Abstract

In this paper, using the theories and methods of ecology and ordinary differential equations, an ecological model with an impulsive control strategy and a distributed time delay is defined. Using the theory of the impulsive equation, small-amplitude perturbations, and comparative techniques, a condition is identified which guarantees the global asymptotic stability of the prey-($x$) and predator-($y$) eradication periodic solution. It is proved that the system is permanent. Furthermore, the influences of impulsive perturbations on the inherent oscillation are studied numerically, an oscillation which exhibits rich dynamics including period-halving bifurcation, chaotic narrow or wide windows, and chaotic crises. Computation of the largest Lyapunov exponent confirms the chaotic dynamic behavior of the model. All these results may be useful for study of the dynamic complexity of ecosystems.

Keywords: Impulsive; Lyapunov exponent; Chaotic; Periodic solution

1. Introduction

The dynamical relationships between predators and their prey have been and will continue to be one of the dominant themes in both ecology and mathematical ecology because of its universal existence and importance [15]. At present it is clear that predator–prey functional responses have ability to influence the dynamical behaviors of predator–prey systems, especially predator–prey systems with some impulse perturbations, all these results are exhibited in these papers [2,1,30,37]. Many evolutionary processes are characterized by the fact that at certain moments they experience an abrupt change of state. These processes are subject to short-term perturbations of negligible duration compared with the duration of the process [13,14]. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of an impulse. It is well known that many biological phenomena involving thresholds, such as bursting rhythm models in medicine and biology, optimal control models in economics and pharmacokinetics, and frequency-modulated systems exhibit impulse effects. Thus impulsive differential equations, or differential equations involving impulsive effects, appear to be a natural description of observed evolution phenomena for several real-world problems [18,3].
In recent decades, the dynamical behaviors of biological systems with impulse perturbations have been studied in detail by these papers [28,9,24,16,10,35,34,19]. Mathematical theoretical works have been pursuing the investigation of the stability of semi-trivial periodic solution and the threshold expression of critical parameters under the condition of all species persistence and some species extinction. Numerical analysis indicates that the biological systems with impulse perturbations possess rich dynamical behaviors. Further, the Lyapunov stability theorem is one of well-known methods to prove species extinction, which has been used in many areas, such as biological control [20], synchronization, fuzzy control [5,7,32,6]. Moreover, it is well known that time delays are an important component of mathematical models in ecology. Usually, time delays in those models are of two types: discrete delays and continuous or distributed time delays [26]. For an impulsive model with distributed time delay, papers [12,29,25,17,36] report investigations of ecological models with distributed time delays and an impulsive control strategy.

The Beddington–DeAngelis functional response is introduced by Beddington [4] and DeAngelis et al. [8], independently. The main difference of this functional response from other functional responses is that it contains an extra term presenting mutual interference by predators [2]. Although a direct link between the predators and prey cannot be established unless quantitative methods are used, the precious works clearly show that the amount of three species are often related, and a change in one species can cause a change in the others, especially predator. Thus we apply Beddington–DeAngelis functional response to describe their relationship with sufficient accuracy in this paper.

This paper presents an ecological model with distributed time delays and an impulsive control strategy. The model can be described by the following differential equations:

\[
\begin{aligned}
\frac{dx(t)}{dt} &= r(x(t))\left(\frac{k_0 - x(t)}{k_1 - x(t)}\right) - b_3x^2(t) - a_3x(t)y(t) - \frac{a_1x(t)z(t)}{b_1 + x(t) + c_1z(t)} \\
\frac{dy(t)}{dt} &= d_1y(t)\int_{-\infty}^{t} F(t - s)x(s)ds - \frac{a_2y(t)z(t)}{b_2 + y(t) + c_2z(t)} - m_1y(t) \\
\frac{dz(t)}{dt} &= \frac{e_1a_1x(t)z(t)}{b_1 + x(t) + c_1z(t)} + \frac{e_2a_2y(t)z(t)}{b_2 + y(t) + c_2z(t)} - m_2z(t)
\end{aligned}
\]

\(t \neq nT,\) \(t = nT,\) \(\Delta x(t) = 0,\) \(\Delta y(t) = 0,\) \(\Delta z(t) = p\)

where \(x(t), y(t),\) and \(z(t)\) are respectively the densities of the lowest level prey, mid-level predator, and top predator at time \(t, \Delta x(t) = x(t^n) - x(t), \Delta y(t) = y(t^n) - y(t), \Delta z(t) = z(t^n) - z(t),\) \(r\) is the intrinsic growth rate, \(a_i (i = 1, 2, 3)\) are the cropping rates, \(e_i (i = 1, 2)\) denote the efficiency with which resources are converted to new consumers, \(r_k (0 \leq k_0/k_1 \leq 1)\) is the carrying capacity of the prey, \(k_1\) is the value of limiting resources, \(b_i (i = 1, 2)\) are saturation constants, \(c_i (i = 1, 2)\) are scaling factors for the impact of predator interference. \(m_i (i = 1, 2)\) are the mortality rates for each predator, \(b_1\) is the rate of intra-specific competition for the prey, and \(d_i\) denotes the product of the per-capita rate of predation and the conversion rate of prey into predators. The function \(F(t)\) satisfies \(\int_{0}^{+\infty} F(s)ds = 1\) and \(F(t) = se^{-\alpha t}, \alpha > d_1 > 0.\) Then \(T\) is the period of the impulsive effect, \(n \in N, N\) is the set of all non-negative integers, and \(p > 0\) is the number of predators released at \(t = nT.\)

To study the system, the first step is to carry out the chain transformation:

\[
q(t) = \int_{-\infty}^{t} F(t - s)x(s)ds.
\]

Because \(\int_{-\infty}^{t} F(t - s)ds = \lim_{\Delta \to -\infty} \int_{A}^{t} de^{-\alpha(t-s)}ds = 1\) and \(\int_{-\infty}^{t} F(t - s)x(s)ds\) is convergent,

\[
\Delta q(t) = \int_{-\infty}^{t} F(t - s)x(s)ds - \int_{-\infty}^{t} F(t - s)x(s)ds = 0, \quad t = nT, \quad n \in N.
\]
Furthermore, system (1.1) becomes
\[
\begin{aligned}
\frac{dx(t)}{dt} &= rx(t) - b_1x(t) - a_1xz(t) - b_3x^2(t) - a_3xy(t) - \frac{a_1xz(t)}{b_1 + x(t) + c_1z(t)} \\
\frac{dy(t)}{dt} &= d_1y(t)q(t) - \frac{a_2yz(t)}{b_2 + y(t) + c_2z(t)} - m_1y(t) \\
\frac{dz(t)}{dt} &= e_1a_1xz(t) + \frac{e_2a_2yz(t)}{b_2 + y(t) + c_2z(t)} - m_2z(t) \\
\frac{dq(t)}{dt} &= \sigma x(t) - \sigma q(t)
\end{aligned}
\]
\[t \neq nT,
\]
(1.2)

From the above discussions, it is clear that the properties of system (1.1) can be obtained by investigating system (1.2); therefore, in the following discussion, system (1.2) will mainly be considered.

2. Mathematical analysis

Let \( R_+ = [0, \infty), R^4_+ = \{ X \in R^4 | X \geq 0 \} \). Denote by \( f = \{ f_1, f_2, f_3, f_4 \} \) the map defined by the right-hand sides of the first, second, third, and fourth equations of system (1.2). Let \( V : R_+ \times R^4_+ \to R_+ \); then \( V \) is said to belong to class \( V_0 \) if it exists. The following are also true:

1. \( V \) is continuous in \((nT, (n + 1)T) \times R^4_+\), and for each \( X \in R^4_+ \), \( n \in N \), \( \lim_{(t,y) \to (nT^+,X)} V(t, y) = V(nT^+, X) \) exists.

2. \( V \) is locally Lipschitzian in \( X \).

**Definition 2.1.** Let \( V \in V_0 \); then for \((t, X) \in (nT, (n + 1)T) \times R^4_+\), the upper right derivative of \( V(t, X) \) with respect to the impulsive differential system (1.2) is defined as:

\[
D^+ V(t, X) = \lim \sup_{h \to 0} \frac{1}{h} [V(t + h, X + hf(t, X)) - V(t, X)].
\]

The solution of system (1.2) is a piecewise continuous function \( X : R_+ \to R^4_+ \); \( X(t) \) is continuous on \((nT, (n + 1)T)\), \( n \in N \), and \( X(nT^+) = \lim_{t \to nT^+} X(t) \) exists. The smoothness properties of \( f \) guarantee the global existence and uniqueness of the solution of system (1.2); for details, see Refs. [18,3].

The following lemma is obvious.

**Lemma 2.2.** Let \( X(t) \) be a solution of system (1.2) with \( X(0^+) \geq 0 \); then \( X(t) \geq 0 \) for all \( t \geq 0 \). Furthermore, \( X(t) > 0 \), \( t > 0 \) if \( X(0^+) > 0 \).

The following argument will use an important comparison theorem on impulsive differential equations.

**Lemma 2.3** ([18]). Suppose that \( V \in V_0 \). Assume further that:
\[
\begin{aligned}
D^+ V(t, X) &\leq g(t, V(t, X)), \quad t \neq nT, \\
V(t, X(t^+)) &\leq \psi_n(V(t, X)), \quad t = nT,
\end{aligned}
\]
(2.1)
where \( g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is continuous in \((nT, (n+1)T] \times \mathbb{R}_+ \) and, for \( u \in \mathbb{R}_+, \ n \in \mathbb{N}, \ \lim_{(t,v) \to (nT^+,u)} g(t, v) = g(nT^+, u) \) exists, and \( \psi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is non-decreasing. Let \( r(t) \) be the maximal solution of the scalar impulsive differential equations:

\[
\begin{cases}
\frac{du(t)}{dt} = g(t, u(t)), \ t \neq nT, \\
u(t^+) = \psi_n(u(t)), \ t = nT, \\
u(0^+) = u_0.
\end{cases}
\]  

(2.2)

existing on \([0, \infty)\). Then \( V(0^+, X_0) \leq u_0 \) implies that \( V(t, X(t)) \leq r(t), \ t \geq 0 \), where \( X(t) \) is any solution of system (1.2).

If the lowest level prey and mid-level predators are absent, that is, \( x(t) = 0 \) and \( y(t) = 0 \), and also \( q(t) = 0 \), then system (1.2) reduces to:

\[
\begin{cases}
\frac{dz(t)}{dt} = -m_2z(t), \ t \neq nT, \\
z(t^+) = z(t) + p, \ t = nT, \\
z(0^+) = z_0.
\end{cases}
\]  

(2.3)

Clearly \( z^*(t) = (p\exp(-m_2(t-nT))/(1-\exp(-m_2T))), t \in (nT, (n+1)T], n \in \mathbb{N}, \) and \( z^*(0^*) = (p\exp(-m_2T))/(1-\exp(-m_2T)) \) is a positive periodic solution of system (2.3). Because \( z(t) = (z(0^*) - (p\exp(-m_2T))\exp(-m_2t) + z^*(t) \) is the solution of system (2.3) with initial value \( z_0 = z(0^*) \geq 0 \), where \( t \in (nT, (n+1)T], n \in \mathbb{N} \), the following lemma can be derived.

**Lemma 2.4.** For a positive periodic solution \( z^*(t) \) of system (2.3) and every solution \( z(t) \) of system (2.3) with \( z_0 \geq 0 \), it follows that \( |z(t) - z^*(t)| \rightarrow 0, t \rightarrow \infty \).

Therefore, the complete expression for the lowest level prey and mid-level predator eradication periodic solution \((0, 0, z^*(t), 0)\) of system (1.2) has been obtained.

The next step is to investigate the stability of the lowest level prey and mid-level predator eradication periodic solution.

**Theorem 2.1.** Let \((x(t), y(t), z(t), q(t))\) be any solution of system (1.2); then \((0, 0, z^*(t), 0)\) is locally asymptotically stable if:

\[
\frac{r k_0}{k_1} T + \frac{a_1}{m_2 c_1} \ln \left[ 1 - \frac{c_1 p(1-\exp(-m_2T))}{b_1(1-\exp(-m_2T)+c_1 p)} \right] < 0
\]

and

\[
-m_1 T + \frac{a_2}{m_2 c_2} \ln \left[ 1 - \frac{c_2 p(1-\exp(-m_2T))}{b_2(1-\exp(-m_2T)+c_2 p)} \right] < 0
\]

**Proof.** The local stability of the periodic solution \((0, 0, z^*(t), 0)\) can be determined by considering the behavior of small-amplitude perturbations of the solution. Define

\[
x(t) = u(t), \ y(t) = v(t), \ z(t) = w(t) + z^*(t), \ q(t) = h(t).
\]  

(2.4)
Substituting Eq. (2.4) into Eq. (1.2), the following linearization of the system results:

\[
\begin{align*}
\frac{du(t)}{dt} &= \left( r_0 - \frac{a_1 z^*(t)}{b_1 + c_1 z^*(t)} \right) u(t) \\
\frac{dv(t)}{dt} &= \left( -m_1 - \frac{a_2 z^*(t)}{b_2 + c_2 z^*(t)} \right) v(t) \\
\frac{dw(t)}{dt} &= \frac{e_1 a_1 z^*(t)}{b_1 + c_1 z^*(t)} u(t) + \frac{e_2 a_2 z^*(t)}{b_2 + c_2 z^*(t)} v(t) - m_2 w(t) \\
\frac{dh(t)}{dt} &= \sigma u(t) - \sigma h(t)
\end{align*}
\]

where \( \lambda_3 = \exp(-m_2 T) < 1 \) and \( \lambda_4 = \exp(-\sigma T) < 1 \), \( \lambda_3, \lambda_4 \) have absolute value less than one; then the periodic solution \((0, 0, z^*(t), 0)\) is locally stable. Therefore, given the eigenvalues \( \Theta \),

\[
\begin{align*}
\lambda_1 &= \exp \left( \int_0^T \left( r_0 - \frac{a_1 z^*(t)}{b_1 + c_1 z^*(t)} \right) dt \right), \\
\lambda_2 &= \exp \left( \int_0^T \left( -m_1 - \frac{e_1 a_1 z^*(t)}{b_1 + c_1 z^*(t)} - m_1 \right) dt \right), \\
\lambda_3 &= \exp(-m_2 T) < 1, \\
\lambda_4 &= \exp(-\sigma T) < 1.
\end{align*}
\]
then according to the Floquet theory of impulsive differential equations, the system \((0, 0, z^*(t), 0)\) is locally asymptotically stable if \(|\lambda_1| < 1\) and \(|\lambda_2| < 1\), that is to say,

\[
rk_0 \frac{T}{k_1} + \frac{a_1}{m_2 c_1} \ln \left[ 1 - \frac{c_1 p (1 - \exp(-m_2 T))}{b_1 (1 - \exp(-m_2 T) + c_1 p)} \right] < 0,
\]

and

\[
-m_1 T + \frac{a_2}{m_2 c_2} \ln \left[ 1 - \frac{c_2 p (1 - \exp(-m_2 T))}{b_2 (1 - \exp(-m_2 T) + c_2 p)} \right] < 0.
\]

This completes the proof. \(\square\)

**Theorem 2.2.** There exists a constant \(M > 0\) such that \(x(t) < M, y(t) < M, z(t) < M,\) and \(q(t) < M\) for each solution \(X(t) = (x(t), y(t), z(t), q(t))\) of system (1.2) for all \(t\) large enough.

**Proof.** Define \(V(t, X(t))\) such that

\[
V(t, X(t)) = e_1 x(t) + e_2 y(t) + z(t) + q(t),
\]

then \(V \in V_0\). Because \((dx(t)/dt) \leq rx(t) - b_3 x^2(t)\) and \(q(t) = \int_{t}^{t+T} F(t-s) x(s) ds\), then \(x(t) \leq (rb_3), q(t) \leq (rb_3)\). By calculating the upper right derivative of \(V(t, X)\) along a solution of system (1.2), the following impulsive differential equation is obtained:

\[
\begin{align*}
D^+ V(t) + LV(t) & \leq Le_1 x(t) + \sigma x(t) + re_1 x(t) \left( \frac{k_0 - x(t)}{k_1 - x(t)} \right) - b_3 e_1 x^2(t) a_3 e_1 x(t)y(t) + d_1 e_2 p(t) y(t) \\
& + (Le_2 - m_1 e_2) y(t) + (L - m_2) z(t) + (L - \sigma) q(t), \quad t \neq nT, \\
V(t^+) & = V(t) + p, \quad t = nT.
\end{align*}
\]

Obviously,

\[
D^+ V(t) + LV(t) \leq Le_1 x(t) + \sigma x(t) + re_1 x(t) - b_3 e_1 x^2(t) + \left( Le_2 + \frac{rd_1 c_2}{b} - m_1 e_2 \right) y(t) + (L - m_2) z(t) + (L - \sigma) q(t).
\]

Let \(0 < L < \min \{ m_2, m_1 - (d_1 rb_3), \sigma \}\); then \(D^+ V(t) + LV(t)\) is bounded. Select \(L_1\) and \(L_2\) such that

\[
\begin{align*}
D^+ V(t) & \leq -L_1 V(t) + L_2, \quad t \neq nT, \\
V(t^+) & = V(t) + p, \quad t = nT,
\end{align*}
\]

where \(L_1\) and \(L_2\) are two positive constants.

According to Lemma 2.3,

\[
V(t) \leq \left( V(0^+) - \frac{L_2}{L_1} \right) \exp(-L_1 t) + \frac{p (1 - \exp(-L_1 T)) \exp(L_1 T) \exp(-L_1 t - nT) + L_2}{L_1},
\]

where \(t \in (nT, (n + 1)T)\). Hence,

\[
\lim_{t \to \infty} V(t) \leq \frac{L_2}{L_1} + \frac{p \exp(L_1 T)}{\exp(L_1 T) - 1}.
\]

Therefore \(V(t, X(t))\) is ultimately bounded, and it is already known that each positive solution of the system is uniformly ultimately bounded. This completes the proof. \(\square\)

**Theorem 2.3.** Let \((x(t), y(t), z(t), q(t))\) be any solution of system (1.2); then \((0, 0, z^*(t), 0)\) is globally asymptotically stable if

\[
\begin{align*}
\frac{rk_0}{k_1} + \frac{a_1}{m_2 c_1} \ln \left[ 1 - \frac{c_1 p (1 - \exp(-m_2 T))}{b_1 (1 - \exp(-m_2 T) + c_1 p)} \right] < 0, \\
-m_1 T + \frac{a_2}{m_2 c_2} \ln \left[ 1 - \frac{c_2 p (1 - \exp(-m_2 T))}{b_2 (1 - \exp(-m_2 T) + c_2 p)} \right] < 0,
\end{align*}
\]
and
\[ p > \frac{(\exp(m_2 T) - 1)(r + \sigma)(b_1 + M + c_1 M)}{a_1} . \]

**Proof.** By Theorem 2.1, we know that \((0, 0, z^*(t), 0)\) is locally asymptotically stable. Following is a proof of its global attraction. Let
\[ V(t) = x(t) + y(t) + q(t). \]

Then
\[
V'(t) = rx(t) \left( \frac{k_0 - x(t)}{k_1 - x(t)} \right) + \sigma x(t) - b_3 x^2(t) - a_3 x(t)y(t) + \frac{a_1 z(t)x(t)}{b_1 + x(t) + c_1 z(t)} \\
+ \left( - \frac{a_2 z(t)}{b_2 + y(t) + c_2 z(t)} - m_1 \right) y(t) + (d_1 - \sigma)q(t).
\]
Thus,
\[
V'(t) \leq \left( r + \sigma - \frac{a_1 z(t)}{b_1 + x(t) + c_1 z(t)} \right) x(t) + \left( - \frac{a_2 z(t)}{b_2 + y(t) + c_2 z(t)} - m_1 \right) y(t) + (d_1 - \sigma)q(t)
\]
and
\[
\frac{dz(t)}{dt} = \begin{cases} 
\frac{e_1 a_1 x(t)z(t)}{b_1 + x(t) + c_1 z(t)} + \frac{e_2 a_2 y(t)z(t)}{b_2 + y(t) + c_2 z(t)} - m_2 z(t) \geq -m_2 z(t), & t \neq nT, \\
\Delta z(t) = p, & t = nT. 
\end{cases}
\]
By Lemmas 2.3 and 2.4, it is known that there exists a \(t_1 > 0\) and an \(\varepsilon > 0\) small enough so that \(z(t) \geq z_1^*(t) - \varepsilon\) for all \(t \geq t_1\). Therefore,
\[
z(t) \geq z_1^*(t) - \varepsilon = \frac{p \exp(-m_2 T)}{1 - \exp(-m_2 T)} - \varepsilon,
\]
and
\[
\Delta \gamma = \frac{p \exp(-m_2 T)}{1 - \exp(-m_2 T)} - \varepsilon.
\]
By Theorem 2.2, there exists a constant \(M > 0\) such that \(x(t) \leq M, y(t) \leq M, z(t) \leq M, q(t) \leq M\) for each solution \(X(t) = (x(t), y(t), z(t), q(t))\) of system (1.2) for all \(t\) large enough. Therefore,
\[
V'(t) \leq \left( r + \sigma - \frac{a_1 \gamma}{b_1 + M + c_1 M} \right) x(t) + \left( - \frac{a_2 \gamma}{b_2 + y(t) + c_2 z(t)} - m_1 \right) y(t) + (d_1 - \sigma)q(t)
\]
\[- b_3 x^2(t) - a_3 x(t)y(t), \]
if \(r + \sigma - (a_1 \gamma/(b_1 + M + c_1 M)) < 0\), that is to say,
\[
p > \frac{(\exp(m_2 T) - 1)}{a_1} \left( \frac{(r + \sigma)(b_1 + M + c_1 M) + \varepsilon}{a_1} \right) > \frac{(\exp(m_2 T) - 1)(r + \sigma)(b_1 + M + c_1 M)}{a_1} .
\]
Thus, when \(t \geq t_1\),
\[
V'(t) \leq \left( r + \sigma - \frac{a_1 \gamma}{b_1 + M + c_1 M} \right) x(t) + \left( - \frac{a_2 \gamma}{b_2 + y(t) + c_2 z(t)} - m_1 \right) y(t) + (d_1 - \sigma)q(t)
\]
\[- b_3 x^2(t) - a_3 x(t)y(t) < 0. \]

It follows that \(V(t) \to 0, x(t) \to 0, y(t) \to 0,\) and \(q(t) \to 0\) as \(t \to \infty\). Note that the limiting case of system (1.2) is exactly system (2.3). From Lemma 2.4, it follows that the lowest level prey and mid-level predator eradication periodic solution, \((0, 0, z^*(t), 0)\), is a global attractor. This completes the proof. \(\square\)
**Theorem 2.4.** The system (1.2) ispermanent if
\[
\frac{rk_0}{k_1} + \frac{a_1}{m_2 c_1} \ln \left[ 1 - \frac{c_1 p(1 - \exp(-m_2)T)}{b_1(1 - \exp(-m_2)T + c_1)} \right] > 0,
\]
\[
-m_1 T + \frac{a_2}{m_2 c_2} \ln \left[ 1 - \frac{c_2 p(1 - \exp(-m_2)T)}{b_2(1 - \exp(-m_2)T + c_2)} \right] > 0,
\]
and
\[
p < \frac{b_1 m_2 T(rk_0/k_1) - b_3 M - a_3 M}{a_1}.
\]

**Proof.** Suppose \(X(t) = (x(t), y(t), z(t), q(t))\) is any solution of system (1.2) with \(X(0) > 0\). From Theorem 2.2, we assume that \(x(t) \leq M, y(t) \leq M, z(t) \leq M\) and \(q(t) \leq M\) with \(t \geq 0\). From Lemma 2.4, we have \(z(t) > z^*(t) - \epsilon\) for all \(t\) large enough and some \(z(t) > (p \exp(-m_2 T)/(1 - \exp(-m_2 T))) - \epsilon \equiv \xi_1\) for \(t\) large enough. Thus we only need to find \(\xi_2\) and \(\xi_3\) such that \(x(t) \geq \xi_2\) and \(y(t) \geq \xi_3\) for \(t\) large enough.

We will let \(\epsilon > 0\) be small enough such that
\[
\eta_1 = \exp \left( \int_{nT}^{(n+1)T} \left[ \frac{k_0}{k_1} - b_3 M - a_3 M - \frac{a_1(v^*_3(t) + \epsilon)}{b_1} \right] dt \right) > 1.
\]

Now, we will prove that there exists a constant \(\xi_2\), such that \(x(t) \geq \xi_2\) for \(t\) large enough.

We will prove that there exists a \(\xi_3 \in (0, +\infty)\) such that \(x(t) \geq \xi_3\) for \(t\) large enough. Otherwise \(x(t) < \xi_2\) for all \(t > 0\). From system (1.1), we have
\[
\begin{align*}
dz(t) &\leq \frac{(e_1 a_1 T^2 + e_2 a_2 M)}{b_1} - m_2 z(t), & t \neq nT, \\
z(t^+) &= z(t) + p, & t = nT, \\
z(0^+) &= z_0. 
\end{align*}
\] (2.8)

Then we have \(z(t) \leq v_3(t) + v_3^*(t)(t \to +\infty)\) where \(v_3(t)\) is the solution of
\[
\begin{align*}
\frac{dv_3(t)}{dt} &\leq \left( \frac{e_1 a_1 T^2 + e_2 a_2 M}{b_1} - m_2 \right) v_3(t), & t \neq nT, \\
v_3(t^+) &= v_3(t) + p, & t = nT, \\
v_3(0^+) &= z_0. 
\end{align*}
\] (2.9)

For \(t \in (nT, (n + 1)T]|n \in N\), there exists
\[
v_3^*(t) = \frac{p \exp(-(m_2 - (e_1 a_1 T^2/b_1) - (e_2 a_2 M/b_2))) (t - nT)}{1 - \exp(-(m_2 - (e_1 a_1 T^2/b_1) - (e_2 a_2 M/b_2)))}.
\]

Therefore there exists \(t \geq T_1\) such that \(z(t) \leq v_3(t) + v_3^*(t) + \epsilon\) and
\[
\begin{align*}
\frac{dx(t)}{dt} &\leq \left( \frac{k_0}{k_1} - b_3 M - a_3 M - \frac{a_1(v_3^*(t) + \epsilon)}{b_1} \right) x(t), & t \neq nT, \\
x(t^+) &= x(t), & t = nT. 
\end{align*}
\] (2.10)

Let \(N_1 \in N\) and \(N_1 T > T_1\), integrating (2.10) on \((nT, \ (n + 1)T, \ n \geq N_1\), we can get
\[
\begin{align*}
x((n + 1)T) &\geq x(nT^+) \exp \int_{nT}^{(n+1)T} \left[ \frac{k_0}{k_1} - b_3 M - a_3 M - \frac{a_1(v_3^*(t) + \epsilon)}{b_1} \right] dt \\
&= x(nT) \exp \int_{nT}^{(n+1)T} \left[ \frac{k_0}{k_1} - b_3 M - a_3 M - \frac{a_1(v_3^*(t) + \epsilon)}{b_1} \right] dt \\
&= x(nT) \eta_1.
\end{align*}
\]
Then \( x((N_1 + k)T) \geq x(N_1T)\eta_1^n \rightarrow \infty, k \to \infty \), which is contradiction to the boundedness of \( x(t) \). Hence there exists a \( t_1 \) such that \( x(t_1) \geq \xi_2 \).

Second if \( x(t_1) \geq \xi_2 \) for all \( t \geq t_1 \), then our aim is obtained. Hence we only need to consider those solutions which leave the region \( \mathcal{R} = \{ x(t) : x(t) < \xi_2 \} \) and reenter it again. Let \( t_2 = \inf_{t > t_1} \{ x(t) : x(t) < \xi_2 \} \), we have \( x(t) \geq \xi_2, t \in (t_1, t_2) \) and \( t_2 \in (n_1T, (n_1+1)T), n_1 \in \mathbb{N} \). It is easy to prove \( x(t_2) = \xi_2 \) since \( x(t) \) is continuous.

We claim that there must exists a \( t_3 \in ((N_1 + 1)T, (n_1 + 1)T + T) \) such that \( x(t_3) \geq \xi_2 \), otherwise \( x(t) < \xi_2 \), \( t \in ((n_1 + 1)T, (n_1 + 1)T + T), n_1 \in \mathbb{N} \). Select \( n_2, n_3 \in \mathbb{N} \) such that
\[
(n_2 - 1)T > \frac{\ln(c/(M+p))}{(e_1a_1\xi_2/b_1) - (e_2a_2M/b_2)}
\]
and \( \exp(x((n_2 + 1)T)\eta_1^n) > 1 \).

We consider (2.9) with \( v_3(t) = z(t) \), we have
\[
v_3(t) = (v_3((n_1 + 1)T^-) - \frac{p}{1 - \exp(-(m_2 - \frac{e_1a_1\xi_2}{b_1} - \frac{e_2a_2M}{b_2})T)}) + v_3^u(t)
\]
for \( t \in (nT, (n + 1)T), n + 1 < n < n_1 + n_2 + n_3 + 1 \). For \( (n_1 + n_2 + 1)T \leq t \leq (n_1 + 1)T + T \), there exists
\[
|v_3(t) - v_3^u(t)| = (M + p)\exp\left(-\left(m_2 - \frac{e_1a_1\xi_2}{b_1} - \frac{e_2a_2M}{b_2}\right)(t - (n_1 + 1)T)\right) < \epsilon,
\]
and \( z(t) \leq v_3(t) \leq v_3^u(t) + \epsilon \), which implies that (2.10) holds for \( (n_1 + n_2 + 1)T \leq t \leq (n_1 + 1)T + T \).

Integrating (2.10) on \( ((n_1 + 1)T, (n_1 + 1)T + T) \), we have \( x(n_1 + n_2 + n_3 + 1)T) \geq x((n_1 + n_2 + 1)T)\eta_1^n \).

There are two possible cases for \( t \in (t_3, (n_1 + 1)T) \):

Case (I): If \( x(t) < \xi_2 \) for all \( t \in (t_3, (n_1 + 1)T), \) then \( x(t) < \xi_2 \) for all \( t \in (t_3, (n_1 + n_2 + 1)T) \), we have
\[
\frac{dx(t)}{dt} \geq x(t) \left[k_0 - b_3\xi_2 - a_3M - \frac{a_1(v_3^u(t) + \epsilon)}{b_1}\right] = tx(t).
\]
(2.11)

Integrating (2.11) on \( (t_3, (n_1 + n_2 + 1)T) \) yields
\[
x((n_1 + n_2 + n_3 + 1)T) \geq \xi_2 \exp(x((n_2 + 1)T)\eta_1^n) > \xi_2, \]
which is a contradiction. Let \( t_4 = \inf_{t > t_3} \{ x(t) \geq \xi_2 \} \) and \( x(t_4) = \xi_2 \) and (2.11) holds on \( [t_3, t_4] \). Integrating (2.11) on \( [t_3, t_4] \) yields
\[
x(t) \geq x(t_3) \exp(\tau(t - t_3)) \geq \xi_2 \exp(x((n_2 + n_3 + 1)T)\eta_1^n) \geq \xi_2.
\]
For \( t > t_4 \), the same argument can be continued since \( x(t) \geq \xi_2 \) and we can find a \( \xi_2 \) such that \( x(t) \geq \xi_2 \) for all \( t > t_1 \).

Case (II): There exists \( t_5 \in (t_2, (n_1 + 1)T) \) such that \( x(t_5) \geq \xi_2 \). Let \( t_6 = \inf_{t > t_5} \{ x(t) \geq \xi_2 \}, \) then \( x(t) < \xi_2 \) for \( t \in [t_2, t_6) \) and \( x(t_6) = \xi_2 \). Integrating (2.11) on \( [t_2, t_6] \), we have \( x(t) \geq x(t_2) \exp(\tau(t - t_2)) > \xi_2 \). This process can be continued since \( x(t_6) \geq \xi_2 \) and we have \( x(t) \geq \xi_2 \) for \( t > t_6 \). Thus in both cases, we conclude \( x(t) \geq \xi_2 \) for \( t > t_1 \).

From system (1.2), we have \( \frac{dy(t)}{dt} > y(t)((m_1 - (a_2M/b_2)) \), thus \( y(t) > y(t_0)\exp((-m_1 - (a_2M/b_2))(t - t_0)) \), it is easy to find a \( \xi_2 \) such that \( y(t) \geq \xi_2 \) for \( t \) large enough.

Set \( \Omega = \{ (x, y, z, q) : \xi_2 \leq x \leq M, \xi_2 \leq y \leq M, \xi_2 \leq z \leq M, \xi_2 \leq q \leq M \} \). Every solution of system (1.2) will eventually enter and remain in region \( \Omega \). Therefore, system (1.1) is permanent. The proof is completed. \( \square \)

3. Numerical analysis

3.1. Bifurcation analysis

To study the dynamics of an ecological model with an impulsive control strategy and a distributed time delay, a solution of system (1.2) with initial conditions in the first quadrant is obtained numerically for a biologically feasible range of parametric values because the corresponding continuous system (1.2) cannot be solved explicitly. In order to investigate the effect of impulsive control strategy, a bifurcation diagram for each population as a function of the release amount \( p \) is studied for the system (1.2).

From Theorem 2.1, it is known that the lowest level periodic and mid-level predator eradication periodic solution, \((0, 0, \bar{z}^z(t), 0)\), is locally asymptotically stable provided that \( p > p_{\text{max}} = 13.153215 \). An example of a lowest level prey and mid-level predator eradication periodic solution of system (1.2) is shown in Fig. 1. It is evident that the variable \( z(t) \)
oscillates in a stable cycle. In contrast, the prey population \((x(t))\) and the predator population \((y(t))\) rapidly decrease to zero when \(p > p_{\text{max}} = 13.153215\). When the value of \(p\) is smaller than \(p_{\text{max}} = 13.153215\), the prey-\((x)\) and predator-\((y)\) eradication periodic solution will becomes unstable. It is possible that the prey \((x)\), mid-range predator \((y)\), and top predator \((z)\) populations can coexist in a stable positive periodic solution.

Now, we study the influence of the impulsive perturbation \(p\) on the system’s inherent oscillations. The bifurcation diagrams of system (1.2), plotted as a function of the bifurcation parameter \(p\), are shown in Fig. 2. Because of the similarity of these bifurcation diagrams, only Fig. 2a is analyzed in detail. As the parameter \(p\) increases from 0 to 2.355725, the system enters a chaotic band with a wide periodic window. In particular, when \(p\) increases beyond 1.768315, the size of the chaotic attractor changes abruptly, constituting a type of attractor crisis. Then a stable periodic window appears. As \(p\) increases further to 1.821435, the periodic solution loses its weak stability, and the system again enters the chaotic band (Fig. 3). When \(p\) increases beyond 2.372345, the size of the chaotic attractor changes abruptly, again constituting a type of attractor crisis. As \(p\) increases further, a period-halving bifurcation leads the system to a stable state (Fig. 4).

### 3.2. The largest Lyapunov exponent

Convincing evidence for deterministic chaos has come from several recent experiments [11, 31, 27, 21, 38]. The results of those studies have confirmed the importance of detecting and exploring chaos. Here, the largest Lyapunov exponents are considered; these have proved to offer the most useful diagnostic for a chaotic system [22, 33]. The largest Lyapunov exponents take into account the average exponential rates of divergence or convergence of nearby orbits in phase space [11]. For a chaotic attractor, the largest Lyapunov exponent \(\lambda\) must be positive. If \(\lambda\) is negative, this implies a stable state or a periodic attractor. By reviewing the bifurcation diagram shown in Fig. 2, the corresponding largest Lyapunov exponent \([0.2 \leq p \leq 10.5]\) can be calculated for system (1.2). The outputs of this calculation are shown in Fig. 5.
3.3. Strange attractors and power spectra

To study the qualitative nature of strange attractors, the commonly used power spectrum method was chosen [23]. The power spectrum was calculated using 4096 points corresponding to the time series of the variable $x(t)$ with a time difference $\Delta t = 0.5$. By calculating the largest Lyapunov exponent for the strange attractor ($a$) (Fig. 6a), it can be established that the value of the largest Lyapunov exponent for the strange attractor ($a$) is 0.23935237. Obviously the strange attractor ($a$) is a chaotic attractor. In the spectrum of the strongly chaotic attractor ($a$) (Fig. 6b), no sharp peaks...
can be distinguished. These results agree with the fact that the strange attractor (a) arises from a weak limiting cycle that is affected by noise.

4. Conclusions and remarks

In this paper, the dynamic complexities of an ecological model with an impulsive control strategy and a distributed time delay are studied numerically and analytically. Using the Floquet theorem and small-amplitude perturbations, it has been proved that the periodic solution \((0, 0, z^*(t), 0)\) is globally asymptotically stable if \((r_k/k_1)T + (a_1/m_2)\ln[1 - (c_1p(1 - \exp(-m_2)T)b_1(1 - \exp(-m_2)T + c_1p)) < 0, -m_1T + (a_2/m_2)\ln[1 - (c_2p(1 - \exp(-m_2)T)b_2(1 - \exp(-m_2)T + c_2p)) < 0 \text{ and } p > (\exp(m_2T) - 1)\sigma(b_1 + M + c_1M)\alpha_1),\) we have obtained that the system is permanent. Numerical analysis indicates that the complex dynamics of system (1.2) depend on the values of the impulsive perturbation \(p\) and of various other parameters. By varying the impulsive perturbation \(p\), a series of bifurcation diagrams has been obtained. The bifurcation diagrams show that system (1.2) exhibits dynamic complexity, including chaotic bands with narrow and wide periodic windows, chaotic crises, and a period-halving bifurcation cascade. Furthermore, using computer-based simulations of the largest Lyapunov exponents, the presence of chaotic dynamics in the system can be confirmed. At the same time, using Fourier spectra, the qualitative nature of strange attractors has been investigated. All these results show that the dynamic behavior of system (1.2) becomes more complex under periodically impulsive perturbations.

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References
