Robust model predictive control for LPV systems using relaxation matrices


Abstract: A method of computing a new model predictive control (MPC) law for linear parameter varying systems with input constraints is proposed. The proposed method improves feasibility and system performance by deriving a new sufficient condition for the cost monotonicity. The control problem is formulated as a minimisation of the upper bound of finite horizon cost function satisfying the sufficient conditions. The relaxation matrices yield less conservative sufficient condition in terms of linear matrix inequalities so that it allows to design a more robust MPC. A numerical example is included to illustrate the effectiveness of the proposed method.

1 Introduction

Model predictive control (MPC), also called moving or receding horizon control, is an effective control method that allows to handle multi-variable linear systems with constraints systematically. Moreover, other advantages are known that MPC can easily handle time-varying system and provide good tracking performance. For instance, see the works such as Cuzzola et al. [1], Lu and Arkun [2], Kothare et al. [3] and Casavola et al. [4], [9]. Linear parameter varying (LPV) system is one of good applications for which MPC can show its full ability. LPV system is a linear system of which elements of state-space matrix are represented by affine functions of some time-varying parameters. Those parameters can be measured in real time. Recently, some results are presented for MPC controller design of LPV system [2, 4–6]. Especially, Lu and Arkun [2] proposed quasi-min–max MPC algorithms for LPV system. The MPC algorithm is called ‘quasi’ because the first stage cost can be computed without uncertainty. Using this method, they developed infinite horizon scheduling MPC algorithm for LPV plant. Wada et al. [6] have proposed an MPC method which is derived using the parameter-dependent Lyapunov function such as that of Cuzzola et al. [1] for LPV systems. However, Lu and Arkun [2] and Wada [6] proposed only an MPC law for polytopic LPV systems. The MPC technique for polytopic LPV systems has the problem of huge on-line computational burdens since the number of linear matrix inequalities (LMIs) grows exponentially with the number of uncertainties and the prediction horizon $N$. Moreover, in the case that system matrices are expressed as rational function of parameters, the polytopic model cannot describe system parameters accurately [7].

Therefore, we consider a linear fractional representation (LFR) to overcome computational complexity. In this paper, we propose a new MPC method for LPV systems with LFR. The MPC technique improves feasibility and performance by deriving a new sufficient condition for the cost monotonicity. The sufficient condition for the cost monotonicity has derived by relaxation matrices. Then, we minimise the upper bound of the cost function satisfying the sufficient condition. The relaxation matrices yield less conservative sufficient conditions in terms of LMIs so that it allows to design a more better MPC. Simulation result demonstrates the benefits of the proposed MPC methodology for the LPV systems.

Throughout the paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ real matrices. $I$ denotes the identity matrix of appropriate dimension. * represents the elements above the main diagonal of a symmetric matrix. $\| \cdot \|$ denotes the norm of a given vector or matrix.

2 Problem statement and preliminary

Consider the following LPV discrete time systems

$$x(k + 1) = A(\theta(k))x(k) + B(\theta(k))u(k)$$

with input constraints

$$- \bar{u} \leq u(k) \leq \bar{u}, \quad \forall k \in [0, \infty)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, the time-varying parameter vector $\theta(k) = [\theta_1(k), \ldots, \theta_l(k)]^T \in \mathbb{R}^l$, which, for all $k > 0$, is restricted to lie in a polytope $\Theta \subseteq \mathbb{R}^l$ and we assume that the parameter vector $\theta(k)$ is measured in real time. That is, $[A(\theta(k)), B(\theta(k))] \in \Omega$ in system (1) where $\Omega$ is some pre-specified set. In this case, system (1) is analysed by model representation of polytopic uncertainty or that of structured feedback uncertainty (LFR) [3, 6, 8, 9].
An equivalent LFR of the system (1) is given by
\[
x(k+1) = Ax(k) + Bu(k) + Bp(k)
\]
\[
q(k) = Cx(k) + D_u u(k) + D_p p(k)
\]
\[
p(k) = \Delta(k) q(k)
\]
where \(x \in \mathbb{R}^n, A, B_u, B, C, D_u, D\) are constant matrices of appropriate dimensions, \(p, q \in \mathbb{R}^l\) are additional variables accounting for the time varying parameter and \(\Delta(k) \in \mathbb{R}^{n \times l}\) is a norm bounded time-varying uncertain matrix in a set, define as
\[
\Delta = \Delta(k)\Delta(k) = \text{diag} (\delta_1 I, \delta_2 I, \ldots, \delta_l I), \quad ||\delta|| \leq 1, i = 1, 2, \ldots, l \text{ for all } k \in [0, \infty),
\]
we assume that parameters \(\delta_i, i = 1, 2, \ldots, l\) are measured in real time. The details can be found in [8, Section 8.2]. To significantly simplify the synthesis, we assume that \(D_u\) is zero.

The goal of this paper is to design a stabilising control \(u(k)\) for system (1) by the MPC strategy. To find such a control, we consider the following performance index
\[
J(k) \triangleq \sum_{j=0}^{\infty} \{x(k+j)q^T Q x(k+j) + u(k+j)q^T R u(k+j)\}
\]
where \(Q \in \mathbb{R}^{n \times n} > 0\) and \(R \in \mathbb{R}^{m \times m} > 0\).

Furthermore, propose one step predictive controller at time \(k+j, j \geq 1\).
\[
\begin{align*}
\tilde{u} & \leq u(k+j)k = u^*(k+j) \leq \bar{u}, & \text{ for } j = 0 \\
\tilde{u} & \leq u(k+j)k = K(k)x(k+j) \leq \bar{u}, & \text{ for } j \geq 1
\end{align*}
\]
that robustly minimises the performance index (5). Then, the infinite horizon cost function (5) can rewritten as
\[
J(k) = J_1(k) + J_2(k)
\]
where
\[
J_1(k) = x(k)q^T Q x(k) + u(k)q^T R u(k)
\]
\[
J_2(k) = \sum_{j=1}^{\infty} \{x(k+j)q^T Q x(k+j) + u(k+j)q^T R u(k+j)\}
\]
In order to impose an upper bound of the cost (9), we define the following a quadratic function
\[
V(k+j) = x(k+j)^T P(k)x(k+j) + u(k+j)^T R u(k+j), \quad \forall k, \forall j \geq 0
\]
where \(P(k) > 0\) for any \(j \geq 0\).

At sampling instance \(k\), it is assumed that \(V\) satisfies the following robust stability constraint for all states and control inputs satisfying system (1), and \([A(k+j) B(k+j) \in \Omega], j \geq 0\)
\[
V(k+j+1) - V(k+j) < -[x(k+j)^T Q x(k+j) + u(k+j)^T R u(k+j)\]
\]
Note that the inequality (11) is called the terminal inequality [3].

By summing (11) from \(j = 1\) to \(j = \infty\) and requiring \(x(\infty) = 0\) or \(V(x(\infty)) = 0\), we obtain the upper bound of the cost function (9) as follows
\[
\max_{[A(k+j) B(k+j) \in \Omega], j \geq 1} \{J_2(k)\}
\]
\[
\leq x(k+1)^T P(k)x(k+1) \leq \gamma_2.
\]
Therefore, the model predictive controller (6), which minimises the cost function (5), can be obtained by solving the following min–max optimisation problem
\[
\max_{u(k), P(k)} \{J_1(k) + J_2(k)\}
\]
The problem given by (13) is equivalent to
\[
\min \max_{\gamma_1(k), \gamma_2(k)} \{\gamma_1(k) + \gamma_2(k)\}
\]
where \(\gamma_1(k)\) and \(\gamma_2(k)\) are upper bound of \(J_1(k)\) and \(J_2(k)\), respectively. That is,
\[
\max_{[A(k+j) B(k+j) \in \Omega], j \geq 0} \{J_1(k) \leq \gamma_1(k)\}
\]
and
\[
\max_{[A(k+j) B(k+j) \in \Omega], j \geq 1} \{J_2(k) \leq \gamma_2(k)\}
\]
Assume that the min–max problem (14) has the solution set \([u^*(k), K^*(k), P^*(k)]\), associated with the minimum cost \(J^*(k)\) at each sampling time \(k\). Then, the following lemma is needed for the cost monotonicity.

**Lemma 1:** If there exist \(P(k) > 0\) and \(K(k)\) satisfying
\[
x(k+j+1) = x(k+j) + [x(k+j)]^T P(k)x(k+j) + u(k+j)k^T R u(k+j) - x(k+j)k^T Q x(k+j)
\]
\[
\leq -[x(k+j)k^T Q x(k+j) + u(k+j)k^T R u(k+j)] + x^T (k+j)K^T (k) R K(k)x(k+j)
\]
then
\[
J^*(k+1) < J^*(k)
\]
**Proof:** Let \([u^*(k), K^*(k), P^*(k)]\) be solution of the optimisation problem at time \(k\) and define the minimum cost as \(J^*(k)\). Then,
\[
\Delta J^*(k) = J^*(k+1) - J^*(k)
\]
\[
= \sum_{j=0}^{\infty} \{ ||x(k+j+1)k+1||_Q^2 + ||u^*(k+j+1)k+1||_R^2 \}
\]
\[
- \sum_{j=0}^{\infty} \{ ||x(k+j)k||_Q^2 + ||u^*(k+j)k||_R^2 \}
\]
\[
\leq \{ ||x(k+1)k+1||_Q^2 + ||u^*(k+1)k+1||_R^2 \}
\]
\[
+ ||x(k+2)k+1||_R^2 - ||x(k)k||_Q^2
\]
\[
+ ||u^*(k)k||_R^2 + ||x(k)k||_R^2 - ||x(k)k||_P^2
\]
If we use \(u^*(k), K^*(k)\) and \(P^*(k)\) instead of \(u^*(k+1), K^*(k+1)\) and \(P^*(k+1)\) at the time \(k+1\), then by
optimality
\[
\Delta J^*(k) \leq \{||x(k + 1)k + 1||^2_R + ||u^*(k + 1)k||^2_R
+ ||x(k + 2)k + 1||^2_R \}
- \{||x(k)k||^2_R + ||u^*(k)k||^2_R + ||x(k + 1)k||^2_R \}
\]
(20)
where \( ||x(k + 1)k + 1||^2_R \leq ||x(k + 1)k||^2_R \), since \( x(k + 1)k + 1 \) is measured state and \( x(k + 1)k \) is the predicted state which is expressed by (1) for any uncertainties \( \Delta(k) \in \Delta, k \geq 0 \).

Therefore, we have
\[
\Delta J^*(k) \leq \{-||x(k)k||^2_R + ||u^*(k)k||^2_R
+ ||x(k + 1)k||^2_R \}
\]
(21)

Since the inequality (11) is satisfied for any \( u^*(k + 1)k = K^*(k)x(k + N)k \), it holds
\[
\Delta J^*(k) \leq -\{-||x(k)k||^2_R + ||u^*(k)k||^2_R \}
\]
(22)

Since \( J^*(k) \geq 0 \) and strictly decreases as time goes to infinity, it plays a role as a Lyapunov function.

3 Main result

In this section, we propose a new MPC technique to design a controller for the system (1). The novel MPC improves feasibility and performance by deriving less conservative condition for the cost monotonicity. It is based on optimisation problem that minimises upper bound on worst value of the cost function \( J(k, k + N) \) subject to cost monotonicity condition. The minimisation problem (14) is solved in two steps. First, we derive a new sufficient condition for the cost monotonicity using relaxation matrices. Then, we minimise the upper bound \( y(k) + y^*(k) \) of the cost function \( J(k, k + N) \), satisfying the sufficient condition. The relaxation matrices yield less conservative sufficient condition in terms of LMIs so that it allows us to design a much better MPC.

Let us define a new variable for simplicity
\[
y(k) \triangleq x(k + 1) - x(k)
\]
(23)

Then, for state feedback \( u(k) = K(k)x(k) \), the state equation of the system (1) is rewritten as
\[
[A + B_uK(k) - I \quad I]
\]
(24)

For non-zero relaxation matrices \( N_1(k) \in \mathbb{R}^{n \times n} \), \( N_2(k) \in \mathbb{R}^{m \times n} \) and \( N_3(k) \in \mathbb{R}^{n \times n} \), the following equality is always satisfied
\[
\begin{bmatrix}
x(k) \\
p(k) \\
y(k)
\end{bmatrix}
^T
\begin{bmatrix}
N_1(k) \\
N_2(k) \\
N_3(k)
\end{bmatrix}
\begin{bmatrix}
A + B_uK(k) - I \quad I
\end{bmatrix}
\begin{bmatrix}
x(k) \\
p(k) \\
y(k)
\end{bmatrix}
= 0
\]
(25)

Using (25), we derive a new sufficient condition satisfying the terminal inequality (11) for any uncertainty \( \Delta(k) \in \Delta \) in the following theorem. The new condition can be less conservative due to scaling effect of these system relaxation matrices.

3.1 New LMI condition for cost monotonicity

In this subsection, a novel LMI condition for cost monotonicity is presented before giving our main result for MPC design.

Theorem 1: The terminal inequality (11) is satisfied for any \( \Delta(k) \in \Delta \), if there exist \( X(k), Y(k), Z(k), H(k), Q(k) = Q(k)^T > 0 \) and a diagonal matrix \( \Lambda > 0 \), satisfying the following LMI
\[
\begin{bmatrix}
-X(k) - X^T(k) & BL(k) - Y(k) \\
* & -\Lambda(k)
\end{bmatrix}
\begin{bmatrix}
* & * \\
* & *
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
D^T & 0 & 0 \\
Q^T(k)C^T(k) & Q^T(k)Q^T(k) & H(k)R^T(k)
\end{bmatrix}
< 0
\]
(26)

\[
0 & 0 & 0 \\
-\Lambda(k) & 0 & 0 \\
* & -I & 0 \\
* & * & -I
\]

where
\[
H(k) = K(k)Q(k), \quad Q(k) \triangleq P^{-1}(k).
\]
(27)

Proof: For a state feedback controller \( u(k + jk) = K(k)x(k + jk) \), the terminal inequality (11) can be written as
\[
\Delta V(k) = x(k + j + 1k)^TP(k)x(k + j + 1k)
- x(k + jk)^TP(k)x(k + jk)
= (x(k + jk) - y(k + jk))^TP(k)x(k + jk)
- y(k + jk))^TP(k)x(k + jk)
= x(k + jk)^TP(k)y(k + jk) + x(k + jk)^T
\]
\[
\begin{bmatrix}
x(k + jk) \\
p(k) \\
y(k + jk)
\end{bmatrix}
^T
\begin{bmatrix}
0 & 0 & P(k) \\
P(k) & 0 & P(k) \\
0 & P(k) & 0
\end{bmatrix}
\begin{bmatrix}
x(k + jk) \\
p(k) \\
y(k + jk)
\end{bmatrix}
< -x(k + jk)^T(Q + K^T(k)R(K)K(k)x(k + jk)
\]
(28)

where \( y(k + jk) = x(k + j + 1k) - x(k + jk) \) form (23).

Also, we obtain a following inequality from the norm bounded uncertainty property \( ||\delta_i|| \leq 1 \) for \( i = 1, 2, \ldots, I \) to
consider a norm bounded time-varying uncertain matrix Δ(k) in system (1).

\[
p(k)^{T}p(k) = (C(x + jk) + Dp(k))^T Δ(k)^T Δ(k)
\]

\[
× (C(x + jk) + Dp(k))
\]

\[
≤ (C(x + jk) + Dp(k))^T
\]

\[
× (C(x + jk) + Dp(k))
\]

Define the following matrix variable to perform S-procedure [8]

\[
\Lambda(k) = \lambda(k)I
\]

where \( \lambda(k) \) is a positive scalar.

Then, we derive the following inequality from (29) by using the variable defined in (30)

\[
\begin{bmatrix}
x(k + jk) \\
p(k) \\
y(k + jk)
\end{bmatrix}
\begin{bmatrix}
C(k)^TΛ^{-1}(k)C(k) \\
D^TΛ^{-1}(k)C(k) \\
-Λ^{-1}(k) + D^TΛ^{-1}(k)D
\end{bmatrix}
\begin{bmatrix}
x(k + jk) \\
P(k) \\
y(k + jk)
\end{bmatrix} ≥ 0
\]

After adding the left-hand side of (25) with (28), perform the S-procedure [8] with (31), then we have

\[
\begin{bmatrix}
0 & 0 & P(k) \\
0 & 0 & 0 \\
P(k) & 0 & P(k)
\end{bmatrix}
\begin{bmatrix}
N_1(k) \\
N_2(k) \\
N_3(k)
\end{bmatrix}
\begin{bmatrix}
\hat{A}(k) \\
B \\
-I
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
C(k)^TΛ^{-1}(k)C(k) \\
D^TΛ^{-1}(k)C(k) \\
-Λ^{-1}(k) + D^TΛ^{-1}(k)D
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
< - \begin{bmatrix}
\hat{Q}(k) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

For the simplicity, we define

\[
M(k) = \begin{bmatrix}
0 & 0 & Q(k) \\
0 & I & 0 \\
N_1(k) & P(k) & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
N_1(k) & 0 & P(k) \\
N_2(k) & I & 0 \\
N_3(k) & 0 & 0
\end{bmatrix}^{-1}
\]

Pre- and post-multiplying \( M(k)^{T} \) and \( M(k) \) in the left-hand side of inequality (33), respectively, then we obtain that

\[
\begin{bmatrix}
X^{T}(k) \\
Y^{T}(k) \\
Z^{T}(k)
\end{bmatrix}
\begin{bmatrix}
P(k) [X(k) & Y(k) & Z(k)] \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\hat{X}(k) - X^{T}(k) & B(k) - Y(k) & 0 \\
0 & B^{T} - Y^{T}(k) & 0 \\
0 & 0 & \hat{Y}(k) \hat{Q}(k) + X^{T}(k) - Z^{T}(k)
\end{bmatrix}
\]

By the Schur complements [8], the inequality (35) is equivalent to

\[
\begin{bmatrix}
-A(k)^{T}Q(k) + B_{1}H(k) - Z(k) + X^{T}(k) & 0 \\
0 & Y^{T}(k) & D^{T} \\
0 & 0 & Z(k) + Z^{T}(k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & Q(k) \\
0 & 0 & 0 \\
-\hat{Q}(k) & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{X}(k) & 0 & 0 \\
Y^{T}(k) & 0 & 0 \\
\hat{Y}(k) & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{Y}(k) & 0 & 0 \\
\hat{Y}(k) & 0 & 0 \\
\hat{Y}(k) & 0 & 0
\end{bmatrix}
\]

\[
< 0
\]

(33)
Perform a congruence transformation with a matrix diag \([I, \Lambda(k), I, I, I, I]\), and define
\[
\tilde{y}(k) = \Lambda(k)y(k).
\]  
(37)

Then, finally we have the inequality (26). This completes the proof.

3.2 Model predictive controller design

A solution of the min–max problem (5) is obtained by solving the minimisation problem (14), which satisfies the terminal inequality (11). In order to design a robust model predictive controller for the system (1), we transformed the minimisation problem (14) into the LMI framework satisfying the new sufficient condition for the terminal inequality.

**Theorem 2:** The min–max problem (13) subject to (6) and (11) is converted to the following LMI optimisation problem

\[
\gamma_1(k) + \gamma_2(k)
\]
subject to

\[
\begin{bmatrix}
-\gamma_1(k) & * & * \\
Q^{1/2}x(k) & -I & * \\
\mathcal{R}^{1/2}u(k) & 0 & -I
\end{bmatrix} \leq 0
\]  
(39)

\[
\begin{bmatrix}
-I & * & * \\
\Psi(k) & -\tilde{Q}(k)
\end{bmatrix} \leq 0
\]  
(40)

\[
-\mathcal{X}(k) - \mathcal{X}^T(k) \quad B - \tilde{Y}(k) \quad (A - I)\tilde{Q}(k) + B_u \\
* & -\Lambda(k) & \tilde{Y}^T(k) \\
* & * & Z(k) + Z^T(k) \\
* & * & * \\
* & * & * \\
0 & \mathcal{X}^T(k) & 0 \quad 0 \quad 0 \\
D^T & \tilde{Y}(k) & 0 \quad 0
\]

\[
\begin{bmatrix}
\tilde{Q}(k)C^T & \tilde{Q}^T(k)Q^{1/2} & H^T(k)\mathcal{R}^{1/2} \\
-\Lambda(k) & 0 & 0 \\
* & -\tilde{Q}(k) & 0 \quad 0 \\
* & * & -\gamma_2 I \quad 0 \\
* & * & * & -\gamma_2 I
\end{bmatrix} < 0
\]  
(41)

and the second inequality \(J_2(k) \leq \gamma_2(k)\) is satisfied, if and only if,

\[
\Psi(k)^TP(k)\Psi(k) \leq \gamma_2(k)
\]  
(45)

where \(\Psi(k) \triangleq A(\theta(k))x(k|k) + B(\theta(k))u(k|k)\).

The inequality (45) is converted to the following LMI by Schur complement [8]

\[
\begin{bmatrix}
-I & \ast & \ast \\
\Psi(k) & -\tilde{Q}(k) & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix} \leq 0
\]  
(46)

where \(\tilde{Q}(k) = \gamma_2P^{-1}(k)\).

Substituting \(\tilde{Q}(k) = \gamma_2P^{-1}(k), \tilde{Y} = \Lambda(k)y(k)\) with diagonal matrix \(\Lambda > 0\) and performing the same procedure as in Theorem 1, a new sufficient condition for terminal inequality is derived as follows.

\[
\begin{bmatrix}
-\mathcal{X}(k) - \mathcal{X}^T(k) \quad B - \tilde{Y}(k) \\
* & -\Lambda(k) \\
* & * \\
* & * \\
* & * \\
* & * \\
0 & \mathcal{X}^T(k) & 0 \quad 0 \quad 0 \\
D^T & \tilde{Y}(k) & 0 \quad 0
\]

\[
\begin{bmatrix}
\tilde{Q}(k)C^T & \tilde{Q}^T(k)Q^{1/2} & H^T(k)\mathcal{R}^{1/2} \\
-\Lambda(k) & 0 & 0 \\
* & -\tilde{Q}(k) & 0 \quad 0 \\
* & * & -\gamma_2 I \quad 0 \\
* & * & * & -\gamma_2 I
\end{bmatrix} < 0
\]  
(47)

For the invariant set

\[
\xi(P(k)) = \{x|x(k+j|k)\mathcal{X}(k+j|k) < \gamma_2, j \geq 1\}
\]

\[
= \{x|x(k+j|k)^T\tilde{Q}(k)^{-1}x(k+j|k) < 1, j \geq 1\}
\]  
(48)

the first input constraint in (6) is expressed as

\[-\tilde{u} \leq u(k) \leq \tilde{u}\]

(49)

and by ellipsoid constraint method [8], the second input constraint in (6) in \([u_i(k+j|k)] = |K_i(k)x(k+j|k)| < \tilde{u}, j \geq 1, i = 1, \ldots, m\), holds that

\[
\max_{j \geq 1} |u_i(k+j|k)|^2 = \max_{j \geq 1} |K_i(k)x(k+j|k)|^2
\]

\[
\leq \max_{j \geq 1} |(H\tilde{Q}^{-1}x(k+j|k))|^2
\]

\[
\leq \|H\tilde{Q}^{-1/2}\|^2
\]

\[
= \|H\tilde{Q}^{-1/2}H^T\|_2
\]  
(50)
Using Schur complement [8], the input constraints
\[ u_i(k+j) \leq \bar{u}_i, \quad j \geq 1, \quad i = 1, 2, \ldots, m \]  
are satisfied, if
\[ \begin{bmatrix} G(k) & H(k) \\ H^T(k) & Q(k) \end{bmatrix} \geq 0, \quad \begin{bmatrix} G_k(k) \end{bmatrix} \leq \bar{u}_i^2. \]  
This completes the proof.

**Remark 1:** Note that the solution of the optimisation problem has a unique minimum at each sampling time \( k \) since the LMIs are convex.

**Remark 2:** The optimisation problem (38) provides the robust MPC law \( u^*(k) \) only at \( k \)-th sampling instant. Therefore, the problem (38) must be solved at every sampling instant.

**Theorem 3:** (Feasibility) Assume that the solutions of the min–max problem (38) are feasible at sampling instant \( k = 0 \), then the solutions of the problem (38) are feasible for all sampling times \( k \geq 1 \).

**Proof:** Let denote that \( u^*(k), \mathcal{K}^*(k), \mathcal{P}(k) \) are the solutions of the min–max problem (38), associated with the minimum cost \( J^*(k) = \gamma_1^*(k) + \gamma_2^*(k) \). We show that if a solution of the problem (38) is feasible at the \( k \)-th sampling time. Since inequalities (44), (46), (47), (49) and (52) are satisfied at time \( k \), we have the following relation
\[ \gamma_1^*(k) + \gamma_2^*(k) \geq x(k)^T Q x(k) + u(k)^T R u(k) \]
\[ \geq \sum_{j=1}^m \{x(k+j)^T Q x(k+j) \]
\[ + u(k+j)^T R u(k+j) \]
\[ + x(k+2)^T P^*(k) x(k+2) \]  
(53)

Since \( x(k+1|k+1) \) is the measured state and \( x(k+1|k) \) is the predicted state which is expressed by state equation (7) for any time varying uncertain variable \( \Delta(k) \in \Delta(k), k \geq 0, x(k+1|k+1)^T Q x(k+1|k+1) \leq x(k+1|k)^T Q x(k+1|k) \), therefore, we have
\[ \gamma_1^*(k) + \gamma_2^*(k) \geq x(k)^T Q x(k) + u(k)^T R u(k) \]
\[ + x(k+2)^T P^*(k) x(k+2) \]  
(54)

The inequality (54) implies that the solution of the problem (38) is feasible at time \( k+1 \), when \( P^*(k+1), u^*(k+1) \) and \( \gamma(k+1) \) are replaced by \( P^*(k), u^*(k) \) and \( \gamma(k) \), respectively. Also, the input constraint (2) are satisfied with

**Table 1:** Simulation results

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<tr>
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</thead>
<tbody>
<tr>
<td>The maximum feasible solution (( \gamma(0) ))</td>
<td>3.2</td>
<td>5.3</td>
<td>153.8</td>
</tr>
<tr>
<td>Upper bound of cost (( \gamma(0) ))</td>
<td>138.2</td>
<td>123.7</td>
<td>80.6</td>
</tr>
</tbody>
</table>

**Table 2:** Comparison of computational complexities (cpu-time/iter)

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</thead>
<tbody>
<tr>
<td>Time/iter</td>
<td>0.2010</td>
<td>0.5692</td>
<td>0.3026</td>
</tr>
</tbody>
</table>

**Fig. 1** State responses for \( x_1(k) \)

\[ u(k+1|k+1) = u^*(k+1|k) \quad \text{and} \quad u(k+1|k+1) = K^*(k) \]

\[ (k+1|k+1) \]. Therefore, a solution of the problem (38) satisfying LMIs (44), (46), (47), (49) and (52) is feasible at all sampling time \( k \geq 1 \).

**Fig. 2** Input trajectories
We assume that $\theta_1$ and $\theta_2$ are measurable time varying parameter belonging to the following sets

$$\theta_1(k) \in [0.5, 2.5], \quad \theta_2(k) \in [1, 2].$$  (56)

The system (55) can be expressed by the equivalent system (4) with following matrices

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.15 & 0.15 & 1 & 0 \\ 0.1 & -0.1 & 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0.067 \\ 0 & 0.067 \\ 0 & -0.33 \end{bmatrix}, \quad C = \begin{bmatrix} -0.15 & 0.15 & 0 & 0 \\ -0.15 & 0.15 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 1 & -0.5 \end{bmatrix}, \quad \theta_i = \tilde{\theta}_i(I + \delta_{\theta_i}),$$

$$\Delta = \begin{bmatrix} \delta_{\theta_1} & 0 \\ 0 & \delta_{\theta_2} \end{bmatrix}, \quad \|\delta_{\theta_i}\| \leq 1, \quad i = 1, 2.$$  (57)

The objective is to design a constrained receding horizon controller for the state track the set point $x_r = [1, 1, 0, 0]^T$ with the following weighting matrices

$$Q = I, \quad R = I.$$  (58)

The feasibility and system performance of the proposed method was compared with previous results from Lu’s method [2] with common terminal weighting matrix, and Wada’s method [6], with polytopic parameter-dependent terminal weighting matrix. For fair comparison, we modified Wada’s method to have the free control input $u$. In order to show the less conservativeness of the proposed method, by varying $\alpha$ equal to maximum value of $\theta_2(k)$ from 0.5 to $\bar{\alpha}$, the maximum feasible solution $\bar{\alpha}$ is found. Also, for $\theta_1(k) = 1.5 + \sin(k)$, $\theta_2(k) = 1.5 + 0.5 \sin(k)$, we compute the upper bound of cost for each technique.

Table 1 shows the comparison of the proposed method with recent other robust MPC methods for LPV systems. One can see that the proposed method provides a feasible solution at the largest range of parameter [0.5 153.8] and the upper bound of cost, 80.6, is the smallest among the methods. Table 2 shows that the proposed method has lower computational burdens than the modified Wada’s method [6].

Figs. 1, 2 and 3 illustrate the simulation results of Lu’s method [2], Wada’s method [6] and our proposed method. The upper bound $\gamma$ of the performance index is more improved than other results, using relaxation matrices for LPV with LFR.

The simulations were performed on a PC with Pentium core2duo processor (speed 2.13 GHz, total memory 2046 MB).

5 Conclusions

In this paper, we proposed new cost monotonicity condition of MPC algorithm for a discrete-time LPV system with LFR. A less conservative results were achieved using relaxation matrices to have freedom of optimisation. The synthesis conditions were represented in the form of a finite number of LMIs. The effectiveness of the proposed methods were shown by comparing with existing results in a numerical example.

6 References