RAINBOW TRIANGLES IN 3-EDGE-COLORED GRAPHS

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SIAM Conference on Discrete Mathematics
Jun 17, 2014
**Problem**

Find a 3-edge-coloring of a complete graph $K_n$ maximizing the number of copies of rainbow colored triangles.
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Color edges randomly, expected density $\frac{2}{9}$. 

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Iterated blow-up of triangle

\[ \frac{1}{4} = \text{denotes graph and/or its density} \]
\[ F(n) = \max \text{ # of } \begin{array}{c}
\text{over all 3-edge-colorings of } K_n
\end{array} \]
$F(n) = \max \# \text{ of } \triangle \text{ over all 3-edge-colorings of } K_n$

**Conjecture (Erdős and Sós; ’72)***

For all $n > 0$,

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

where $a + b + c + d = n$; $a, b, c, d$ are as equal as possible, and $F(0) = 0$. 

![Diagram](attachment:diagram.png)
$F(n) = \max \text{ # of over all 3-edge-colorings of } K_n$

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\text{over all 3-edge-colorings of } K_n
\end{array}
\end{array}$

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0.4 = \begin{array}{c}
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\end{array}
\end{array} $$
Flag algebras application

Construction: $0.4 \leq \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{triangle.png}}
\end{array}
\end{array}$

- get a matching upper bound $\approx 0.4$
- round the result
- get subgraphs with 0 density
- get extremal construction (stability)
Flag algebras application

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Flag algebras (on 6 vertices) give only $\leq 0.4006$, not enough for rounding.
Flag algebras application

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Flag algebras (on 6 vertices) give only $\leq 0.4006$, not enough for rounding.

The iterative extremal construction is causing troubles....
**Not iterated extremal constructions**

**Theorem (Turán)**

\# of edges over $K_1$-free graphs is maximized by

**Theorem (Hatami, Hladký, Král, Norine, Razborov)**

\# of $C_5$s over triangle-free graphs is maximized by

**Theorem (Cummings, Král, Pfender, Sperfeld, Treglown, Young)**

\# of monochromatic triangles over 3-edge-colored graphs is minimized by

And more... [http://flagmatic.org](http://flagmatic.org)  
$(n$ large enough$)$
Iterated extremal constructions

**Theorem (Falgas-Ravry, Vaughan)**

The number of \( \left\langle \right\rangle \) and \( \left( \right) \) is maximized by

**Theorem (Huang)**

The number of \( \downarrow \) is maximized by

**Theorem (Hladký, Král, Norine)**

The number of \( \downarrow \) is maximized by
Our main result

\[ F(n) = \max \text{ # of } \begin{array}{c} \text{over all coloring of } K_n \end{array} \]

Theorem (Balogh, Hu, L., Pfender, Volec, Young)

For all \( n > n_0 \),

\[ F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd, \]

where \( a + b + c + d = n; a, b, c, d \text{ are as equal as possible.} \)
Sketch of proof

Goal: maximizing gives edge-coloring like
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- pick a properly 3-edge-colored $K_4$
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Goal: maximizing gives edge-coloring like

- pick a properly 3-edge-colored $K_4$

How to pick the properly 3-edge-colored $K_4$?

($|X_i|$ is close to $0.25n$, few wrongly colored edges, small trash)
**Sketch of proof**

Goal: maximizing gives edge-coloring like

- pick a properly 3-edge-colored $K_4$
- partition the rest
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Sketch of proof

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- $|X_i|$ close to 0.25n, few wrongly colored edges, small trash
**Sketch of proof**

Goal: maximizing gives edge-coloring like

- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
Sketch of proof

Goal: maximizing \( X_1 \) gives edge-coloring like \( X_2 \)

- pick a properly 3-edge-colored \( K_4 \)
- partition the rest
- correct edges between \( X_i \)'s
- no orange trash
Sketch of proof

Goal: maximizing gives edge-coloring like

- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
- no **orange** trash
- balance sizes of $X_i$s
Goal: maximizing gives edge-coloring like

- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
- no orange trash
- balance sizes of $X_i$s
**Sketch of proof**

Goal: maximizing $g$ gives edge-coloring like

- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
- no orange trash
- balance sizes of $X_i$s

How to pick the properly 3-edge-colored $K_4$?

($|X_i|$s close to 0.25n, few wrongly colored edges, small trash)
How to pick $K_4$?

Use Flag Algebras!
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Use Flag Algebras!

Try 1: Pick maximizing

\[
|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n - 5)
\]

Balancing needed...
How to pick $K_4$?

Use Flag Algebras!

Try 1: Pick maximizing

$(n - 5) \geq$
How to pick $K_4$?

Use Flag Algebras!

Try 1: Pick maximizing

$$(n - 5) \geq \frac{1}{\binom{n}{4}} \sum (n - 5)$$
How to pick $K_4$?

Use Flag Algebras!

Try 1: Pick maximizing

\[
(n - 5) \geq \frac{1}{\binom{n}{4}} \sum \binom{n - 5}{4} = \frac{2 \binom{n}{5}}{\binom{n}{4}}
\]
How to pick $K_4$?

Use Flag Algebras!

Try 1: Pick maximizing

$$(n - 5) \geq \frac{1}{\binom{n}{4}} \sum \left( n - 5 \right) = \frac{2 \binom{n}{5}}{\binom{n}{4}} = \frac{2}{5} (n - 5)$$

Balancing needed...
How to pick $K_4$?

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Try 1: Pick maximizing

$$\sum \frac{n}{4} \geq \frac{n}{5} \cdot (n-5)$$

Result for $K_n$:

$$|X_1| + |X_2| + |X_3| + |X_4| > 0.988 (n-5)$$

FA: $\geq 0.4$ then $> 0.23516$, $< 0.0952$
How to pick $K_4$?

Use Flag Algebras!

Try 1: Pick $\text{maximizing} > 0.988$

$$\sum_{(n - 5)} \geq \frac{1}{\binom{n}{4}} \sum \binom{n}{4} (n - 5) = \frac{2\binom{n}{5}}{\binom{n}{4}} = \frac{2\binom{n}{5}}{5} (n - 5)$$

FA: $\geq 0.4 \text{ then } > 0.23516, < 0.0952$
**How to pick $K_4$?**

Use Flag Algebras!

**Try 1:** Pick maximizing $> 0.988$

$$(n - 5) \geq \frac{1}{\binom{n}{4}} \sum (n - 5) = \frac{2\binom{n}{5}}{\binom{n}{4}} = \frac{2}{5}(n - 5)$$

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$$|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n - 5)$$
**How to pick $K_4$?**

Use Flag Algebras!

Try 1: Pick maximizing $> 0.988$

\[
(n - 5) \geq \frac{1}{\binom{n}{4}} \sum (n - 5) = \frac{2\binom{n}{5}}{\binom{n}{4}} = \frac{2}{5}(n - 5)
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Result for $K_n$:

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|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n - 5)
\]

Balancing needed...
**How to pick $K_4$?**

Use Flag Algebras!

Try 2: Pick maximizing

$$+ \quad + \quad - \quad \frac{26}{9}$$

$n^2 F = \text{wrongly colored edges.}$
**How to pick $K_4$?**

Use Flag Algebras!

Try 2: Pick maximizing

$$\begin{align*}
\text{FA: } & \frac{4}{15} \left( \begin{array}{c}
\text{ } + \\
\text{ } + \\
\text{ } - \frac{26}{9}
\end{array} \right) > 0.002629
\end{align*}$$
How to pick $K_4$?

Use Flag Algebras!

Try 2: Pick maximizing

$$\begin{align*}
\sum_{1 \leq i < j \leq 4} |X_i| |X_j| - \frac{26}{9} > 0.0276
\end{align*}$$

$$\begin{align*}
FA: \quad \frac{4}{15} \left( \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 3}
\end{array}
\end{array} \right) - \frac{26}{45} > 0.002629
\end{align*}$$
**How to pick $K_4$?**

Use Flag Algebras!

Try 2: Pick maximizing

$$\begin{pmatrix} + & + & - \\ \end{pmatrix} > 0.0276$$

FA: $$\frac{4}{15} \left( \begin{pmatrix} + & + & - \\ \end{pmatrix} \right) - \frac{26}{45} > 0.002629$$

Final equation:

$$2 \sum_{1 \leq i<j \leq 4} |X_i||X_j| - |F| - \frac{26}{9} \sum_{1 \leq i \leq 4} |X_i|^2 > 0.0276n^2$$

$F =$ wrongly colored edges.
How the first step worked

\[ 2 \sum_{1 \leq i < j \leq 4} |X_i||X_j| - |F| - \frac{26}{9} \sum_{1 \leq i \leq 4} |X_i|^2 > 0.0276n^2 \]

Implies:

\[ 0.244n < |X_i| < 0.256n \]
\[ |Trash| < 0.006n \]
\[ |F| < 0.00008 \binom{n}{2} \]

\( F = \) wrongly colored edges.
Theorem

# of rainbow $K_3$s is maximized by

if on $4^k$ vertices.
More results

**Theorem**

# of rainbow $K_3$s is maximized by if on $4^k$ vertices.

![Diagram of four vertices with edges in different colors]

**Theorem**

# of induced $C_5$s is maximized by if on $5^k$ vertices.

![Diagram of five vertices forming a cycle]

(for all $k$)
More results

**Theorem**

# of rainbow $K_3$s is maximized by

if on $4^k$ vertices.

**Theorem**

# of induced $C_5$s is maximized by

if on $5^k$ vertices.

**Theorem**

# of induced oriented $C_4$s is maximized by

if on $4^k$ vertices.
More results

Theorem

# of rainbow $K_3$s is maximized by if on $4^k$ vertices.

Theorem

# of induced $C_5$s is maximized by if on $5^k$ vertices.

Theorem

# of induced oriented $C_4$s is maximized by if on $4^k$ vertices.

(for all $k$)
Thank you for listening!