On the Control of Open Quantum Systems in the Weak Coupling Limit

Domenico D’Alessandro¹, Edmond Jonckheere² and Raffaele Romano³.

Abstract

When dealing with the control of quantum systems in interaction with an external environment, one of the most commonly used models is the Lindblad-Kossakowski equation for the density matrix. Its derivation relies on various approximations, which are usually justified in terms of weak interaction between the system and its surroundings and special properties of the bath interacting with the system. This equation contains dissipative corrections, accounting for the interaction with the environment, whose expression strongly depends on the adopted Markov approximation. In the case of coherently controlled Lindblad-Kossakowski equation, it is usually assumed that this correction is independent of the control Hamiltonian. However, this procedure is not consistent with the rigorous derivation of the Markov approximation in the standard case, the so-called weak coupling limit, in which the dissipative contribution depends on the coherent part of the dynamics.

In this paper we discuss the rigorous derivation of the Lindblad-Kossakowski equation in the weak coupling limit regime, following the original derivation of Davies [8], in the case where the control is present, and explicitly displaying the dependence of the model on the control. We then consider the special case of a 2-level system in a bosonic bath, and show explicitly how the model depends on the parameters in the Hamiltonian which contains the control.

We also discuss the impact of this more rigorous modeling procedure in the current research on methods to find controls that decouple the system from the environment. We indicate directions and problems for further research.

1 Introduction

In the last decades there has been a large interest in the control of quantum systems whose interaction with the environment cannot be neglected. Such systems are named open quantum systems and their control presents a series of challenges and a structure which is not a simple extension of what is known for the case of closed systems [4]. The effect of the interaction with the environment leads to a degradation of the ‘quantum’ nature of the evolution which jeopardizes the correct implementation of quantum information processing. Therefore methods to control the dynamics in order to shield a system from the environment are currently under investigation.

Generally, a well motivated model of open quantum dynamics is the Markovian Quantum Master Equation for the density matrix \( \rho \) describing the system. This is given by

\[
\dot{\rho} = [-iH, \rho] + \sum_{j,k} \left( V_j \rho V_k^\dagger - \frac{1}{2} \left\{ V_k^\dagger V_j, \rho \right\} \right),
\]

where the operators \( V_j \) are the so-called Lindblad-Kossakowski operators, cf., e.g., [3]. Here \( H \) is the closed system Hamiltonian \( H_0 \) augmented with the so-called Lamb shift term, \( H_{\text{Lamb}} \), which depends on the interaction between the system and the environment. The derivation of the equation (1) is the work of many authors including Nakajima, Zwanzig, Prigogine, Resibois, Lindblad, Kossakowski and Davies. This work proceeded from phenomenological arguments to rigorous mathematical proofs (see e.g., [6], [7], [8], and the references in [19]). Equation (1) is valid under the assumption that the coupling with the external environment is small and that memory effects are negligible, the so-called Born and Markov approximations. More specifically, there are two main approaches to rigorous Markov approximations, valid under different conditions: the so-called weak coupling limit and the so-called singular coupling limit. In the first case,
which is usually better motivated from a physical point of view, the Lindblad-Kossakowski operators depend on the Hamiltonian $H_0$. A survey on the master equation is presented in [9].

When dealing with coherent control of open quantum systems, the Hamiltonian $H_0$ depends on the control and typically authors in the control theory literature modify equation (1) by simply replacing $H_0$ with a time varying Hamiltonian, $H_c := H_c(u)$, depending on the control (see e.g., the papers [2], [15], [18], [22], in the special issue of the IEEE Transactions on Automatic Control dedicated to quantum control, and the references therein). While this procedure is justified in the singular coupling limit, it is not in the weak coupling limit, since in the latter case the $V_j$’s operators in (1) depend on the Hamiltonian itself, which in turn depends on the control (which, in principle, has yet to be designed). By using an example, in this paper we will show that this dependence can be very significant. While this fact has been already discussed to some extent in the physics literature (cf., e.g., [21] and references therein), our goal is to point it out to the control community, and to suggest directions for future research. These include methods to shield the system from the environment, such as the decoherence splitting manifold (DSM) approach (cf., [12]).

The paper is organized as follows. In section 2 we describe the quantum master equation and define the above mentioned approximations. Our treatment follows mainly the work of Davies in [8] (cf. also [6], [7], and [5]) and it is presented in a way that highlights the dependence of the Lindblad-Kossakowski operators on the Hamiltonian and therefore the controls. We stress the role of the Generalized Master Equation, an integral equation of the Volterra type, which describes the evolution before any approximation is applied. The two cases of singular coupling limit and weak coupling limit are discussed, pointing out that in the second case the Lindblad-Kossakowski operators depend on the control. In section 3 we consider the model of a 2-level system in an external environment which consists of a continuous set of harmonic oscillators (bosonic bath). We derive the generator of the dynamics, and explicitly show the aforementioned dependence. In section 4 we discuss how this dependence may affect current research in finding controls to decouple the system dynamics from the environment (see e.g., [11], [16]), and in particular we discuss the effect of taking into account the rigorous derivation of the master equation in the creation of Decoherence Splitting Manifolds. We give some concluding remarks in section 5.

We would like to emphasize at the outset that, although the arguments presented in this paper limit the applicability of some of the results published in the literature, they also open new scenarios for the development of the theory of control of open quantum systems. The dependence of the model of decoherence on the control offers a new way to shield a quantum systems from the detrimental effects of the interaction with the environment as the control enters not only the coherent part of the model, but also the additional terms modeling the interaction with the environment.

2 The Quantum Master Equation

2.1 Preliminaries

We consider a system $S$ and a bath $B$ in a composite state described by a density operator $\rho_T$ on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. Our objective is to obtain a differential equation (1) for the state of the system $S$ only, which is $\rho_S := Tr_B(\rho_T)$. We assume that the states of the system $S$ and bath $B$ are initially uncorrelated so that

$$\rho_T(0) = \rho_S(0) \otimes \rho_B,$$

for an equilibrium state of the bath $\rho_B$, satisfying $([H_B, \rho_B] = 0)$. The dynamics of the total system $S + B$ is determined by an Hamiltonian operator $H_{TOT}(t)$, given by the sum of the term $\hat{H}_S(t) \otimes 1$, which describes the dynamics of the system alone, the term $1 \otimes H_B$, which describes the dynamics of the bath alone, and finally the term $\epsilon \hat{H}_{SB}$, which describes the interaction between system and bath. The parameter $\epsilon$ is used to describe the strength of the interaction, which, in the limit considered below, is assumed to be small. We have therefore

$$H_{TOT}(t) := \hat{H}_S(t) \otimes 1 + 1 \otimes H_B + \epsilon \hat{H}_{SB}.$$
Without loss of generality, we write the interaction Hamiltonian as $\hat{H}_{SB} := \sum_j V_{Sj} \otimes V_{Bj}$. Adding and subtracting $\epsilon \sum_j \text{Tr}(V_{Bj}\rho_B) V_{Sj} \otimes 1 = \epsilon \text{Tr}_B(\hat{H}_{SB} 1 \otimes \rho_B) \otimes 1$ in (3), we define

$$H_S^\varepsilon(t) := \hat{H}_S(t) + \epsilon \sum_j \text{Tr}(V_{Bj}\rho_B) V_{Sj} = \hat{H}_S(t) + \epsilon \text{Tr}_B(\hat{H}_{SB} 1 \otimes \rho_B),$$

$$H_{SB} := \hat{H}_{SB} - \sum_j \text{Tr}(V_{Bj}\rho_B) V_{Sj} \otimes 1 = \hat{H}_{SB} - \text{Tr}_B(\hat{H}_{SB} 1 \otimes \rho_B) \otimes 1,$$

so that we can rewrite the Hamiltonian $H_{TOT}(t)$ in (3) as

$$H_{TOT}(t) = H_S^\varepsilon(t) \otimes 1 + 1 \otimes H_B + \epsilon H_{SB}.$$  

The term $\epsilon \text{Tr}_B(\hat{H}_{SB} 1 \otimes \rho_B)$ in (4) is the so-called *Lamb shift*. It represents a shift in the Hamiltonian of the system due to the interaction with the bath. Notice that the Hamiltonian $\hat{H}_S(t)$ (and therefore $H_S^\varepsilon(t)$), considered in (6), can be time-dependent, since it contains the time-varying control fields; all other Hamiltonians are assumed constant. Notice also that $\text{Tr}(H_{SB} \rho_S \otimes \rho_B) = 0$.

The dynamics of $\rho_T$ follows the Liouville-Schrödinger equation

$$\dot{\rho}_T = [-iH_{TOT}(t), \rho_T] = [-iH_S^\varepsilon(t) \otimes 1, \rho_T] + [-i1 \otimes H_B, \rho_T] + \epsilon [-iH_{SB}, \rho_T].$$

As customary, $ad_A$ denotes the operator $ad_A(B) := [A, B]$, and we use the short notation $ad_S^\varepsilon(t), ad_B$ and $ad_{SB}$ for $ad_{-iH_S^\varepsilon(t)\otimes 1}, ad_{-i1\otimes H_B}$, and $ad_{-iH_{SB}}$, respectively, so that (7) can be compactly written as

$$\dot{\rho}_T = (ad_S^\varepsilon(t) + ad_B)(\rho_T) + \epsilon ad_{SB}(\rho_T).$$

Following the technique of Nakajima [17] and Zwanzig [24], we define a projection operator $P$ (cf. [19]), as

$$P[\rho] = \text{Tr}_B(\rho) \otimes \rho_B,$$

so that $P(\rho_T) = \rho_S \otimes \rho_B$, where $\rho_B$ is the same as in (2). We denote by $Q := 1 - P$. It follows from the definitions and properties of $ad_S^\varepsilon(t), ad_B$ and $ad_{SB}$, and $\rho_B$ that

$$Pad_S^\varepsilon(t) = ad_S^\varepsilon(t)P, \quad Qad_S^\varepsilon(t) = ad_S^\varepsilon(t)Q, \quad \forall t \geq 0$$

$$Pad_B = ad_B P = 0, \quad Qad_B = ad_B Q = ad_B,$$

$$Pad_{SB} P = 0.$$

### 2.2 The Generalized Master Equation

The *Generalized Master Equation* (GME) is a Volterra integral equation representing the dynamics of $\rho_S$. It does not involve any approximation, and it represents the starting point to obtain a Markovian dynamics, in the form of a differential equation as in (1).

By writing $ad_{SB} = (P + Q)ad_{SB}(P + Q)$, and recalling (12), we can rewrite equation (8) as

$$\dot{\rho}_T = (ad_S^\varepsilon(t) + ad_B + \epsilon Qad_{SB}Q)(\rho_T) + \epsilon Qad_{SB}P + Pad_{SB}Q)(\rho_T).$$
Following [6, 7, 8], we introduce evolution operators (or propagators) associated with the linear operators appearing in (13). $\Phi_0(t, s)$ is the propagator associated with $ad_S^0(t) + ad_B$, and it would describe the evolution of the state $\rho_T$ if there was no interaction between system and bath. $\Phi_1(t, s)$ is the propagator associated with $ad_S^1(t) + ad_B$; it is the same as $\Phi_0(t, s)$ but it takes into account the Lamb-shift term in the free dynamics of the system $S$. $\Phi_2(t, s)$ is the propagator associated with $ad_S^2(t) + ad_B + \epsilon Qad_{SB}Q$, cf.(13). $\Phi_3(t, s)$ is the (full) propagator associated with $ad_S^3(t) + ad_B + \epsilon Qad_{SB}Q + \epsilon(Qad_{SB}P + Pad_{SB}Q)$. Notice that because of (10), (11), we have

$$P\Phi_2(t, s) = \Phi_2(t, s)P = P\Phi_1(t, s) = \Phi_1(t, s)P. \quad (14)$$

The propagator $\Phi_3^\epsilon$ satisfies the following integral equation (cf. (13))

$$\Phi_3^\epsilon(t, s) = \Phi_2^\epsilon(t, s) + \epsilon \int_s^t \Phi_2^\epsilon(t, r)(Qad_{SB}P + Pad_{SB}Q)\Phi_3^\epsilon(r, s)dr. \quad (15)$$

From this, by using (14), we calculate $P\Phi_3^\epsilon(t, s)P$ and $Q\Phi_3^\epsilon(t, s)P$; we obtain respectively

$$P\Phi_3^\epsilon(t, s)P = P\Phi_1^\epsilon(t, s)P + \epsilon \int_s^t \Phi_1(t, r)Pad_{SB}Q\Phi_3^\epsilon(r, s)Pdr, \quad (16)$$

$$Q\Phi_3^\epsilon(t, s)P = \epsilon \int_s^t \Phi_2(t, r)Qad_{SB}P\Phi_3^\epsilon(x, s)Pdx. \quad (17)$$

Replacing (17) into (16), and setting $s = 0$, we obtain

$$P\Phi_3^\epsilon(t, 0)P = P\Phi_1^\epsilon(t, 0)P + \epsilon^2 \int_0^t \Phi_1(t, r)Pad_{SB} \left[ \int_0^r \Phi_2(r, x)Qad_{SB}P\Phi_3^\epsilon(x, 0)Pdx \right] dr, \quad (18)$$

where the term in square brackets is $Q\Phi_3^\epsilon(u, 0)P$. We can slightly simplify (18) by using (12) to replace $Qad_{SB}P$ with $ad_{SB}P$, and obtain

$$P\Phi_3^\epsilon(t, 0)P = P\Phi_1^\epsilon(t, 0)P + \epsilon^2 \int_0^t \Phi_1(t, r)Pad_{SB} \left[ \int_0^r \Phi_2(r, x)ad_{SB}P\Phi_3^\epsilon(x, 0)Pdx \right] dr. \quad (19)$$

The operator $P\Phi_3^\epsilon(t, 0)P$ in (19) describes the evolution of the state of $\rho_S$ of $S$. Since the initial state of the total system $S + B$ is given by (2), $P$ acts on it as the identity. $\Phi_3^\epsilon(t, 0)P\rho_T(0)$ gives the state of the system $S + B$ at time $t$, and the application of $P$ recovers the state of the system $S$: the final result is $\rho_S(t) \otimes \rho_B$. Changing the order of integration in (19) and replacing $\Phi_1^\epsilon(t, r)$ with $\Phi_1^\epsilon(t, x)(\Phi_1^\epsilon(r, x))^{-1}$, we obtain

$$P\Phi_3^\epsilon(t, 0)P = P\Phi_1^\epsilon(t, 0)P + \epsilon^2 \int_0^t \Phi_1^\epsilon(t, x) \left[ \int_x^t (\Phi_1^\epsilon(r, x))^{-1}Pad_{SB}\Phi_2^\epsilon(r, x)ad_{SB}dr \right] P\Phi_3^\epsilon(x, 0)Pdx. \quad (20)$$

Defining as $L = L(\epsilon, t, x)$ the kernel

$$L(\epsilon, t, x) := \epsilon^2 \int_x^t (\Phi_1^\epsilon(r, x))^{-1}Pad_{SB}\Phi_2^\epsilon(r, x)ad_{SB}dr, \quad (21)$$

4
Equation (20) can be written in the form

\[ P\Phi_S(t, 0)P = P\Phi_S^t(t, 0)P + \int_0^t \Phi^t_1(t, x)L(\epsilon, t, x)P\Phi_S^t(x, 0)Pdx, \]  

(22)

which, when applied to \( \rho_S(0) \otimes \rho_B \), gives the Generalized Master Equation (GME)

\[ \rho_S(t) \otimes \rho_B = \Phi_S^t(t, 0)[\rho_S(0) \otimes \rho_B] + \int_0^t \Phi_S^t(t, x)L(\epsilon, t, x)[\rho_S(x) \otimes \rho_B]dx. \]

(23)

The GME is an exact representation of the dynamics of the state of the system. There is no approximation involved at this stage. The GME is a Volterra integral equation [10] and shows how the state \( \rho_S \) at time \( t \) depends on its former evolution. In other words, the state \( \rho_S(t) \) is a weighed sum of the whole history of \( \rho_S \), and, consequently, the evolution is non-Markovian.

2.3 The Weak Coupling Limit

The work of Davies [6], [7] and Davies and Spohn [8] aimed to identify an appropriate generator (i.e., a differential equation) whose solution is a good approximation, in a precise sense, of the real trajectory as \( \epsilon \to 0 \). We base our analysis on the main result (Theorem 2) of [8], which describes the generator, and is valid for 'slow' controls. Several variations are available (cf., [1], [9], [19]). This result assumes that the Lamb shift \( tr_B(\hat{H}_{SB}1 \otimes \rho_B) \) in (4) vanishes, so that \( \hat{H}_S(t) = \hat{H}_S^t(t) := H_S(t) \) and \( \Phi_0(t, s) = \Phi_1(t, s) \), independently of \( \epsilon \). Assuming that the integral converges, consider the operator \( \hat{L} \) defined as

\[ \hat{L}(t)[\rho_S] := tr_B\left( \int_0^{+\infty} e^{-ad_S(t)r} \otimes e^{-ad_B r} ad_{SB} e^{ad_S(t)r} \otimes e^{ad_B r} ad_{SB} [\rho_S \otimes \rho_B]dr \right) \]

(24)

for every \( \rho_S \). We have denoted by \( ad_S(t) \) be the operator \( \rho_S \to [-iH_S(t), \rho_S] \), which can be written as

\[ \hat{ad}_S(t) := \sum_{n=1}^{n_S^2-1} -i\lambda_j(t)\Pi_j(t), \]

(25)

where \( n_S \) is the dimension of the system \( S \) (which we assume finite), and \( \Pi_j(t) \) are the projections onto the one dimensional eigenspaces of \( \hat{ad}_S(t) \).\(^\text{4}\) If we assume that the control \( u = u(t) \) (and therefore \( H_S(t) \)) is an analytic function of \( t \), it can be shown that both \( \Pi \) and \( \lambda_j \) can be taken as analytic functions. We assume this is the case in the following (cf. [13]). We also define

\[ K = K(t) = -\sum_{j=1}^{n_S^2-1} \Pi_j(t)\Pi_j'(t) \]

(26)

Theorem 1 Consider the solution \( \rho^\epsilon \) of the linear differential equation

\[ \frac{d\rho^\epsilon}{dt} = \left( \hat{ad}_S(t) + K(t) + \epsilon^2 L^2(t) \right)[\rho^\epsilon(t)], \]

(27)

\(^\text{4}\)Notice also that \( e^{ad_S(t)r} \otimes e^{ad_B r} = \Phi_0(r, 0) \) if \( H_S \) does not depend on \( t \).
where

\[ L^2(t) := \sum_{j=1}^{n_S^2-1} \Pi_j(t) \bar{L}(t) \Pi_j(t) \]  

under the above assumptions. Then \( \rho^\epsilon(t) \) tends to the solution \( \rho_S(t) \) in (23) as \( \epsilon \to 0 \) in the following sense: for a fixed \( \tau_0 \)

\[ \lim_{\epsilon \to 0} \sup_{0 \leq t \leq \epsilon^{-2}\tau_0} \| \rho_S(t) - \rho^\epsilon(t) \| = 0. \]  

(29)

For future reference, we find it convenient to rewrite the term \( \bar{L}(t)[\rho_S] \) as

\[ \bar{L}(t)[\rho_S] = -\sum_{j,k} \int_0^{+\infty} \left[ Tr \left( \rho_B V_{Bj}(r) V_{Bk} \right) \left( V_{Sj}(t, r) V_{Sk} \rho_S - V_{Sk} V_{Sj}(t, r) \rho_S \right) \right] dr \]

(30)

where we have used the definition \( ad_{SB}[\cdot] = [-iH_{SB}, \cdot] \) and \( H_{SB} = \sum_j V_{Sj} \otimes V_{Bj} \), and we have used the notation \( V_{Sj}(t, r) := e^{ad_S(t)r} V_{Sj} \), and \( V_{Bj}(r) := e^{ad_B[r]} V_{Bj} \). From the expression (30) and (28), it is possible to show that, in the special case where \( H_S \) is time invariant, the generator in (27) is of the form (1) (where the Hamiltonian \( H \) is augmented with an extra correction of the form \( \epsilon^2 H_1 \otimes 1 \)). For sake of brevity we omit the details (see, e.g., [19] and references therein). However, a special case will be shown in the next section (where we actually will allow the Hamiltonian \( H_S \) to be time varying).

### 2.4 The Singular Coupling Limit

We go back now to the generalized master equation (23) and the kernel operator \( L = L(\epsilon, t, x) \) in (21). When developed using the definition of \( \Phi_1^\epsilon \) and \( \Phi_2^\epsilon \) and neglecting terms in \( \epsilon \) of order 3 or higher, this expression contains functions which measure the correlation of observables on the bath at two different times. Such functions are called autocorrelation functions (cf., e.g., [3]). More precisely, the autocorrelation functions arise from the trace over the bath of the double commutator appearing in equation (20), obtained since the operator \( ad_{SB}[\cdot] \) is applied twice. The presence of the time propagator lead to the evaluation of the average of interaction operators taken at different times. A Markovian approximation of the dynamics can be obtained under the assumption that the bath is fast. From the mathematical point of view, this means that the bath autocorrelation functions are delta-functions in time, that is, the bath has no memory. This approach is called **Singular Coupling Limit** and this leads to a rigorous Markovian approximation of the dynamics, whose details are out of the scope of this work. The assumption of singular autocorrelation functions automatically suppresses any non-Markovian contribution in the dynamics of the relevant system, and the dissipative part of the generator of the dynamics is independent of its Hamiltonian part. Therefore, any work concerning the control of open quantum systems in terms of Lindblad-Kossakowski operators which are independent on the Hamiltonian part is ultimately addressing only those systems where the singular coupling procedure is well justified. While there are scenarios where this approach is valid, the corresponding results have a limited validity, which does not cover the entirety of Markovian dynamics. On the other side, the weak coupling procedure is more satisfactory from this perspective, since it does not require drastic assumptions on the autocorrelations of the bath, but only a weak coupling between system and environment and therefore it is more widely applied.
3 A case study

In this section we derive the generator of the Markovian dynamics in the weak coupling limit as given in (27), for a system coupled with a bosonic bath through the Jaynes-Cummings Hamiltonian. We assume that the system is given by a single qubit, or a pair of them.

3.1 One qubit

In this case the interaction Hamiltonian is given by

$$H_{SB} = \sum_j g(\omega_j) \left( \sigma^- \otimes b^\dagger(\omega_j) + \sigma^+ \otimes b(\omega_j) \right),$$

where \(\omega_k\) is the angular frequency of the \(k\)-th bosonic mode (harmonic oscillator), and \(\epsilon_g(\omega_k)\) its coupling to the qubit. As usual, \(b^\dagger(\omega_k)\) and \(b(\omega_k)\) are creation and annihilation operators for the \(k\)-th mode, satisfying

$$[b(\omega_k), b(\omega_l)] = [b^\dagger(\omega_k), b^\dagger(\omega_l)] = 0; \quad [b(\omega_k), b^\dagger(\omega_l)] = \delta_{kl},$$

and \(\sigma^+, \sigma^-\) the qubit rising and lowering operators, defined by

$$\sigma_{\pm} = \frac{1}{\sqrt{2}} (\sigma_x \pm i \sigma_y).$$

The free Hamiltonians are given by

$$H_S(t) = \frac{1}{2} \omega_0(t) \hat{\sigma}_z(t), \quad H_B = \sum_k \omega_k \left( b^\dagger(\omega_k) b(\omega_k) + \frac{1}{2} \right),$$

where \(\hat{\sigma}_z(t) = h_x(t) \sigma_x + h_y(t) \sigma_y + h_z(t) \sigma_z\) with \(h_x^2(t) + h_y^2(t) + h_z^2(t) = 1\). Our goal here is to investigate how the dissipative part in (27) depends on the Hamiltonian \(H_S(t)\) and therefore the control in a coherent control scheme. We will see that the dependence is very significant. For further reference we find it convenient to write

$$h_x(t) = \sin \theta(t) \sin \phi(t), \quad h_y(t) = \sin \theta(t) \cos \phi(t), \quad h_z(t) = \cos \theta(t),$$

and define \(\hat{\sigma}_x(t), \hat{\sigma}_y(t)\) such that the standard \(su(2)\) algebra is satisfied at any time: \([\hat{\sigma}_k(t), \hat{\sigma}_l(t)] = 2i \hat{\sigma}_{kl}(t),\) where \((k, l, m)\) is a cyclic permutation of \((x, y, z)\). The standard Pauli matrices are related to these redefined (control dependent) Pauli matrices through an \(SO(3)\) transformation \(U(t)\) such that

$$\sigma_k = \sum_l U_{kl}(t) \hat{\sigma}_l(t), \quad \hat{\sigma}_k(t) = \sum_l U_{lk}(t) \sigma_l,$$

where \(k, l \in \{x, y, z\}\). According to (35), we can write

$$U(t) = \begin{pmatrix} -\sin \phi(t) \cos \theta(t) & -\cos \phi(t) \cos \theta(t) & \sin \theta(t) \\ \cos \phi(t) & -\sin \phi(t) & 0 \\ \sin \phi(t) \sin \theta(t) & \cos \phi(t) \sin \theta(t) & \cos \theta(t) \end{pmatrix}.$$ (37)

Moreover, defining

$$\hat{\sigma}_\pm = \frac{1}{\sqrt{2}} (\sigma_x \pm i \sigma_y)$$

(38)

\(^{5}\)a similar analysis can be performed with an arbitrary interaction term, and the Jaynes - Cummings Hamiltonian is a prototypical example.
in accordance with (33), we can also write
\[ \sigma_k = \sum_l V_{kl}(t)\hat{\sigma}_l(t), \quad \hat{\sigma}_k(t) = \sum_l V_{lk}^\ast(t)\sigma_l, \]
where \( k, l \in \{+, -, z\} \), and
\[
\mathcal{V}(t) = \frac{i}{2} \begin{pmatrix}
\cos \theta(t)e^{-i\phi(t)} + e^{i\phi(t)} & -\cos \theta(t)e^{i\phi(t)} + e^{-i\phi(t)} & -\sqrt{2}t \sin \theta(t) \\
\cos \theta(t)e^{-i\phi(t)} - e^{i\phi(t)} & -\cos \theta(t)e^{i\phi(t)} - e^{-i\phi(t)} & -\sqrt{2}t \sin \theta(t) \\
-\sqrt{2} \sin \theta(t)e^{i\phi(t)} & \sqrt{2} \sin \theta(t)e^{-i\phi(t)} & -2i \cos \theta(t)
\end{pmatrix}.
\] (40)

Now, by following the Davies’ prescriptions summarized in the previous section, we can evaluate the different contributions appearing in the dissipative part of the generator of the dynamics in the weak coupling limit in (27). We assume that the equilibrium state of the bath is given by the vacuum state \( \rho_B = |0\rangle \langle 0| \), that is, no oscillator is initially excited. Since in our case \( V_{Bj} = b(\omega_j) \) or \( b^\dagger(\omega_j) \), it turns out that \( Tr(\rho_B V_{Bj}) = 0 \) for all \( j \), and there is no Lamb-shift. Define
\[
b(\omega_j, t) := e^{-ad_{Bt}}b(\omega_j) = e^{iH_{Bt}}b(\omega_j)e^{-iH_{Bt}} \quad \text{(cf. (24)).}
\]
From (32) and (34), it follows that
\[
[H_B, b(\omega_j)] = -\omega_j b(\omega_j), \quad [H_B, b^\dagger(\omega_j)] = \omega_j b^\dagger(\omega_j).
\] (41)

This gives
\[
b(\omega_j, t) = e^{-i\omega_j t}b(\omega_j), \quad b^\dagger(\omega_j, t) = e^{i\omega_j t}b^\dagger(\omega_j),
\] (42)
and these are the expressions for \( V_{Bj}(t) \) as used in (30). With a view to the application in (30) we notice the equations
\[
Tr(\rho_B b(\omega_k)b(\omega_j, t)) = Tr(\rho_B b(\omega_k, t)b(\omega_j)) = 0,
\]
\[
Tr(\rho_B b^\dagger(\omega_k)b^\dagger(\omega_j, t)) = Tr(\rho_B b^\dagger(\omega_k, t)b^\dagger(\omega_j)) = 0,
\]
\[
Tr(\rho_B b(\omega_k)b(\omega_j, t)) = Tr(\rho_B b(\omega_k, t)b(\omega_j)) = 0,
\]
\[
Tr(\rho_B b(\omega_k)b^\dagger(\omega_j, t)) = e^{-i\omega_j t}\delta_{jk}, \quad Tr(\rho_B b(\omega_k)b^\dagger(\omega_j, t)) = e^{i\omega_j t}\delta_{jk}.
\]
Using these in (30), we calculate
\[
\hat{L}(t)[\rho_S] = -\sum_j g_j^2(\omega_j) \int_0^{+\infty} \left( \left( \sigma_+(t, r)\sigma_- - \sigma_-\sigma_+(t, r) \right)e^{-i\omega_j r} + \rho_S e^{i\omega_j r} \right) dr,
\]
where
\[
\sigma_+(t, r) \equiv e^{iH_S(t)r}\sigma_+ e^{-iH_S(t)r} = \mathcal{V}_{++}(t)e^{i\omega_j r}\sigma_+ + \mathcal{V}_{+-}(t)e^{-i\omega_j r}\sigma_- + \mathcal{V}_{+z}(t)\sigma_z(t)
\] (44)
and
\[
\sigma_-(t, r) \equiv e^{iH_S(t)r}\sigma_- e^{-iH_S(t)r} = \mathcal{V}_{+\dot{+}}(t)e^{i\omega_j r}\sigma_+ + \mathcal{V}_{+\dot{-}}(t)e^{-i\omega_j r}\sigma_- + \mathcal{V}_{+z}(t)\sigma_z(t).
\] (45)
Of course, \( \sigma_-(t, r) = \sigma_+(t, r) \), since \( \mathcal{V}_{+\dot{+}}(t) = \mathcal{V}_{+\dot{+}}(t) \), \( \mathcal{V}_{+\dot{-}}(t) = \mathcal{V}_{+\dot{-}}(t) \), and \( \mathcal{V}_{+z}(t) = \mathcal{V}_{+z}(t) \) are real numbers. In (44) and (45) we have found it convenient to express \( \sigma_+ \) and \( \sigma_- \) in the (time-dependent) basis of the eigenstates of the
The next step is to average this operator following the Davies’ prescription (28) and obtain $L \equiv \sum_j \ldots \rightarrow \int_0^{+\infty} \ldots \rho(\omega)d\omega$, where $\rho(\omega)$ is the density of modes with angular frequency $\omega$. For the present purposes we don’t need to specify the form of this density, but we simply define
\[ \int_0^{+\infty} g^2(\omega)e^{\pm i\omega r} \rho(\omega)d\omega \equiv c(r) \pm is(r). \] (47)

Moreover, we define
\[ \int_0^{+\infty} c(r)e^{\pm i\omega_0(t)r} dr \equiv \gamma_c(\omega_0, t) \pm i\gamma_s(\omega_0, t), \quad \int_0^{+\infty} c(r)dr \equiv \gamma_0, \] (48)
\[ \int_0^{+\infty} s(r)e^{\pm i\omega_0(t)r} dr \equiv \delta_c(\omega_0, t) \pm i\delta_s(\omega_0, t), \quad \int_0^{+\infty} s(r)dr \equiv \delta_0, \] (49)
and assume that all these integrals converge. Therefore, by using (44), (45), and (48) we can write for $\tilde{L}$ in (30)
\[ \tilde{L}(t)[\rho_S] = -\left[ V_{++}(t) \left( \gamma_c(\omega_0, t) + i\gamma_s(\omega_0, t) - i\delta_c(\omega_0, t) + \delta_s(\omega_0, t) \right) \left( \hat{\sigma}_+(t)\sigma_-\rho_S - \sigma_-\rho_S\hat{\sigma}_+(t) \right) \right. \]
\[ + V_{+-}(t) \left( \gamma_c(\omega_0, t) - i\gamma_s(\omega_0, t) - i\delta_c(\omega_0, t) - \delta_s(\omega_0, t) \right) \left( \hat{\sigma}_-(t)\sigma_-\rho_S - \sigma_-\rho_S\hat{\sigma}_-(t) \right) \]
\[ + V_{++}(t)(\gamma_0 - i\delta_0) \left( \hat{\sigma}_z(t)\sigma_-\rho_S - \sigma_-\rho_S\hat{\sigma}_z(t) \right) \left( \hat{\sigma}_+(t)\rho_S - \rho_S\hat{\sigma}_+(t) \right) \]
\[ + V_{+-}(t) \left( \gamma_c(\omega_0, t) + i\gamma_s(\omega_0, t) + i\delta_c(\omega_0, t) - \delta_s(\omega_0, t) \right) \left( \rho_S\sigma_-\rho_S + \hat{\sigma}_+\rho_S\sigma_+ \right) \]
\[ + V_{-+}(t) \left( \gamma_c(\omega_0, t) - i\gamma_s(\omega_0, t) + i\delta_c(\omega_0, t) + \delta_s(\omega_0, t) \right) \left( \rho_S\sigma_-\rho_S - \hat{\sigma}_-(t)\rho_S\sigma_+ \right) \]
\[ + V_{-+}(t)(\gamma_0 + i\delta_0) \left( \rho_S\sigma_-\rho_S - \hat{\sigma}_z(t)\rho_S\sigma_+ \right). \] (50)

The next step is to average this operator following the Davies’ prescription (28) and obtain $L^\natural(t)$. In this model, the eigenprojections $\Pi_n(t)$ are explicitly given by
\[ \Pi_j(t)[\rho_S] = \frac{1}{2} Tr \left( \rho_S\hat{\sigma}_j^\dagger(t) \right) \hat{\sigma}_j(t), \] (51)
with $j \in \{+, -, z\}$, and the respective eigenvalues are $\omega_0(t)$, $-\omega_0(t)$, 0. We notice that
\[ \sum_n \Pi_n(t) L(t) \Pi_n(t) = \frac{1}{2T} \int_{-T}^T \int_{-T}^T e^{-iH_S(t)r} L(t) e^{iH_S(t)r} dr, \] (52)
and, moreover,
\[ \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^T e^{\pm i\omega_0(t)r} du = \frac{1}{2T} \int_{-T}^T e^{\pm 2i\omega_0(t)r} dr = 0. \] (53)
Therefore, if we expand $\sigma_+$ and $\sigma_-$ in (49) as in (44) and (45), the only non-vanishing averages contain both $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$, or $\hat{\sigma}_z(t)$ twice, and we obtain

$$L^2(t)[\rho_S] = -\left[\mathcal{V}_{++}(t)\mathcal{V}_{--}(t)\left(\gamma_c(\omega_0, t) + \delta_s(\omega_0, t)\right)\left(2\hat{\sigma}_-(t)\rho_S\hat{\sigma}_+(t) - \{\rho_S, \hat{\sigma}_+(t)\hat{\sigma}_-(t)\}\right) + \mathcal{V}_{+-}(t)\mathcal{V}_{-+}(t)\left(\gamma_c(\omega_0, t) - \delta_s(\omega_0, t)\right)\left(2\hat{\sigma}_+(t)\rho_S\hat{\sigma}_-(t) - \{\rho_S, \hat{\sigma}_-(t)\hat{\sigma}_+(t)\}\right) + \mathcal{V}_{+z}(t)\mathcal{V}_{-z}(t)\gamma_0(2\hat{\sigma}_z(t)\rho_S\hat{\sigma}_z(t) - 2\rho_S) + 2i\mathcal{V}_{++}(t)\mathcal{V}_{--}(t)\left(\gamma_s(\omega_0, t) - \delta_c(\omega_0, t)\right)[\hat{\sigma}_+(t)\hat{\sigma}_-(t), \rho_S] - \mathcal{V}_{+-}(t)\mathcal{V}_{-+}(t)\left(\gamma_s(\omega_0, t) + \delta_c(\omega_0, t)\right)[\hat{\sigma}_-(t)\hat{\sigma}_+(t), \rho_S]\right].$$

(53)

In this expression, the first three lines are terms of the Lindblad-Kossakowski form, while the last two contributions are dissipative corrections to the coherent part of the dynamics. By using (40), we can also write

$$\begin{align*}
\mathcal{V}_{++}(t)&\mathcal{V}_{--}(t) = \frac{1}{2}\left(\cos^2 \frac{\theta(t)}{2} + \cos \theta(t) \cos \phi(t)\right), \\
\mathcal{V}_{+-}(t)&\mathcal{V}_{-+}(t) = \frac{1}{2}\left(\cos^2 \frac{\theta(t)}{2} - \cos \theta(t) \cos \phi(t)\right), \\
\mathcal{V}_{+z}(t)&\mathcal{V}_{-z}(t) = \frac{1}{2} \sin^2 \theta(t),
\end{align*}$$

(54)

and these relations can further simplify (53). For instance, the Hamiltonian part can be more compactly rewritten as

$$2i\left(\gamma_s(\omega_0, t) \cos \theta(t) \cos \phi(t) - \delta_c(\omega_0, t) \cos \theta(t) \cos \phi(t)\right)[\hat{\sigma}_z(t), \rho_S].$$

(55)

The next step is the evaluation of the term $K(t)$ (26) in the Markovian generator. In the present case, it is given by the sum of three contributions:

$$-K(t)[\rho_S] = \Pi_0(t)\Pi'_0(t)[\rho_S] + \Pi_+(t)\Pi'_+(t)[\rho_S] + \Pi_-(t)\Pi'_-(t)[\rho_S],$$

(56)

which we now compute separately. From (50) we can write

$$\Pi_0(t)[\rho_S] = \frac{1}{2}\left(\rho_S\hat{\sigma}_z(t)\right)\hat{\sigma}_z(t) + \frac{1}{2}\left(\rho_S\hat{\sigma}_z(t)\right)\hat{\sigma}_z(t),$$

(57)

and then

$$\Pi_0(t)\Pi'_0(t)[\rho_S] = \frac{1}{2}\left(Tr\left(\rho_S\hat{\sigma}_z(t)\right) + \frac{1}{2}Tr\left(\rho_S\hat{\sigma}_z(t)\right)Tr\left(\hat{\sigma}_z(t)\hat{\sigma}_z(t)\right)\right)\hat{\sigma}_z(t) = \frac{1}{2}Tr\left(\rho_S\hat{\sigma}_z(t)\right)\hat{\sigma}_z(t),$$

(58)

since

$$Tr\left(\hat{\sigma}_z(t)\hat{\sigma}_z(t)\right) = 0$$

(59)

as a consequence of $\hat{\sigma}_z^2(t) = 1$. On the other hand, it is possible to prove that

$$\Pi_+(t)\Pi'_+(t)[\rho_S] = \frac{1}{2}Tr\left(\rho_S\hat{\sigma}_z(t)\right)\hat{\sigma}_z(t) + \frac{1}{4}Tr\left(\rho_S\hat{\sigma}_z(t)\right)Tr\left(\hat{\sigma}_z(t)\hat{\sigma}_z(t)\right)\hat{\sigma}_z(t) - \frac{1}{4}Tr\left(\rho_S\hat{\sigma}_z(t)\right)\hat{\sigma}_z(t)$$

(60)
since
\[ Tr\left( \hat{\sigma}_-(t)\hat{\sigma}_+^\dagger(t) \right) = Tr\hat{\sigma}_z^\dagger(t) = 0, \] (61)
and similarly
\[ \Pi_- (t)\Pi'_-(t) [\rho_S] = \frac{1}{2} Tr\left( \rho_S\hat{\sigma}_z^\dagger(t) \right) \hat{\sigma}_-(t). \] (62)
It is possible to prove that (58), (60) and (62) are time-dependent Lindblad-Kossakowski terms. For instance, from
\[ \Pi_+(t) [\rho_S] = \frac{1}{4} \left( \hat{\sigma}_+(t)\hat{\sigma}_-(t)\rho_S\hat{\sigma}_-(t)\hat{\sigma}_+(t) \right), \] (63)
and
\[ \left( \hat{\sigma}_+(t)\hat{\sigma}_-(t) \right)' = \hat{\sigma}_z^\dagger(t), \quad \left( \hat{\sigma}_-(t)\hat{\sigma}_+(t) \right)' = -\hat{\sigma}_z^\dagger(t), \] (64)
we have that
\[ \Pi'_+(t) [\rho_S] = \frac{1}{4} \left( \hat{\sigma}_z^\dagger(t)\rho_S\hat{\sigma}_-(t)\hat{\sigma}_+(t) - \hat{\sigma}_+(t)\hat{\sigma}_-(t)\rho_S\hat{\sigma}_z^\dagger(t) \right), \] (65)
leading to
\[ \Pi_+(t)\Pi'_+(t) [\rho_S] = \frac{1}{4} \left( \hat{\sigma}_+(t)\hat{\sigma}_-(t)\hat{\sigma}_z^\dagger(t)\rho_S\hat{\sigma}_+(t)\hat{\sigma}_-(t) - \hat{\sigma}_+(t)\hat{\sigma}_-(t)\rho_S\hat{\sigma}_z^\dagger(t)\hat{\sigma}_-(t)\hat{\sigma}_+(t) \right), \] (66)
which is clearly a time-dependent Lindblad-Kossakowski contribution. In fact, the anti-commutator part appearing in (1) in this case vanishes, since \( \hat{\sigma}_z^\dagger(t) = 0 \). By an analogous computation we can prove that
\[ \Pi_-(t)\Pi'_-(t) [\rho_S] = \frac{1}{4} \left( \hat{\sigma}_-(t)\hat{\sigma}_+(t)\hat{\sigma}_z^\dagger(t)\rho_S\hat{\sigma}_+(t)\hat{\sigma}_-(t) - \hat{\sigma}_-(t)\hat{\sigma}_+(t)\rho_S\hat{\sigma}_z^\dagger(t)\hat{\sigma}_-(t)\hat{\sigma}_+(t) \right), \] (67)
and the same conclusion holds, since \( \hat{\sigma}_z^\dagger(t) = 0 \). The explicit computation for \( \Pi_0(t)\Pi'_0(t) [\rho_S] \) is more lengthy and is omitted.

The above computations show how the generator in (27) depends on the coherent Hamiltonian and therefore the control. All Kossakowski-Lindblad operators are ‘rotated’ according to the operator \( \hat{\sigma}_z(t) \) in (34), which depends on the control.

### 3.2 Two qubits

In this case the interaction Hamiltonian is given by
\[ H_{SB} = \sum_j g(\omega_j) \left( (\sigma_- \otimes I + I \otimes \sigma_-) \otimes b^\dagger(\omega_j) + (\sigma_+ \otimes I + I \otimes \sigma_+) \otimes b(\omega_j) \right), \] (68)
under the assumption that the two qubits are subject to the same coupling to a common environment. The free Hamiltonian of the system is given by
\[ H_S(t) = \frac{1}{2} \omega_0(t)(\hat{\sigma}_z(t) \otimes I + I \otimes \hat{\sigma}_z(t)), \] (69)
where we assume that the two quits have the same natural frequency \( \omega_0 \), and, as before, \( \hat{\sigma}_z(t) = h_z(t)\sigma_x + h_y(t)\sigma_y + h_z(t)\sigma_z \) with \( h_x^2(t) + h_y^2(t) + h_z^2(t) = 1 \). A more general scenario can be considered by choosing qubit-dependent couplings, frequencies, and control strengths in (68) and (69), leading to a more complicated but conceptually similar derivation.
The computation of the previous section can be adapted to this case; therefore, with the same notations, we can calculate (30) as

\[
\tilde{L}_2(t)[\rho_S] = - \sum_j g_j^2(\omega_j) \int_0^{+\infty} \left[ \left( \sigma_+(t,r) \sigma_- \otimes I_{\rho_S} - \sigma_- \otimes I_{\rho_S} \sigma_+(t,r) \otimes I \right) e^{-i\omega_j r} + \\
(\rho_S \sigma_+ \sigma_-(t,r) \otimes I - \sigma_-(t,r) \otimes I_{\rho_S} \sigma_+ \otimes I) e^{+i\omega_j r} + \\
(I \otimes \sigma_+(t,r) \sigma_- - I \otimes \sigma_- \sigma_+ I \otimes \sigma_+ (t,r)) e^{-i\omega_j r} + \\
(\rho_S I \otimes \sigma_+ \sigma_-(t,r) - I \otimes \sigma_- \sigma_+ I \otimes \sigma_+ (t,r)) e^{+i\omega_j r} + \\
(\sigma_+(t,r) \otimes \sigma_- \rho_S - I \otimes \sigma_- \rho_S \sigma_+(t,r) \otimes I) e^{-i\omega_j r} + \\
(\rho_S \sigma_+ \otimes \sigma_- (t,r) - I \otimes \sigma_- (t,r) \rho_S \sigma_+ \otimes I) e^{+i\omega_j r} + \\
(\sigma_- \otimes \sigma_+ (t,r) \rho_S - \sigma_- \otimes I_{\rho_S} \sigma_+ (t,r) e^{-i\omega_j r} + \\
(\rho_S \sigma_-(t,r) \otimes \sigma_+ - \sigma_-(t,r) \otimes I_{\rho_S} \sigma_+ e^{+i\omega_j r} \right] dr.
\]

(70)

The subscript 2 in \(\tilde{L}_2(t)\) refers to the two-qubits case we are considering here. We will use the same notation for the other operators appearing in the generator of the dynamics. Notice that the first and second rows in (70) are the dissipative term associated to the first qubit alone, and similarly the third and fourth rows for the second qubit. The remaining four rows are the dissipative coupling between the two qubits, and this is the contribution we are going to focus on, since the computation for the other terms has been detailed in the former section. We write

\[
\tilde{L}_2(t) = \tilde{L}(t) \otimes I + I \otimes \tilde{L}(t) + \tilde{L}_{\text{coup}}(t)
\]

and we compute

\[
\tilde{L}_{\text{coup}}(t)[\rho_S] = - \left[ V_{\text{coup}}(t) \left( \gamma_c(\omega_0, t) + i\gamma_s(\omega_0, t) - i\delta_c(\omega_0, t) + \delta_s(\omega_0, t) \right) \cdot \\
(\hat{\sigma}_+(t) \otimes \sigma_- \rho_S - I \otimes \sigma_- \rho_S \hat{\sigma}_+(t) \otimes I + \sigma_- \otimes \hat{\sigma}_+(t) \rho_S - \sigma_- \otimes I_{\rho_S} I \otimes \hat{\sigma}_+(t)) \\
+ V_{-\text{coup}}(t) \left( \gamma_c(\omega_0, t) - i\gamma_s(\omega_0, t) - i\delta_c(\omega_0, t) - \delta_s(\omega_0, t) \right) \cdot \\
(\hat{\sigma}_-(t) \otimes \sigma_- \rho_S - I \otimes \sigma_- \rho_S \hat{\sigma}_-(t) \otimes I + \sigma_- \otimes \hat{\sigma}_-(t) \rho_S - \sigma_- \otimes I_{\rho_S} I \otimes \hat{\sigma}_-(t) \\
+ V_{\text{coup}}(t) \left( \gamma_0 - i\delta_0 \right) \cdot \\
(\hat{\sigma}_z(t) \otimes \sigma_- \rho_S - I \otimes \sigma_- \rho_S \hat{\sigma}_z(t) \otimes I + \sigma_- \otimes \hat{\sigma}_z(t) \rho_S - \sigma_- \otimes I_{\rho_S} I \otimes \hat{\sigma}_z(t) \\
+ V_{-\text{coup}}(t) \left( \gamma_c(\omega_0, t) + i\gamma_s(\omega_0, t) + i\delta_c(\omega_0, t) - \delta_s(\omega_0, t) \right) \cdot \\
(\rho_S \sigma_+ \otimes \hat{\sigma}_z(t) - I \otimes \hat{\sigma}_z(t) \rho_S \sigma_+ \otimes I + \rho_S \hat{\sigma}_z(t) \otimes \sigma_+ - \hat{\sigma}_z(t) \otimes I_{\rho_S} I \otimes \sigma_+ \\
+ V_{-\text{coup}}(t) \left( \gamma_c(\omega_0, t) - i\gamma_s(\omega_0, t) + i\delta_c(\omega_0, t) + \delta_s(\omega_0, t) \right) \cdot \\
(\rho_S \sigma_+ \otimes \hat{\sigma}_z(t) - I \otimes \hat{\sigma}_z(t) \rho_S \sigma_+ \otimes I + \rho_S \hat{\sigma}_z(t) \otimes \sigma_+ - \hat{\sigma}_z(t) \otimes I_{\rho_S} I \otimes \sigma_+ \\
+ V_{\text{coup}}(t) \left( \gamma_0 + i\delta_0 \right) \cdot \\
(\rho_S \sigma_+ \otimes \hat{\sigma}_z(t) - I \otimes \hat{\sigma}_z(t) \rho_S \sigma_+ \otimes I + \rho_S \hat{\sigma}_z(t) \otimes \sigma_+ - \hat{\sigma}_z(t) \otimes I_{\rho_S} I \otimes \sigma_+ \right].
\]

(72)
Now, we have to average \( \tilde{L}_2(t) \) following the Davies’ prescription (28) and obtain \( L_2^Z(t) \). The eigenprojections \( \Pi_n(t) \) are given by

\[
\Pi_{j,k}(t)[\rho_S] = \frac{1}{4} Tr \left( \rho_S \hat{\sigma}^\dagger_j(t) \otimes \hat{\sigma}^\dagger_k(t) \right) \hat{\sigma}_j(t) \otimes \hat{\sigma}_k(t),
\]

(73)

with \( j, k \in \{+, -, z\} \), and the eigenvalues are \( \omega_0^2(t), -\omega_0^2(t), 0 \). Following the derivation presented in the previous section (based on ergodic averages), we obtain

\[
L_2^Z(t) = L^2(t) \otimes I + I \otimes L^2(t) + L^Z_{\text{coup}}(t),
\]

(74)

with

\[
L^Z_{\text{coup}}(t)[\rho_S] = - \left[ V_{++}(t)V_{--}(t) \left( \gamma_c(\omega_0, t) + \delta_s(\omega_0, t) \right) \cdot \left( 2I \otimes \hat{\sigma}_+(t) \rho_S \hat{\sigma}_+(t) \otimes I - \{ \rho_S, \hat{\sigma}_+(t) \otimes \hat{\sigma}_-(t) \} + 2\hat{\sigma}_-(t) \otimes I \rho_S I \otimes \hat{\sigma}_+(t) - \{ \rho_S, \hat{\sigma}_-(t) \otimes \hat{\sigma}_+(t) \} \right) \right] + \left[ V_{++}(t)V_{--}(t) \left( \gamma_c(\omega_0, t) - \delta_s(\omega_0, t) \right) \cdot \left( 2I \otimes \hat{\sigma}_-(t) \rho_S \hat{\sigma}_-(t) \otimes I - \{ \rho_S, \hat{\sigma}_-(t) \otimes \hat{\sigma}_+(t) \} + 2\hat{\sigma}_+(t) \otimes I \rho_S I \otimes \hat{\sigma}_-(t) - \{ \rho_S, \hat{\sigma}_+(t) \otimes \hat{\sigma}_-(t) \} \right) \right] + i \left[ \gamma_0 \left( 2I \otimes z(t) \rho_S \hat{\sigma}_z(t) \otimes I + 2\hat{\sigma}_z(t) \otimes I \rho_S I \otimes \hat{\sigma}_z(t) - 2\{ \rho_S, \hat{\sigma}_z(t) \otimes \hat{\sigma}_z(t) \} \right) \right] + 2\gamma_0 \left[ \left( \gamma_0 \left( 2I \otimes z(t) \rho_S \hat{\sigma}_z(t) \otimes I + 2\hat{\sigma}_z(t) \otimes I \rho_S I \otimes \hat{\sigma}_z(t) - 2\{ \rho_S, \hat{\sigma}_z(t) \otimes \hat{\sigma}_z(t) \} \right) \right] + 2\gamma_0 \left[ \left( \gamma_0 \left( 2I \otimes z(t) \rho_S \hat{\sigma}_z(t) \otimes I + 2\hat{\sigma}_z(t) \otimes I \rho_S I \otimes \hat{\sigma}_z(t) - 2\{ \rho_S, \hat{\sigma}_z(t) \otimes \hat{\sigma}_z(t) \} \right) \right] + 2\gamma_0 \left[ \left( \gamma_0 \left( 2I \otimes z(t) \rho_S \hat{\sigma}_z(t) \otimes I + 2\hat{\sigma}_z(t) \otimes I \rho_S I \otimes \hat{\sigma}_z(t) - 2\{ \rho_S, \hat{\sigma}_z(t) \otimes \hat{\sigma}_z(t) \} \right) \right] + 2\gamma_0 \left[ \left( \gamma_0 \left( 2I \otimes z(t) \rho_S \hat{\sigma}_z(t) \otimes I + 2\hat{\sigma}_z(t) \otimes I \rho_S I \otimes \hat{\sigma}_z(t) - 2\{ \rho_S, \hat{\sigma}_z(t) \otimes \hat{\sigma}_z(t) \} \right) \right]
\]

(75)

containing time-dependent Lindblad and Hamiltonian contributions. To conclude, we have to compute the operator \( K_2(t) \).

4 Decoherence control

4.1 Decoherence Splitting Manifold (DSM)

A **Decoherence Splitting Manifold (DSM)** is a subset of the set of density operators \( \{ \rho_S = \rho_S^\dagger, \rho_S \geq 0, \text{Tr}(\rho_S) = 1 \} \) in which some (possibly multiple) eigenvalues \( \lambda_{k \in K} \) of \( \rho \) are preserved, along with their multiplicities \( m_{k \in K} \), while the other eigenvalues \( \lambda_{k \notin K} \) are not specified. The only requirement is that their multiplicities \( m_{k \in K} \) remain constant. Such a subset will be denoted as \( D_{\Lambda_K, m_K} \), where \( \Lambda_K = \text{block diag} \{ \lambda_k I_{m_k} : k \in K \} \) denotes the diagonal matrix of preserved eigenvalues, while \( m_K \) denotes the specifications on the multiplicities of the remaining eigenvalues. It can be shown that \( D_{\Lambda_K, m_K} \) is a real-analytic manifold [12]. This manifold is referred to as *splitting*, because any density \( \rho \in D_{\Lambda_K, m_K} \) has its eigenvalues *splitting* between, on the one hand, those eigenvalues \( \Lambda_K \) that are preserved and, on the other hand, those eigenvalues \( \Lambda_K \) allowed to drift. Every density matrix \( \rho_S \) with eigenvalue specifications belongs to one DSM. While in unitary dynamics, all the eigenvalues of the density matrix \( \rho_S \) are preserved, in dissipative open system dynamics these eigenvalues drift. However, if the dynamics is such as to keep the system on a DSM, some of the eigenvalues are preserved, so that some information on the initial density matrix is preserved. The **DSM control problem** consists of finding a coherent control to keep \( \rho_S \) on a DSM. This is a less stringent requirement than the creation of *Decoherence Free Subspaces* (cf [16], [14]) because instead of relying solely on common eigenvectors to the jump operators it offers more flexibility as to where the information is protected [12, Sec. 1, C-F].
Assume $\rho_S(t) \in D_{K,m_K}$ for every $t$. Then we can project it to a sub-density matrix which has unitary evolution. Let

$$\rho_S(t) = V(t) \begin{pmatrix} \Lambda_K & 0 \\ 0 & \Lambda_{\bar{K}}(t) \end{pmatrix} V^\dagger(t)$$

(76)

where the eigenvalues in $\Lambda_K$ are preserved, while the remaining eigenvalues in $\Lambda_{\bar{K}}(t)$ could be evolving, with the only restriction that there are no eigenvalue crossings. If $V_K(t)$ is the matrix of eigenvectors for the eigenvalues of $\Lambda_K$. We define the sub-density

$$\varrho_{DPS}(t) := \frac{1}{\text{Trace}(\Lambda_K)} V_K(t) \Lambda_K V_K^\dagger(t)$$

(77)

where $\Lambda_{\bar{K}}(t)$ could be evolving, with the only restriction that there are no eigenvalue crossings. If $V_K(t)$ is the matrix of eigenvectors for the eigenvalues of $\Lambda_K$. We define the sub-density

$$\varrho_{DPS}(t) := \frac{1}{\text{Trace}(P_{DPS}(t) \varrho(t) P_{DPS}(t))} P_{DPS}(t) \varrho(t) P_{DPS}(t)$$

(77)

where

$$P_{DPS}(t) = V_K(t) V_K^\dagger(t).$$

If we differentiate (77), we get

$$\dot{\varrho}_{DPS}(t) = -i[H_{\text{EFF}}(t), \varrho_{DPS}(t)],$$

(78)

where $H_{\text{EFF}}$ is some “effective” Hamiltonian,

$$H_{\text{EFF}}(t) := -iV_K(t) V_K^\dagger(t) = iV_K(t) V_K^\dagger(t)$$

(79)

which is easily seen to be Hermitian.

### 4.2 State space master equation—singular coupling limit

Consider now the evolution of $\rho_S$ given in (1). It is convenient to express the matrix $\rho_S$ in terms in the vector of coherences representation, i.e., using the vector $\vec{x} := [x_1, \ldots, x_{n_S}]^T$, where $\rho_S = \frac{1}{n_S} \mathbb{1} + \sum_{j=1}^{n_S} x_j \Sigma_j$ for some orthonormal basis $\{\Sigma_j\}, j = 1, \ldots, n_S$ in the space of $n_S \times n_S$ Hermitian matrices. With this notation, consider the quantum master equation (1) where, in the coherent part, we assume $m$ control variables $u_1, \ldots, u_m$, and we assume for simplicity, justified by the singular coupling limit, that the control appears linearly. The quantum master equation can be written in the form

$$\dot{\vec{x}} = A \vec{x} + \sum_{j=1}^{m} B_j \vec{x} u_j + G \vec{x},$$

(80)

for appropriate $(n_S^2 - 1) \times (n_S^2 - 1)$ matrices $A$, $B_j$, $j = 1, \ldots, m$, and $G$.

The condition that $\rho_S$ belongs to the DSM can be expressed in terms of appropriate functions of the vector $\vec{x}$ equal to zero. These functions represent the characteristic polynomial of $\rho_S$, $\det(\rho_S - \lambda \mathbb{1})$ and (possibly) some of its derivatives with respect to $\lambda$, equal to zero. The constancy of the multiplicities on the complementary eigenvalues is somewhat more complicated. The lack of numerical specifications on those eigenvalues require then to be eliminated using the Tarski-Seidenberg quantifier elimination, resulting in additional polynomial constraints. Taking the Jacobian matrix of these constraints $J_{\vec{x}}$, the condition for $\rho_S(t)$ to belong to a DSM at all $t \geq 0$, is

$$J_{\vec{x}}(A \vec{x}(t) + \sum_{j=1}^{m} B_j \vec{x}(t) u_j + G \vec{x}(t)) = 0.$$ 

(81)
However it is a general fact that on the DSM
\[ J_x A \vec{x} = 0, \quad \text{and} \quad J_x B_j \vec{x} = 0, \quad \forall j = 1, \ldots, m. \] (82)

This is a simple consequence of the fact that the dynamics without the dissipative part \( G \vec{x} \) is unitary and therefore preserves all eigenvalues including the ones in the set \( \Lambda_K \) defining the DSM \( D_{\Lambda_K,m_K} \). Therefore the condition (81) reduces to
\[ J_x (G \vec{x}(t)) = 0. \] (83)

This tells us that it is not possible to simply use the control to cancel all the terms in (81). The control however affects \( x(t) \) in (83) and one could in principle try to use it to induce a trajectory satisfying (83) and therefore preserving the desired eigenvalues.

The above considerations assume that the master equation takes the form (80) with \( G \) constant, independent of the control \( u \). As we have discussed in the previous sections, this assumption is only justified in the singular coupling limit. A feature of the weak coupling limit is that \( G \) in (80) depends explicitly on the controls (in addition to the possibility of terms depending on the derivatives of the controls.) In the weak coupling limit, disregarding the possible \( \dot{u}_j \) terms, the situation might in fact be more favorable as it might be possible to solve the equation \( J_x (A \vec{x}(t) + \sum_{j=1}^{m_0} B_j \vec{x}(t) u_j + G(u) \vec{x}(t)) = 0 \) for \( u \) as a function of \( \vec{x} \). We examine this in more details in the section that follows.

### 4.3 State space master equation—weak coupling limit

Consider the weak coupling limit master equation (27) of Theorem 1. The term \( \text{ad}_S(t) \rho(t) \) maps to the state space as \( A \vec{x}(t) + \sum_j B_j \vec{x}(t) u_j(t) \). The term \( e^{2L^2} \rho(t) \) maps to some polynomial terms in the controls \( u_j(t) \). Indeed, in \( \tilde{L}(t) \), note that by Cayley-Hamilton theorem, \( \exp(\text{ad}_S(t)) \) is polynomial in \( \text{ad}_S(t) \), hence in \( u_j(t) \). The term \( K(t) \rho(t) \), on the other hand, is totally nonclassical, as it involves \( \Pi_j(t) \), where the \( \Pi_j \)'s are the eigenprojections of \( \text{ad}_S(t) \). Since \( \text{ad}_S(t) \) contains the controls \( u_j(t) \), \( K(t) \) could contain some \( \dot{u}_j(t) \). We now examine this issue in more details.

Take the following system with no drift but with control Hamiltonian:
\[ H_S = u_{1z}(\sigma_z \otimes I) + u_{2z}(I \otimes \sigma_z), \]
that is, both spins are subject to a \( z \)-magnetic field control. Not surprisingly (since \( \sigma_z \otimes I \) and \( I \otimes \sigma_z \) commute), the eigenprojections \( \Pi_j \) of \( \text{ad}_S \) do not depend on the magnetic field controls. The eigenvalues, however, do depend on the controls, they in fact are linear in the controls, but this does not produce time-derivatives of the controls in the state-space master equation.

Next, take
\[ H_S = u_{1x}(\sigma_x \otimes I) + u_{2z}(I \otimes \sigma_z), \]
that is, the two qubits are subject to an \( x \)-magnetic field control on spin 1 together with a \( z \)-magnetic field control on spin 2. Not surprisingly (since \( \sigma_x \otimes I \) and \( I \otimes \sigma_z \) still commute), the eigenprojection operators \( \Pi_j \) do not depend on the controls, hence \( \Pi_j \) = 0, and the right-hand side of the state-space master equation depends neither on \( u_{1x} \) nor on \( u_{2z} \).

The eigenvalues of \( \text{ad}_S \) are still linear in the controls.

However, taking the more complicated control pattern
\[ HS = u_{1x}(\sigma_x \otimes I) + u_{2z}(I \otimes \sigma_z) + u_{1z}(\sigma_z \otimes I), \]
where we added a \( z \)-magnetic field control on spin 2 relative to the preceding case, we get some of the eigenprojections \( \Pi_j(t) \) depending on the controls. This is not surprising, since now \( \sigma_x \otimes I \) and \( \sigma_z \otimes I \) no longer commute. However, not completely trivial is the fact that the eigenvalues are no longer linear in the controls, and that some eigenvalues are even complicated rational functions of the controls, with the same for some components of \( \Pi_j \). Therefore, not only does the right-hand side of the master equation depend on the time-derivatives of the controls, but the same equation is not even bilinear!
5 Concluding Remarks

The goal of this paper has been to point out that the model of the quantum master equation often used in controllability analysis of open quantum systems is valid only in the restrictive singular coupling limit, where an infinitely fast dynamics for the bath is assumed. In the more common weak coupling limit, the coherent part of the dynamics and therefore the control appears in the whole dynamical model including the part modeling dissipation.

In order to show this we have recalled the classical results of E. Davies [6] [7], [8], starting from the generalized master equation and giving the form of the generator of the approximate Markovian dynamics. We have worked out the explicit calculation for a system consisting of a spin $\frac{1}{2}$ particle in a bosonic bath showing how, under the assumptions of Davies’ results, the model depends on the control.

The manipulation of open quantum systems present several challenges for control theorists. In particular the design of the control has to take into account the whole modeling procedure and what kind of assumptions on the system and bath are made. However, as we have pointed out in section 4, for the DSM problem, this might become an opportunity to solve important problems. The exploration of these techniques will be the subject of further research.

Acknowledgement

This work is supported by ARO MURI grant W911NF-11-1-0268.

References


