FRACTIONAL FOURIER TRANSFORM
AND FRACTIONAL DIFFUSION-WAVE EQUATIONS

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Abstract
The integral transform method based on joint application of a fractional generalization of the Fourier transform and the classical Laplace transform is applied for solving Cauchy-type problems for the time-space fractional diffusion-wave equations expressed in terms of the Caputo time-fractional derivative and the generalized Weyl space-fractional operator. The solutions, representing the probability density function, are obtained in integral form where the kernels are Green’s functions represented in terms of the Fox H-functions. It is shown that the results derived include some well known results as particular cases.

1. INTRODUCTION

Fractional calculus is nowadays a significant topic in mathematical analysis as a result of its broad range of applications. Operators for fractional differentiation and integration have been used in various fields such as: hydraulics of dams, potential fields, diffusion problems and waves in liquids and gases [1]. The use of half-order derivatives and integrals leads to a formulation of certain electrochemical problems which is more useful than the classical approach in terms of Fick’s law of diffusion [2]. The main advantage of the fractional calculus is that the fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. In special treaties like [3], [4], [5] and [6], the mathematical aspects and applications of the fractional calculus are extensively discussed.

The modeling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [6], [7]. An illustration of this is the fractional-order diffusion equation studied by Metzler, Glöckle and Nonnenmacher [8], as well as the fractional diffusion equation [9], [10] in the form

\[
\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = a^2 \frac{\partial^2 u}{\partial z^2}, \quad 0 < \beta \leq \frac{1}{2}
\]
introduced by Nigmatullin. The equation (1) is known as the fractional diffusion-wave equation \[11], \[12]. When \(2\beta = 1\) the equation becomes the classical diffusion equation, and if \(2\beta = 2\), it becomes the classical wave equation. The case \(0 < 2\beta < 1\) regards the so-called ultraslow diffusion whereas the case \(1 < 2\beta < 2\) corresponds to the intermediate processes \[13]\.

A space-time fractional diffusion equation, obtained from the standard diffusion equation by replacing the second order space-derivative by a fractional Riesz derivative, and the first order time-derivative by a Caputo fractional derivative, has been treated by Saichev and Zaslavsky \[14\], Uchajkin and Zolotarev \[15\], Gorenflo, Iskenderov and Lucko \[16\], Scalas, Gorenflo and Mainardi \[17\], Metzler and Klafter \[18\]. The results obtained in \[16\] are further developed in \[19\], where the fundamental solution of the corresponding Cauchy problem is found by means of joint application of Fourier and Laplace transforms. Based on Mellin-Barnes integral representation, the fundamental solutions of the problem are expressed in terms of Fox H-functions \[20\].

The Fourier-Laplace transform method was developed in a number of papers by Saxena et al. \[21\], \[22\], \[23\] and Haubold et al. \[24\]. The same approach was also implemented in \[25\], where solutions of generalized fractional partial differential equations involving the Caputo time-fractional derivative and the Weyl space-fractional derivative are obtained.

We employ in this paper a fractional generalization of the Fourier transform that acts on a fractional derivative as the conventional Fourier transform does when applied on a standard derivative. By means of a joint application of this transform and Laplace transform, we study the Cauchy-type problems for the time-space fractional diffusion-wave equation expressed in terms of the Caputo time-fractional derivative of order \(\gamma\) and a generalized Weyl space-fractional operator. We distinguish also the cases of ultraslow diffusion \(0 < \gamma < 1\) and the intermediate processes \(1 < \gamma < 2\) to obtain the space-time probability density function in terms of the Fox H-functions. We show that the results obtained include some of the already known results as particular cases.
2. PRELIMINARIES

For a function $u$ of the class $S$ of rapidly decreasing test functions on the real axis $R$, the Fourier transform is defined as

$$\hat{u}(\omega) = F[u(x); \omega] = \int_{-\infty}^{\infty} e^{i\omega x} u(x) dx, \quad \omega \in R \quad (2)$$

whereas the inverse Fourier transform has the form

$$u(x) = F^{-1}[\hat{u}(\omega); x] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{u}(\omega) d\omega, \quad x \in R \quad (3)$$

Denote by $V(R)$ the set of functions $v(x) \in S$ satisfying

$$\frac{d^n v}{dx^n} \bigg|_{x=0} = 0, \quad n = 0, 1, 2, ...$$

Then the Fourier pre-image of the space $V(R)$

$$\Phi(R) = \{ \varphi \in S; \varphi \in V(R) \}$$

is called the Lizorkin space. As it is stated in [26], the space $\Phi(R)$ is invariant with respect to the fractional differentiation and integration operators.

For a function $u \in \Phi(R)$, we employ the fractional generalization of the Fourier transform called Fractional Fourier Transform (FRFT) of order $\alpha$ $(0 < \alpha \leq 1)$ as introduced in [26],

$$\tilde{u}_\alpha(\omega) = F_\alpha[u(x); \omega] = \int_{-\infty}^{\infty} e_\alpha(\omega, x) u(x) dx, \quad \omega \in R \quad (4)$$

where

$$e_\alpha(\omega, x) = \begin{cases} e^{-i\frac{1}{\alpha} |\omega|^{\frac{2}{\alpha}} x}, & \omega \leq 0 \\ e^{i\frac{1}{\alpha} |\omega|^{\frac{2}{\alpha}} x}, & \omega > 0 \end{cases} \quad (5)$$

If $\alpha = 1$, the kernel (5) of the FRFT (4) reduces to the kernel of (2) and thus

$$\tilde{u}_\alpha(\omega) = F_{1}[u(x); \omega] = F[u(x); k] = \hat{u}(k) \quad (6)$$

where

$$k = \begin{cases} -|\omega|^{\frac{1}{2}}, & \omega \leq 0 \\ |\omega|^{\frac{1}{2}}, & \omega > 0 \end{cases} \quad (7)$$
Therefore, if
\[ F_\alpha[u(x); \omega] = F[u(x); k] = \hat{u}(k), \]
then
\[ u(x) = F^{-1}_\alpha[\hat{u}_\alpha(\omega); x] = F^{-1}[\hat{u}(k); x] \tag{8} \]

For our considerations in this paper we adopt the Caputo fractional derivative defined as [27]
\[ D_\alpha^\alpha f(t) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha}} d\tau, & n-1 < \alpha < n \\
\frac{d^n f(t)}{dt^n}, & \alpha = n
\end{cases} \tag{9} \]

where \( n > 0 \) is integer.

This definition was introduced by Caputo in the late sixties of the twentieth century and utilized by him and Mainardi in the theory of linear viscoelasticity [28].

The method we follow makes the rule of the Laplace transform
\[ L[f(t); s] = \int_0^\infty e^{-st} f(t) dt \tag{10} \]
of Caputo derivative of key importance [6],
\[ L[D_\alpha^\alpha f(x); s] = s^\alpha L[f(t); s] - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} s^{\alpha-k-1}, \quad n-1 < \alpha \leq n \tag{11} \]

For \( 0 \leq \sigma < 1, u \in \Phi(R) \) and \( \beta \in \mathbb{R} \), we consider the following generalized Weyl fractional operator
\[ D_\beta^{\sigma+1} u(x) = (1 - \beta) D_+^{\sigma+1} u(x) - \beta D_-^{\sigma+1} u(x) \tag{12} \]

where
\[ D_+^{\sigma+1} u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^2}{dx^2} \int_0^x (x-t)^{-\alpha} u(t) dt \]
\[ D_-^{\sigma+1} u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^2}{dx^2} \int_x^\infty (t-x)^{-\alpha} u(t) dt \tag{13} \]

are the left-sided and the right-sided Weyl fractional operators of order \( \sigma + 1 \) respectively [5].

The choice \( \sigma = 0 \) leads to a standard derivative since (12) becomes
\[ D_0^\beta u(x) = (1 - \beta) D_0^{1+} u(x) - \beta D_0^{1-} u(x) = (1 - \beta) \frac{du}{dx} + \beta \frac{du}{dx} = \frac{du}{dx}. \]

We take also the advantage of the rule of integration by parts [5], according which for functions \( u \in \Phi(R) \) and \( v \in \Phi(R) \),

\[ \int_{-\infty}^{\infty} v(x) D_0^{1+} u(x) \, dx = \int_{-\infty}^{\infty} u(x) D_0^{1-} v(x) \, dx \quad (14) \]

The one-parameter generalization of the exponential function was introduced by Mittag-Leffler [29] as

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0 \]

Its further generalization is credited to Agarval [30] who defined the two parameter function of the Mittag-Leffler type in the form

\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \beta > 0 \quad (15) \]

The effect of the application of the Laplace transform (10) on the function (15) is provided by the formulas [6, 1.2.2, (1.80)]

\[ L \left[ t^{\alpha n+\beta-1} \frac{d^m}{dt^m} E_{\alpha,\beta}(\pm at^\alpha) \right] = m! \frac{s^{\alpha-\beta}}{(s^{\alpha} + \alpha)^{m+1}}, \quad \text{Res} > \left| a \right|^\alpha \quad (16) \]

By Fox’s H-function one means a generalized hypergeometric function represented by the Mellin-Barnes type integral

\[ H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left( \frac{a_p, A_p}{b_q, B_q} \right) = H_{p,q}^{m,n} \left[ \frac{(a_1, A_1), \ldots, (a_p, A_p)}{(b_1, B_1), \ldots, (b_q, B_q)} \right] \]

\[ = \frac{1}{2\pi i} \int_{\gamma} \Theta(\xi) z^{-\xi} d\xi, \]

where

\[ \Theta(\xi) = \frac{[\prod_{j=1}^{m} \Gamma(b_j + B_j \xi)] [\prod_{l=1}^{p} \Gamma(1 - a_l - A_l \xi)]}{[\prod_{j=1}^{m+1} \Gamma(1 + a_j + A_j \xi)] [\prod_{l=1}^{p+1} \Gamma(a_l + A_l \xi)]} \]
and the contour of integration $L$ can be defined as in [21]. In terms of the usual notations $N_0 = \{0,1,2,\ldots\}$, $R = (-\infty, \infty)$, $R_+ = (0, \infty)$ and $C$ being the complex numbers field; the orders $m$, $n$, $p$, $q \in N_0$ with $1 \leq n \leq p$, $1 \leq m \leq q$, $A_i$, $B_j \in R_+$, $a_i$, $b_j \in R$ or $C$ ($i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$) such that

$A_i(b_j + k) \neq B_j(a_i - i - 1)$, $k$, $l \in N_0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$

The empty product is always interpreted as unity.

It has been established in [31] that, if $\alpha \in C$, for $Re \alpha > 0$

$$E_{\alpha, \beta}(z) = H_{1, 2}^{1, 1} \left[ -z \left| \begin{array}{c} (0, 1) \\ (1 - \beta, \alpha) \end{array} \right. \right]$$

(17)

If we set in (17) $\beta = 1$, we see that

$$E_{\alpha, 1}(z) = E_\alpha(z) = H_{1, 2}^{1, 1} \left[ -z \left| \begin{array}{c} (0, 1) \\ (0, 0, 0, \alpha) \end{array} \right. \right]$$

(18)

According to [32], [33], the cosine transform of the $H$-function is given by

$$\int_0^{\infty} t^{\rho-1} \cos(kt) H_{p,q}^{m,n} \left[ a t^\mu \left| \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] dt$$

$$= \frac{\pi}{k^\rho} H_{q+1, p+2}^{n+1, m} \left[ \frac{k^\mu}{\alpha} \left| \begin{array}{c} (1 - b_q, B_q), (1 + p \mu, 1) \\ (p, 1 - a_p, A_p), \left( \frac{1 + p \mu}{2}, \mu \right) \end{array} \right. \right]$$

(19)

where

$$Re \left[ p + \mu \min_{1 \leq j \leq m} \left( \frac{b_j}{b_j} \right) \right] > 1; \quad k^\mu > 0;$$

$$Re \left[ p + \mu \min_{1 \leq i \leq n} \left( \frac{a_i - 1}{A_i} \right) \right] < \frac{3}{2}, \quad |\arg a| < \frac{\pi}{2} a;$$

$$\Theta > 0, \quad \Theta = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j$$

We also refer to the following property [34]

$$H_{p,q}^{m,n} \left[ x^\delta \left| \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] = \frac{1}{\delta} H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right]$$

(20)

where $\delta > 0$. [8]
3. FRFT OF GENERALIZED WEYL OPERATOR

The application of the Fourier transform (2) in solving fractional differential equations leads in most of the cases to multi-valued complex factors that the transform produces when applied on a fractional derivative [18], [19]. To avoid such complications, we employ the FRFT (4) instead, showing that it acts on a fractional derivative exactly the same way as (2) does once applied on a standard derivative. To describe the effect of the application of the FRFT (4) on the generalized Weyl operator (12) we use that if \( x \in R, \ \omega \in R, \ \omega \neq 0 \) and \( 0 < \sigma < 1 \)[26],

\[
I_4^{\sigma} e^{i\omega x} = e^{i\omega x} |\omega|^{-\sigma} \left[ \cos \frac{\omega}{2} - i \text{sign}(\omega) \sin \frac{\omega}{2} \right]
\]  

(21)

and

\[
I_4^{\sigma} e^{i\omega x} = e^{i\omega x} |\omega|^{-\sigma} \left[ \cos \frac{\omega}{2} + i \text{sign}(\omega) \sin \frac{\omega}{2} \right]
\]

(22)

**Lemma 3.1:** Let \( \omega \in R, \ \omega \neq 0 \) and \( 0 < \sigma < 1 \). Then

\[
D_+^{\sigma+1} e^{i\omega x} = -\omega^2 |\omega|^{\sigma-1} \left[ \sin \frac{\omega}{2} - i \text{sign}(\omega) \cos \frac{\omega}{2} \right] e^{i\omega x}
\]

**Proof:** From (22) it follows readily that

\[
D_+^{\sigma+1} e^{i\omega x} = \frac{d^2}{dx^2} \{ I_4^{1-\sigma} e^{i\omega x} \}
\]

\[
= \frac{d^2}{dx^2} \left[ e^{i\omega x} |\omega|^{\sigma-1} \left[ \sin \frac{\omega}{2} - i \text{sign}(\omega) \cos \frac{\omega}{2} \right] \right]
\]

\[
= -\omega^2 |\omega|^{\sigma-1} \left[ \sin \frac{\omega}{2} - i \text{sign}(\omega) \cos \frac{\omega}{2} \right] e^{i\omega x}
\]

In like manner, by using (21) it is possible to prove also the following auxiliary statement [35].

**Lemma 3.2:** Let \( \omega \in R, \ \omega \neq 0 \) and \( 0 < \sigma < 1 \). Then

\[
D_+^{\sigma+1} e^{i\omega x} = -\omega^2 |\omega|^{\sigma-1} \left[ \sin \frac{\omega}{2} + i \text{sign}(\omega) \cos \frac{\omega}{2} \right] e^{i\omega x}
\]

**Theorem 3.1:** If \( 0 < \alpha \leq 1, \ 0 \leq \sigma < 1 \) and \( u \in \Phi(R) \), then for any \( \beta \in R \),

\[
F_\alpha [D_+^{\sigma+1} u(x); \omega] = c_\beta (\sigma) |\omega|^{\sigma+1} F_\alpha [u(x); \omega]
\]

where
Proof: If $\alpha = 1$ and $\sigma = 0$, according to (6),

$$F[D_\beta^{\sigma+1}u(x); \omega] = \mathcal{F}\left[(1 - \beta) \frac{du}{dx} + \beta \frac{du}{dx}; \omega\right] = \mathcal{F}\left[\frac{du}{dx}; \omega\right] = -i\omega F[u(x); \omega]$$

and the assertion of the theorem agrees with the classical result for the Fourier transform (2).

Consider now the case $0 < \alpha < 1$, $0 < \sigma < 1$ and $\omega = 0$. Since $\Phi(R)$ is closed with respect to fractional differentiation, it becomes clear from (4) and (13) that

$$F_\alpha[D_\beta^{\sigma+1}u(x); 0] = \int_{-\infty}^{\infty} \left[(1 - \beta) D_-^{\sigma+1}u(x) - \beta D_+^{\sigma+1}u(x)\right] dx = D_\beta^{\sigma}u(x)\bigg|_{-\infty}^{\infty} = 0$$

Let $0 < \alpha < 1$, $0 < \sigma < 1$ and $\omega > 0$. Then (4), (5), (14), Lemma 3.1 and Lemma 3.2 yield

$$F_\alpha[D_\beta^{\sigma+1}u(x); \omega] = \int_{-\infty}^{\infty} e^{i\omega \frac{1}{\alpha}x} D_\beta^{\sigma+1}u(x) dx$$

$$= (1 - \beta) \int_{-\infty}^{\infty} u(x) D_-^{\sigma+1} e^{i\omega \frac{1}{\alpha}x} dx - \beta \int_{-\infty}^{\infty} u(x) D_+^{\sigma+1} e^{i\omega \frac{1}{\alpha}x} dx$$

$$= (1 - \beta) \int_{-\infty}^{\infty} u(x) \left\{-|\omega|^\frac{\sigma+1}{\alpha} \left[\sin \frac{\sigma \pi}{2} + i\cos \frac{\sigma \pi}{2}\right] e^{i\omega \frac{x}{\alpha}} \right\} dx$$

$$= -\beta \int_{-\infty}^{\infty} u(x) \left\{-|\omega|^\frac{\sigma+1}{\alpha} \left[\sin \frac{\sigma \pi}{2} - i\cos \frac{\sigma \pi}{2}\right] e^{i\omega \frac{x}{\alpha}} \right\} dx$$

$$= |\omega|^\frac{\sigma+1}{\alpha} \left[(2\beta - 1)\sin \frac{\sigma \pi}{2} - i\cos \frac{\sigma \pi}{2}\right] F_\alpha[u(x); \omega]$$

In the same way we consider the case $0 < \alpha < 1$, $0 < \sigma < 1$, $\omega < 0$ and by (4), (5), (14), Lemma 3.1 and Lemma 3.2 we get

$$F_\alpha[D_\beta^{\sigma+1}u(x); \omega] = |\omega|^\frac{\sigma+1}{\alpha} \left[(2\beta - 1)\sin \frac{\sigma \pi}{2} + i\cos \frac{\sigma \pi}{2}\right] F_\alpha[u(x); \omega]$$

that accomplishes the proof.

Remark: If $\beta = \frac{1}{2}$ and $\alpha = \sigma + 1$, Theorem 3.1 reduces to the result obtained by Lucko at al. [26].
4. FRACTIONAL DIFFUSION EQUATION

Here we apply the FRFT (4) to solve the Cauchy-type problem for the nonhomogeneous fractional diffusion equation of the form

$$D_t^\gamma u(x,t) - \mu^2 D_\beta^{\sigma+1} u(x,t) = q(x,t), \quad x \in \mathbb{R}, \quad t > 0$$  \hspace{1cm} (23)

subject to the initial condition

$$u(x,t)|_{t=0} = f(x)$$  \hspace{1cm} (24)

where $0 < \gamma \leq 1$, $f(x) \in \Phi(R)$ and $\mu$ is a diffusivity constant.

**Theorem 4.1:** Let $0 < \gamma \leq 1$, $0 < \sigma \leq 1$ and for $t > 0$, $u(x,t) \in \Phi(R)$ and $q(x,t) \in \Phi(R)$. Then for any $\beta \in R$, the Cauchy-type problem (23) – (24) is solvable and for $x \neq 0$ the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x - \xi, t)f(\xi)d\xi + \int_0^t \left( \int_{-\infty}^{\infty} G_2(x - \xi, t - \tau)q(\xi, \tau)d\xi \right)d\tau$$

where

$$G_1(x, t) = \frac{1}{(\sigma + 1)x} H_{5.1}^{2,1} \left[ \frac{|x|}{(-\mu^2 c_\beta(\sigma)t^\gamma)^{\sigma+1}} \left( \frac{1}{1, \frac{1}{\sigma+1}}, \left( 1, \frac{\gamma}{\sigma+1} \right), \left( 1, \frac{1}{1,2} \right) \right) \right]$$

$$G_2(x, t) = \frac{1}{(\sigma + 1)x} H_{5.3}^{2,1} \left[ \frac{|x|}{(-\mu^2 c_\beta(\sigma)t^\gamma)^{\sigma+1}} \left( \frac{1}{1,1, \frac{1}{\sigma+1}}, \left( \frac{\gamma}{\sigma+1}, \frac{1}{1,\sigma+1} \right), \left( 1, \frac{1}{1,2} \right) \right) \right]$$

**Proof:** Denote $L[u(x,t); s] = \bar{u}(x,s)$ and $F_\alpha[u(x,t); \omega] = \tilde{u}_\alpha(\omega,t)$, where $0 < \alpha \leq 1$. First consider the case $0 < \gamma \leq 1$ and $0 < \sigma < 1$. According to (11) and Theorem 3.1, the application of the Laplace transform (10) followed by the FRFT (4) to the equation (23) and the initial condition (24) leads to the Laplace-FRFT transform of the solution

$$\bar{u}_\alpha(\omega, s) = \frac{s^{\gamma-1}}{s^{\gamma - \mu^2 c_\beta(\sigma)|\omega\| \alpha^{\sigma+1}}} \tilde{f}_\alpha(\omega) + \frac{\bar{q}_\alpha(\omega, s)}{s^{\gamma - \mu^2 c_\beta(\sigma)|\omega\| \alpha^{\sigma+1}}}$$  \hspace{1cm} (25)

By means of (6), (8) and (16), the equation (25) converts into
\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) E_{\nu,1}\left[\mu^2 c_B(\sigma)|k^{\sigma+1}t^\gamma\right] dk \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left\{ \int_{0}^{t} (t-\tau)^{\nu-1} E_{\nu,\nu}\left[\mu^2 c_B(\sigma)|k^{\sigma+1}(t-\tau)^\gamma\right] \hat{q}(k, \tau) d\tau \right\} dk \]

Then the convolution theorem for (2) implies

\[ u(x, t) = \int_{-\infty}^{\infty} G_1(x-\xi, t) f(\xi) d\xi \]

\[ + \int_{0}^{t} (t-\tau)^{\nu-1} \left\{ \int_{-\infty}^{\infty} G_2(x-\xi, t-\tau) q(\xi, \tau) d\xi \right\} d\tau, \]

where

\[ G_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\nu,1}\left[\mu^2 c_B(\sigma)|k^{\sigma+1}t^\gamma\right] dk \quad (26) \]

and

\[ G_2(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\nu,\nu}\left[\mu^2 c_B(\sigma)|k^{\sigma+1}t^\gamma\right] dk \quad (27) \]

The formulas (18) and (20) with \( \delta = \frac{2}{\sigma+1} \) allow us from (26) to obtain

\[ G_1(x, t) = \frac{2}{(\sigma+1)\pi} \int_{0}^{\infty} \cos kx H_{1,\nu}^{\nu,1}\left[ k^2 (-\mu^2 c_B(\sigma)t^\gamma)^{\frac{2}{\sigma+1}} \right] \left[ \frac{0}{\frac{2}{\sigma+1}}, \frac{2}{\sigma+1} \right] dk \]

Taking into account (19) and (20), we get

\[ G_1(x, t) = \frac{1}{(\sigma+1)x} H_{2,3}^{2,1}\left[ \frac{|x|}{(-\mu^2 c_B(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}} \right] \left[ \frac{1}{(1, \frac{1}{\sigma+1}, \frac{1}{\sigma+1}, \frac{1}{\sigma+1})}, \frac{1}{(1, \frac{1}{\sigma+1}, \frac{1}{\sigma+1}, \frac{1}{\sigma+1})} \right] \]

In a like manner, by (17), (19) and (20), we obtain from (27),

\[ G_2(x, t) = \frac{1}{(\sigma+1)x} H_{3,3}^{2,1}\left[ \frac{|x|}{(-\mu^2 c_B(\sigma)t^\gamma)^{\frac{1}{\sigma+1}}} \right] \left[ \frac{1}{(1, \frac{1}{\sigma+1}, \frac{1}{\sigma+1}, \frac{1}{\sigma+1})}, \frac{1}{(1, \frac{1}{\sigma+1}, \frac{1}{\sigma+1}, \frac{1}{\sigma+1})} \right] \]
We accomplish the proof with the remark that the validity of the statement in the case $0 < \gamma \leq 1$ and $\sigma = 1$ was confirmed by the results obtained in [21] and [36].

**Corollary 4.1** ([35], [37]): If $0 < \gamma \leq 1$, $\sigma = 1$, $\beta = 0$, $f(x) \in \Phi(R)$ and $q(x, t) \equiv 0$, the solution of the Cauchy-type problem (23) – (24) is given by the integral

$$u(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t) f(\xi) d\xi,$$

where

$$G(x, t) = \frac{1}{2\pi} \frac{1}{H_{1,1}^{4,1}} \left[ \left( \frac{1}{2} \right), \left( 1, \frac{1}{2} \right), \left( 1, \frac{1}{2} \right) \right] \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right).$$

By (17), (19) and the formula [37, p. 611, (5)]

$$F^{-1} \left[ \frac{\pi}{\sqrt{\alpha e^{-\frac{\omega^2}{4\alpha^2}}}} \right] = e^{-\alpha x^2},$$

it is possible to see that the solution provided by Theorem 4.1 contains as particular case the fundamental solution of the classical diffusion problem.

**Corollary 4.2** [35]: If $\gamma = 1$, $\sigma = 1$, $\beta = 0$, $f(x) \in \Phi(R)$ and $q(x, t) \equiv 0$, the solution of the Cauchy-type problem (23) – (24) is of the integral form

$$u(x, t) = \frac{1}{\sqrt{4\pi \mu t}} \int_{-\infty}^{\infty} e^{-\frac{(x - \xi)^2}{4\mu t}} f(\xi) d\xi.$$

5. FRACTIONAL WAVE EQUATION

Let consider the Cauchy-type problem for the equation (23), assuming that $1 < \gamma \leq 2$ and $0 < \sigma \leq 1$, subject to the initial conditions

$$u(x, t) |_{t=0} = f(x), \quad u_t(x, t) |_{t=0} = g(x), \quad x \in R \quad (28)$$

**Theorem 5.1:** Let $1 < \gamma \leq 2$, $0 < \sigma \leq 1$ and for $t > 0$, $u(x, t) \in \Phi(R)$ and $q(x, t) \in \Phi(R)$. Then for any $\beta \in R$, the Cauchy-type problem (23) – (28) is solvable and for $x \neq 0$ its solution is given by
\[ u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)f(\xi)d\xi + \int_{-\infty}^{\infty} G_2(x-\xi,t)g(\xi)d\xi + \int_{0}^{t} (t-\tau)^{\gamma-1} \left\{ \int_{-\infty}^{\infty} G_3(x-\xi,t-\tau)q(\xi,\tau)d\xi \right\} d\tau \]

where

\[ G_1(x,t) = \frac{1}{(\sigma + 1)x} H_{3,3}^{2,1} \left[ \frac{|x|}{(-\mu^2 \beta(\sigma)t^\gamma)^{1/\gamma+1}} \right] \begin{pmatrix} 1, \frac{1}{\sigma+1}, \left(1, \frac{\gamma}{\sigma+1}\right), \left(1, \frac{1}{2}\right) \\ (1,1), \left(1, \frac{1}{\sigma+1}\right), \left(1, \frac{1}{2}\right) \end{pmatrix} \]

\[ G_2(x,t) = \frac{t}{(\sigma + 1)x} H_{3,3}^{2,1} \left[ \frac{|x|}{(-\mu^2 \beta(\sigma)t^\gamma)^{1/\gamma+1}} \right] \begin{pmatrix} 1, \frac{1}{\sigma+1}, \left(2, \frac{\gamma}{\sigma+1}\right), \left(1, \frac{1}{2}\right) \\ (1,1), \left(1, \frac{1}{\sigma+1}\right), \left(1, \frac{1}{2}\right) \end{pmatrix} \]

\[ G_3(x,t) = \frac{1}{(\sigma + 1)x} H_{3,3}^{2,1} \left[ \frac{|x|}{(-\mu^2 \beta(\sigma)t^\gamma)^{1/\gamma+1}} \right] \begin{pmatrix} 1, \frac{1}{\sigma+1}, \left(\gamma, \frac{\gamma}{\sigma+1}\right), \left(1, \frac{1}{2}\right) \\ (1,1), \left(1, \frac{1}{\sigma+1}\right), \left(1, \frac{1}{2}\right) \end{pmatrix} \]

**Proof:** As in Theorem 4.1, let us consider first the case \( 1 < \gamma \leq 2 \) and \( 0 < \sigma < 1 \).

Because of (11) and Theorem 3.1, the application of the Laplace transform (10) followed by the FRFT (4) of order \( 0 < \alpha \leq 1 \) to the equation (23) and the initial conditions (28) leads to the following joint Laplace-FRFT transform of the solution

\[ \mathcal{L}u(\omega,s) = \frac{s^{\gamma-1}}{s^\gamma - \mu^2 \beta(\sigma)|\omega|^\alpha} \mathcal{L}f(\omega) + \frac{s^{\gamma-2}}{s^\gamma - \mu^2 \beta(\sigma)|\omega|^\alpha} \mathcal{L}g(\omega) \]

\[ + \frac{s^{\gamma-2}}{s^\gamma - \mu^2 \beta(\sigma)|\omega|^\alpha} \mathcal{L}q(\omega,s) \]

By (6), (8), (16) and the convolution theorem for the Fourier transform (2), the equation (29) becomes

\[ u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)f(\xi)d\xi + \int_{-\infty}^{\infty} G_2(x-\xi,t)g(\xi)d\xi \]
where

\[ G_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixk} E_{y,1} \left[ \mu^2 c_\beta(\sigma) |k|^{\sigma+1} t^\gamma \right] dk, \]

\[ G_2(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixk t} E_{y,2} \left[ \mu^2 c_\beta(\sigma) |k|^{\sigma+1} t^\gamma \right] dk, \]

\[ G_3(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixk} E_{y,\gamma} \left[ \mu^2 c_\beta(\sigma) |k|^{\sigma+1} t^\gamma \right] dk. \]

From Theorem 4.1 we know that

\[ G_1(x, t) = \frac{1}{(\sigma + 1)x} H_{5,3}^{2,1} \left[ \frac{|x|}{(-\mu^2 c_\beta(\sigma) t) t^{\frac{1}{\sigma+1}}} \right] \left[ (1, \frac{1}{\sigma + 1}), \left( \frac{\gamma}{\sigma + 1} \right), (1, \frac{1}{2}) \right] \]

and

\[ G_3(x, t) = \frac{1}{(\sigma + 1)x} H_{5,3}^{2,1} \left[ \frac{|x|}{(-\mu^2 c_\beta(\sigma) t) t^{\frac{1}{\sigma+1}}} \right] \left[ (1, \frac{1}{\sigma + 1}), \left( \frac{\gamma}{\sigma + 1} \right), (1, \frac{1}{2}) \right] \]

The application of (17) and (20) with \( \delta = \frac{2}{\sigma+1} \) implies

\[ G_2(x, t) = \frac{2t}{(\sigma + 1) x \pi} \int_0^\infty \cos k x H_{1,2}^{1,1} \left[ \frac{k^2 (-\mu^2 c_\beta(\sigma) t) t^{\frac{2}{\sigma+1}}} \right] \left[ (0, \frac{2}{\sigma + 1}), (-1, \frac{2\gamma}{\sigma + 1}) \right] dk \]

Then from (19) and (20) with \( \delta = \frac{1}{2} \) it follows that

\[ G_2(x, t) = \frac{t}{(\sigma + 1)x} H_{5,3}^{2,1} \left[ \frac{|x|}{(-\mu^2 c_\beta(\sigma) t) t^{\frac{1}{\sigma+1}}} \right] \left[ (1, \frac{1}{\sigma + 1}), (\frac{\gamma}{\sigma + 1}), (1, \frac{1}{2}) \right] \]
The validity of the theorem for the case $1 < \gamma \leq 2$ and $\sigma = 1$ is provided by the results obtained in [37, 6.7, (b)].

**Corollary 5.1** ([35], [38]): If $\gamma = 2$, $\sigma = 1$, $\beta = 0$, $f(x) \in \Phi(R)$ and $g(x) \in \Phi(R)$, the Cauchy-type problem (23) – (28) has a solution of the form

$$u(x, t) = \frac{1}{2} [f(x - \mu t) + f(x + \mu t)] + \frac{1}{2\mu} \int_{x-\mu t}^{x+\mu t} g(\eta) d\eta$$

$$+ \frac{1}{2\mu} \int_{0}^{t} \left[ \int_{x-\mu(t-\tau)}^{x+\mu(t-\tau)} q(\eta, \tau) d\eta \right] d\tau$$

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**REFERENCES**


