Theory Links:
Applications to Automated Theorem Proving

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We develop the notion of theory link, which is a generalization of ordinary link to a set of literals that are simultaneously unsatisfiable relative to a given set of clauses. We show that theory links may be 'activated' in much the same manner as ordinary links when inferencing with respect to the given set of clauses. Several link deletion results are shown to hold for theory links, and some examples are presented using first-order theory links.

1. Introduction

In (Murray & Rosenthal, 1985a, 1985b, 1985c, 1987) we developed a graphical representation of NNF quantifier-free predicate calculus formulas and a new rule of inference, path resolution, which employs this representation. Stickel (1983, 1985a, 1985b) introduced theory resolution in which inferences depend on the existence of a 'black box' to implement a theory. Stickel designed theory resolution to be "a method of incorporating specialized reasoning procedures in a resolution theorem prover so that the reasoning task will be effectively divided into two parts: special cases ... are handled efficiently by specialized reasoning procedures, while more generalized reasoning is handled by resolution."

Path resolution operations hinge on the discovery of subgraphs (called resolution chains) which have the special property that all their c-paths contain a link. Many results from path resolution go through when we consider a generalization of link which we call a theory link. Intuitively, an ordinary link is a set of two c-connected (conjoined) literals such that under no assignment can both be true; a theory link is a set of n c-connected literals such that under no T-assignment, i.e., an assignment satisfying the axioms of theory T, can all be true. These specialized theory links can then be used in resolution-like procedures.

Finding a large resolution chain is hard in general, being essentially a subdeduction, i.e., the theorem proving problem on a (possibly non-explicit) subformula. One major advantage to the use of theory links is that they often represent large
chains. Moreover, the theory link stores the sub-deduction, and thus the chain and inherited versions of it need never be discovered again.

A simple example of a set of theory links is as follows. Suppose the statements "elephants are mammals" and "mammals are animals" comprise a theory expressed as clauses. Each clause yields a theory link, and it is immediate from the sequel that "elephants are animals" yields one also. Frequently the theory links may be incorporated directly into a formula in such a way that subsequently the theory need never be consulted.

Several authors have developed related ideas. Bibel (1982) derived a special case of theory links in order to deal with equality. For example, if \( a = b \), he would link the literals \( P(a) \) and \( P(b) \). Dixon (1973) "compiled" axioms and the resulting code operated in effect like theory links. The linked inference principle of Wos et al. (1982) makes use of "linking clauses" that operate somewhat like theory links.

A brief summary of required background material is presented in Section 2; for a more detailed exposition see (Murray & Rosenthal, 1985a, 1987). Three equivalent formulations of theory link are presented in Section 3, and lifting is discussed in Section 4. In Section 5 we compare this work to Stickel's Theory Resolution. Section 6 is a brief review of some results concerning link deletion followed by a discussion relating those results to theory links. In section 7 we present sample deductions involving both ordinary and theory links. We make use of techniques originally developed for semantic graphs and ordinary links.

The symbol ' \( \Box \) ' is used to indicate the end of a proof.

2. Preliminaries

We briefly summarize semantic graphs and path resolution, including only those results necessary for the introduction of theory links.

A semantic graph is empty, a single node, or a triple \((N, C, D)\) of nodes, c-arcs, and d-arcs, where a node is a literal occurrence, a c-arc is a conjunction of two non-empty semantic graphs, and a d-arc is a disjunction of two non-empty semantic graphs. We use the notation \((G, \alpha)\) for the c-arc containing \( G \) and \( \alpha \), and, similarly, \((G, H)\) for the d-arc.

The construction of a graph may be thought of as a sequence of such arcs. There will always be exactly one arc \((X, Y)\) with the property that every other arc is an arc in \( X \) or in \( Y \). We call this arc the final arc of the graph, and we call \( X \) and \( Y \) the final subgraphs. Since this arc completely determines \( G \), we frequently write \( G = (X, Y) \).

The notion of fundamental subgraph is often useful: if \( G = (X, Y) \) and the final arc of \( Y \) is not of type \( \alpha \), then \( Y \) is a fundamental subgraph of \( G \); otherwise the fundamental subgraphs of \( Y \) are fundamental subgraphs of \( G \).

A semantic graph may be thought of as a binary [n-ary] tree in which each node represents an explicit [fundamental] subgraph, and the children of a node are its final [fundamental] subgraphs. The root is of course the entire graph, and the leaves are the literals.

If \( a = (X, Y) \) is an arc in a graph, and if \( A \) and \( B \) are nodes in \( X \) and \( Y \), respectively, then we say that \( a \) is the arc connecting \( A \) and \( B \). If \( a \) is a c-arc, we say that \( A \) and \( B \) are c-connected, and if \( a \) is a d-arc, \( A \) and \( B \) are d-connected. A c-path is a
maximal collection of c-connected nodes, and a d-path is a maximal collection of d-connected nodes. The semantics of a graph may be characterized by its paths: it is easy to verify that a c-path and a d-path have exactly one node in common, and that a graph is satisfied by an interpretation I iff I satisfies (every literal on) some c-path, and the graph is falsified iff some d-path is falsified by I.

The following example illustrates some of these notions. Consider the formula

\[(D \lor (A \iff B)) \land ((C \land E) \lor \neg A \lor (\neg B \land P))\]

The corresponding graph is

```
  D -- C \rightarrow E
    \downarrow
  \overline{A} \rightarrow \overline{B} \rightarrow \overline{A}
    \downarrow
  B \rightarrow A \rightarrow \overline{B} \rightarrow P
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Examples of c-paths are \{B, A, \overline{A}\} and \{D, \overline{B}, P\}, and some d-paths are \{D, \overline{B}, A\} and \{C, \overline{A}, P\}.

In (Murray & Rosenthal, 1985a, 1987), the subgraph of a given graph with respect to a given set of nodes is precisely defined. Intuitively, it is a graph which contains the given nodes and those arcs associated with those nodes. Certain classes of subgraphs, the blocks, turn out to be especially important. There are three types of blocks: the c-blocks, the d-blocks, and the full-blocks.

A c-block C is a subgraph of a semantic graph with the property that any c-path p which includes at least one node from C must pass through C; that is, the subset of p consisting of the nodes which are in C must be a c-path through C. A d-block is similarly defined with d-paths, and a full block is a subgraph which is both a c-block and a d-block. We define a strong c-block in a semantic graph G to be a subgraph C of G with the property that every c-path through G contains a c-path through C. (If G is in CNF then C is one or more clauses.) A strong d-block is similarly defined.

Recall that a link in a formula in CNF is a pair of literals from different clauses that can be made complementary by an appropriate substitution. A link in a semantic graph is similarly defined for a pair of c-connected nodes. A formula in CNF is unsatisfiable iff every c-path contains a link; the same is true for a semantic graph. (A formula satisfying this condition is said to be spanned by its links.)

A resolution subgraph R in a semantic graph G is a subgraph with the property that every c-path through it contains a link. If a c-path p through the entire graph is satisfiable, it cannot possibly pass through the resolution subgraph. Thus p must miss part of R. The c-blocks of R are the parts of R with the property that one of them must be missed by p. Associated with each c-block is an auxiliary subgraph: that part of G that must be hit by a c-path that misses the c-block. Intuitively, the path resolvent of R in G contains the disjunction of the auxiliary subgraphs.

We define WS(H,G), the weak split graph of H in G as follows:

Let the fundamental subgraphs of G that meet H be \(F_1, ..., F_k\) and let \(F_{k+1}, ..., F_n\) be those that do not. Then
WS(\emptyset, G) = G \quad \text{and} \quad WS(G, G) = \emptyset.

WS(H, G) = WS(H_{F_1}, F_1) \lor \cdots \lor WS(H_{F_n}, F_n)
\quad \text{if the final arc of } G \text{ is a } d\text{-arc}

WS(H, G) = WS(H_{F_1}, F_1) \lor \cdots \lor WS(H_{F_k}, F_k)
\quad \text{if the final arc of } G \text{ is a } c\text{-arc}

The strong split graph of } H \text{ in } G \text{ is defined in a similar manner except that the last equation becomes}

SS(H, G) = SS(H_{F_1}, F_1) \lor \cdots \lor SS(H_{F_k}, F_k) \land F_{k+1} \land \cdots \land F_n

The weak or strong split graph of a resolution chain will be referred to as a path resolvent. When the graph is in CNF, both operations yield the same result. Intuitively, \( WS(H, G) \) is the disjunction of the auxiliary subgraphs of the maximal c-blocks of } H. (Certain redundancies are automatically removed by weak split. See (Murray & Rosenthal, 1985a), Theorems 4 and 5 for a precise statement.) On the other hand, strong split is essentially formed with the nodes lying on c-paths that miss the c-blocks of } H. (It is surprising that these two notions in general lead to different inferences, although they are the same in CNF.) When we write WS\((H_{F_1}, F_1)\) in the above definition, } } H_{F_1} \text{ denotes the subgraph of } G \text{ relative to the nodes in } H \text{ that are in } F_1.

Considering the example above, the subgraph
\[ A \rightarrow B \rightarrow l \]
forms a resolution subgraph. Its path resolvent (in this case both weak and strong) is
\[ C \rightarrow E \]
\[ \downarrow \]
\[ D \]
\[ \downarrow \]
\[ \overline{A} \rightarrow \overline{B} \]

3. Theory Links in the Ground Case

Suppose we express a propositional theory } T \text{ as a semantic graph. Let } H \text{ be a semantic graph with the property that } G = (T, H)_c \text{ is unsatisfiable. We assume that both } T \text{ and } H \text{ are satisfiable; since } G \text{ is unsatisfiable, } H \text{ is obviously } T\text{-unsatisfiable in the sense of Stickel (1985b). Some c-paths through } H \text{ may contain links, but there must be at least one linkless c-path } p_H \text{ through } H \text{ because } H \text{ is satisfiable. Yet the literals on such a c-path } p_H \text{ must be } T\text{-unsatisfiable, and it is likely that some proper subset } Q_{p_H} \text{ of those literals is } T\text{-unsatisfiable. One way to compute } Q_{p_H} \text{ is to extend } p_H \text{ to a c-path } p_Tp_H \text{ in every possible way, i.e., form the c-path } p_Tp_H \text{ for every}


satisfiable c-path $p_H$ through $T$. By recording a minimal set of nodes on $p_H$, each member of which is linked to some linkless $p_T$, we can determine $Q_{p_H}$. If we now compute $Q_{p_H}$ for each linkless $p_H$, then in some sense the $Q$'s and the ordinary links of $H$ provide sufficient evidence that $H$ is $T$-unsatisfiable.

Our intent then is to define, in a computationally feasible way, a collection of such $Q$'s so that any $T$-unsatisfiable semantic graph $H$ is spanned by its links and the $Q$'s. Such $Q$'s will generically be called theory links. Three characterizations of theory links are discussed and shown to be equivalent in power.

3.1 T-links

Assume the axioms of a theory $T$ are expressed as the $m$ clauses $C_1, C_2, \ldots, C_m$. (We assume only $T$ to be in CNF.) Let $R(T)$ be the union of $T$ and all possible binary resolvents of clauses in $T$, and let $R^n(T) = R(R^{n-1}(T))$. Then $T^*$ is the set of all clauses obtainable from $T$ by (ordinary binary) resolution; i.e., $T^* = \bigcup_{j=1}^{\infty} R^j(T)$. Of course, $T^*$ is finite in the ground case. We let $T_i$ denote the theory axiomatized by the first $i$ clauses of $T$.

A T-link is defined to be a set $Q$ of c-connected nodes such that $Q \subseteq C$, where $C$ is a clause in $T^*$. In other words, $Q$ is a set of c-connected nodes that are complements of the nodes in some clause in $T^*$. The following lemma is obvious.

Lemma 1. Any T-link $Q$ is $T$-unsatisfiable. \(\square\)

Theorem 1. Given a ground theory $T$ defined by the $m$ clauses $C_1, \ldots, C_m$ and a $T$-unsatisfiable semantic graph $H$, $H$ is spanned by its links and T-links; i.e., every linkless c-path in $H$ contains a T-link.

Proof: Let $p_H$ be a linkless c-path through $H$. Since $p_H$ is $T$-unsatisfiable, $p_H$ is a logical consequence of $T$. Resolution is of course complete for consequence finding in the sense that if clause $M'$ is a logical consequence of $T$, then some $M$ that subsumes $M'$ can be derived from $T$ by resolution. Any such $M$ for $p_H$ does the trick: $Q = \overline{M}$ is the required T-link. \(\square\)

The above proof is clear and concise, but it is not very constructive, and it relies crucially on the completeness of resolution for the derivation of logical consequences. We now present an alternative proof that is somewhat more constructive, and that contains a proof of the completeness result required above. The proof is derived directly from the structure of the semantic graph. In fact, for the special case where $T$ is unsatisfiable, it is (with a few modifications) an interesting technique for establishing ground resolution refutation completeness.

Alternative Proof: By induction on $m$, the number of clauses in $T$. For $m=1$, $T^* = T = C_1$. Let $C_1 = r_1 \lor r_2 \lor \cdots \lor r_n$ and let $p_H$ be a linkless c-path through $H$. For $1 \leq i \leq s$, $p_H \cup r_1$ contains a link from $r_i$ to $r_j$ on $p_H$ (since $C_1 \rightarrow H$ is spanned.) Therefore, $p_H$ contains $\{r_1, r_2, \ldots, r_s\}$ which is a T-link.

Now we show that if the theorem holds for theories with $m = k$, then it also holds when $m = k+1$. Let $G = T_{k+1} \rightarrow H$, and let $p_H$ be a linkless c-path through $H$. We must show that $p_H$ contains a $T_{k+1}$-link. Let $p_{T_k}$ be a linkless c-path through $T_k$ such that $p_{T_k} \cup p_H$ is also linkless. (If no such $p_{T_k}$ exists, $T_k \cup p_H$ is
unsatisfiable. Then \( p_H \) is \( T_k \)-unsatisfiable and hence contains a \( T_k \)-link, and we are done.) Now, extending \( p_{T_k} \cup p_H \) through any literal of \( C_{k+1} \) must pick up at least one link since the entire graph \( G \) is spanned. Such a link is either to \( p_{T_k} \) or to \( p_H \).

Let \( C_{k+1} = r_1 \lor \cdots \lor r_s \lor r_{s+1} \lor \cdots \lor r_{t+k} \), where for \( 1 \leq i \leq s \), \( r_i \) is linked to node \( r_i \) on \( p_H \), and for \( 1 \leq i \leq t \), \( r_{s+i} \) is linked to \( r_{s+i} \) on \( p_{T_k} \) (but not to \( p_H \). If \( t = 0 \) we are done since in that case \( C_{k+1} \) is contained in \( p_H \); also, \( s \neq 0 \) since \( p_H \) is not \( T_k \)-unsatisfiable. The situation is diagramed in Figure 1.

Consider \( p_H \cup r_{s+i} \) where \( 1 \leq i \leq t \). Any such path does not contain a link. Therefore, by the induction hypothesis, it must contain a \( T_k \)-link since \( C_{k+1} \land H \) is \( T_k \)-unsatisfiable, and, as we noted above, this \( T_k \)-link is not entirely on \( p_H \). So, let the \( T_k \)-link for \( p_H \cup r_{s+i} \) be \( M_i \land r_{s+i} \), where \( M_i \) is the set of \( c \)-connected nodes from \( p_H \) that are in the \( T_k \)-link. These \( T_k \)-links tell us that the clauses \( \{ r_{s+i} \lor \overline{M_i} \}, 1 \leq i \leq t \), are in \( T_k^* \). Furthermore, these clauses are distinct; i.e., for \( i \neq j \), \( r_{s+i} \) and \( r_{s+j} \) occur in distinct clauses of \( T_k^* \). If this were not so, then \( r_{s+i} \) would be in \( M_j \) implying that \( r_{s+i} \) is on \( p_H \). But we know that \( r_{s+i} \) is on \( p_{T_k} \), and therefore \( p_{T_k} \) and \( p_H \) would be linked, which is impossible.
Thus we have $C_{k+1}$ and the $t$ clauses we know to be in $T_k^*$ as shown in Figure 2.

A sequence of $t$ resolution steps on the links shown will produce the clause $Q = (\overline{M_1} \lor \cdots \lor \overline{M_t} \lor r_1 \lor \cdots \lor r_d)$, which is in $T_{k+1}^*$. (More precisely, a resolution on one link followed by resolutions on inherited versions of the other $t-1$ is required. Of course, a single path resolution or clash resolution will do.) The complements of all literals in $Q$ are on $p_k$, and so $\overline{Q}$ provides the required $T_{k+1}$-link.

### 3.2 Strong T-links and resolvent T-links

Path resolution and semantic graphs lead us to two other natural ways of characterizing sets of T-unsatisfiable nodes. Given sets $L_1, L_2$ of theory links, we say $L_1$ subsumes $L_2$ if every member of $L_2$ is a superset of some member of $L_1$. Recall that a strong c-block is a subgraph with the property that every c-path in the graph goes through the subgraph; in CNF such a subgraph must be a collection of clauses. Define a **strong T-link** to be a set $Q$ of c-connected nodes such that $S \rightarrow Q$ is a resolution chain for some strong c-block $S$ in $T$. Let $^{ST}T$ denote the set of all strong T-links.

**Theorem 2.** Given a ground theory $T$, $^{ST}T$ is subsumed by $T^*$.

**Proof:** Let $Q$ be a strong T-link with $S$ its associated strong c-block in $T$. By the definition of strong c-block, no c-path through $Q$ can be extended through $T$ unless it passes through $S$, and hence through $S \rightarrow T$, which is a resolution chain, guaranteeing the T-unsatisfiability of $Q$. By Theorem 1, $Q$ must contain a T-link $Q'$; this is true for any strong T-link $Q$, and the proof is complete.

We now give a third characterization of theory links. Define a **resolvent T-link** to be either the negation of a clause in $T$ or a set $Q$ of c-connected nodes such that $\overline{Q}$ is a path resolvent of some resolution chain in $T$. (Since $T$ is in CNF, it is irrelevant
Theorem 3. Given a ground theory \( T \), every element of \( R^T \) is in \( S^T \), and hence \( R^T \) is subsumed by \( S^T \).

Proof: Let \( R \) be a resolution chain whose path resolvent is \( \overline{Q} \). Consider the strong c-block \( S \) consisting of the clauses in \( T \) that meet \( R \). The literals in \( \overline{Q} \) are exactly the literals in \( S - R \). We will be done if we can show that \( S \rightarrow Q \) is a resolution chain (making \( Q \) a strong T-link), so consider a c-path \( p \) through \( S \rightarrow Q \). If \( p \) goes through \( R \) it must have a link; if it does not, then \( p \) contains a node from \( \overline{Q} \) and this node is linked to its complement on \( Q \).

We have now established that \( T^* \) subsumes \( S^T \), and that \( S^T \) subsumes \( R^T \). Theorem 4 completes the cycle, assuring us that each characterization will give us a set of theory links sufficient for establishing the T-unsatisfiability of any formula.

Theorem 4. Given a ground theory \( T \), \( T^* \) is subsumed by \( R^T \).

Proof: Let \( Q \) be a T-link. The semantic graph \( T \rightarrow Q \) is unsatisfiable. Define \( L \) to be the set of nodes in \( T \) that are linked to some node in \( Q \). Consider first the case where \( L \) contains some clause \( C \) from \( T \). Then \( C \subseteq Q \), and \( \overline{C} \) is the desired resolvent T-link.

Otherwise, let \( R \) be the subgraph of \( T \) relative to the nodes in \( T - L \). Since the above case does not apply, \( R \) meets each clause in \( T \), so any c-path \( p_R \) through \( R \) is a c-path through \( T \). By the definition of \( L \), \( p_R Q \) cannot contain a link to \( Q \). But it is a c-path through \( T \rightarrow Q \) and must therefore contain some link. Thus this link lies entirely within \( R \). Therefore \( R \) is a resolution chain, its path resolvent is \( \overline{L} \), and \( L \subseteq Q \) is a resolvent T-link.

Theorems 2-4 imply the existence of a set of essential theory links: those that are not subsumed by any other theory link (of any of the three types).

The following interesting result on consequence completeness for path resolution in the ground case is now immediate for a graph \( G \) in CNF: Any (single clause) logical consequence of \( G \) is in \( G \) or is subsumed by a path resolvent of \( G \). Looked at another way, Theorem 4 guarantees that for \( i \geq 0 \), the clause obtained in the \( i \)th step of a resolution derivation is subsumed by a one-step path resolvent from the original graph.

3.3 Minimizing theory links

It is obvious that if \( Q \) is a theory link, then any superset of \( Q \) is unsatisfiable in the theory. Therefore, when we compute a set of theory links, we may throw out any member that is a superset of another member. This is essentially subsumption checking in the ground case. The characterization used to compute theory links may affect the amount of subsumption checking required since the three characterizations may yield distinct sets of theory links (of course all three are equivalent in power). Suppose for example that \( T = \{\{A, B, C\}, \{\overline{A}, \overline{B}, D\}\} \). The clause \( L = \{A, B, C, D\} \) is in \( T^* \), but \( \overline{L} \) is not a resolvent T-link; neither are any of the infinitely many supersets of \( \overline{L} \), yet they are all strong T-links. Contained in \( \overline{L} \) is of course the resolvent T-link corresponding to the first clause in \( T \).

Let \( R \) and \( R' \) be resolution chains in a ground formula in CNF, and let \( C \) and \( C' \) be the subgraphs consisting of all clauses met by \( R \) and \( R' \), respectively. We say that \( R' \) is a subchain of resolution chain \( R \) if both of the following conditions hold:
The next lemma is obvious; it suggests one way to reduce the theory links computed: Search for maximal resolution chains, i.e., those that are subchains of no other resolution chain, and compute resolvent T-links.

**Lemma 2.** Let \( R \) and \( R' \) be resolution chains and let \( P \) and \( P' \) be the path resolvents generated by \( R \) and \( R' \), respectively. If \( R' \) is a subchain of \( R \), then \( P' \) is subsumed by \( P \). □

Building only maximal chains is a non-trivial problem, and, taken to its extreme, amounts to the entire theorem proving task for unsatisfiable ground formulas. The situation is further complicated in the first order case.

### 4. First order theory links

Some of our ground level results lift directly into first order logic; others lift in somewhat modified form.

During construction of the mgsu of a resolution chain, some care must be taken regarding the familiar process of standardizing variables apart. If \( \chi \) is any variable, two occurrences of \( \chi \) cannot be standardized apart if they appear in d-connected nodes. In CNF this is a sufficient condition for determining whether variables may be standardized apart; in semantic graphs (NNF), this is not the case. What is required is the transitive closure of the relation 'are d-connected', which provides all the occurrences of \( \chi \) that are in fact the same variable.

Let \( R \) and \( R' \) be (not necessarily ground) resolution chains as defined above; similarly let \( C \) and \( C' \) be subgraphs consisting of clauses met by \( R \) and \( R' \) respectively. Let the mgsu’s of \( R \) and \( R' \) be \( \sigma \) and \( \sigma' \) respectively. For any substitutions \( \alpha \) and \( \beta \), we denote by \( \alpha|\beta \) the restriction of \( \alpha \) to \( \beta \), i.e., the substitution constructed from the components of \( \alpha \), whose variables are also variables changed by \( \beta \). We say that \( R' \) is a first order subchain of resolution chain \( R \) if conditions [a] and [b] from Section 3.3 hold, and if \( \sigma(\sigma'|\sigma) = \sigma'|\sigma \); i.e., if \( \sigma \) is as general as \( \sigma' \) restricted to the variables of \( \sigma \).

**Lemma 3.** Let \( R' \) be a first order subchain of resolution chain \( R \). If \( P' \) and \( P \) are the path resolvents generated by \( R' \) and \( R \) respectively, then \( P' \) is subsumed by \( P \). □

The conditions of Lemma 3 are sufficient but not necessary. The requirement that [a] and [b] hold guarantees that \( P' \) is formed from a superset of the nodes that form \( P \). But even if the nodes that form \( P' \) are a subset of the nodes that form \( P \), there may exist a substitution \( \theta \) such that \( P\theta \subseteq P' \).

### 4.1 Lifting

Lemma 1 and Theorem 1 concerning T-links lift directly in much the same way that resolution does. In general, \( T^* \) may be infinite, but a finite subset of \( T^* \) is always sufficient to demonstrate the T-unsatisfiability of a formula \( H \). Incorporating sufficient T-links into \( H \) is semi-decidable; it becomes decidable in cases where \( T^* \) is known to be finite, such as when \( T \) is ground, or when, as in the second example of Section 7,
T = T* because T is linkless. Furthermore, when T \cup H is function-free, its Herbrand universe is finite, and generating enough T-links is decidable whether or not T* is finite.

Theorems 2 and 4 relate T-links to strong T-links and to resolvent T-links. They lift from any one instance of T: Suppose R is a resolution chain in \{T\theta_1, T\theta_2, ..., T\theta_n\}, a set of ground instances of a theory T. Unless n = 1, we cannot be sure that the chain lifts to T. Of course, R does lift if we have n copies of T. Thus the resolvent T-links do lift from any single instance of T, and hence, in view of Theorem 3, the same is true of strong T-links.

Theorem 3 remains true at the general level. But the reason is that the essential strong T-links lost in lifting are exactly the essential resolvent T-links similarly lost.

4.2 Unifiers for theory links

Unifying an ordinary link is straightforward since the only atoms requiring inspection are present in the potential link; unifying a theory link may be more difficult since atoms in the theory may also require inspection. Of course, the first time a theory link is detected, unification with atoms in the theory is unavoidable. However, after an inference, it is desirable to determine whether an inherited set of literals forms a theory link without further reference to the theory. For example, if the literal A from a theory link \{A, B, C\} is inherited, it may be instantiated in such a way that B, C, and the inherited version of A cannot be unified to form a theory link. We wish to avoid repeated consultations with the theory to determine whether this has occurred. Theorem 5 below gives a condition under which this determination may be made directly.

Suppose T is a first order theory in CNF. Any logical consequence of one instance of T is an instance of some path resolvent Q of a resolution chain R in T. Let S be the strong c-block consisting of the clauses in T that meet R. Then the nodes in the set (S-R) are exactly the nodes used to form Q. In particular, if r is the mgsu for R, then Q = (S-R)r. A first order theory link Q' corresponding to Q is a set of c-connected nodes such that some instance of \overline{Q'} is also an instance of Q. If this is the case, Q and \overline{Q'} are unifiable with mgsu r; i.e., Qr = (S-R)\sigma r = \overline{Q'} r. We call \omega the most general simultaneous T-unifier (mgsTu) of Q', if \omega is the restriction of r to the variables in Q'; we will write \omega = r|Q'. We focus on \omega, not on r, since we need only know the mapping from Q to Qr = \overline{Q'} \omega. The mapping from Q to Qr (which is different from Q\omega) is unimportant except that it exists.

Using the above notation, suppose we have a theory T = {{P(u)}, {P(x)}, B(x,g(a))}. One consequence of T is Q = {{P(u)}, {P(x)}, B(x,g(a))}. We have R = {P(u), P(x)}, S = T, and \sigma = \{u/x\}. Let Q' consist of the single node B(f(w),z); then the unifier for Q and \overline{Q'} is r = \{f(w)/x, g(a)/z\} and the mgsTu for Q' is \omega = r|Q' = \{g(a)/z\}.

Suppose now that a resolution chain R is activated resulting in path resolvent P, and let Q' be the inherited version of an instance Q' of the theory link Q. As noted above, the inherited image Q' need not be a theory link, but if it is, its mgsTu may be different from that of Q'. Let \omega be the mgsTu of Q', and let \rho be the unifier for R. Let O' = Q' \cap Q' and I' = Q' \setminus O'. (Thus the nodes in O' are 'original' nodes, or nodes 'outside' P, and I' consists of inherited nodes, or nodes 'inside' P.) Thus
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Q' \cup O' \cup I' \text{ and } Q_1' = O' \cup I' \rho. \text{ Then } Q_1' \text{ is a theory link if } \rho \text{ is compatible (Kowalski, 1975) with } \omega \text{ in the sense that there exists a } \gamma \text{ such that } Q_\gamma = Q_1' \gamma; \text{ i.e., for } Q_1' \text{ to be a theory link, } Q_1' \text{ must be unifiable with } Q. \text{ This defines the compatibility of } \rho \text{ and } \omega, \text{ but we would like a more convenient test for compatibility, ideally based on } \rho \text{ and } \omega \text{ alone.}

**Theorem 5.** Using the notation above, if V1, ..., Vn are the variables in I', let V = <V1, ..., Vn> be the corresponding (ordered) sequence. Then \rho is compatible with \omega if I' \rho and O' have no variables in common and the elements of V \rho and V \omega are simultaneously pairwise unifiable. In that case, if \alpha is the mgsu for V \rho and V \omega, then \omega | O' \alpha is the mgsTu for Q1' .

**Proof:** Note first that there is an obvious 1-1 correspondence between the nodes of QI' and the nodes of Q. If n_\sigma is a node in QI', n_\sigma (\omega | O')\alpha = n_\sigma \omega \alpha. But \omega is the mgsTu of QI', so n_\sigma \omega = n_\sigma \alpha for the corresponding node n_\sigma in Q. Hence, n_\sigma (\omega | O')\alpha = n_\sigma \alpha.

Consider now a node n_\rho in QI'y. It contains no variables in O', so n_\rho \rho (\omega | O') = n_\rho \rho. Therefore n_\rho \rho (\omega | O')\alpha = n_\rho \rho \alpha which, from the theorem hypothesis, we know to be n_\rho \omega \alpha. Since n_\rho \omega = n_\rho \alpha for the node n_\rho in Q corresponding to n_\rho, we have n_\rho \rho \rho (\omega | O')\alpha = n_\rho \omega \alpha = n_\rho \alpha; \text{ i.e., the substitution } (\omega | O')\alpha \text{ maps any node in } QI' \text{ to the same instance to which } \alpha \text{ maps the corresponding node in } Q.

Since Q is in the theory, its variables are distinct from those of both Q' and Q1', so (Q1')\omega | O' \alpha \text{ is the mgsu of } QI' and Q. Restricting this substitution to QI' produces } (\omega | O' \alpha \text{ again, which is the mgsTu for QI'}.}

In order to apply Theorem 5, it is convenient to assume that the variables of the path resolvent may be renamed apart from those of Q'. This is always possible when P is conjoined (as a new fundamental subgraph) with the entire semantic graph. Such renaming may not be possible when the resolvent is conjoined with some full block that is not large enough to be fundamental in the entire graph. In such a case, compatibility may be tested by a direct attempt to unify QI' with Q.

Let us suppose that Q' in our previous example appears instantiated in a path resolvent formed with mgsu \rho = \{g(y)/w, v/z\}. We have O' = \emptyset, I' = Q', and I' \rho = B(f(g(y)), v). The sequence V of variables in I' is <w, z>; V \rho = <g(y), v>, and V \omega = <w, g(a)>. The mgsu of V \rho and V \omega is } \alpha = \{g(y)/w, g(a)/v\}. The inherited theory link is Q_I' = B(f(g(y)), v) and (\omega | O')\alpha = \alpha \text{ is its mgsTu. The most general common instance of QI' and Q is Q_I' \alpha = B(f(g(y)), g(a))}. \text{ If we compose } \tau | Q = \{f(w)/x\} \text{ with (the empty } \omega \text{ | O'} \text{ and then with } \alpha \text{ we get } \{f(g(y))/x, g(a)/v\} \text{ which is the mgsu of QI' and Q.}

5. Theory Links and Theory Resolution

The work of Stickel (1983, 1985a, 1985b) on theory resolution is closely related to our work on theory links. One important contrast in these approaches occurs in the assumptions made about how T-unsatisfiable sets of literals (or sets of clauses) are recognized. Theory resolution assumes a 'black box' for this recognition and different categories of inference arise as a function of the power of the black box. We have instead assumed that the black box can be expressed as a formula in CNF. The consequences of the formula can then be incorporated into a knowledge base in the form of
theory links, and used in the inference process.

Let us suppose that a T-decision procedure for theory resolution can be expressed as a set of clauses T. Let H be some knowledge base on which we wish to do deduction relative to T-interpretations. Path resolving on a single T-link in H corresponds to a total narrow theory resolution. Path resolving on a resolution chain built from several T-links (not all on the same c-path) corresponds to a total wide theory resolution.

Within our framework, partial theory resolution can be stated as the following theorem.

**Theorem 6.** Suppose H is a semantic graph containing a set q of c-connected nodes. Suppose further that Q is a theory link with \( \text{mgsTu} \omega \), and that \( q \subseteq Q \). If \( r \) is the set of c-connected literals appearing in \( Q - q \), then

\[
( \text{WS}(t, H) \lor \neg r ) \omega
\]

may be soundly inferred (with respect to the theory).

The proof is straightforward and left to the reader. □

For well known theories (like equality) whose usefulness is almost universal, a highly developed and streamlined black box may be the best answer. But there may be situations in which a theory, representable as a set of clauses, is being learned or acquired. Adding new clauses to the theory-knowledge base would give rise to new theory links in the assertional knowledge, i.e., that part of the knowledge base on which a system performs deduction.

The 'density' of ordinary links in a particular theory may influence the practicality of pre-computing theory links. But theory links need not always be pre-computed; linkless c-paths in a partially constructed resolution chain may be checked for theory links in order to complete the chain. Of course, such links can be more permanently recorded after their discovery. It might be helpful to record the theory clauses on which a theory link is based. If a dynamic system should remove one such clause, the dependent theory link could then be immediately removed also.

Theory links may also be useful in pure theorem proving applications. Within this context, any subset of the formulas describing a problem may be chosen as the 'theory.' One obvious choice would be a (perhaps maximal) subset which yields a finite number of theory links.

### 6. Link Deletion

Deleting a link after activation reduces the size of the search space since not only will the given link never be used again, but also it will never be inherited. In (Murray & Rosenthal, 1985b, 1985c) we developed several link deletion results for ordinary links. In this section, we briefly summarize those results. The key idea in those theorems is that under certain conditions, the deletion of links after activation does not affect the spanning property; that is, if a semantic graph is spanned by a set of links before activation of a chain, it will still be spanned if certain links are deleted after activation.

Recall that in CNF a strong c-block is a union of clauses. This is essentially true in an arbitrary semantic graph: A strong c-block is a union of d-paths through the
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graph. As a result, any subgraph $R$ of a semantic graph $G$ has the form:

$$C_1 \cup C_2 \cup \ldots \cup C_n \cup C_s$$

where $C_s$ is a strong c-block, and the others are c-blocks that are not strong. (It is certainly possible that $C_s$ is empty; moreover the $C_i$'s need not be disjoint. The technical term for this decomposition is 'proper c-family'. See (Murray & Rosenthal, 1985a).)

Suppose now that $R$ is a resolution chain, and that it has the form:

$$C_1 \rightarrow C_2 \rightarrow \ldots \rightarrow C_n \rightarrow C_s$$

(In general, resolution chains need not have this form.) If $R$ is activated, some of its links may be deletable. In (Murray & Rosenthal, 1985c) we proved the following result assuming that $R$ has been activated using weak split.

**Theorem:** If $n=2$, and if $C_1$ and $C_2$ are strong with respect to c-connected full blocks $M_1$ and $M_2$ respectively, then every link that meets both $M_1$ and $M_2$ is deletable. \(\square\)

The following considerably stronger result holds if strong split is used. (In (Murray & Rosenthal, 1985c), a kind of hybrid inference rule, proper split, is developed. It uses strong split 'near' the chain and weak split 'away' from the chain.)

**Theorem:** If $n=2$, then every link from $C_1$ to $C_2$ may be deleted when $R$ is activated. \(\square\)

These two theorems apply at the ground level. Their proofs are purely structural and may be applied to (ground) theory links as well. The key to both proofs is the observation (first made by Bibel (1981) for single (ordinary) link chains in CNF) that a c-path that contains a deleted link must go through both $M_1$ and $M_2$; the structure of the chain then guarantees that such a c-path will pick up an inherited link. The reason that $n=2$ is necessary in these theorems is that the link must meet each non-strong c-block. The identical results apply to theory links that meet every non-strong c-block in the chain.

The two theorems lift (for theory links as well as for ordinary links) with additional restrictions on unifiers: A sufficient condition for a first order link to be deletable is that its unifier be identical to the simultaneous unifier of the chain being activated; i.e., the entire chain is present in any instance of the graph in which the link in question is present.

7. Examples

Consider again the theory consisting of the clauses $C_1 = \{E(x), M(x)\}$ and $C_2 = \{M(x), A(x)\}$ representing "elephants are mammals" and "mammals are animals," respectively. The resolvent of $C_1$ and $C_2$ is $C_3 = \{E(x), A(x)\}$. This yields $T^* = \{C_1, C_2, C_3\}$.

Suppose we know that elephants like peanuts and that dumbo is an elephant.
We have the graph in Figure 3.

Note that \( E(\text{dumbo}) \) is a subset of an instance of the theory link \( C_3 \). Theorem 6 applies, yielding partial theory resolvent \( A(\text{dumbo}) \). The ordinary resolvent of the link \( \{E(x), E(\text{dumbo})\} \) is \( \text{Likes}(\text{dumbo}, \text{peanuts}) \), and we have proven that dumbo is an animal that likes peanuts.

The next example has been discussed by Stickel (1985b). His theory resolution inference rule has been used in the KLAUS (Haas & Hendrix, 1980) and KRYPTON (Brachman et al., 1985; Pigman, 1984) systems to generate refutations for this knowledge base. Suppose we have the following information about boys and girls: boys are persons whose sex is male, and girls are persons whose sex is female. We represent this as a theory \( T \) of four clauses.

\[
T = \{ \{B(x) \rightarrow P(x)\}, \{G(x) \rightarrow P(x)\}, \{\neg B \rightarrow M(y)\}, \{S(x,y) \rightarrow \neg F(y)\} \}
\]

The semantic graph in Figure 4 defines \( \text{NoSons} \) (as Persons all of whose children are \( \text{Girls} \)) and \( \text{NoDaughters} \); it also declares that every \( \text{Person} \) has a \( \text{Sex} \), Males and Females are disjoint, and finally that chris has neither sons nor daughters and yet has a Child. This semantic graph is \( T \)-unsatisfiable. The \( T \)-links are represented by the double curves, and the ordinary links by the single curves. There are only four \( T \)-links since \( T^* = T \).
We now illustrate a path resolution refutation using a combination of ordinary links and theory links. The (ordinary) links numbered 1 and 2 in the diagram form a 'nice' resolution chain: there are two maximal c-blocks, one of which is strong; each link meets both c-blocks; and the mgu of link 1 equals the mgsu of the chain, allowing deletion of the link. Had we used weak split, the path resolvent would be $G(a)$; We instead used strong split, producing $G(a) \land (\overline{ND(c)} \lor B(a)))$ and allowing for the deletion of link 1. The Pure Lemma (Murray & Rosenthal, 1985b, 1985c) then permits removal of the two fundamental subgraphs that meet link 1 and of their links. The resulting graph is shown in Figure 5.

![Figure 5. An inherited theory link, Q_1'].

Notice that $Q_1'$ (in Figure 5) is a potential theory link. Theorem 5 may be used to confirm that it is in fact a theory link. (The same is true for the three other theory links in Figure 5; we leave verification to the reader.) First consider $Q_1'$, the parent of $Q_1$. The mgsTu of $Q'$ is $\omega = \{v/z, f(v)/w\}$. The mgsu of the chain consisting of links 1 and 2 is $\rho = \{c/u, a/z\}$. Hence, $Q_1' = O' \cup I'\rho$, where $O' = \{S(v,f(v)), \overline{F(w)}\}$, $I' = \{G(z)\}$, and $I'\rho = G(a)$. Thus $O'$ and $I'\rho$ have no variables in common. To apply Theorem 5, we must also show that if $V$ is the sequence $<z>$ of variables from $I$, then $V\rho$ and $V\omega$ are simultaneously pairwise unifiable. But this is immediate since $V\rho = <a>$ and $V\omega = <v>$ are unifiable by mgsu $\alpha = \{a/v\}$. Therefore $Q_1'$ is a theory link whose mgsTu is $(\omega \cup O')\alpha = \{f(a)/w, a/v\}$.

Observe that this inherited theory link represents a resolution chain (with mgsu $\{a/x, f(a)/y, a/v, f(a)/w\}$ consisting of three ordinary links involving a clause from the theory. Figure 6 illustrates this chain along with the fundamental subgraphs that
This particular chain would be fairly easy to find, but in general, determination of large chains is difficult. Activation of this single theory link is equivalent to the activation of this chain. In this way, the theory link allows us to store, inherit, and recall such subdeductions as required.

We now resolve on link 3 and theory link 4 (which comprise what might be called a 'hybrid' resolution chain). The path resolvent is $S(a, fa)$. Link 4, the theory link, is deletable, and the resulting graph is shown in Figure 7. We show only the inherited theory links since they are all that are required to complete the proof.

The inherited theory links ($5'$ and $Q_{II}'$) form a five-node resolution chain with mgsTu $\{fa/w\}$ (in which link $5'$ is deletable). The path resolvent is $\overline{ND(c)}$, which inherits a link (to $ND(c)$) that spans the entire graph.

Note that the last link could have been added to the chain in the previous step producing a contradiction then. In fact, the whole proof could have been done in one step since one instance of this graph is contradictory.

A more difficult problem that has received considerable attention recently is Schubert's Steamroller. It can be handled by a variety of techniques including the theory resolution of Stickel (1985b) and by the many-sorted provers of Cohn (1985) and Walther (1984). In (Murray & Rosenthal, 1986) we present a 9-step proof which uses path resolution and theory links. The first twelve clauses (which assure the existence of, and describe the 'sorts' of, foxes, birds, etc.) are treated as a theory, and
only finitely many theory links are generated. The search space is reduced by applying both link deletion and the Pure Lemma.

8. Conclusions

We have introduced the notion of theory link and related it to Stickel's theory resolution. Many techniques and results from the study of semantic graphs and path resolution are directly applicable to performing inferences with theory links. In particular, the inferencing mechanisms designed for path resolution are completely adequate to handle resolution chains built in an arbitrary way. Link deletion results are also applicable. We have demonstrated that in the ground case any clause C which is a logical consequence of a set G of clauses can be essentially derived from G in one step by path resolution.

Theory links may be useful in two ways. First, given a theory (for example, a set of axioms for a knowledge base) that is to be used over and over again, the theory may be preprocessed to create a set of theory links. If the complete set of theory links is finite (which is often the case), the theory links will be adequate for deductive purposes, and the theory itself will never need to be consulted.

Theory links may also be applied to pure theorem proving. A set of clauses (for example all two-literal and unit clauses without function symbols) may be chosen as the theory. The advantage gained is that a single theory link appearing in the rest of the formula will typically represent a chain consisting of at least two ordinary links. Such a resolution chain actually represents a sub-deduction which may in general be hard to find, and storing this chain as a theory link is likely to be advantageous.

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