b = c + d = (a + e) + d = a + (e + d)$. It follows $b = e + d$ by (PEA3). Thus, $d \leq b$.

If $d \leq b$, then $d \leq b \leq a + b = c + d$. Since (PEA6) is equivalent to (PEA6'), we have $b = m + d$ for some $m \in L$. We have $(a + m) + d = a + (m + d) = a + b = c + d$. Therefore, $a + m = c$ and so $a \leq c$. $\square$

3. Dual pseudo-BL algebras

Definition 3.1 (DiNola et al. [1], Flandor et al. [4] and Georgescu and Leustean [7]). An algebra structure $(L; \leq, \otimes, \rightarrow, \leadsto, 0, 1)$ is called a pseudo-BL algebra, if the following conditions hold:

(PBL1) $(L; \leq, 0, 1)$ is a lattice with a smallest element 0 and a largest element 1.

(PBL2) $(L; \otimes, 1)$ is a monoid, that is, $\otimes$ is an associative binary operation, and $x \otimes 1 = 1 \otimes x = x$, for any $x \in L$.

(PBL3) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ and only if $y \leq x \leadsto z$.

(PBL4) $x \land y = (x \rightarrow y) \otimes x = x \otimes (x \leadsto y)$ for any $x, y, z \in L$.

(PBL5) $(x \rightarrow y) \lor (y \rightarrow x) = 1 = (x \leadsto y) \lor (y \leadsto x)$, for any $x, y \in L$.

Definition 3.2. A structure $(L; \leq, \otimes, \rightarrow, \leadsto, 0BL, 1BL)$ is a pseudo-BL algebra, where $a \leq b$ iff $b \leq _BL a$, $a \oplus b = a \otimes b$, $a \ominus b = b \rightarrow a$, $a - b = b \Rightarrow a$, $0 = 1BL$, $1 = 0BL$.

The following Lemma directly gives the definition of dual pseudo-BL algebra.

Lemma 3.3. An algebra structure $(L; \leq, \oplus, \ominus, \rightarrow, 0, 1)$ is a dual pseudo-BL algebra, if and only if the following conditions hold:

(DPBL1) $(L; \leq, 0, 1)$ is a lattice with a smallest element 0 and a largest element 1.

(DPBL2) $(L; \oplus, 0)$ is a monoid, that is, $\oplus$ is an associative binary operation, and $x \oplus 0 = 0 \oplus x = x$, for $x \in L$.

(DPBL3) $z \leq x \lor y$ iff $z \ominus y \leq x$ iff $z - x \leq y$. 
In [4], pseudo-BL algebras and dual pseudo-BL algebras are, respectively, called left pseudo-BL algebras and right pseudo-BL algebras.

For dual pseudo-BL algebras, we obtain the following result.

**Proposition 3.4.** If \((L; \leq, \oplus, \ominus, 0, 1)\) is a dual pseudo-BL algebra, then, for any \(a, b, c \in L\),

1. \((a \ominus b) - c = (a - c) \ominus b\).
2. \((c \ominus a) \ominus b = c \ominus (b \oplus a); (c - a) - b = c - (a \oplus b)\).
3. If \(a \leq b\), then \((b \ominus a) \oplus a = a \oplus (b - a) = b\).
4. \(a \ominus a = a - a = 0\).
5. If \(a \oplus b = c\), then \(a \leq c\) and \(b \leq c\).
6. If \(a \leq b \leq c\), then \(b - a \leq c - a, b \ominus a \leq c \ominus a, c - b \leq c - a\) and \(c \ominus b \leq c \ominus a\).
7. If \(a \leq b \leq c\), then \((c \ominus b) \oplus (b \ominus a) = c \ominus a\) and \((b - a) \ominus (c - b) = c - a\).

**Proof.** (1) Let \(t = (a \ominus b) - c\). Then \((a \ominus b) - c \leq t\). By (DPBL3), we have \(a \ominus b \leq c \oplus t\). By (DPBL3) again, we have \(a \leq (c \oplus t) \oplus b = c \oplus (t \oplus b), a - c \leq t \oplus b\). Thus, \((a - c) \ominus b \leq t\), this means that \((a - c) \ominus b \leq (a \ominus b) - c\). Similarly, let \(m = (a - c) \ominus b\), we also have \((a \ominus b) - c \leq m = (a - c) \ominus b\) by (DPBL3). Therefore, \((a \ominus b) - c = (a - c) \ominus b\).

(2) Let \((c \ominus a) \ominus b = t\). Then, by (DPBL3) we have \((c \ominus a) \ominus b \leq t\), \((c \ominus a) \leq t \oplus b\), \(c \leq (t \oplus b) \oplus a = t \oplus (b \oplus a), c \ominus (b \oplus a) \leq t = (c \ominus a) \ominus b\).

Conversely, let \(m = c \ominus (b \oplus a)\). Then, by again (DPBL3) we have \(c \ominus (b \oplus a) \leq m, c \leq m \oplus (b \oplus a), c \ominus a \leq m \oplus b, (c \ominus a) \ominus b \leq m = c \ominus (b \oplus a)\).

By above observation, we have \((c \ominus a) \ominus b = c \ominus (b \oplus a)\).

Similarly, we also have \((c - a) - b = c - (a \oplus b)\).

(3) If \(a \leq b\). Then \((b \ominus a) \oplus a = a \oplus (b - a) = b \vee a = b\) by (DPBL4).

(4) It is obvious by (DPBL5).

(5) Suppose \(a \oplus b = c\). Since \(a - a = 0 \leq b\), it follows that \(a \leq a \oplus b = c\) by (DPBL3). Similarly, we also have \(b \leq c\).

(6) If \(a \leq b \leq c\). Then \(c = (c \ominus a) \oplus a\) by (3), and so \(b \leq c = (c \ominus a) \oplus a\). Therefore, \(b \ominus a \leq c \ominus a\) by (DPBL3).

Similarly, we have \(b - a \leq c - a\).

By (5) we have \((c \ominus a) \ominus a \leq (c \ominus a) \ominus (b - a)\).

Using (DPBL3) we get \(c \ominus b \leq c \ominus a\).

Similarly, we have \(c = a \oplus (c - a) \leq (b - a) \oplus (a \oplus (c - a)) = b \oplus (c - a), \) and so \(c - b \leq c - a\).

(7) If \(a \leq b \leq c\). Then \((b \ominus a) \oplus a = b\) by (3). From this and above (2), we have \(c \ominus b = c \ominus ((b \ominus a) \oplus a) = c \ominus (b \ominus a)\).

And so \((c \ominus b) \oplus (b \ominus a) = ((c \ominus a) \ominus (b \ominus a)) \oplus (b \ominus a)\).

Thus, \((c \ominus b) \oplus (b \ominus a) = c \ominus a\).

Similarly, we have \(b - a \ominus (c - b) = (b - a) \ominus ((c - (a \oplus (b - a)))) = (b - a) \ominus ((c - a) - (b - a)) = c - a\).

The operation of pseudo-BL algebras and the one of pseudo-effect algebras are different, that is, the binary operation of pseudo-BL algebras is total, but the addition of pseudo-effect algebras is partial. Thus, in order to consider the relationship between pseudo-BL algebras and pseudo-weak MV-effect algebras, it is important to interconvert partial addition and total addition.

**Definition 3.5.** (i) Let \((L; \leq, \oplus, \ominus, 0, 1)\) be a dual pseudo-BL algebra, and \(a, b \in L\). If \(a = \min\{x \in L| x \oplus b = a \oplus b\}\), then \(a\) is called minimal in the sum \(a \oplus b\). Similarly, if \(b = \min\{y \in L| a \oplus y = a \oplus b\}\), then \(b\) is also called minimal in the sum \(a \oplus b\). The definition of the partial addition “+” as following, for any \(a, b \in L\), \(a + b = a \oplus b\), if
4. Relationship between pseudo-BL algebras and pseudo-weak MV-effect algebras

Remark. (1) In commutative case, the above definition is same as Definition 4.1 of [16]. (2) It follows, when we say that $\oplus$ is the total addition belonging to $+$ for a pseudo-weak MV-effect algebra, it is implicitly claimed the total addition always exits, i.e. satisfying condition (Ex).

Proposition 3.6. Let $(L; \leq, \oplus, \ominus, -, 0, 1)$ be a dual pseudo-BL algebra, and $+$ is the partial addition belonging to $\oplus$. Then, the following conditions are equivalent for any $a, b \in L$:

(1) $a + b$ is defined and $a + b = c$;

(2) $c \ominus b = a$ and $c - a = b$.

Proof. (1) $\Rightarrow$ (2): Suppose $a + b = c$. Then $a \otimes b = c$, and so $c \leq a \oplus b$. Thus, we have $c \oplus b \leq a$, $c - a \leq b$ by (DPBL3). From $a \otimes b = c$ we have $a \leq c$ and $b \leq c$ by Proposition 3.4(5). So, we have the following result by Proposition 3.4(3):

$$c = (c \ominus b) \oplus b, \quad c = a \ominus (c - a).$$

Because $a$ and $b$ are minimal in $a \oplus b$, we obtain $a \leq c \ominus b$ and $b \leq c - b$. Consequently, $c \ominus b = a$ and $c - a = b$.

(2) $\Rightarrow$ (1): According to $b = c - a$, we have $b = c - a \leq c$ (since $c \leq a \oplus c$ by Proposition 3.4(5), so $c - a \leq c$ by (DPBL3)). Thus, from $c \ominus b = a$ and $b \leq c$ we have $a \otimes b = (c \ominus b) \oplus b = c$. For any $x, y \in L$ such that $x \oplus b = c$ and $a \oplus y = c$. By (DPBL3), $c \ominus b \leq x, c - a \leq y$ and so $a \leq x, b \leq y$. Thus, $a$ as well as $b$ are minimal in the sum $a \oplus b$. So, $a + b$ is defined and equals to $c$.

Proposition 3.7. Let $(L; \leq, +, 0, 1)$ be a pseudo-weak MV-effect algebra, $\oplus$ is the total addition belonging to $+$, then the following results hold:

(1) For any $a, b \in L$ such that $a \leq b$, then $a \oplus c \leq b \oplus c$ and $c \oplus a \leq a \oplus b$.

(2) For any $a, b \in L$ such that $a \leq b$, there are smallest element $x \in L$ and $y \in L$ such that $a \oplus x = b$, $y \oplus a = b$.

(3) For any $a, b, c \in L$ such that $c \leq a \oplus b$ there are $a_1 \leq a$ and $b_1 \leq b$ such that ($\alpha$)$(c = a_1 \oplus b_1)$ and ($\beta$)$a_1 = a$ in case $a \leq c$, or $b_1 = b$ in case $b \leq c$.

(4) For any $a, b \in L$, there are $a_1, b_1, c \in L$ such that $a = a_1 \oplus c, b = b_1 \oplus c$, and $a_1 \wedge b_1 = 0$.

Proof. (1) For any $m \leq a, n \leq c$ such that $m + n$ be defined, we have $m \leq a \leq b$ and $n \leq c$. By the definition of $\oplus$, it follows that $m + n \leq b \oplus c$. Thus $a \oplus c \leq b \oplus c$. Similarly, we have $c \oplus a \leq a \oplus b$.

(2), (3), (4): It is similar to the proof of [16, Theorem 4.3], they also hold in the non-commutative case.

4. Relationship between pseudo-BL algebras and pseudo-weak MV-effect algebras

Definition 4.1. A pseudo-BL algebra $(L; \leq, \otimes, \rightarrow, \neg, 0, 1)$ is called a pseudo-BL algebra with condition (W), if it satisfies the following condition:

$$(W)x = (x \rightarrow y)\rightarrow y \iff x = (x \rightarrow y) \rightarrow y \quad \forall x, y \in L.$$ 

Definition 4.2. A dual pseudo-BL algebra $(L; \leq, \oplus, \ominus, -, 0, 1)$ is called a dual pseudo-BL algebra with condition (W), if it satisfies the following condition:

$$(W) a = c \ominus (c - a) \iff a = c - (c \ominus a) \quad \forall a, b \in L.$$ 

Remark. (1) The condition (W) in Definition 4.1 is dual of the condition (W) in Definition 4.2, so we use a same name. Where, “W” means “well”. (2) The condition (W) is used in the proof of following Theorem 4.5 (please see (PEA2))(c) in the proof of Theorem 4.5).
(3) In fact, the condition (W) appears in other literatures. For example, in the quantale theory (the notion of quantale was introduced by C.J. Mulvey to give a non-commutative extension of the locale concept that could be applied to spaces related to the foundations of quantum theory), the following notion is introduced (see [17, Definition 1.3] or the definition before Theorem 1 in [14], also see (Definition 17) in “Quantales” by Grzegorz Bancerek from http://mizar.org/JFM/Vol16/quantall.html): A system $A = (A, \lor, \otimes)$ is called a quantale if $(A, \lor)$ is a complete lattice, $(A, \otimes)$ is a semigroup and $\otimes$ distributes on both sides over arbitrary $\lor$’s. An element $d$ of a quantale $A$ is called dualizing if

$$(x \to_r d) \to_l d = x = (x \to_l d) \to_r d$$

for any $x \in A$ where $\to_r d(\to_l d$, respectively) denotes the right adjoint to the endofunctor $d \otimes - (\otimes d$, respectively). It is easy to see that “condition (W)” is similar to “dualizing”.

(4) Moreover, the notion of “good pseudo-BL algebras” in [8] and the notion of “pseudo-BCK lattice with condition (pC)” in [11] are all near the condition (W).

There are two examples to show that some dual pseudo-BL algebras satisfy condition (W).

**Example 4.3 (Hájek [9]).** Let $\mathbb{NS}[0,1]$ (non-standard interval [0,1]) be the ordered set whose elements are pairs $(a, b)$ such that $a = 0$ and $b \geq 0$ or $0 < a < 1$ and $b$ arbitrary or $a = 1$ and $b \leq 0$ (b running on real set). The ordering is lexicographic: $(a, b) \leq (c, d)$ if and only if $a < c$ or $(a = c$ and $b \leq d)$. The ordered set $\mathbb{NS}[0, 1]$ endowed with the operations

$$(a, b) \otimes (c, d) = \max \left\{ (0, 0), \left( \frac{1}{2} (a + c - 1 + ac), \frac{b(c + 1)}{2} + d \right) \right\},$$

$$(a, b) \to (c, d) = (a, b) \sim (c, d) = 1 \quad \text{for } (a, b) \leq (c, d) \quad \text{otherwise} :$$

$$(a, b) \to (c, d) = \left( \frac{2c - a + 1}{1 + a}, \frac{2d - 2b}{1 + a} \right), \quad (a, b) \sim (c, d) = \left( \frac{2c - a + 1}{1 + a}, \frac{-b(c + 1)}{1 + a} + d \right).$$

Then $(\mathbb{NS}[0, 1]; \leq, \otimes, \to, \sim, 0, 1)$ is a pseudo-BL algebra. It is easy to verify that for any $(a, b), (c, d) \sim \mathbb{NS}[0, 1] (a, b) = [(a, b) \to (c, d)] \sim (c, d)$ if and only if $(a, b) = [(a, b) \sim (c, d)] \to (c, d)$. So we know that the pseudo-BL algebra of $(\mathbb{NS}[0, 1]; \leq, \otimes, \to, \sim, 0, 1)$ satisfies condition (W).

**Remark.** The preceding remark show that “condition (W)” is similar to the property “dualizing”. However, “dualizing” is definitely much stronger than “condition (W)”. In fact, let $(\mathbb{NS}[0, 1]; \leq, \otimes, \to, \sim, 0, 1)$ be the pseudo-BL algebra in Example 4.3, we know that $(\mathbb{NS}[0, 1]; \leq, \otimes, \to, \sim, 0, 1)$ satisfies condition (W), but for some $d \sim \mathbb{NS}[0, 1]$, the condition $(x \to d) \sim d = x = (x \sim d) \to d$ is not true for any $d \sim \mathbb{NS}[0, 1]$. For example, let $d = (1, -0.5)$, we can get the following results:

$$[(1, -0.5) \to (0.5, 0.5)] \sim (0.5, 0.5) = (1, -0.5) = [(1, -0.5) \sim (0.5, 0.5)] \to (0.5, 0.5);$$

$$[(0, 1) \to (0.5, 0.5)] \sim (0.5, 0.5) \neq (0, 1) \neq [(0, 1) \sim (0.5, 0.5)] \to (0.5, 0.5), \text{ but } [(0, 1) \to (0.5, 0.5)] \sim (0.5, 0.5) = [(0, 1) \sim (0.5, 0.5)] \to (0.5, 0.5);$$

$$[(0.6, 1) \to (0.5, 0.5)] \sim (0.5, 0.5) \neq (0, 1) \neq [(0.6, 1) \sim (0.5, 0.5)] \to (0.5, 0.5), \text{ and } [(0.6, 1) \to (0.5, 0.5)] \in (0.5, 0.5) \neq [(0.6, 1) \sim (0.5, 0.5)] \to (0.5, 0.5).$$

This means that the condition (W) is different and new.

**Example 4.4 (Flondor et al. [4]).** Let $(G; \lor, \land, +, -) = 0$ be an arbitrary $I$-group and let “$\perp$” be a symbol distinct from the elements of $G$. If $G^- = \{ x' \in G | x' \leq 0 \}$, then we define on $G_L = \{ \perp \} \cup G^-$ the following structure:

$$x' \otimes y' = \begin{cases} x' + y' & \text{if } x', y' \in G^- \\ \perp & \text{otherwise} \end{cases}$$
If we put $\perp \leq x'$ for any $x' \in G_L$, then $(G_L; \leq)$ becomes a lattice with first element “$\perp$”, and last element “$0$”. Then the structure $(G_L; \vee_L = \vee, \wedge_L = \wedge, \ominus, \rightarrow, \Rightarrow, 0_L = \perp, 1_L = 0)$ is a pseudo-BL algebra. Then, the structure $(G_L; \leq_L, \ominus, \ominus, -0, \perp)$ is dual pseudo-BL algebra, where $a \leq_L b$ iff $b \leq b$ and $a \ominus b = a \ominus b = a \rightarrow a$, $a - b = b = a$. We can verify that $(G_L; \leq_L, \ominus, \ominus, -0, \perp)$ is dual pseudo-BL algebra with condition (W).

**Theorem 4.5.** Let $(L; \leq, \ominus, \ominus, -0, 1)$ be a dual pseudo-BL algebra with condition (W), and let $+$ be the partial addition belonging to $\ominus$. Then $(L; \leq, +, 0, 1)$ is a pseudo-weak MV-effect algebra with condition (Ex) and the total addition belonging to $+\ominus$ coincides with $\ominus$.

**Proof.** Firstly, we show that the total addition belonging to $+$ exists and coincides with $\ominus$. We denote the total addition to $+$ by $\oplus$. For any $a, b \in L$. Setting set $A = [a' + b]' a' \leq a, b' \leq b$ and $a' + b'$ is defined. We call that $a \oplus b$ is defined, that is, the maximum of the set $A$ exists. Setting $a \oplus b = c$, let $a' = c \ominus(c - a)$, and $b' = c - a$. By (DPBL3), we have $c \leq a \ominus (c - a)$ from $c - a \leq c - a$, and so $c \ominus(c - a) \leq a$. Thus $a' \leq a$. Since $a \oplus b = c$ and $a \leq a \ominus b$, we have $c = a \ominus b$ by (DPBL3). Thus, $b' \leq a$. On the one hand, we have $b' = c - a \leq c - a'$ from $a' \leq a$ and Proposition 3.4.6. On the other hand, we have $c \ominus(c \ominus b') \leq b'$ from $c \ominus b' \leq c - b'$, and so $c - (c \ominus b') \leq b'$. Therefore, $c - a' \leq a'$. Consequently, $\ominus a' = b'$, and we know that $a' + b'$ is defined and equals to $c$ by Proposition 3.6. It is obvious, that, for any $a' \ominus a, b' \ominus b, a' + b' = a'' \ominus b'' \leq a \ominus b = c = a' + b'$. Thus, $a' + b'$ is the maximum of set $A$. Consequently, $\ominus a'$ exists and is the same as $\ominus$.

(PEA1) is obvious by (DPBL1).

(PEA2) (a): Let $(a + b) + c$ be defined. Setting $d = a + b$ and $e = (a + b) + c$. By Proposition 3.6, we have $b = d - a, a = d \ominus b, d = e \ominus c$ and $c = e - d$. Thus, by Proposition 3.4, it follows:

$$b = (e \ominus c) - a = (e - a) \ominus c, \quad c = e - (a + b) = e - (a \ominus b) = (e - a) - b.$$ 

Thus, by Proposition 3.6, we see that $b + c$ is defined, and also $b + c = e - a$. Then, we show that $a + (b \ominus c)$ is defined. Indeed, it follows from Proposition 3.4.2 that $e \ominus (b \ominus c) = (e \ominus c) \ominus b = d \ominus b = a$. Therefore, by Proposition 3.4.6, we have $c = e - (a \ominus b)$ is defined and equals to $e$ by again Proposition 3.4.3.

Now, suppose $b + c$ and $a + (b + c)$ be defined. We show that $a + b$ and $(a + b) + c$ are defined. Let $b + c = d, a + d = e$. By Proposition 3.6, we also have $b = d \ominus c, c = d - b, d = e - a$ and $a = e \ominus d$. Thus, by Proposition 3.4, we have $b = d \ominus c = (e - a) \ominus c = (e \ominus c) - a, a = e \ominus d = e \ominus (b \ominus c) = (e \ominus c) - b$. So, $a + b$ is defined and equals to $e \ominus c$. We also have $e - (a \ominus b) = (e - a) - b = d - b = c$. From $a \ominus b = e \ominus c$ and $e = (a \ominus b)$, we have $a + (b + c) + c$ is defined and equals to $e$.

Consequently, (PEA2) (a) holds.

(PEA2) (b) is clear by (DPBL2) and Definition 3.5(i).

(PEA2) (c): Let $a + b = c$. Then, $b = c - a$ and $a = c \ominus b$. So, we have $b = c - (c \ominus b)$ and $a = c \ominus (c - a)$. By condition (W), we also have $b = c \ominus (c - b)$ and $a = c - (c \ominus a)$. Setting $e = c - b$ and $d = c \ominus a$. Thus, $b = c \ominus e$ and $a = c - d$. By Proposition 3.6, we have $b + e = c = d + a$.

(PEA3) (a): Let $a, b, c \in L, a + c$ and $b + c$ be defined, and $a \leq b$. By (DPBL4), we have $b = b \vee a = (b \ominus a) \ominus a$, and so $b + c = b \ominus c = ((b \ominus a) \ominus a) \ominus c = (b \ominus a) \ominus (a \ominus c) = (b \ominus a) \ominus (a + c)$. Thus, by Proposition 3.4(5), we have $a + c \leq b + c$. Conversely, let $a + c \leq b + c$, that is, $a + c \leq b + c$. Thus, by (DPBL3), we have $(a + c) \ominus c \leq b$. By Proposition 3.6, we also have $(a + c) \ominus c = (a + c) \ominus c = a$. Therefore, $a \leq b$.

By the same way, we also have (PEA3) (b).

(PEA4): Suppose $a \leq b$. Let $\overline{a} = b \ominus (b - a)$, and $x = b - a$. So, we have $\overline{a} = b \ominus x$. By (DPBL3), it follows from $b - a \leq b - a$ that $b \leq a \ominus (b - a)$, and so $b \ominus (b - a) \leq a$. Thus $\overline{a} \leq a$. From $b \ominus (b - a) \leq b \ominus (b - a)$, we have
Thus, for any $a = \bar{a} = -a$. By Proposition 3.4(6) and $\bar{a} \leq a \leq b$, we have $x = b - a \leq b - \bar{a}$. It follows that $b - \bar{a} = b - (b \ominus (b - a)) \leq b - a = x$. Consequently, we have $\bar{a} + x = b$. For any $a' \leq a$ such that $a' + y = b$ for some $y \in L$, we have $a' = b \ominus y = b \ominus (b - a) = \bar{a}$ from $b - a \leq b - a'$ and Proposition 3.4(6). Thus, $\bar{a}$ is the largest element $\bar{a} \leq a$.

(PEA5): Setting $a_1, a_2 \leq b_1, b_2$. Since $L$ is a lattice, $a_1 \vee a_2$ exist. Let $c = a_1 \vee a_2$. Clearly, $a_1, a_2 \leq c \leq b_1, b_2$.

(PEA6): Suppose that $a, b, c \in L$ such that $c \leq a + b$. Let $a' = a \wedge c$. Then, $a_1 \leq a, a_1 \leq c$. Let $a_1 = c \ominus (c - a')$ and $b_1 = c - c$. By the Proof of (PEA4) above, we have $a_1 + b_1 = c$. From $c - c \leq c' - a_1$, we have $c \leq c' \ominus (c - a')$ and so $c \ominus (c - a') \leq b_1$. Thus, $a_1 \leq b_1 \leq a$. According to the Proof of RDP in [15, Theorem 3.3], we have $b_1 \leq b$. In case $a \leq c$, we have $a = a'$ from $a' = a \wedge c$. So, $a_1 = c \ominus (c - a)$. Setting $d = a + b$, by Proposition 3.4, we have $a + d \ominus (d - a) = (c - a) \ominus (d - a)$, and $b_1 \ominus (c - a) = a_1 \leq a$. Thus, $c = a + b_1$.

(PEA7): (a) For any $a, b \in L$, let $a_2 = a - a, a_1 = a \ominus a, b_2 = b - a$, and $b_1 = b \ominus b$. So, by Proposition 3.4(3), we have $a_1 \oplus a_2 = (a \ominus (a - b)) \ominus (a - b) = a$ from $a - b \leq a$. Thus $a_1 \oplus a_2 = a \oplus a_2 = a$. So, there are $a_1 \leq a_1$ and $a_2 \leq a_2$ such that $a_1 \oplus a_2 = a$, by the above observation. And, $a_1 = a \ominus (a - b)$ and $b_1 \leq b_1$. Similarly, there are $b_1 \leq b_1$ and $b_2$ such that $b_1 + b_2 = b$. Consequently, we have $a_1 \leq a_1 \leq b, b_1' \leq b_1, a_2 \leq a_2$ and $a_2 \wedge b_2 \leq a_2 \wedge b_2 = (a - b) \wedge (b - a) = 0$.

By the same way, we have (PEA7) (b).

Theorem 4.6. Let $(L; \leq, +, 0, 1)$ be a pseudo-weak MV-effect algebra with condition (Ex), and let $\oplus$ be the total addition belonging to $\oplus$. Then $(L; \leq, \ominus, \ominus, 0, 1)$ is a dual pseudo- BL algebra with condition (W) and the partial addition belonging to $\oplus$ coincides with $\oplus$. Thus, $a \ominus b \subseteq \min\{x | x \ominus b = a \ominus b\}$ and $a \ominus b \subseteq \min\{x | x \ominus b = a \ominus b\}$.

Proof. According to the Proposition 3.7(2) and the following proof of lattice, we see that $\ominus$ and $-$ are well decided for any $a, b \in L$.

(DPBL1): We show that $L$ is a lattice. For any $a, b \in L$, by Proposition 3.7(4), there are $a_1, b_1, c \in L$ such that $a = a_1 \ominus c, b = b_1 \ominus c$ and $a_1 \wedge b_1 = 0$. By the definition of $\ominus$, we have $c = 0 \leq c \leq a_1 \ominus c = a$ and $c = 0 \leq c \leq b_1 \ominus c = b$. Let $x \leq a, b$. From $x \leq a = a_1 \ominus c$, we have $x = m \ominus n$ such that $m \leq a_1, n \leq c$ by Proposition 3.7(3). From $n \leq x$, there is a smallest element $m'$ such that $m' \ominus n = x$ by Proposition 3.7(2). So, $m' \leq m$. There is $r \leq L$ such that $r \leq n$ for $n \leq c$ by Proposition 3.7(2). Thus, $n \leq x \leq b_1 \ominus c = b_1 \ominus r \leq r$. By again Proposition 3.7(3), there is a $y \leq b_1 \ominus r$ such that $y = x \ominus n$. By the minimality of $m'$, it follows that $m' \leq y$. From $m' \leq y \leq b_1 \ominus r$, there are $b_1' \leq b_1$ and $r' \leq r$ such that $m' = b_1' \ominus r'$. From $b_1' \leq b_1$ and $b_2' \leq b_2$, we have $b_1' \leq a_1 \wedge b_1 = 0$. Thus, $b_1' = 0$ and $m' = 0 \ominus r' = r' \leq r$. Consequently, $x = m' \ominus n \leq r \ominus n = c$ by Proposition 3.7(1). So, $c = a \ominus b$.

From $b = b_1 \ominus c$, we have a smallest element $\bar{b}_1$ such that $b = \bar{b}_1 \ominus c$ by Proposition 3.7(2). We show that $a \ominus b = \bar{b}_1 \ominus a \ominus c \ominus \bar{b}_1$. Let $d = b \ominus a \ominus c \ominus \bar{b}_1$. From $d = \bar{b}_1 \ominus a \ominus c \ominus \bar{b}_1 = \max\{u + v + t : u \leq \bar{b}_1, v \leq a_1, t \leq c\}$, we have $b, c \leq d$. Suppose $a, b \leq y$. By Proposition 3.7(2), we have $s \leq L$ such that $y = s \ominus a \ominus s \ominus a \ominus c$. From $c \leq s \leq a \ominus c$, we obtain that $b = b_1 \ominus c$ and $b_1' \leq a \ominus c$. According to the minimality of $\bar{b}_1$, we have $\bar{b}_1 \leq b_1 \wedge a_1 = 0$, from $b_1' \leq a \ominus a_1$, we have $\bar{b}_1 = s \ominus a_1$ for some $s \leq a_1 \ominus a_1$. Thus, $a_1 = \bar{b}_1 \ominus a_1 = 0$ from $a_1 \ominus a_1$. It follows that 0 = s \leq a_1 \ominus a_1. Therefore, $d = \bar{b}_1 \ominus a_1 \ominus c \leq a \ominus a_1 \ominus c = y$ by Proposition 3.7(1). This means that $a \ominus b = d = \bar{b}_1 \ominus a_1 \ominus c$.

Consequently, $L$ is a lattice.

(DPBL2): For any $a, b, c \in L$. Let set $A = \{u + z | u \leq a \ominus b, z \leq c, u + z \leq u + z\}$ and set $B = \{x | x \leq y \ominus z | c \leq x, y \ominus z \leq c, x \leq y \ominus z\}$. By definition of $\ominus$, we have $x \leq y \ominus a \ominus b$ for any $x \leq a, y \leq b$ and $x \leq y \ominus b$ be defined. Thus, for any $m \leq B, m \leq x \vee y \leq z$. Let $x \leq y = u$, then $m = u \leq c$ and $x \leq y \leq a \ominus b, z \leq c$. We have $m \leq A$.

Conversely, for any $n \leq A, n = u \leq z \leq c$. By again the definition of $\ominus$, there are $a' \leq a$ and $b' \leq b$ such that $a' + b' = a \ominus b$. Thus, $a' \leq a' + b'$ by (PEA6), there are $a_1$ and $b_1$ such that $a_1 \leq a', b_1 \leq b'$ and $a_1 \ominus b_1 = 1$. Therefore, $n = u \leq z = a_1 \ominus b_1 + z$, and so $n \in B$. By the above observation, it follows $A = B$.

Similarly, $B$ also equals to $C = \{x \leq u \leq y \leq a \ominus b, z \leq c, x \leq u \leq y \leq a \ominus b\}$. Thus, we have $A = C$. Consequently, $(a \ominus b) \ominus c = a \ominus (b \ominus c)$. Clearly, we have $x \ominus y = 0 \leq x = x$ by the definition of $\ominus$. 

Theorem 4.7. Let \((L; \leq, \oplus, \ominus, \sim, \otimes, \odot, 0, 1)\) be a dual pseudo-BL algebra, then \((L; \leq, +, 0, 1)\) is a pseudo-weak MV-effect algebra with condition \((Ex)\) if and only if \((L; \leq, \oplus, \ominus, \sim, \otimes, \odot, 0, 1)\) satisfies \((W)\), where \(+\) is same as Definition 3.5(i). Conversely, let \((L; \leq, +, 0, 1)\) be a pseudo-weak MV-effect algebra, then \((L; \leq, \oplus, \ominus, \sim, \otimes, \odot, 0, 1)\) is a dual pseudo-BL algebra with condition \((W)\) if and only if the addition belonging to \(+\) always exists, where \(\oplus, \ominus\) and \(\sim\) are same as Theorem 4.6.

5. Pseudo-MV algebras and pseudo-MV effect algebras

Definition 5.1 (Di Nola et al. [1], Georgescu and Popescu [6] and Georgescu and Leustean [7]). A pseudo-MV algebra is a structure \((M; \otimes, \oplus, _-, \sim, 0, 1)\) of type \((2, 2, 1, 1, 0, 0)\) such that the following axioms satisfied for all \(x, y, z \in M:\)

(PM1) \(x \otimes (y \otimes z) = (x \otimes y) \otimes z;\)
(PM2) \(x \otimes 1 = 1 \otimes x = x;\)
(PM3) \(x \otimes 0 = 0 \otimes x = 0;\)
(PM4) \(0^* = 1, 0^\sim = 1;\)
(PM5) \((x^\sim \otimes y^\sim) = (x^\sim \otimes y^\sim)\);  
(PM6) \(x \otimes (x^\sim \oplus y) = y \otimes (y^\sim \ominus x) = (x\ominus y^\sim) \otimes (x^\sim \oplus y) = (y\ominus x^\sim) \otimes x;\)
(PM7) \( x \bigoplus (x^\sim \otimes y) = (x \otimes y^\sim) \oplus y \);

(PM8) \( (x^\sim)^\sim = x \);

where \( \ominus x \equiv (x^\sim \otimes y^\sim) \).

**Definition 5.2** (*Di Nola et al. [1], Georgescu and Popescu [6] and Georgescu and Leustean [7]*). A dual pseudo-MV algebra is a structure \((M; \oplus, *, \sim, 0, 1)\) of type \((2, 2, 1, 1, 0, 0)\) such that the following axioms satisfied for all \(x, y, z \in M:\)

(DPM1) \( x \oplus (y \oplus z) = (x \oplus y) \oplus z \);

(DPM2) \( x \oplus 0 = 0 \oplus x = x \);

(DPM3) \( x \oplus 1 = 1 \oplus x = 1 \);

(DPM4) \( 1^\sim = 0, 1^\sim = 0 \);

(DPM5) \( (x^\sim \oplus y^\sim)^\sim = (x^\sim \oplus y^\sim) \sim \);

(DPM6) \( x \oplus (x^\sim \ast y) = y \oplus (y^\sim \ast x) = (x \ast y^\sim) \oplus y = (y \ast x^\sim) \oplus x \);

(DPM7) \( x \ast (x^\sim \oplus y) = (x \oplus y^\sim) \ast y \);

(DPM8) \( (x^\sim)^\sim = x \);

where \( x \ast y \equiv (x^\sim \otimes y^\sim) \sim \).

**Remark.** In this paper, pseudo-MV algebras and dual pseudo-MV algebras are, respectively, left pseudo-MV algebras and right pseudo-MV algebras (see [4,11,20]).

**Lemma 5.3.** (i) Let \((M; \oplus, *, \sim, 0, 1)\) be a dual pseudo-MV algebra. Then, \((M; \leq, \oplus, \otimes, \sim, 0, 1)\) is a dual pseudo-BL algebra with condition (W), where for any \(x, y \in M\), \(y \ominus x \equiv y \ast x^\sim, y - x \equiv x^\sim \ast y \).

(ii) Let \((M; \ominus, *, \sim, 0, 1)\) be a pseudo-MV algebra. Then, \((M; \leq, \ominus, \rightarrow, \sim, 0, 1)\) is a pseudo-BL algebra with condition (W), where for any \(x, y \in M\), \(x \rightarrow y \equiv y \ominus x^\sim, x \sim y \equiv x^\sim \ominus y \).

**Proof.** According to Propositions 7.7 and 7.8 in [4], we need only to show that \((M; \leq, \oplus, \otimes, \sim, 0, 1)\) and \((M; \leq, \ominus, \rightarrow, \sim, 0, 1)\) satisfy condition (W). \(
\)

For any \(x, y \in M\), it follows from Proposition 6.2(2) in [4] that

\[
x \ominus (x - y) = x \ast (y^\sim \ast x^\sim) = x \ast (x \ominus y) = (y \ominus x^\sim) \ast x = (x \ast y^\sim) \sim x = x - (x \ominus y).
\]

Thus, \((M; \leq, \ominus, \rightarrow, \sim, 0, 1)\) satisfies condition (W).

For any \(x, y \in M\), it follows from Proposition 6.7 in [4] that

\[
(x \sim y) \ast \sim y = (y \ominus x^\sim) \ominus \sim y = (x \ominus y^\sim) \ominus y = (y \ominus (x^\sim \ominus x)) = y \ominus (y^\sim \ominus y^\sim) = (x \sim y) \rightarrow y.
\]

Thus, \((M; \leq, \ominus, \rightarrow, \sim, 0, 1)\) satisfies condition (W). \(
\)

**Definition 5.4** (*Dvurečenskij and Vetterlein [3]*). Let \((E; \leq, +, 0, 1)\) be pseudo-effect algebra.

(a) We say that \(E\) fulfils the weak Riesz decomposition property (RDP) for short, if for any \(a, b_1, b_2 \in E\) such that \(a \leq b_1 + b_2\), there are \(d_1, d_2 \in E\) such that \(d_1 \leq b_1, d_2 \leq b_2\) and \(a = d_1 + d_2\).

(b) We say that \(E\) fulfils the strong Riesz decomposition property (RDP2) for short, if for any \(a_1, a_2, b_1, b_2 \in E\) such that \(a_1 + a_2 = b_1 + b_2\), there are \(d_1, d_2, d_3, d_4 \in E\) such that \(i) d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2\), and \(ii) d_2 \land d_3 = 0\).

**Proposition 5.5.** Let \((E; \leq, +, 0, 1)\) be a pseudo-effect algebra, then

(i) If \(E\) is lattice-ordered, then \(E\) fulfils (PEA5);

(ii) \(E\) fulfils RDP0 if and only if it fulfils (PEA6);

(iii) If \(E\) fulfils RDP2, then it fulfils (PEA7).
Proof. (i) For any \(a_1, a_2, b_1, b_2 \in E\) such that \(a_1, a_2 \leq b_1, b_2\). Let \(c = a_1 \vee a_2\). Thus, \(a_1, a_2 \leq c \leq b_1, b_2\).

(ii) We only show that for any \(a \leq c \leq a + b\), there is \(b_1 \in E\) such that \(c = a + b_1\) and \(b_1 \leq b\). If \(a \leq c\), there is \(b_1 \in E\) such that \(c = a + b_1\) by (PEA4). Thus, \(a + b_1 = c \leq a + b\). Then, it follows from (PEA3) that \(b_1 \leq b\).

(iii) For any \(a, b \in E\), there are \(m, n\) such that \(a + m = b + n\). By RDP2, there are \(d_1, d_2, d_3, d_4 \in E\) such that \(a = d_1 + d_2, b = d_1 + d_3\) and \(d_2 \wedge d_3 = 0, d_1 \leq a, d_1 \leq b\). □

**Definition 5.6.** A pseudo-effect algebra \((E; \preceq, +, 0, 1)\) is called pseudo-MV-effect algebra, if it satisfies (PEA5), (PEA6) and (PEA7).

By the definition of pseudo-weak MV-algebras, we know that every pseudo-MV effect algebras is a pseudo-weak MV-effect algebra. Thus, if a pseudo-MV effect algebra fulfils condition (Ex), it is called a pseudo-MV effect algebra with condition (Ex).

**Theorem 5.7.** Let \((E; \preceq, +, 0, 1)\) be a pseudo-MV effect algebra with condition (Ex), and let \(\oplus\) be the total addition belonging to \(+\). Then, \((E; \oplus, *, \cdot, \sim, 0, 1)\) is a dual pseudo MV-algebra, where for any \(x, y \in E\), \(x \sim \equiv u\) if \(u + x = 1, x \preceq v\) if \(v + x = 1\), \(x \equiv y\) if \(x \wedge y = 0, x \equiv y\) if \(x \vee y = 0\) and \(y \cdot x \equiv (x \oplus y)^\sim\).

Proof. By the proof of Theorem 4.6, we know that \(E\) is lattice ordered. By Proposition 5.5(ii), RDP0 holds in \(E\). Thus, according to Theorem 3.2 in [3], \((E; \oplus, *, \cdot, \sim, 0, 1)\) is a dual pseudo-MV algebra (Note that the notion of pseudo-MV algebra in [3] conform to the notion of dual pseudo-MV algebra in this paper).

**Theorem 5.8.** Let \((M; \oplus, *, \sim, \sim, 0, 1)\) be a dual pseudo-MV algebra, then \((M; \preceq, +, 0, 1)\) be a pseudo-MV effect algebra with condition (Ex), where \(+\) is the partial addition belonging to \(\oplus\).

Proof. At first, we show that \((M; \preceq, \ominus, \sim, 0, 1)\) is a dual pseudo-BL algebra with (W) by Lemma 5.3, where for any \(x, y \in M\), \(y \ominus x \equiv y \cdot x^\sim, y \cdot x \equiv x^\sim \cdot y\). According to Theorem 4.5, \((M; \preceq, +, 0, 1)\) is a pseudo-weak MV-effect algebra with condition (Ex). Thus, we only prove that \((M; \preceq, +, 0, 1)\) satisfies (PEA4).

For any \(a, b \in M\) such that \(a \leq b\), \(b \ominus (b - a) = b * (a^\sim \cdot b) = b * (b \ominus a) = b \wedge a = a\) by Proposition 6.2(2) in [4]. Thus, \(\overline{a} = a\) in the proof of (PEA4') in Theorem 4.5. So, \(b = \overline{a} + x = a + x\). Then, by (PEA2)(c), there are \(y\) such that \(y + a = b\). Conversely, suppose \(a + x = b\). It follows from (PEA3) and \(0 \leq x\) that \(a = a + 0 \leq a + x = b\).

Therefore, (PEA4) holds in \((M; \preceq, +, 0, 1)\), i.e. \((M; \preceq, +, 0, 1)\) is a pseudo-MV-effect algebra with condition (Ex).

6. Conclusion

We have established the mutual relation between pseudo-BL algebras and pseudo-weak effect algebras by restricting the total operation to a partial one and extending the partial operation to a total one in a natural manner. Some sufficient conditions from one algebraic structure to another algebraic structure are given. Especially, the sufficient and necessary condition for dual pseudo-BL algebra to be a pseudo-weak MV-effect algebra is obtained. Since every pseudo-BL algebra is a non-commutative residuated lattice, every pseudo-weak MV-effect is a pseudo-effect algebra, so the conclusions obtained in present paper may be generalized to the corresponding ones between non-commutative residuated lattice and pseudo-effect algebra, we will investigate the topic in another paper.

Moreover, in present paper we introduced the notion of pseudo-BL algebra with condition (W), and give some examples. But we do not know if there exist pseudo-BL algebras that are not pseudo-BL algebra with condition (W), this is an open problem. It is similar with the following open problem in [8,11]: whether there exist pseudo-BL algebras that are not good.

The following figures (Figs. 1 and 2) show important results in [15,16] and present paper:

Remark. (1) In the figures above, “\(\rightarrow\)” means implication, “\(\leftrightarrow\)” means categorically equivalent, “\(\dashv \rightarrow\)” with (Ex) means that the total addition belonging \(\oplus\) always exists. (2) From figures above we know that weak MV-effect algebras can be called BL-effect algebra, and pseudo-weak MV-effect algebras can be called pseudo-BL effect algebra.
Fig. 1. BL-algebras and effect algebras.

Fig. 2. Pseudo-BL algebras and pseudo-effect algebras.

References