Cusp-Free, Time-Invariant, 3D Feedback Control Law for a Nonholonomic Floating Robot

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Abstract

In this paper, the feedback control of a nonholonomic 3D vehicle is considered; namely, the problem of steering an underactuated rigid body to a target position along a desired direction is addressed. A simple time-invariant strategy is determined on the basis of standard vector kinematics and a Lyapunov-like stability analysis. The resulting control law guarantees almost global exponential convergence of the configuration error to zero with paths that do not exhibit any cusps, thus satisfying a major requirement for the application of such results on real systems that are not allowed or desired to move in both the forward and backward directions.

KEY WORDS—nonholonomic systems, position control, underactuated vehicles, time-invariant control

Notations

- Vectors are underlined lowercase letters (e.g., $\underline{u}$).
- Matrices are in boldface capital letters (e.g., **J**).
- The scalar product is denoted by a dot (e.g., $\underline{x} \cdot \underline{y}$).
- The vector product is denoted by a "wedge" symbol (e.g., $\underline{x} \wedge \underline{y}$).
- Frames are denoted with lowercase letters in between a $<>$ symbol (e.g., $<a>$).
- The transpose operation is denoted with a right-hand-side capital T superscript for both vectors and matrices (e.g., $\mathbf{J}^T$ or $\underline{x}^T$).
- The projection of a vector on a given reference frame is a column vector denoted with a lowercase left superscript (e.g., ${}^a\underline{u} = ({}^a u_x, {}^a u_y, {}^a u_z)^T$ is the projection of vector $\underline{u}$ on frame $<a>$).
- A dot on top of a variable denotes its time derivative (e.g., $\dot{\alpha} = \frac{d}{dt}\alpha$).
- Z denotes the set of integer numbers.

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1. Introduction

The issue of controlling nonholonomic vehicles has received wide attention in the past few years when their application in robotics has been stressed, starting from the simple unicycle up to the more complex systems such as trailers or 3D mobile robots. Much effort has been devoted to the determination of control laws that accomplish the so-called parking maneuver, which ensures the attainment of a target position and orientation starting from any initial conditions. Indeed, such controllers may be exploited not only for automatic docking purposes but also for via-point-based navigation (Aicardi et al. 1995). The major difficulty in designing such controllers arises from the limitation imposed by Brockett's theorem (Brockett 1983), which prevents the existence of a smooth stationary feedback stabilizing law. In particular, if a dynamic system is given in the form

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i \quad ; \quad x(t) \in \mathcal{D} \subset R^n \tag{1}$$

with the vectors $g_i(x)$ being linearly independent in the origin, there exists a stationary solution to the stabilization problem of system (1) in the origin if and only if $m = n$; namely, there are as many control inputs as states. For example, the well-known Cartesian unicycle kinematic model

$$\begin{aligned} \dot{x} &= u\cos\theta \\ \dot{y} &= u\sin\theta \\ \dot{\theta} &= \omega \end{aligned}$$

is of the form (1), but $m = 2 < n = 3$; thus, it may not be stabilized in the origin by a stationary state feedback law. As noted by Brockett, the matter is completely different if the set $\{g_i(x)\}$ drops dimension precisely at $x = 0$. In this sense, the only interesting kinds of distributions are distributions with singularities. This observation may be exploited by adopting a singular description of the system model. As a consequence, controllers for the class of systems subject to Brockett's result have been designed by either smooth, but time-varying, or discontinuous, but stationary, strategies (see Kolmanowsky and McClamroch 1995). This holds also for the problem of driving a 3D underwater vehicle in a desired position and orientation having as control inputs

- the angular velocity vector
- one component of the linear velocity vector

if a kinematic model of the vehicle is considered and

- the applied torque vector
- a force vector with a fixed direction in the body fixed reference frame

if a dynamic model of the vehicle is considered. A smooth time-varying solution to the parking problem for the dynamic model of an autonomous underwater vehicle (AUV) is outlined in Pettersen and Egeland (1999), whereas an example of a piecewise smooth time-invariant solution for the kinematic model is given in Egeland, Dalsmo, and Sørdalen (1996). The controller suggested in Pettersen and Egeland (1999) guarantees local exponential stability, whereas the controller in Egeland, Dalsmo, and Sørdalen (1996) achieves global exponential convergence of the configuration error to zero but not stability in the sense of Lyapunov.

In general, with reference to systems having a representation subject to Brockett's theorem, the literature appears to have suggested solutions for two different kinds of problems: (1) global convergence plus local stability (in the sense of Lyapunov) achieved with smooth time-varying strategies and (2) global convergence of the configuration error to zero (*regulation* but not Lyapunov stability) achieved with piecewise smooth stationary strategies. This second problem, although weaker than the first, is nevertheless relevant from a practical and engineering perspective for the following reasons: (i) it allows one to better address long-range motion tasks because the resulting paths are free from the typical oscillations induced by all known time-varying solutions, (ii) it can be solved with piecewise smooth and stationary controllers, and (iii) it does not exclude the possibility of addressing the local stability issue locally, for example, switching to a different controller able to guarantee local Lyapunov stability only in a neighborhood of the target configuration.

Moreover, it should be noted that the above-cited solutions to the 3D parking maneuver, and the other solutions known in the literature such as, for example, Egeland, Berglund, and Sørdalen (1994), give rise to paths that exhibit cusps (a point at which two arcs meet from the same direction terminating with a common tangent). This occurs because it is implicitly assumed that the given vehicle may move both forward and backward. In the great majority of practical cases (if not in all of them), this assumption is incorrect, since all known slender-body AUVs are designed to move only forward, and even if the system is able to move in both directions (as some AUVs and most open-frame vehicles), usually it is not preferred because of practical considerations such as, for example, sensor allocation or thruster performance. Indeed, the presence of cusps in the paths resulting from the implementation of the above-cited laws greatly limits the range of their practical use. Another important requirement in real applications such as automatic docking is that the target configuration be approached along a straight line; namely, the path's curvature should be asymptotically null. In spite of its importance, this point is not explicitly addressed in any of the existing results known to us.

The objective of this paper was to design a stationary piecewise smooth controller able to (almost) globally exponentially drive the kinematic model of an underactuated 3D floating robot in a given target point along a desired direction while satisfying the following requirements: (1) absence of cusps or unnatural control-induced oscillations in the resulting paths, (2) asymptotically null path curvature in the approaching phase, and (3) inducement of natural, qualitatively nice paths. Following a well-established methodology (Badreddin and Mansour 1993; Casalino et al. 1994; Astolfi 1994, 1996, 1999; Aicardi et al. 1995; Indiveri 1999), the controller is designed using a polar-like description of the model that allows one to easily obtain the above-listed objectives. Because of the singularity of the polar-like model in the origin, Brockett's theorem does not apply, which leads to a control law that fully exploits this opportunity. The resulting law is discontinuous, and if remapped in the Cartesian domain it is regional, not global. Indeed, if the configuration of the vehicle is described by the vector $(x, y, z, \phi, \vartheta, \psi)^T \in R^6$, where (x, y, z) is the Cartesian position and (ϕ, ϑ, ψ) is the Euler angle orientation of the system, the domain of definition and basin of attraction of the closed-loop system is $\mathcal{D} = \{R^6 \setminus \{0, 0, 0, \phi, \vartheta, \psi)^T\} \quad \forall \quad \phi, \vartheta, \psi \in R\}$. This means that as long as the system is not already positioned in the target point at the initial time, it will converge to the target exponentially with exponentially null curvature and will always move in only one direction. Because the only position not belonging to the basin of attraction is the origin, with a slight abuse of notation the solution might be called "almost" global rather than regional. It should be noted that from a practical point of view, the difference between $\mathcal{D}$ and R^6 is not relevant. In fact, if the vehicle is already located in the origin but with the wrong orientation it requires only to rotate on the spot to reach the target configuration. On the other hand, when a docking maneuver is required, it is usually because the vehicle is *not* already located in the desired position. It should be also noted that as far as the angular variables are concerned, the solution is global in the sense that the proposed law converges and is well defined for *any* initial orientation. This is indeed an important aspect, since one could have designed a cusp-free controller combining previously known results, namely, the cusp-free discontinuous control for nonholonomic systems in chained form outlined in Astolfi (1996) and the chained-form description of the kinematic model of a 3D nonholonomic vehicle presented in Egeland, Berglund, and Sørdalen (1994). Unfortunately, any controller designed on the basis of these findings cannot be defined on the whole range of possible vehicle orientations. Indeed, the above-cited chained-form description of the kinematic model is defined *only* if one of the direction cosines describing the orientation of the vehicle is always strictly positive. As a consequence, any controller designed on the basis of that model is bound to be defined only on a limited set of all possible vehicle orientations, whereas the solution presented in this paper is global as far as orientation is concerned.

The contribution of this work consists of the design of a stationary, piecewise smooth law that guarantees exponential convergence of the configuration error to zero, constant linear motion direction (absence of cusps), and exponentially null path curvature for the kinematic model of an underactuated floating robot as an unmanned underwater vehicle (UUV). In Section 2, the problem is formulated and the adopted model is described. In Section 3, the steering control law is derived and the convergence properties are discussed. Sections 4 and 5 address implementation and robustness issues, whereas simulation examples of the behavior of the proposed strategy are reported in Section 6. Conclusions are outlined in Section 7.

2. Problem Formulation

The kinematics of the surge axis of a 3D floating robot will be described with polar-like variables. In particular, with reference to Figure 1, the following quantities are defined:

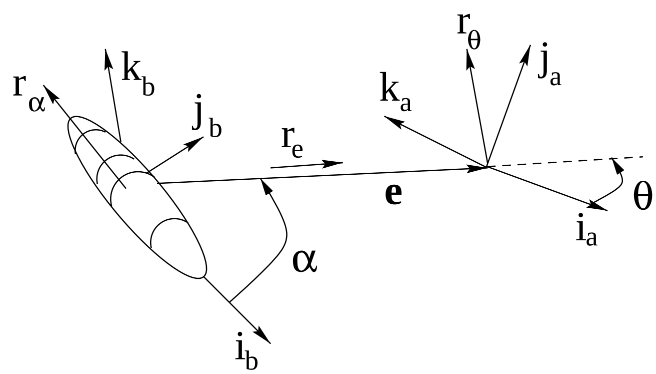

Fig. 1. The model.

$$\underline{r}_e \; : \; \|\underline{r}_e\| = 1 \tag{2}$$
$$\underline{r}_\alpha \; : \; \|\underline{r}_\alpha\| = 1 \; ; \; \underline{r}_\alpha \cdot \underline{i}_b = \underline{r}_\alpha \cdot \underline{r}_e = 0 \tag{3}$$
$$\underline{r}_\theta \; : \; \|\underline{r}_\theta\| = 1 \; ; \; \underline{r}_\theta \cdot \underline{i}_a = \underline{r}_\theta \cdot \underline{r}_e = 0 \tag{4}$$
$$\underline{e} = e \, \underline{r}_e \quad \forall \quad e > 0 \tag{5}$$
$$\underline{\alpha} = \alpha \, \underline{r}_\alpha \quad \forall \quad e > 0 \quad \forall \quad \alpha \neq n\pi \quad \forall \, n \in Z \setminus \{0\} \tag{6}$$
$$\underline{\alpha} = 0 \quad \forall \quad e > 0 \quad , \quad \alpha = 0 \tag{7}$$
$$\underline{\theta} = \theta \, \underline{r}_\theta \quad \forall \quad e > 0 \quad \forall \quad \theta \neq n\pi \quad \forall \, n \in Z \setminus \{0\} \tag{8}$$
$$\underline{\theta} = 0 \quad \forall \quad e > 0 \quad , \quad \theta = 0. \tag{9}$$

Note that the set of chosen variables appears redundant in number. In fact, the number (five) of needed variables can be extracted from $\underline{e}$, $\underline{\alpha}$, and $\underline{\theta}$. Considering that $\underline{\alpha} \cdot \underline{i}_b = 0$ and $\underline{\theta} \cdot \underline{i}_a = 0$, the vectors $\underline{\alpha}$ and $\underline{\theta}$ can be represented with four numbers only. Moreover, taking into account that only the norm of $\underline{e}$ is sufficient to complete the system description, the set of five variables required by standard mechanical considerations is recovered. Nevertheless, $\underline{e}$, $\underline{\alpha}$, and $\underline{\theta}$ will be considered in the following because their differential equations can be determined regardless of the frame they are projected onto. Note also that when $e = 0$, none of the remaining variables are defined; thus, the adopted polar-like system description is singular in the origin. The nonholonomic constraints of the vehicle can be written as

$$\underline{u} = u \, \underline{i}_b \tag{10}$$
$$\underline{\omega} \cdot \underline{i}_b = 0, \tag{11}$$

where $\underline{u}$ and $\underline{\omega}$ are the linear and angular velocities of the UUV, respectively. These constraints imply that the only actuated degrees of freedom are surge, pitch, and yaw. The addressed problem may be thus stated as follows.

PROBLEM 1. Determine a stationary feedback law for $\underline{u}$ and $\underline{\omega}$ such that $\underline{u} = u \, \underline{i}_b$, $\underline{\omega} \cdot \underline{i}_b = 0$, and $(\alpha, \theta, e) \to (0, 0, 0)$ exponentially.

Note that because in $e = 0$ none of the remaining variables are defined the target configuration $(0, 0, 0)$ to which (α, θ, e)

is required to tend will be called the *limit* point. Within this formulation, eq. (11) implies that roll is not actuated. Most often in the literature, the parking task is formulated assuming that all three components of $\underline{\omega}$ are actuated and by assigning a target value to the roll angle. As will be apparent in Section 3, the proposed solution to Problem 1, as well as *any* other possible solutions to Problem 1, may be immediately extended to cover this case as soon as the constraint (11) is relaxed. Thus, the formulation given in the statement of Problem 1 and its solution are by no means less stringent than the alternative formulation that neglects the constraint (11) and assigns a target value to the roll angle. In the remainder of this paper, a solution to Problem 1 will be designed with the additional constraint $u(t) \geq 0 \quad \forall t$. To this extent, consider the kinematics of e and $\underline{e}$:

$$\dot{\underline{e}} = -u\,\underline{i}_b \tag{12}$$
$$\dot{e} = -u\cos\alpha. \tag{13}$$

The control for u is chosen as (Indiveri 1999)

$$u = \gamma\, e \tag{14}$$

so that the closed-loop equation of e is

$$\dot{e} = -\gamma\, e\cos\alpha\,. \tag{15}$$

The vector $\underline{e}$ has an angular velocity given by

$$\underline{\omega}_e = \underline{r}_\alpha \frac{u\sin\alpha}{e} \quad ; \quad e \neq 0, \tag{16}$$

which is caused by the linear velocity of the UUV $\underline{u}$ and is independent of its angular velocity $\underline{\omega}$ (see Fig. 1). Note that the singularity occurring in eq. (16) when $e = 0$ is canceled by our choice of u (14), namely, in closed-loop $\underline{\omega}_e$, is

$$\underline{\omega}_e = \underline{r}_\alpha\,\gamma\,\sin\alpha. \tag{17}$$

When $\underline{r}_\alpha$ is well defined, the time derivative of α is

$$\dot{\alpha} = -\underline{r}_\alpha\cdot(\underline{\omega} - \underline{\omega}_e) = -\underline{r}_\alpha\cdot\underline{\omega} + \gamma\,\sin\alpha. \tag{18}$$

The vector $\underline{\alpha}$ has time derivative

$$\dot{\underline{\alpha}} = \dot{\alpha}\,\underline{r}_\alpha + \alpha\,\dot{\underline{r}}_\alpha, \tag{19}$$

where $\underline{r}_\alpha$ is a unit vector by definition and

$$\dot{\underline{r}}_\alpha = \tilde{\underline{\omega}}\wedge\underline{r}_\alpha, \tag{20}$$

where $\tilde{\underline{\omega}}$ is the angular velocity that changes the plane $(\underline{i}_b, \underline{e})$. With reference to Figure 1, note that $\tilde{\underline{\omega}}$ will be induced by $\underline{\omega}$ only and not by $\underline{u}$. In particular, $\tilde{\underline{\omega}}$ will be the projection of $\underline{\omega}$ on the axis normal to the plane $(\underline{r}_\alpha, \underline{i}_b)$, namely,

$$\tilde{\underline{\omega}} = \underline{\omega} - (\underline{\omega}\cdot\underline{r}_\alpha)\underline{r}_\alpha - (\underline{\omega}\cdot\underline{i}_b)\underline{i}_b. \tag{21}$$

Because eq. (11) must hold, the above reduces to

$$\tilde{\underline{\omega}} = \underline{\omega} - (\underline{\omega}\cdot\underline{r}_\alpha)\underline{r}_\alpha, \tag{22}$$

which implies

$$\dot{\underline{r}}_\alpha = \underline{\omega}\wedge\underline{r}_\alpha\,. \tag{23}$$

On the whole, we may write

$$\dot{\underline{\alpha}} = \underline{r}_\alpha(-\underline{r}_\alpha\cdot\underline{\omega} + \gamma\,\sin\alpha) + \tilde{\underline{\omega}}\wedge\underline{\alpha} \tag{24}$$
$$= \underline{r}_\alpha(-\underline{r}_\alpha\cdot\underline{\omega} + \gamma\,\sin\alpha) + \underline{\omega}\wedge\underline{\alpha}\,. \tag{25}$$

If $\alpha = n\pi, n \in Z$, then $\underline{r}_\alpha$ is not defined, so neither are the above time derivatives. Nevertheless, if such a configuration should occur, the application of an angular velocity normal to $\underline{i}_b$ (eq. (11)) will generate a plane containing both $\underline{i}_b$ and $\underline{e}$ such that $\dot{\underline{\alpha}}$ would be given by

$$\dot{\underline{\alpha}} = -\underline{\omega} \quad ; \quad \alpha = n\pi, n \in Z. \tag{26}$$

Moreover, it will be shown that the closed-loop evolution of α excludes the occurrence of $\alpha = n\pi \quad \forall \quad n \in Z \setminus \{0\}$ for all initial conditions $\alpha_{t=0} \in (-\pi, \pi)$. As far as $\dot{\theta}$ and $\dot{\underline{\theta}}$ are concerned, note that they are induced by the linear velocity $\underline{u}$ and not by the angular velocity $\underline{\omega}$. In particular, if $\underline{r}_\theta$ and $\underline{r}_\alpha$ are well defined,

$$\dot{\theta} = \underline{r}_\theta\cdot\underline{\omega}_e = (\underline{r}_\alpha\cdot\underline{r}_\theta)\gamma\,\sin\alpha \tag{27}$$

$$\dot{\underline{\theta}} = \dot{\theta}\,\underline{r}_\theta + \theta\,\dot{\underline{r}}_\theta \tag{28}$$

$$\dot{\underline{r}}_\theta = \bar{\underline{\omega}}\wedge\underline{r}_\theta, \tag{29}$$

where $\bar{\underline{\omega}}$ is the angular velocity, induced by $\underline{u}$ and not by $\underline{\omega}$, which changes the plane $(\underline{r}_e, \underline{r}_\theta)$. More specifically, $\bar{\underline{\omega}}$ is given by the projection of $\underline{\omega}_e$ on the axis normal to the plane $(\underline{r}_\theta, \underline{r}_e)$, namely,

$$\bar{\underline{\omega}} = \underline{\omega}_e - (\underline{\omega}_e\cdot\underline{r}_\theta)\,\underline{r}_\theta - (\underline{\omega}_e\cdot\underline{r}_e)\underline{r}_e, \tag{30}$$

which because of eq. (17) and because $\underline{r}_\alpha\cdot\underline{r}_e \equiv 0$ reduces to

$$\bar{\underline{\omega}} = (\underline{r}_\alpha - \underline{r}_\theta\,(\underline{r}_\alpha\cdot\underline{r}_\theta))\gamma\,\sin\alpha. \tag{31}$$

Thus,

$$\dot{\underline{r}}_\theta = \gamma\,\sin\alpha\,\underline{r}_\alpha\wedge\underline{r}_\theta \tag{32}$$

and

$$\dot{\underline{\theta}} = \underline{r}_\theta\,(\underline{r}_\alpha\cdot\underline{r}_\theta)\,\gamma\,\sin\alpha + \bar{\underline{\omega}}\wedge\underline{\theta}. \tag{33}$$

For the case in which $\theta = n\pi, n \in Z$, standard kinematic considerations imply that

$$\dot{\underline{\theta}} = \underline{\omega}_e = \underline{r}_\alpha\,\gamma\,\sin\alpha, \tag{34}$$

which is well defined *also* for the case in which $\alpha = m\pi \ \ \forall \ \ m \in Z$ as in such case $\underline{\omega}_e \equiv \underline{0}$.

Summarizing, the open-loop dynamics of $\underline{\alpha}$, $\underline{\theta}$, $\underline{e}$ are given by

$$\dot{\underline{\alpha}} = -\underline{r}_\alpha (\underline{\omega} \cdot \underline{r}_\alpha) + \underline{\omega} \wedge \underline{\alpha} + \underline{r}_\alpha \frac{u}{e} \sin\alpha \quad (35)$$

$$\dot{\underline{\theta}} = \left(\underline{r}_\theta (\underline{r}_\alpha \cdot \underline{r}_\theta) + \underline{r}_\alpha \wedge \underline{\theta}\right) \frac{u}{e} \sin\alpha \quad (36)$$

$$\dot{\underline{e}} = -u\, \underline{i}_b. \quad (37)$$

3. Steering Law Derivation

To guarantee the convergence of θ and α to zero, the following Lyapunov-like function is defined:

$$V = \frac{1}{2}\left(\underline{\alpha} \cdot \underline{\alpha} + h\,\underline{\theta} \cdot \underline{\theta}\right) \quad ; \quad h > 0. \quad (38)$$

Note that eq. (38) is a Lyapunov-like function rather than a Lyapunov function because the limit point ($e = 0$) in the vicinity at which the state trajectories are analyzed is not an equilibrium point in the standard sense, since it does not belong to the domain of definition of the system given by eqs. (35), (36), and (37). Differentiating with respect to time, we have

$$\dot{V} = \underline{\alpha} \cdot \dot{\underline{\alpha}} + h\,\underline{\theta} \cdot \dot{\underline{\theta}}. \quad (39)$$

Note that while the choice of $u = \gamma e$ can be made for every value of the proposed variables, more attention must be given when determining the steering law with regard to the angular velocity (i.e., when considering $\dot{V}$). Consider first the case corresponding to the existence of the two unit vectors $\underline{r}_\alpha$ and $\underline{r}_\theta$.

3.1. $\alpha \neq n\pi \ \ \forall\, n \in Z$ and $\theta \neq m\pi \ \ \forall\, m \in Z$

In this case, $\dot{V}$ takes the form

$$\begin{aligned} \dot{V} &= \underline{\alpha} \cdot \left(\underline{r}_\alpha \left(-\underline{r}_\alpha \cdot \underline{\omega} + \gamma \sin\alpha\right) + (\tilde{\underline{\omega}} \wedge \underline{\alpha})\right) \\ &+ h\,\underline{\theta} \cdot \left(\underline{r}_\theta \left(\underline{r}_\theta \cdot \underline{r}_\alpha \gamma \sin\alpha\right) + (\bar{\underline{\omega}} \wedge \underline{\theta})\right), \end{aligned} \quad (40)$$

since $\underline{x} \cdot (\underline{y} \wedge \underline{x}) = 0 \ \ \forall\ \underline{x}, \underline{y}, \underline{\alpha} \cdot \underline{r}_\alpha = \alpha$, and $\underline{\theta} \cdot \underline{r}_\theta = \theta$. Thus,

$$\dot{V} = \alpha \left(\gamma \sin\alpha - \underline{r}_\alpha \cdot \underline{\omega}\right) + \gamma h \left(\underline{r}_\alpha \cdot \underline{r}_\theta\right) \theta \sin\alpha. \quad (41)$$

Then, we can make $\dot{V}$ negative by choosing (if possible)

$$\underline{\omega} \cdot \underline{r}_\alpha = K\alpha + \gamma \sin\alpha + \gamma h \theta (\underline{r}_\alpha \cdot \underline{r}_\theta) \frac{\sin\alpha}{\alpha} \quad : \quad K > 0. \quad (42)$$

Recalling (6), eq. (42) can be rewritten as

$$\underline{\omega} \cdot \underline{r}_\alpha = K\underline{\alpha} \cdot \underline{r}_\alpha + \gamma \frac{\sin\alpha}{\alpha} \underline{\alpha} \cdot \underline{r}_\alpha + \gamma h \frac{\sin\alpha}{\alpha} \underline{\theta} \cdot \underline{r}_\alpha\,. \quad (43)$$

In the desire to find a steering control $\underline{\omega}$ that satisfies both eq. (43) and the constraint (11) $\underline{\omega} \cdot \underline{i}_b = 0$, it should be noted that the "natural" choice

$$\underline{\omega} = K\underline{\alpha} + \gamma \left(\underline{\alpha} + h\,\underline{\theta}\right) \frac{\sin\alpha}{\alpha} + \underline{\omega}_p \qquad \forall\ \underline{\omega}_p \parallel (\underline{r}_\alpha \wedge \underline{i}_b) \quad (44)$$

indeed satisfies (43) but generally is not normal to $\underline{i}_b$ because of the term proportional to $\underline{\theta}$. Note that the above choice would yield for $\dot{V}$

$$\dot{V} = -K\alpha^2 \leq 0. \quad (45)$$

An alternative choice for $\underline{\omega}$ that *is* normal to $\underline{i}_b$ and still satisfies both eq. (43) and eq. (45) can be obtained by simply projecting the vector given by (44) on the body fixed reference $<b>$ as follows:

$$\begin{aligned} {}^b\underline{\omega} &= K\,{}^b\underline{\alpha} + \gamma \left({}^b\underline{\alpha} + h\,{}^b\tilde{\underline{\theta}}\right) \frac{\sin\alpha}{\alpha} + {}^b\underline{\omega}_p \\ &\forall\ \underline{\omega}_p \parallel (\underline{r}_\alpha \wedge \underline{i}_b), \end{aligned} \quad (46)$$

where ${}^r\underline{x}$ usually denotes the projection of the vector $\underline{x}$ on the frame $<r>$ and $\tilde{\underline{\theta}}$ is defined as

$${}^b\tilde{\underline{\theta}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} {}^b\underline{\theta}. \quad (47)$$

Note that by its very definition, $\tilde{\underline{\theta}}$ guarantees that

$$\tilde{\underline{\theta}} \cdot \underline{i}_b = 0 \quad (48)$$

$$\tilde{\underline{\theta}} \cdot \underline{r}_\alpha = \theta\, \underline{r}_\theta \cdot \underline{r}_\alpha. \quad (49)$$

Again, we stress that (46) has components only on the plane $(\underline{j}_b, \underline{k}_b)$ of the body fixed frame $<b>$ and, hence, can be implemented by the UUV; moreover, the time derivative of the candidate Lyapunov-like (38) function still remains the same as that reported in eq. (45).

3.2. α and/or $\theta = n\pi \ \ \forall\ \ n \in Z$

The steering control law (46) is not defined when either (or both) $\underline{r}_\alpha$ or $\underline{r}_\theta$ is not defined. In such cases, eq. (46) may still be implemented imposing

$$\underline{r}_\alpha \equiv \underline{r}_\theta;\ \text{if}\ \alpha = n\pi,\ n \in Z\ \text{and}\ \theta \neq m\pi \ \ \forall\, m \in Z \quad (50)$$

$$\underline{r}_\theta \equiv \underline{r}_\alpha;\ \text{if}\ \alpha \neq n\pi \ \ \forall\, n \in Z\ \text{and}\ \theta = m\pi,\ m \in Z \quad (51)$$

$$\underline{r}_\alpha \equiv \underline{r}_\theta \equiv \underline{k}_b;\ \text{if}\ \alpha = n\pi,\ n \in Z\ \text{and}\ \theta = m\pi,\ m \in Z, \quad (52)$$

which allows us to define vectors $\underline{\alpha}$ and $\underline{\theta}$ in any case. Of course, this will introduce a discontinuity in $\underline{\omega}$.

3.3. The Closed-Loop Dynamics

Recalling eqs. (15), (18), (23), (27), (32), (35), (36), (37), and (46), the closed-loop dynamics of the state vector $(\alpha, \theta, e)^T$ is given by

$$\dot{\alpha} = -K\alpha - \gamma h\theta(\underline{r}_\alpha \cdot \underline{r}_\theta)\frac{\sin\alpha}{\alpha} \quad (53)$$

$$\dot{\theta} = (\underline{r}_\alpha \cdot \underline{r}_\theta)\gamma \sin\alpha \quad (54)$$

$$\dot{e} = -\gamma e \cos\alpha. \quad (55)$$

REMARK 1. Considering the structure of the above dynamics and eq. (45), the state $(\alpha, \theta, e)^T$ is bounded at all times (i.e., it does not admit a finite escape time). Moreover,

$$\alpha|_{t=t_0} \in (-\pi, \pi) \;\Rightarrow \alpha \in (-\pi, \pi) \;\; \forall \;\; t \geq t_0.$$

The first part of the above remark is obvious. The second follows from the observation that $\alpha(t)$ is continuous, and, according to eq. (53),

$$\dot{\alpha}|_{\alpha=\pm\pi} = \mp K\pi.$$

Thus, if α is initialized in the open set $(-\pi, \pi)$, it will not be able to leave it. In fact, from eq. (53) it follows that

$$\frac{d}{dt}\left(\frac{1}{2}\alpha^2\right)\bigg|_{\alpha=\pm\pi} = (\alpha\dot{\alpha})|_{\alpha=\pm\pi} = -K\pi^2,$$

ensuring that on the boundary of the open set $(-\pi, \pi)$, $|\alpha|$ must decrease. Another important observation is that if $e = 0$, then α and θ are not defined; thus, neither $\underline{r}_\alpha$, $\underline{r}_\theta$, nor the steering law $\underline{\omega}$ (46) is defined. Indeed, as mentioned in the introduction, the domain of definition of the system is

$$\mathcal{D} = \{R^6 \setminus \{0, 0, 0, \phi, \vartheta, \psi)^T\} \;\; \forall \;\; \phi, \vartheta, \psi \in R\},$$

where $(x, y, z, \phi, \vartheta, \psi)^T \in R^6$ describes the system, (x, y, z) is its Cartesian position, and (ϕ, ϑ, ψ) is its Euler angle orientation. It is thus important to note that if the system is started inside $\mathcal{D}$, it will be bound to stay in $\mathcal{D}$ at all finite times; namely, the set described by $e = 0$ where α, θ are not defined will never be "crossed" during the closed-loop evolution.

REMARK 2. If $e|_{t=t_0} \neq 0$, then $e(t) \neq 0$ at all finite times.

Remark 2 is a consequence of eq. (55), according to which

$$e(t) = e|_{t=t_0} \exp\left(-\gamma \int_{t_0}^{t} \cos\alpha(\tau)d\tau\right). \quad (56)$$

It can be shown that $\mathcal{D}$ is also the basin of attraction toward the origin for the closed-loop dynamics of $(\alpha, \theta, e)^T$ and that convergence is exponential. Given eqs. (45), (53), (54), and (55) and LaSalle's invariance principle (Khalil 1996; Slotine and Li 1991), it follows that $(\alpha, \theta, e)^T$ will converge to the largest invariant set $\mathcal{I}$ where (45) $\dot{V} = 0$, namely, to

$$\mathcal{I} = \{(0, \theta, 0)^T : \dot{\theta} = 0 \quad \forall \quad \theta \in R\} \cup \{(0, 0, 0)^T\}. \quad (57)$$

To prove the exponential convergence of $(\alpha, \theta, e)^T$ to the origin $(0, 0, 0)^T \in \mathcal{I}$, two cases may be distinguished:

(a) There exists a finite time at which $\underline{r}_\alpha$ and $\underline{r}_\theta$ are parallel.

(b) $\underline{r}_\alpha$ and $\underline{r}_\theta$ are never parallel at any finite time.

The occurrence of either of the two cases depends on the initial conditions. Note that eqs. (50), (51), and (52) imply that if either α or θ should cross multiples of π (including 0) at any finite time, then case (a) would automatically occur; namely, it would be imposed. If $\underline{r}_\alpha$ and $\underline{r}_\theta$ should happen to be parallel at a certain instant t^*, it would mean that at that time t^*, the target $\underline{i}_a$ and $\underline{i}_b$ (the vehicle) would lie on the same plane. It will now be shown that if this does happen, the motion will also remain planar for all future times.

REMARK 3. If $\underline{\omega}_p \parallel \underline{r}_\alpha \wedge \underline{i}_b$ is always chosen such that

$$\underline{r}_\alpha = \pm\, \underline{r}_\theta \;\; \Rightarrow \;\; \underline{\omega}_p = \underline{0}, \quad (58)$$

then if $\underline{r}_\alpha \parallel \underline{r}_\theta$, that is, $\underline{r}_\alpha = \pm\, \underline{r}_\theta \;\Rightarrow\; (\underline{r}_\alpha \cdot \underline{r}_\theta) = \pm 1$, at any finite time t^*, the motion will remain planar for all future times $t \geq t^*$.

This follows from the observation that

$$\dot{\underline{r}}_\alpha = \gamma h \frac{\sin\alpha}{\alpha}(\tilde{\underline{\theta}} \wedge \underline{r}_\alpha) + (\underline{\omega}_p \wedge \underline{r}_\alpha) \quad (59)$$

$$\dot{\underline{r}}_\theta = \gamma \sin\alpha\, \underline{r}_\alpha \wedge \underline{r}_\theta \quad (60)$$

obtained by eqs. (23), (32), and (46). If $\underline{r}_\theta = \pm\, \underline{r}_\alpha$, then by its very definition (47)$\tilde{\underline{\theta}} = \theta\, \underline{r}_\theta = \pm\, \theta\, \underline{r}_\alpha$, and assuming that (58) holds, both eqs. (59) and (60) are identically null. Thus, if $\underline{r}_\alpha \parallel \underline{r}_\theta$ or if either α or θ crosses a multiple of π (including 0) at any finite time t^* (which implies $\underline{r}_\alpha \parallel \underline{r}_\theta$ because of (50), (51), and (52)), the control strategy ((14), (46), and (58)) guarantees that the motion will remain planar for all $t \geq t^*$ because $\underline{r}_\alpha = \pm\, \underline{r}_\theta$ is an equilibrium of eqs. (59) and (60). This is an important aspect because in the planar case, $(\alpha, \theta, e)^T$ is exponentially converging to zero. This follows directly from the analysis of the closed-loop dynamics of $(\alpha, \theta, e)^T$ in a neighborhood of $\alpha = 0$ (which always occurs because of (45)) when $(\underline{r}_\alpha \cdot \underline{r}_\theta) = \pm\, 1$. Namely, from (53), (54), and (55), when $(\underline{r}_\alpha \cdot \underline{r}_\theta) = \pm\, 1,\; \alpha \approx 0$,

$$\dot{\underline{z}} = \mathbf{A}\,\underline{z} \;\; : \;\; \underline{z} = (\alpha, \theta, e)^T, \quad (61)$$

where

$$\mathbf{A} = \begin{bmatrix} -K & \mp\gamma h & 0 \\ \pm\gamma & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \quad (62)$$

is Hurwitz $\forall\ \gamma, K, h > 0$. Another interesting feature of the planar case is that choosing the gains such that

$$h > 1 \ ; \ 2\gamma < K < (h+1)\gamma \tag{63}$$

guarantees the curvature

$$\kappa = \frac{\underline{\omega} \cdot \underline{r}_\alpha}{u}$$

to be exponentially converging to zero; namely, the target configuration is approached along a straight line. The proof is given in Casalino et al. (1994). It follows immediately from the analysis, in a neighborhood of $\alpha = 0$, of the dynamic behaviors of u and of $\underline{\omega}$ as given by eqs. (14) and (46) in the planar case (i.e., when $\underline{r}_\alpha \parallel \underline{r}_\theta$ and $\underline{\omega}_p = \underline{0}$ as by eq. (58)).

Summarizing, if during the time evolution of the system either α or θ should be a multiple of π (including 0), because of the steering discontinuity given by (50), (51), or (52), the motion would become planar and the state $(\alpha, \theta, e)^T$ exponentially convergent. This holds for any $\underline{\omega}_p \parallel \underline{r}_\alpha \wedge \underline{i}_b$ such that eq. (58) is satisfied. On the other hand, if the system is started in a configuration such that $\underline{r}_\alpha$ and $\underline{r}_\theta$ are never parallel and α and θ never cross multiples of π at any finite time (case (b)), nothing can be concluded on the convergence to the origin unless, as will be shown, $\underline{\omega}_p$ is chosen such that $\underline{r}_\alpha$ and $\underline{r}_\theta$ are indeed always exponentially parallel.

3.4. Design of $\underline{\omega}_p$

From the above discussion, $\underline{\omega}_p$ must always lie parallel to $\underline{r}_\alpha \wedge \underline{i}_b$ and must satisfy eq. (58). Consider the candidate Lyapunov-like function

$$V_p = \frac{1}{2}\ (\underline{r}_\alpha - \underline{r}_\theta)\cdot(\underline{r}_\alpha - \underline{r}_\theta) = 1 - (\underline{r}_\alpha \cdot \underline{r}_\theta) \Rightarrow \tag{64}$$

$$\begin{aligned} \dot{V}_p &= (\underline{r}_\alpha - \underline{r}_\theta)\cdot(\dot{\underline{r}}_\alpha - \dot{\underline{r}}_\theta) \qquad (65)\\ &= (\underline{r}_\alpha - \underline{r}_\theta)\cdot(\underline{\omega} \wedge \underline{r}_\alpha - \gamma\ \sin\alpha\ \underline{r}_\alpha \wedge \underline{r}_\theta)\\ &= (\underline{r}_\alpha - \underline{r}_\theta)\cdot\left(\gamma\, h\,\frac{\sin\alpha}{\alpha}\ (\tilde{\underline{\theta}} \wedge \underline{r}_\alpha) + \underline{\omega}_p \wedge \underline{r}_\alpha\right)\\ &= -\gamma\, h\,\frac{\sin\alpha}{\alpha}\ (\tilde{\underline{\theta}} \wedge \underline{r}_\alpha)\cdot\underline{r}_\theta - (\underline{\omega}_p \wedge \underline{r}_\alpha)\cdot\underline{r}_\theta. \end{aligned}$$

Consider now the projection $\underline{x}_p$ of $\tilde{\underline{\theta}}$ on the axis normal to the plane $(\underline{i}_b, \underline{r}_\alpha)$ (i.e., on the same axis where $\underline{\omega}_p$ must lie):

$$\begin{aligned} \underline{x}_p &\equiv \tilde{\underline{\theta}} - (\tilde{\underline{\theta}}\cdot\underline{r}_\alpha)\,\underline{r}_\alpha - (\tilde{\underline{\theta}}\cdot\underline{i}_b)\,\underline{i}_b \qquad (66)\\ &= \tilde{\underline{\theta}} - (\tilde{\underline{\theta}}\cdot\underline{r}_\alpha)\,\underline{r}_\alpha \Rightarrow \underline{x}_p \parallel \underline{r}_\alpha \wedge \underline{i}_b \qquad (67)\\ \underline{x}_p \wedge \underline{r}_\alpha &= \tilde{\underline{\theta}} \wedge \underline{r}_\alpha, \qquad (68) \end{aligned}$$

where $(\tilde{\underline{\theta}} \cdot \underline{i}_b) = 0$ in accordance with eq. (48). Then, $\dot{V}_p$ reduces to

$$\dot{V}_p = -\left[\left(\gamma\ h\,\frac{\sin\alpha}{\alpha}\ \underline{x}_p + \underline{\omega}_p\right)\wedge\underline{r}_\alpha\right]\cdot\underline{r}_\theta, \tag{69}$$

which can be made negative semidefinite by

$$\underline{\omega}_p = -\gamma\ h\,\frac{\sin\alpha}{\alpha}\ \underline{x}_p + \mu(\underline{r}_\alpha \wedge \underline{i}_b)(\underline{i}_b \cdot \underline{r}_\theta) : \mu > 0. \tag{70}$$

After simple vector manipulations, eq. (70) reduces eq. (69) to

$$\dot{V}_p = -\mu(\underline{i}_b \cdot \underline{r}_\theta)^2 \le 0. \tag{71}$$

Note that by its very construction, $\underline{\omega}_p$ as given by eq. (70) is parallel to $\underline{r}_\alpha \wedge \underline{i}_b$ and satisfies the constraint (58) that is required in order not to spoil the planar motion once $\underline{r}_\alpha$ is parallel to $\underline{r}_\theta$. This is because $\underline{r}_\alpha = \pm\underline{r}_\theta \Rightarrow \tilde{\underline{\theta}} = \theta\,\underline{r}_\theta = \pm\theta\,\underline{r}_\alpha$ (refer to eq. (47)), which implies $\underline{x}_p = \underline{0}$ and because $\underline{r}_\alpha \parallel \underline{r}_\theta \Rightarrow \underline{i}_b\cdot\underline{r}_\theta = \underline{i}_b\cdot\underline{r}_\alpha = 0$.

Thus, by construction, $\underline{\omega}_p$ does not affect the derivative of the Lyapunov function (38) nor does it spoil the planar motion once $\underline{r}_\alpha$ should parallel to $\underline{r}_\theta$; rather, it orients $\underline{r}_\alpha$ parallel to $\underline{r}_\theta$. It achieves this objective exponentially. This can be proven as follows. Note that according to (71), the only equilibria of V_p are given by the configurations $\underline{r}_\theta\cdot\underline{i}_b = 0$ that occur only if $\underline{r}_\theta = \underline{r}_\alpha$ or $\underline{r}_\alpha = -\underline{r}_\theta$. Therefore, either the desired configuration $\underline{r}_\alpha \parallel \underline{r}_\theta$ is already fulfilled at the initial time t_0 or it is not and in this case at time $t = t_0 : (\underline{r}_\theta\cdot\underline{i}_b)^2 = \epsilon$ for some $\epsilon > 0$. In the first case, the motion is initially planar and will remain so as discussed in the previous section. In the second hypothesis, V_p will decrease continuously as $(\underline{r}_\alpha\cdot\underline{r}_\theta) \to 1$. With reference to Figure 2, the exponential convergence of V_p follows from the observation that

$$\begin{aligned} V_p &= 1 - (\underline{r}_\alpha\cdot\underline{r}_\theta) = 1 - \cos\beta \qquad (72)\\ \dot{V}_p &= -\mu(\underline{r}_\theta\cdot\underline{i}_b)^2 = -\mu\,(\cos\beta\ \sin\alpha)^2. \qquad (73) \end{aligned}$$

Thus, in a neighborhood of $\beta = 0$, $\alpha = 0$ (which in view of eqs. (45) and (71) will always be approached as long as $t = t_0 : (\underline{r}_\theta\cdot\underline{i}_b)^2 = \epsilon$ for some $\epsilon > 0$), the sum $\dot{V}_p + V_p$ tends to $-\mu\alpha^2$, namely, $\exists\, t^* > 0 : \forall\ t > t^*$

$$\dot{V}_p \le -V_p \quad \forall \quad \mu > 0. \tag{74}$$

As a consequence,

$$\exists\,\lambda, \sigma > 0 : 1 - (\underline{r}_\alpha\cdot\underline{r}_\theta) \le \sigma\ \exp(-\lambda\,(t - t_0)). \tag{75}$$

Replacing eq. (70) in eq. (46), the complete steering law is given by

$$\begin{aligned} \underline{\omega} = K\,\underline{\alpha} + \gamma\,(\underline{\alpha} + h\,(\tilde{\underline{\theta}}\cdot\underline{r}_\alpha)\,\underline{r}_\alpha)\frac{\sin\alpha}{\alpha} \\ + \mu\,(\underline{r}_\alpha \wedge \underline{i}_b)(\underline{i}_b\cdot\underline{r}_\theta), \end{aligned} \tag{76}$$

where $\tilde{\underline{\theta}}$ is defined by eq. (47).

To conclude, it must now be proven that with the choice of $\underline{\omega}$ given by eq. (76), the state $(\alpha, \theta, e)^T$ converges exponentially to zero even if the initial conditions should be such that

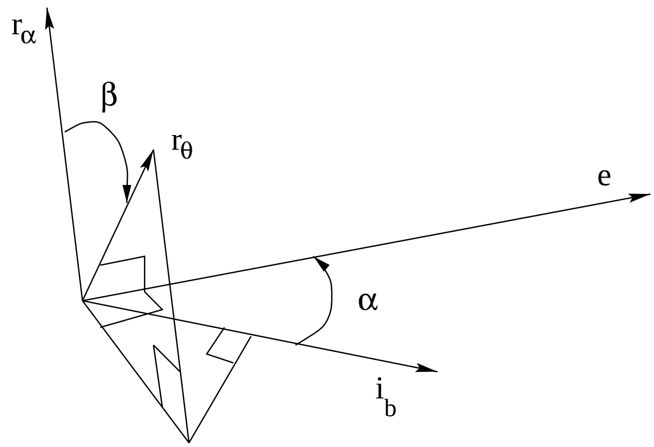

Fig. 2. Geometric interpretation.

$\underline{r}_\alpha$ is never parallel to $\underline{r}_\theta$ at any finite time (case (b)). Suppose that the motion is not already initially planar (case (b)), namely, $(\underline{r}_\alpha \cdot \underline{r}_\theta)^2 < 1$. In this case, once α has converged to a neighborhood of $\alpha = 0$, which will always be globally approached because $\alpha = 0$ belongs to the invariant set $\mathcal{I}$ (57), the dynamics of $(\alpha, \theta, e)^T$ as given by eqs. (53), (54), and (55) can be written as

$$\underline{z} = (\alpha, \theta, e)^T \, , \, \alpha \approx 0 \ \Rightarrow \tag{77}$$
$$\dot{\underline{z}} = \mathbf{A}_1 \, \underline{z} + \underline{b}, \tag{78}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} -K & -\gamma\, h & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix}$$
$$\underline{b} = \gamma \begin{pmatrix} h\,\theta \\ -\alpha \\ 0 \end{pmatrix} (1 - (\underline{r}_\alpha \cdot \underline{r}_\theta)). \tag{79}$$

Because α and θ are bounded (Remark 1) and $1 - (\underline{r}_\alpha \cdot \underline{r}_\theta) \to 0$ exponentially (eq. (75)), $\underline{b}$ as given by eq. (79) has a bounded and exponentially decaying norm, namely,

$$\exists\, \xi > 0 \ : \ \|\underline{b}\| \leq \|\underline{b}\|_{max} \, \exp(-\xi\, t) \quad \forall \quad t, \tag{80}$$

where $\|\underline{b}\|_{max}$ is the maximum of $\|\underline{b}\|$. The above together with the observation that $\mathbf{A}_1$ defined in (79) is Hurwitz are sufficient to conclude that the solution of the linear system (78) is exponentially converging to zero. Indeed, the solution of (78) is

$$\underline{z}(t) = \Phi(t, t_0)\underline{z}_0 + \int_{t_0}^{t} \Phi(t, \tau)\underline{b}(\tau)d\tau \tag{81}$$

$$\frac{d}{dt}\Phi(t, t_0) = \mathbf{A}_1 \Phi(t, t_0) : \Phi(t_0, t_0) = \mathbf{I} \tag{82}$$

$$\mathbf{A}_1 \text{is Hurwitz} \Rightarrow \exists\, \lambda > 0 :$$

$$\|\Phi(t, t_0)\| \leq \exp(-\lambda(t - t_0)) \tag{83}$$

$$\begin{aligned} \|\underline{z}(t)\| &\leq \|\Phi(t, t_0)\underline{z}_0\| \\ &\quad + \int_{t_0}^{t} \|\Phi(t, \tau)\| \, \|\underline{b}(\tau)\| d\tau \end{aligned} \tag{84}$$

$$\begin{aligned} &\leq \|\underline{z}_0\| \, \exp(-\lambda(t - t_0)) + \\ &\quad + \|\underline{b}\|_{max} \exp(-\lambda\, t) \int_{t_0}^{t} \exp((\lambda - \xi)\, \tau) d\tau \\ &= \|\underline{z}_0\| \, \exp(-\lambda(t - t_0)) + \\ &\quad + \frac{\|\underline{b}\|_{max} \exp(-\xi\, t_0)}{\lambda - \xi} \Big[\exp(-\xi\, (t - t_0)) \\ &\quad - \exp(-\lambda\, (t - t_0))\Big], \end{aligned}$$

thus, exponentially converging to zero. The above and the fact that if the motion should be planar exponential convergence has been already proven allow us to state the following main result.

THEOREM 1. The stationary control law given by eqs. (14) and (76), namely,

$$\underline{u} = \gamma\, e\, \underline{i}_b \tag{85}$$

$$\begin{aligned} \underline{\omega} &= K\, \underline{\alpha} + \gamma\, (\underline{\alpha} + h\, (\tilde{\underline{\theta}} \cdot \underline{r}_\alpha)\, \underline{r}_\alpha) \frac{\sin \alpha}{\alpha} \\ &\quad + \mu\, (\underline{r}_\alpha \wedge \underline{i}_b)(\underline{i}_b \cdot \underline{r}_\theta), \end{aligned} \tag{86}$$

together with eqs. (47), (50), (51), and (52) solve Problem 1 “almost” globally (i.e., for *any* initial configuration such that $e|_{t=t_0} \neq 0$). The closed-loop linear velocity has a constant sign, avoiding cusps in the paths. Moreover, if $\underline{r}_\alpha \parallel \underline{r}_\theta$ at some finite time during the closed-loop evolution, the motion will remain planar for all future times, and if the relevant gains satisfy eq. (63), the curvature of the path will be exponentially decaying to zero.

In accordance with the discussion outlined in Section 2, suppose that the constraint (11) is relaxed, namely, that all three components of $\underline{\omega}$ are available as control inputs. In this hypothesis, one may extend the formulation of Problem 1 to include a target roll angle. The problem would consist of designing a law for $\underline{u}$ and $\underline{\omega}$ such that eq. (10) is satisfied and the body fixed reference $< b >$ is asymptotically (exponentially) overlapping with the target fixed frame $< a >$. Calling ${}^b\mathbf{R}_a$

the rotation matrix between $<b>$ and $<a>$, such a problem may be formally stated as follows.

PROBLEM 2. Determine a feedback law for $\underline{u}$ and $\underline{\omega}$ such that $\underline{u} = u\ \underline{i}_b$ and ${}^b\mathbf{R}_a \to \mathbf{I}_{3\times3}$, $e \to 0$ exponentially.

The control law given by eq. (85) and

$$\underline{\omega}_2 \quad = \quad \underline{\omega} - \sigma\,\rho\,\underline{i}_b \ :\ \sigma > 0, \tag{87}$$

where $\underline{\omega}$ is given by eq. (86) and ρ is the roll angle, solves Problem 2 for *any* initial configuration such that $e|_{t=t_0} \neq 0$. Note that an alternative stationary piecewise smooth solution to Problem 2 has already been presented in Egeland, Dalsmo, and Sørdalen (1996), but the vehicle was assumed to move both forward and backward; thus, the resulting paths were not cusp-free and their curvature was not guaranteed to tend to zero but rather tended to some finite constant value.

4. Implementation Issues

To implement the proposed strategy, e, $\underline{\theta}$, and $\underline{\alpha}$ must be evaluated. c^T, the transpose of the matrix or vector c in ${}^a\underline{\eta} = ({}^a\underline{\eta}_1^T, {}^a\underline{\eta}_2^T)^T = (x, y, z, \phi, \vartheta, \psi)^T$, denotes the generalized position of the local reference $<b>$ with respect to reference $<a>$, where ${}^a\underline{\eta}_2$ is a suitable set of rotation angles (e.g., Euler or roll, pitch, and yaw angles). Adopting the roll, pitch, and yaw angles to describe the UUV's orientation, its velocity projected on the local reference $<b>$ will be given by ${}^b\underline{\nu} = ({}^b\underline{\nu}_1^T, {}^b\underline{\nu}_2^T)^T = (u, v, w, p, q, r)^T$; therefore, the kinematics can be written as

$${}^a\dot{\underline{\eta}} = \mathbf{J}({}^a\underline{\eta}_2)\ {}^b\underline{\nu}. \tag{88}$$

The velocity vector ${}^b\underline{\nu}_2$ is given by the control signal $\underline{\omega}$ of eq. (86) projected onto frame $<b>$. The linear velocity u given by eq. (14) corresponds to the surge velocity u (i.e., the first component of vector ${}^b\underline{\nu}$). Standard (Fossen 1994) considerations allow one to write the Jacobian $\mathbf{J}({}^a\underline{\eta}_2)$ as

$$\mathbf{J}({}^a\underline{\eta}_2) = \begin{bmatrix} \mathbf{J}_1({}^a\underline{\eta}_2) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{J}_2({}^a\underline{\eta}_2) \end{bmatrix},$$

where $\mathbf{J}_1({}^a\underline{\eta}_2)$ is the rotation matrix between reference $<b>$ and $<a>$ and $\mathbf{J}_2({}^a\underline{\eta}_2)$ is the matrix describing the relationship between the absolute velocity ${}^a\underline{\eta}_2$ and the local velocity ${}^b\underline{\nu}_2$. Note that the rotation matrix $\mathbf{J}_1({}^a\underline{\eta}_2)$ between reference $<b>$ and $<a>$ is sometimes denoted as ${}^a\mathbf{R}_b$, and thus $\mathbf{J}_1^T({}^a\underline{\eta}_2) = {}^b\mathbf{R}_a$. If the position of the target is measured by the UUV through sensor readings (e.g., sonar, acoustic positioning systems or vision devices), the components $({}^be_x, {}^be_y, {}^be_z)$ of ${}^b\underline{e}$ will be directly available. Moreover, ${}^b\underline{\alpha} = \alpha\ {}^b\underline{r}_\alpha$ may be computed as

$$\alpha = \operatorname{acos}\left({}^be_x/e\right) \quad ; \quad e > 0 \tag{89}$$

$${}^b\underline{r}_\alpha = \frac{\underline{i}_b \wedge {}^b\underline{e}}{\|\underline{i}_b \wedge {}^b\underline{e}\|} \quad ; \quad \underline{i}_b = (1, 0, 0)^T, \tag{90}$$

where acos is the inverse of the cosine function taking values in $[0, \pi]$. Knowing the UUV's attitude vector ${}^a\underline{\eta}_2$, the projection of $\underline{e}$ on frame $<a>$ can be computed as ${}^a\underline{e} = \mathbf{J}_1({}^a\underline{\eta}_2)\ {}^b\underline{e}$, and thus ${}^b\underline{\theta} = \theta\ {}^b\underline{r}_\theta$ can be computed as

$$\theta = \operatorname{acos}\left({}^ae_x/e\right) \quad ; \quad e > 0 \tag{91}$$

$${}^b\underline{r}_\theta = \mathbf{J}_1^T({}^a\underline{\eta}_2)\frac{\underline{i}_a \wedge {}^a\underline{e}}{\|\underline{i}_a \wedge {}^a\underline{e}\|} \quad ; \quad \underline{i}_a = (1, 0, 0)^T. \tag{92}$$

5. Robustness Issues

The use of polar-like variables to overcome the limitations imposed by Brockett's theorem still raises concerns with regard to the robustness of the resulting controllers; this is because polar-like variables are singular in the origin. Thus, it may be properly claimed that the equilibrium is not stable in the sense of Lyapunov, and hence it appears reasonable to question the validity of the classical results on stability under perturbations based on Lyapunov's theory. However, in the above claim, the stress should be on the first portion of the sentence rather than the second; indeed, the origin is *not* an equilibrium point simply because it does not belong to the domain of existence of the system. In the specific case discussed in this paper, the variables α and θ are *not* defined when $e = 0$. The classical robustness results for linear or nonlinear exponentially stable systems are based on the existence of Lyapunov functions having time derivatives that may be shown to remain nonpositive even in the presence of perturbations of a certain structure (Khalil 1996). In the present case, exponential convergence has been guaranteed precisely on the basis of the Lyapunov-like function (38). As long as e can be guaranteed to remain non-null (i.e., as long as the function V given by (38) is defined), the present controller will enjoy all the robustness properties of any exponentially stable system. This follows from the observation that the whole machinery of the Lyapunov-based perturbation analysis (Khalil 1996) may be applied to V given by (38) in its domain of existence. For a more general and thorough discussion of the robustness properties of discontinuous controllers for nonholonomic systems in chained form, refer to Jiang (2000). More specifically, consider the case in which the norms of the control signals $\underline{\omega}$ and $\underline{u}$ as given by eqs. (86) and (85) are affected by a bounded noise, namely, that the input to the system is

$$\underline{\omega}^* \quad = \quad \underline{\omega}\,(1 + \tilde{\xi}_1) \tag{93}$$

$$u^* \quad = \quad \gamma\, e + \tilde{\xi}_2, \tag{94}$$

where $\underline{\omega}$ is given by eq. (86) and

$$0 \ \leq\ |\tilde{\xi}_1(t)| \ \leq\ \xi_1 < 1 \tag{95}$$

$$0 \ \leq\ |\tilde{\xi}_2(t)| \ \leq\ \xi_2 \tag{96}$$

for some constant positive ξ_1, ξ_2. Replacing $\underline{\omega}^*$ and u^* in eqs. (35) and (36), the time derivative of the Lyapunov-like function $V = 1/2\,(\underline{\alpha} \cdot \underline{\alpha} + h\,\underline{\theta} \cdot \underline{\theta})$ results upper bounded as follows:

$$\dot{V} \leq \alpha \left(-\left(K\,(1-\xi_1) - \gamma\,\xi_1 - \frac{\xi_2}{e} \right) \alpha + \gamma\,h|\theta_{max}|\xi_1 + h|\theta_{max}|\frac{\xi_2}{e} \right) \tag{97}$$

which shows that when

$$|\alpha| > h\,|\theta_{max}|\,\frac{\gamma\,\xi_1 + \xi_2/e}{|K\,(1-\xi_1) - \gamma\,\xi_1 - \xi_2/e|}, \tag{98}$$

the function $V = 1/2\,(\underline{\alpha} \cdot \underline{\alpha} + h\,\underline{\theta} \cdot \underline{\theta})$ is always decreasing. Note that as far as the angular variables are concerned, the influence of $\tilde{\xi}_2$ decreases as e grows. Replacing u^* in the dynamic eq. (37) of $\underline{e}$, it may likewise be noted that

$$e\dot{e} = -e\,\cos\alpha\,(\gamma\,e + \tilde{\xi}_2) \tag{99}$$

and that e will continuously decrease when $|\alpha| < \pi/2$ and $e > \xi_2/\gamma$. The overall situation is graphically depicted in Figure 3, where

$$A = -F = h\,|\theta_{max}| \tag{100}$$

$$\eta = h\,|\theta_{max}|\,\frac{\gamma\,\xi_1}{|K\,(1-\xi_1) - \gamma\,\xi_1|}. \tag{101}$$

The shadowed region (which extends to infinity along the e axis) represents the region where e decreases. It may be concluded that the state will be asymptotically bounded in the region $ACDF$. This analysis reveals that the proposed solution is probably best suited to drive the given underactuated system in a "vicinity" of the target configuration. This task will be accomplished robustly, exponentially, always moving in the same direction (no cusps) and with vanishing curvature. The region $ACDF$ may be thought of as a deadband zone where the control inputs u and $\underline{\omega}$ are set to zero or, eventually, where a different control strategy may be adopted.

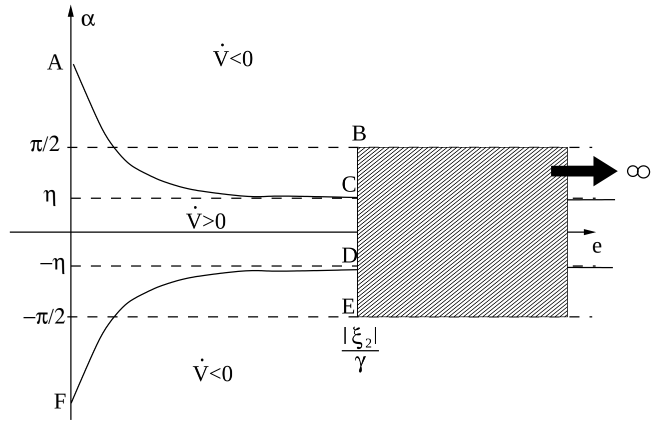

Fig. 3. Robustness properties.

6. Examples

The proposed controller was extensively simulated in both the ideal noise-free setting and with the addition of a zero mean Gaussian noise on the states α, θ, and e. Two specific examples are reported. The initial position of the UUV is $(x_0, y_0, z_0) = (-1, 1, 1)$, its initial orientation (roll, pitch, yaw) is $(\phi_0, \vartheta_0, \psi_0) = (0, 0, 3\pi/4)$, and the gains are fixed to $K = 2.05$, $\gamma = 1$, $h = 1.1$ and $\mu = 1$ in both cases. The differential equations are integrated through the Euler method with a fixed time step $\delta t = 0.005$. The first simulation refers to the noise-free case and gives rise to the results reported in Figure 4. The second simulation refers to the same initial conditions and gains but with additional zero mean Gaussian noises on the measured values of α, θ, and e having standard deviations $\sigma = 0.0017$ [rad](= 1/10 [Deg]) on α and θ and $\sigma = 0.02$ on e. The results of this case are reported in Figure 5. The paths resulting in both cases are plotted in Figure 6. As expected from the previous robustness analysis, they are practically identical for large values of e, whereas their difference becomes apparent closer to the target where the path corresponding to the noisy case (dashed line) deviates.

7. Conclusions

In this paper, the control of the position and orientation of a 3D nonholonomic floating vehicle was considered. A stationary piecewise smooth feedback law guaranteeing exponential convergence to a target configuration was designed. The use of appropriate state variables avoided the obstruction due to Brockett's theorem. The robustness properties of the suggested solution were analyzed both theoretically and via simulated examples. The proposed control law is similar in nature to the one reported in Aicardi et al. (1995) for a unicycle-like vehicle moving in a plane, but it is different from that and other 3D solutions reported in the literature (e.g., Egeland, Dalsmo, and Sørdalen 1996) because the present control scheme does not generate any cusp in the paths, thus satisfying, as in Indiveri (1999) for the 2D case, a major requirement for most practical applications. Moreover, the target configuration is approached along a path with an asymptotically null curvature.

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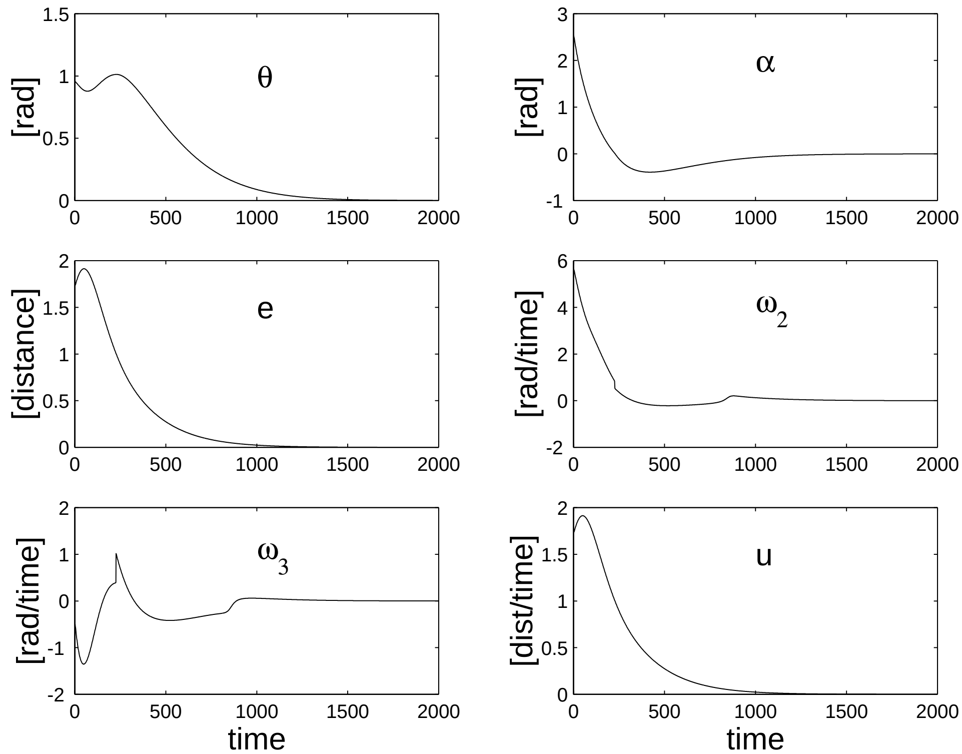

Fig. 4. Noise-free simulation.

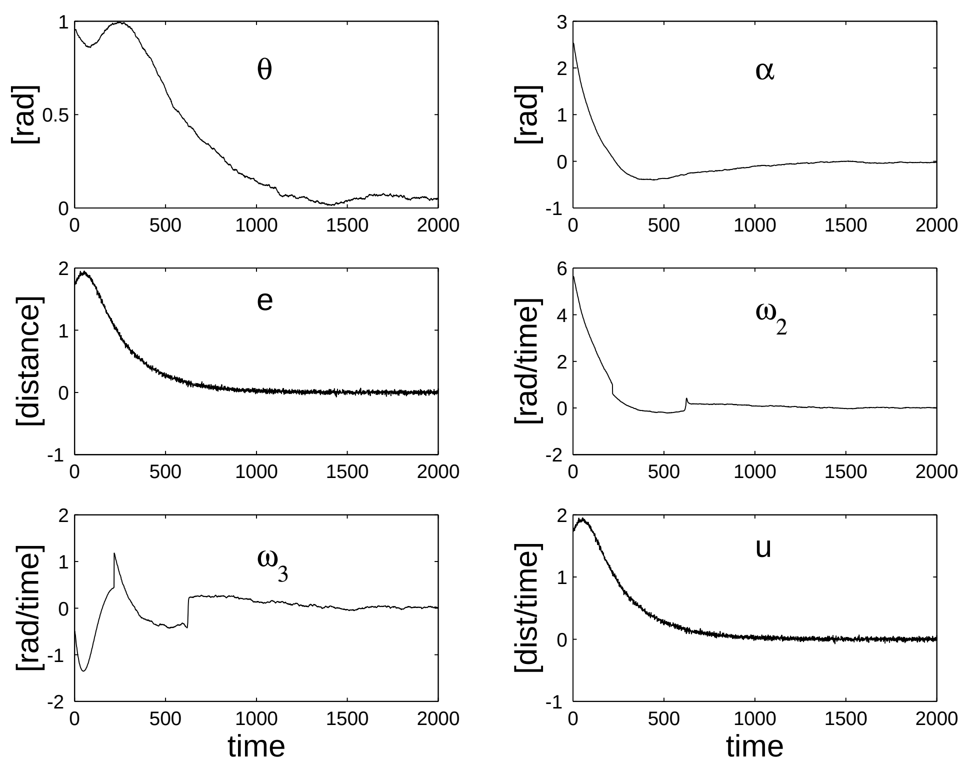

Fig. 5. Simulation with additive noise on the state.

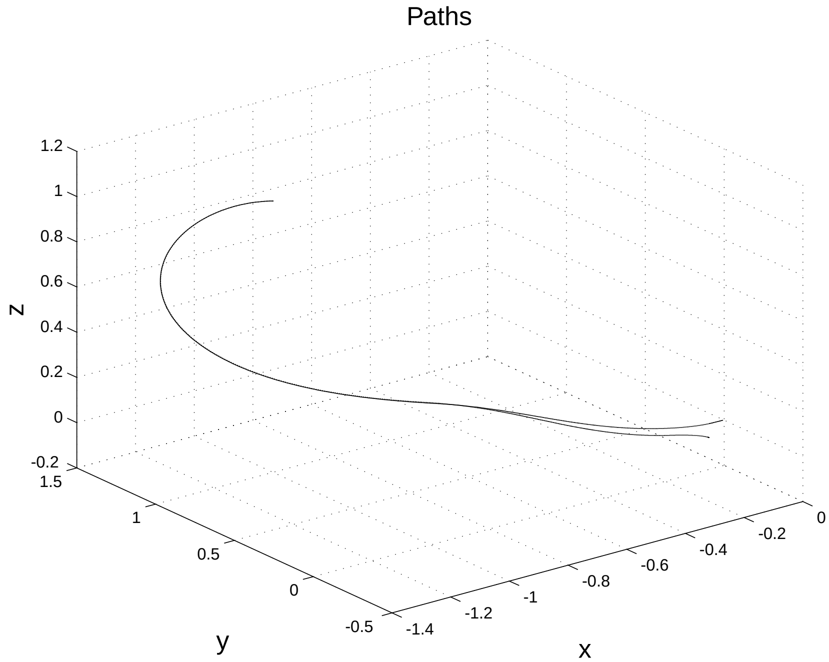

Fig. 6. Paths.

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