An Adaptive Robust Nonlinear Motion Controller Combined With Disturbance Observer

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Abstract—Parameter adaptation and disturbance observer (DOB) have been considered as two contrastively different approaches to handle uncertainties in motion control problems. The purpose of this brief is to merge both techniques into one control design with theoretically guaranteed performance. It is shown that the DOB compensates low-passed components of the lumped uncertainties without the necessity of parameterization, whereas the adaptive mechanism is only automatically activated in the cases where the fast-changing components of the uncertainties beyond the pass-band of the DOB can be parameterized by unknown parameters. It is thus shown theoretically how the DOB and adaptive mechanism play in a cooperative way so that the controller is more effective than the individual ones. Simulation results are provided to support the theoretical results.

Index Terms—Adaptive control, disturbance observer (DOB), input-to-state-stability, motion control, robust control.

I. INTRODUCTION

In the motion control problems of nonlinear mechanical systems, adaptive control, disturbance observer (DOB) control, and sliding mode control (SMC) are typical but contrastively different approaches to handle the uncertainties [1], [2].

So far, many papers have been published on the DOB-based motion controllers [3], [4]. The key point of a DOB is to pass the external disturbances and model mismatch, lumped as an error term, through a low-pass filter and then to compensate the external disturbance and model mismatch by the output of the low-pass filter. Usually, the DOB-based motion controllers are designed according to linear control theory. Most recently, a novel robust nonlinear motion controller with the DOB has been proposed, where the input-to-state stability (ISS) property of the overall nonlinear control system is guaranteed [5]. Compared to the adaptive control technique, the DOB-based control enjoys the advantages of simplicity, compensation ability of unparameterizable uncertainties, reliable transient performance. However, the DOB-based controllers cannot sufficiently compensate some fast-changing uncertainties beyond the pass-band of the DOB [1].

On the other hand, the adaptive control techniques have also been widely used in motion control [1], [6]–[8]. If the uncertainties such as friction and high-frequency ripples can be parameterized, adaptive laws are adopted to achieve a small tracking error after the transient phase. However, the adaptive control techniques may exhibit unsatisfactory transient performance and cannot handle unparameterizable external disturbances through parameter adaptation.

In contrast to the adaptive control or the DOB control, the SMC drives the system states to a stable sliding surface and keeps the states on it by a switching function. The SMC has a simple structure and high robustness against the uncertainties. Since the nonlinear system analysis methodologies of both the adaptive control and SMC are well established, there have been many works where these two techniques are combined to achieve better performance [9]. Besides these, there also have been some works that combine the DOB and SMC [10], [11], where a linear nominal model is considered, and the modeling error and external disturbances are lumped as a disturbance term. We denote here this lumped disturbance term as \( w \). By using the DOB, an estimate of \( w \), i.e., \( \hat{w} \), is obtained. And by the DOB compensation, the lumped disturbance is reduced to \( w - \hat{w} \), and a modest switching gain of the SMC is sufficient. The lumped uncertainty \( w \), however, may include signal dependent uncertainties. Moreover, the DOB itself introduces an extra loop into the system, and hence we cannot easily say \( w \) and \( \hat{w} \) are bounded prior to control design. In this context, it is considered that there still exist possibilities to revise this approach. As a demerit of the SMC, in the neighborhood of the sliding surface, the switching function may chatter due to noise or finite sampling frequency.

The combination of the DOB and adaptive control, however, has not been well studied, due to the fact that the stability of the DOB-based controllers for nonlinear systems has not been widely studied in the literature whereas the adaptive control systems usually require strict Lyapunov-like stability analysis. Therefore, it is of not only theoretical but also practical interests to investigate what performance can be achieved by exploiting the advantages of the two different techniques.

In this brief, we propose a motion controller that merges both advantages into one control design so that it can handle broader classes of uncertainties with theoretically guaranteed performance. It is shown that the DOB compensates low-passed components of the lumped uncertainties, whereas the computationally demanding adaptive mechanism is only automatically activated in the cases where the fast-changing components of the
uncertainties beyond the pass-band of the DOB can be parameterized by unknown parameters. It is thus clarified theoretically how the DOB and adaptive mechanism play in a cooperative way so that the proposed controller is more effective than the individual ones. Finally, simulation results are provided to support the theoretical results.

II. STATEMENT OF THE PROBLEM

Consider the following single-input–single-output (SISO) nonlinear mechanical system:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= F(x) + d(x, t) + G(x)u
\end{align*}
\]

where \(x = [x_1, x_2]^T\), \(x_1\) and \(x_2\) are the position and velocity, respectively, \(u\) is the control input, \(G(x)\) and \(F(x)\) are the modelable nonlinear functions, and \(d(x, t)\) denotes unmodeled nonlinearities and disturbances.

Denoting the nominal nonlinearities based on prior knowledge as \(F_0(x)\) and \(G_0(x)\), we have

\[
F(x) = F_0(x) + \Delta_F(x), \quad G(x) = G_0(x) + \Delta_G(x)
\]

where the modeling errors \(\Delta_F(x)\) and \(\Delta_G(x)\) can be approximated by networks with linear parameters

\[
\hat{\Delta}_F(x, \omega_F) = \phi_F(x)\omega_F, \quad \hat{\Delta}_G(x, \omega_G) = \phi_G(x)\omega_G
\]

where the signal vectors \(\phi_F(x), \phi_G(x)\) and parameter vectors \(\omega_F, \omega_G\) are defined as

\[
\phi_F(x) = [\phi_{F1}(x), \ldots, \phi_{FN_F}(x)]^T,
\phi_G(x) = [\phi_{G1}(x), \ldots, \phi_{GN_G}(x)]^T
\]

\[
\omega_F = [\omega_{F1}, \ldots, \omega_{FN_F}]^T,
\omega_G = [\omega_{G1}, \ldots, \omega_{GN_G}]^T
\]

The reference trajectory \(y_c(t)\) is appropriately chosen as a sufficiently smooth function such that \(\dot{y}_c\) and \(\ddot{y}_c\) are known and

\[
D_{y_c} = \left\{ y_c, \dot{y}_c, \ddot{y}_c \middle| [y_c, \dot{y}_c]^T \in \Omega_c, \|y_c\| \leq \bar{y}_c, \|\dot{y}_c\| \leq \bar{y}_c \right\}.
\]

Remark 1: Assumption 1 is for the adaptive laws with parameter projection [12]. Assumption 2 means the present method is only applicable to the systems where \(G(x)\) is bounded away from zero on the desired domain of operation, such that the control input is not divided by zero [9]. Assumption 3 is a mild assumption that implies the estimated function \(\hat{G}(x, \hat{\omega}_G)\) and the error function \(\hat{G}(x, \hat{\omega}_G)\) do not grow in a higher order than \(G(x)\) itself. \(\hat{F}(x)\) and \(\hat{d}(x, t)\) in Assumption 4 are bounding functions based on the prior knowledge for design of robust damping terms [13]. Assumption 4 means the disturbance terms do not grow faster than their corresponding bounding functions. Finally, we should mention that the proposed controller is confined to the physical systems that satisfy the above assumptions. For more general systems that do not meet these assumptions, modifications may be necessary.

III. CONTROLLER DESIGN

The proposed controller is designed in a backstepping procedure, composed of the following two steps.
Step 1) Define the position error and velocity error signals respectively as

\[ z_1 = x_1 - y_r, \quad z_2 = x_2 - \alpha_1 \]  

(15)

where \( \alpha_1 \) is the virtual input to stabilize \( z_1 \). Then from (1a) we have subsystem \( S1 \) as the following:

\[ S1 : \dot{z}_1 = \alpha_1 + z_2 - \dot{y}_r \]  

(16)

where \( \alpha_1 \) is designed based on the PI control technique

\[ \alpha_1 = -c_{1p}z_1 - c_{1i} \int_0^t z_1 \, dt + \dot{y}_r \]  

(17)

where \( c_{1p}, c_{1i} > 0 \).

Step 2) From (1b), (6), and (15), we have subsystem \( S2 \)

\[ S2 : \dot{z}_2 = \hat{F}(x, \hat{w}_F) - \delta_1 + \hat{G}(x, \hat{w}_G) u 
+ \eta_F(x, w_F) - \hat{\phi}_F^T(x) \hat{w}_F 
+ \eta_G(x, w_G) u - \hat{\phi}_G^T(x) \hat{w}_G u \]  

(18)

Denoting the error terms \( d(x, t) + \eta_F(x, w_F) + \eta_G(x, w_G) u \) as the lumped disturbance \( w \), we have

\[ w = z_2 - \left( \hat{F}(x, \hat{w}_F) + \hat{G}(x, \hat{w}_G) u - \delta_1 \right) \]  

(19a)

\[ = d(x, t) + \eta_F(x, w_F) + \eta_G(x, w_G) u \]  

(19b)

\[ = d(x, t) + \eta_F(x, w_F) - \hat{\phi}_F^T(x) \hat{w}_F 
+ \eta_G(x, w_G) u - \hat{\phi}_G^T(x) \hat{w}_G u \]  

(19c)

However, since \( z_2 \) by direct differentiation is usually contaminated with high frequency noise, we have to pass (19a) through a low-pass filter \( Q(s) \) to obtain the estimate of \( w \) as

\[ \hat{w} = Q(s)w. \]  

(20)

This is the so-called DOB studied extensively in the literature. Here, we adopt a simple first-order filter \( Q(s) \)

\[ Q(s) = \frac{1}{1 + \lambda s}, \quad Q(s) = 1 - Q(s) = \frac{\lambda s}{1 + \lambda s}. \]  

(21)

If the disturbance and model mismatch are fast-changing and hence significantly beyond the pass-band of the DOB, the error \( w - \hat{w} \) may cause significant performance degrading of the control system. In [14], it is commented that if the nominal mass is smaller, a smaller constant \( \lambda \) is recommended. However, a smaller \( \lambda \) may make the DOB’s output noisy. On the other hand, too large a nominal mass may cause large control efforts.

We are thus motivated to propose the following adaptive robust controller incorporating the DOB

\[ u = u_1 + u_0 + u_r + u_e \]  

\[ u = \frac{\alpha_2}{G(x, \hat{w}_G)} \]  

where

\[ \alpha_2 = -c_2z_2 + \delta_1 - \hat{F}(x, \hat{w}_F) \]  

(23)

and \( c_2, \delta_2, \delta_3, \delta_2, \delta_3, \delta_4, \delta_5 > 0 \) are control gains. \( u_1 \) is a feedback controller with model compensation; \( u_0 \) is a compensating term by the DOB’s output; \( u_2, u_3, u_4, u_5 \) are nonlinear damping terms to counteract \( \Delta_F(x) - \Delta_G(x, \hat{w}_G) \) and \( d(x, t) \), respectively; \( u_6 \) is a nonlinear damping term to ensure boundedness of \( z_2 \) when \( \hat{w} \) is used; \( u_7 \) is a nonlinear damping term to ensure boundedness of \( z_2 \) when \( u_4 \) is used.

As will be seen in (24) and (25), \( u_r \) is introduced to compensate the terms \( \epsilon_G \) and \( \epsilon_F \) defined in (25). Notice that \( \epsilon_G \) and \( \epsilon_F \) stem from the fact that the adaptive laws are not applicable directly to the terms \( \bar{Q}(s) \bar{\phi}_F^T(x) \hat{w}_G \) and \( \bar{Q}(s) \bar{\phi}_G^T(x) \hat{w}_F \), but are applicable to the terms \( \{ \bar{Q}(s) \bar{\phi}_F^T(x) \hat{w}_G \} \hat{w}_G \) and \( \{ \bar{Q}(s) \bar{\phi}_G^T(x) \hat{w}_F \}. \) Empirically, however, the amplitudes of \( e_G \) and \( e_F \) are trivial, since \( \hat{w}_G \) and \( \hat{w}_F \) do not change so fast due to adaptive laws with integral calculations.

Applying \( u \) to \( S2 \) described by (18), we have

\[ z_2 = -c_2z_2 + \hat{G}(x, \hat{w}_G) u_r + \eta_F(x, w_F) + d(x, t) \]  

\[ + \eta_G(x, w_G) u - (\hat{w} + \epsilon_G + \epsilon_F) \]  

\[ = -c_2z_2 + \hat{G}(x, \hat{w}_G) u_r + \bar{Q}(s)\eta_F(x, w_F) \]  

\[ + [\bar{Q}(s)\bar{\phi}_F^T(x) \hat{w}_F - \{ \bar{Q}(s) \bar{\phi}_G^T(x) u \}] \hat{w}_G \]  

(24)

where

\[ e_G = \{ \bar{Q}(s)\bar{\phi}_G^T(x) \} \hat{w}_G - \bar{Q}(s) \bar{\phi}_F^T(x) \hat{w}_F \]  

\[ = Q(s) \phi_G(x) u - \{ Q(s)\phi_F^T(x) \} \hat{w}_F \]  

\[ = Q(s) \phi_G(x) u - \{ Q(s)\phi_F^T(x) \} \hat{w}_G \]  

(25)
Although $u$ is included in $\varepsilon_G$, since $Q(s)$ is strictly proper so that the phase of $Q(s)[s]$ is delayed, $\varepsilon_G$ is implementable.

To let the adaptive parameters stay in the prescribed range, we adopt the adaptive laws with projection [12], shown in (26) at the bottom of the page, where $n=1, \ldots, N_F$, $\gamma_F \geq 0$, and $\phi_{Fn}(x)$ is the $n$th element of $\phi_F(x)$ defined in (4), shown in (27) at the bottom of the page. where $n=1, \ldots, N_G$, $\gamma_G \geq 0$, and $\phi_{Gn}(x)$ is the $n$th element of $\phi_G(x)$ defined in (4).

It can be verified that the adaptive laws satisfy

\[
\begin{align*}
\dot{w}_F \leq \dot{w}_{F_t} \leq \dot{w}_F, & \quad \dot{w}_G \leq \dot{w}_{G_t} \leq \dot{w}_G \\
\ddot{w}_F - \gamma_F \{Q(s)[\phi_F(x)]\} \omega_{F_t} & \leq 0 \\
\ddot{w}_G - \gamma_G \{Q(s)[\phi_G(x)u]\} \omega_{G_t} & \leq 0.
\end{align*}
\]  

(28)

Remark 2: Since the complementary filter $\bar{Q}(s)$ is a high-pass filter, in the case of slowly-changing signals, the amplitudes of $\{\bar{Q}(s)[\phi_{Fn}(x)]\}_{n=1}^{N_F}$ and $\{\bar{Q}(s)[\phi_{Gn}(x)u]\}_{n=1}^{N_G}$ are trivial so that the computationally demanding adaptive laws become to be “lazy” or unnecessary. On the other hand, when the passband of the DOB is less broad, the adaptive laws work more significantly. Therefore, we can conclude that the DOB and the adaptive laws play in a complementary and cooperative way to account for uncertainties.

IV. STABILITY ANALYSIS

In this section, we first show the ISS property of each subsystem that specifies the boundedness and transient performance. Then we will show our main theoretical contribution in this brief, i.e., how the adaptive laws and DOB play in a cooperative way; what control performance can be achieved if these two different techniques are merged into one control design. Finally, we summarize the theoretical results for the overall error system.

A. ISS Property Analysis of Each Subsystem

Applying $\alpha_1$ to subsystem $S_1$ (16), we have

\[
\dot{z}_1 = z_2 - c_1p \bar{z}_1 - c_1i \int_0^t z_1 dt
\]  

(29)

and the state-space form

\[
\dot{z}_{1a} = A z_{1a} + B z_2
\]  

(30)

where $z_{1a} = \left[ \int_0^t z_1 dt, z_1 \right]^T$

\[
A = \begin{bmatrix} 0 & 1 \\ -c_{1i} & -c_{1p} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]  

(31)

The ISS property is described in the following lemma [15].

Lemma 1: If the virtual input $\alpha_1$ is applied to the subsystem $S_1$, and if $z_2$ is made uniformly bounded at the next step, then the subsystem $S_1$ is ISS, i.e., for $\exists \alpha_0, \alpha_1 \rho_0 > 0$

\[
|z_{1a}(t)| \leq \alpha_0 e^{\rho_0 t} |z_{1a}(0)| + \frac{\alpha_0}{\rho_0} \sup_{0 \leq \tau \leq t} |z_2(\tau)|.
\]

To establish the ISS property of the subsystem $S_2$, we first rewrite (24) by using (22)

\[
\begin{align*}
\dot{z}_2 &= -c_2z_2 + \bar{Q}(x)u_T + \eta_F(x, \bar{w}_F) + d(x, t) \\
& \quad + \eta_G(x, \bar{w}_G)u_T - \frac{G(x)}{G(x, \bar{w}_G)}(\bar{w} + \varepsilon_G + e_F) \\
& = -c_2z_2 + \bar{G}(x, \bar{w}_G)u_T + w - (\bar{w} + \varepsilon_G + e_F).
\end{align*}
\]  

(32)

Then we have by using (22), (23), and (32)

\[
\begin{align*}
\frac{d}{dt} \left( \frac{z_2^2}{2} \right) &= -c_2 \frac{z_2^2}{2} + \left[ c_2 \frac{D_2}{2} + D_2 \right] z_2^2 + d z_2 \\
& \leq -c_2 \frac{z_2^2}{2} + \left[ c_2 \frac{D_2}{2} + D_2 \right] z_2^2 + \mu_2 
\end{align*}
\]  

(33)

where

\[
\mu_2(t) = \frac{|d_2|}{c_2 + D_2} + \frac{\eta_2(x, \bar{w}_G)}{G(x, \bar{w}_G)} \alpha_20 + \eta_1(x, \bar{w}_F) + d(x, t) - \frac{G(x)}{G(x, \bar{w}_G)}(\bar{w} + \varepsilon_G + e_F)
\]

\[
\begin{align*}
D_2 &= \frac{\bar{G}(x)}{G(x, \bar{w}_G)} \\
& \times \left( \kappa_2 \bar{F}(x) + \kappa_2 \beta_2 d \right) + \kappa_2 \bar{d}(x, t) + \kappa_2 \|\bar{d}\| + \kappa_2 \varepsilon_G + e_F.
\end{align*}
\]  

(35)

According to Assumptions 1–5, it is obvious that each term in the numerator of $\mu_2$ is counteracted by a nonlinear damping
term in the denominator which grows at least as the same order as the corresponding term in the numerator, so that $\mu_2$ is uniformly bounded. Furthermore, from (33) we have
\[
|z_2| \geq \mu_2(t) \Rightarrow \frac{d}{dt} \left( \frac{|z_2|^2}{2} \right) \leq -c_2 z_2^2
\]  
and hence $z_2$ is bounded as
\[
|z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_2(\tau).
\]  
Provided the boundedness of $z_2$, we show how the DOB’s output $\hat{w}$ can bring improvement. Rewrite (33) by using (32)
\[
\frac{d}{dt} \left( \frac{|z_2|^2}{2} \right) \leq -c_2 z_2^2 - \left( c_2 z_2 + D_{2w} \right) |z_2| |z_2 - \mu_2|.
\]  
where
\[
\mu_2 = \frac{|w - \hat{w}| + |K_G + \epsilon F|}{\frac{1}{2} + D_{2w}}
\]  
\[
D_{2w} = \kappa_2 \frac{P T(x) + \kappa_2 \alpha_2 d + \kappa_2 \alpha_3 \langle \tau \rangle}{\hat{\omega}} + \kappa_2 \frac{\hat{\omega}^2}{c} + \kappa_2 \epsilon F + e_G.
\]  
At low-frequencies, we can expect $w - \hat{w}$ to be close to zero. Any nonzero $w - \hat{w}$ at high-frequencies is counteracted by $c_2 z_2 + D_{2w}$ so that $z_2$ is quite robust against $w - \hat{w}$.

**Remark 3:** As mentioned previously, the terms $e_G$ and $e_F$ defined in (25) are inevitably due to the time-varying effects of the adaptively updated parameters, when the DOB is adopted. That is, $e_G$ and $e_F$ are due to the interference between the DOB and adaptive laws. In the case of $\gamma_F = \gamma_G = 0$, i.e., the adaptive laws are switched off, we have $e_G = e_F = 0$, and thus $u_{d5}, u_e$ in (22) can be removed. On the other hand, when the DOB is not used, i.e., $Q(s) = 0$, we have $\hat{\omega} = 0, e_F = e_G = 0$, and thus $u_{d5}, u_e$ in (22) can be removed.

Finally, (38) leads to
\[
|z_2| \geq \mu_2(t) \Rightarrow \frac{d}{dt} \left( \frac{|z_2|^2}{2} \right) \leq -c_2 z_2^2
\]  
and hence follows.

**Lemma 2:** Let Assumptions 1–5 hold. If the control input $u$ is applied to the subsystem $S_2$, then the subsystem $S_2$ is ISS and the error signal $z_2(t)$ is uniformly bounded as
\[
|z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_2(\tau).
\]

**B. Tracking Error Bounds Achieved by DOB and Adaptive Laws**

We are now ready to show how the DOB and adaptive laws work cooperatively to improve the tracking error bounds.

We first consider the subsystem $S_1$ in (30). Since there exists a $P > 0$ satisfying $A^T P + PA = -Q$ for any positive definite symmetric matrix $Q$, we have
\[
\frac{d}{dt} \left( \frac{z_{1a}^2}{2} \right) = -\frac{1}{2} z_{1a}^T Q z_{1a} + z_{1a}^T P B z_{22}
\]  
\[
\leq -\frac{\lambda_{Q_{\min}}}{4} |z_{1a}|^2 + \frac{1}{\lambda_{Q_{\min}}} |PB|^2 |z_{22}|^2
\]  
where $\lambda_{Q_{\min}}$ is the minimal eigenvalue of $Q$. Then we have the following.

**Lemma 3:** If $\alpha_3$ is applied to the subsystem $S_1$, and if $z_2$ is made uniformly ultimately bounded with ultimate bound $\bar{z}_2$ at the next step, the error signal $z_{1a}(t)$ is uniformly ultimately bounded such that
\[
|z_{1a}(t)| \leq \frac{|PB|^2}{\lambda_{Q_{\min}}} \bar{z}_2^2, \text{ as } t \geq T_1 > 0
\]  
and the mean square of $z_{1a}(t)$ satisfies
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |z_{1a}(t)|^2 dt \leq \frac{4 |PB|^2}{\lambda_{Q_{\min}}} \lim_{T \to \infty} \frac{1}{T} \int_0^T |z_2|^2 dt.
\]

Furthermore, to analyze how the adaptive laws help to improve $\bar{z}_2$, we impose one more assumption.

**Assumption 6:** The approximation errors of the networks are sufficiently small on the desired domain of operation $\Omega_X$, i.e., there exist $\hat{\hat{u}}_F$ and $\hat{\hat{u}}_G$ satisfying
\[
\sup_{x \in \Omega_X} |\hat{\eta}_F(x, \hat{\hat{u}}_F)| \leq \epsilon_F, \quad \sup_{x \in \Omega_X} |\hat{\eta}_G(x, \hat{\hat{u}}_G)| \leq \epsilon_G
\]  
where $\epsilon_F$ and $\epsilon_G$ are sufficiently small positive numbers, and $\hat{\hat{u}}_F$ and $\hat{\hat{u}}_G$ are defined in (8).

Then we define the following Lyapunov function.
\[
V_2 = \frac{z_2^2}{2} + \frac{\hat{\hat{u}}_F^T \hat{\hat{w}}_F}{2 \gamma_F} + \frac{\hat{\hat{u}}_G^T \hat{\hat{w}}_G}{2 \gamma_G}, \quad \gamma_F, \gamma_G > 0.
\]  
By using the results of (24) and (28), we have
\[
\hat{V}_2 \leq -(e_2 + D_{2u}) \frac{|z_2|^2}{2} + \frac{|Q(s)w^a|^2}{2}
\]  
\[
\leq -(e_2 + D_{2u}) \frac{|z_2|^2}{2} + b_{2n}^2
\]  
\[
\leq -(e_2 + D_{2u}) \left( V_2 - \frac{1}{2} \bar{z}_2^2 \right)
\]  
where
\[
M_2^2 = (\hat{\hat{u}}_F - \hat{\hat{u}}_F)(\hat{\hat{w}}_F - \hat{\hat{w}}_F)^T \geq \hat{\hat{w}}_F^T \hat{\hat{w}}_F
\]  
\[
M_2^2 = (\hat{\hat{u}}_G - \hat{\hat{w}}_G)(\hat{\hat{w}}_G - \hat{\hat{w}}_G)^T \geq \hat{\hat{w}}_G^T \hat{\hat{w}}_G
\]  
\[
\delta_{2n} = \sqrt{(e_2 + D_{2u})^2 + \frac{|Q(s)w^a|^2}{2}}
\]  
\[
w^a = (x, t) + u_m^a + \epsilon_G(x, \hat{\hat{u}}_G) u
\]  
(45)

The notations involved in (45) are defined in (6)–(10), (21), and (39). Notice that owing to the adaptive laws, the ultimate model mismatch satisfies $|w_m^a| \leq \epsilon_F + \epsilon_G |u|$. Equation (44) implies that
\[
\frac{1}{2} |z_2|^2 \leq V_2 \leq \frac{1}{2} \left( \sup_{0 \leq \tau \leq T_1} |\delta_{2n}(\tau)| \right)^2 \text{ as } t \geq T_2 u > 0.
\]  
From (44b) we have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |z_2|^2 dt \leq \frac{2}{e_2} \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{2n}^2 dt.
\]  
(47)

The results can be summarized in the following lemma.
Lemma 4: Let Assumptions 1–6 hold. If the control input \( u \) (22) and the adaptive laws (26) and (27) are applied to the subsystem \( \mathcal{S}_2 \), then the following results hold.

1) The adaptive parameters satisfy
\[
\begin{align*}
\mathbf{w}_F & \leq \mathbf{w}_F \leq \mathbf{w}_F, & \mathbf{w}_G & \leq \mathbf{w}_G \leq \mathbf{w}_G, \\
\end{align*}
\]
for all \( t \geq 0 \).

2) The ultimate bound of \( z_2 \) is obtained as
\[
|z_2| \leq \left[ \sup_{0 \leq \tau \leq \infty} |\delta_{2u}(\tau)| \right] \text{ as } t \geq 3T_u > 0.
\]

3) The mean square of \( z_2 \) satisfies
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |z_2|^2 dt \leq \frac{2}{\epsilon_2} \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{2m}^2 dt.
\]

Remark 4: we can make the contributions from \( M_F \) and \( M_G \) to the ultimate error bound \( \delta_{2u} \) sufficiently small by taking large values of the adaptive gains \( \gamma_G \) and \( \gamma_F \).

Remark 5: Investigation of (44)–(47) reveals clearly the ultimate error bound and mean square error bound achieved by the cooperative effects of the DOB and adaptive laws. \( \epsilon_F \) and \( \epsilon_G \) defined in Assumption 6 imply the best approximation error achieved by the networks. Since \( d(\mathbf{x},t) \) is an unparameterizable disturbance term, we cannot handle it by parameter adaptation. Therefore, \( u^a \) is what we can achieve by the adaptive laws. However, \( \bar{Q}(s)u^a \) clearly implies that the low-frequency components of \( u^a \) can be further removed owing to the DOB. Therefore, \( \bar{Q}(s)u^a \) can be made sufficiently small if the pass-band of \( \bar{Q}(s) \) is chosen sufficiently broad.

C. Stability of the Overall Error System

Lemmas 1 and 2 imply that the overall error system is a cascade of two ISS subsystems. Define the error signal vector
\[
\mathbf{z}(t) = [z_{1u}(t), z_2(t)]^T.
\]
Then along the same lines of the [13, proof of lemma C4], we can derive the following results:
\[
|\mathbf{z}(t)| \leq \beta_1 e^{-\rho_1 t} |\mathbf{z}(0)| + \beta_2 \left[ \sup_{0 \leq \tau \leq t} \mu_{2u}(\tau) \right]
\]
where
\[
\begin{align*}
\beta_1 &= \sqrt{2} \left( \lambda_0^2 + 3 \frac{\lambda_0^2}{\rho_0} + 3 \frac{\lambda_0}{\rho_0} + 3 \right) \\
\rho_1 &= \min \left\{ \frac{\rho_0}{2}, \frac{c_2}{4} \right\}, & \beta_2 &= \frac{\lambda_0^2}{\rho_0} + \frac{\lambda_0}{\rho_0} + 1
\end{align*}
\]
and \( \rho_0 \) and \( \lambda_0 \) are defined in Lemma 1.

Furthermore, from Lemmas 3 and 4, we have the ultimate position tracking error bound as
\[
|z_{1u}(t)| \leq \frac{2|\bar{P}|}{\kappa_{Q_{\min}}} \left[ \sup_{0 \leq \tau \leq \infty} |\delta_{2u}(\tau)| \right] \text{ as } t \geq 3T_u > 0
\]
and we have the mean square position tracking error bound as
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |z_{1u}|^2 dt \leq \frac{8|\bar{P}|^2}{c_2^2 \kappa_{Q_{\min}}} \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{2m}^2 dt.
\]

VI. SIMULATION EXAMPLES

The motion control problem of a linear motor with friction and periodic ripple disturbances has been widely reported in the literature [6]–[8]. In this brief, we use the same example as in [7], where an adaptive sliding mode controller was studied. Comparative simulation results will be given to confirm the performance of the proposed method.
and its velocity \( \dot{y}_r \), and the unmodelable external disturbance \( f_{\text{ext}}/b \).

**A. Description of the System Model**

The system model in [7] is described as follows:

\[
\dot{x}_2 = \frac{a \dot{x}_1 + u - f_{\text{fric}} - f_{\text{ripple}} - f_{\text{ext}}}{b}
\]

where \( u \) is the control voltage, \( x_1 \) is the position, \( x_2 = \dot{x}_1 \) is the velocity, \( a = -123 \), and \( b = 0.69 \) [7].

The periodic ripple disturbance is given as

\[
f_{\text{ripple}} = A_r \sin \left( \frac{2\pi x_1}{P} + \varphi \right)
\]

where \( \omega = 2\pi/P = 314 \), \( A_r = 8.5 \), \( \varphi = 0.05\pi \). It is assumed here that only the magnetic pitch \( P = 0.02 \) [m] is known.

The friction model is given as

\[
f_{\text{fric}} = \left[ f_c + (f_s - f_c)e^{-(x_2/\lambda)^2} \right] \text{sign}(x_2) + f_{c2}x_2
\]

where \( f_s = 20 \), \( f_c = 10 \), \( \lambda = 0.1 \), \( f_{c2} = 10 \).

The reference trajectory \( y_r \) and its derivative are shown in Fig. 1. The controller is implemented at a sampling period of \( T = 0.2 \) [ms]. To verify the noise effects, a uniformly distributed noise between \(-10^{-6} \) [m] and \( 10^{-6} \) [m] is added to the measurement of \( x_1 \). The measurement of \( x_2 \) is obtained by pseudodifferentiation.

**B. Results of the Adaptive Sliding Mode Controller**

The adaptive sliding mode controller is given as follows [7].

1) Define the tracking errors

\[
e(t) = x_r(t) - x_1(t), \quad \dot{e}(t) = \dot{x}_r(t) - \dot{x}_1(t).
\]

2) Define the PID sliding surface

\[
s = 12000 \int_0^t e(\tau)d\tau + 200e(t) + \dot{e}(t).
\]

3) Introduce the dead-zone to the sliding surface

\[
s_\Delta(t) = s(t) - \delta \text{sat} \left( \frac{s(t)}{\delta} \right), \quad \delta = 0.05\lambda
\]

4) Construct an adaptive sliding mode controller

\[
u = -d_2x_2 - \hat{b}u_e + \hat{A}_r \cos(\omega x_1) + \hat{\delta} \text{sign}(s_\Delta)
\]

\[
u = -12000e - 200\dot{e} - \dot{x}_d - 90s_\Delta.
\]

Notice that \( f_{c2}x_2 \) and \( ax_2 \) have been lumped as one term such that \( ax_2 - f_{c2}x_2 = d_2x_2 \).

5) Construct the adaptive laws

\[
\dot{\hat{d}} = -k_\alpha \hat{s}_\Delta, \quad \dot{\hat{b}} = -k_b u_e \hat{s}_\Delta
\]

\[
\dot{\hat{f}}_c = k_f \text{sign}(x_2) \hat{s}_\Delta, \quad \dot{\hat{A}}_r = k_r \cos(\omega x) \hat{s}_\Delta
\]

\[
\dot{\hat{A}}_r = k_r \sin(\omega x) \hat{s}_\Delta, \quad \dot{\hat{f}} = k_f \hat{s}_\Delta
\]

where \( k_\alpha = k_b = k_{fc} = k_{r1} = k_{r2} = k_f = 50 \).

In [7], the term \((f_s - f_c)e^{-(x_2/\lambda)^2}\) included in the friction term is not estimated by parameter adaptation, but is counteracted by the sliding mode control term.

The results are shown in Fig. 2. It can be verified that the results reflect the theoretical conclusion in [7] quite clearly: the sliding surface signal \( s \) does converge to the dead zone shown by the dashed lines. However, we have found that it is difficult to obtain acceptable results when the dead zone width \( \delta \) is further reduced. As well known, the sliding mode type controller may suffer from the chattering problem due to the switching action. It is uneasy to reduce the chattering without degradation of precision.
C. Results of the Proposed Controller

We restate the plant model (53) according to the problem setting (1), where
\[
F(x) = \frac{\alpha r_1}{b} + \frac{f_c \text{sign}(x_2) - f_c x_2}{b}, \quad G(x) = \frac{1}{b}
\]
\[
d(x, t) = -\frac{(f_s - f_c) e^{-(x_2/b)_2} \text{sign}(x_2)}{b} - \frac{f_{\text{load}}}{b}
\]
where \( F_p(x_1) = \frac{-f_{\text{ripple}}}{b} \) is the periodic position dependent ripple disturbance modeled in (54). Since the term \( \frac{(f_s - f_c) e^{-(x_2/b)_2} \text{sign}(x_2)}{b} \) is not estimated by the adaptive laws in [7], to perform a fair comparative study, we also do not estimate this term and we handle this term by the nonlinear damping terms and the DOB.

The nominal functions are given as
\[
G_0(x) = \frac{1}{b_0} \frac{1}{\sqrt{b_0}}, \quad F_0(x) = 0
\]
where \( b_0 \) denotes the nominal value of \( b \).

The known nonlinear bounding functions required in Assumption 4 are given as
\[
\bar{F}(x) = [x_2] + 1, \quad \bar{d}(x, t) = e^{-x_2^2} + 1
\]
\[
\tilde{\Delta}_G(x) \quad \text{in (5) is modeled by a single parameter as } \tilde{\Delta}_G(x, w_G) = w_G = (1/b - 1/\bar{b}_0).
\]

In [7], it is assumed that the form of the ripple disturbance is known. In contrast, in our method, the ripple disturbance is approximated by a periodic RBF network of period \( P \). Define the truncated variable \( [x_1] \in [0, P] \) as
\[
[x_1] = x_1 - P \cdot \text{floor}\left(\frac{x_1}{P}\right)
\]
where \( \text{floor}(\cdot) \) is a function that rounds the entry to the greatest integer less than it. Then \( F_p(x_1) \) is modeled as
\[
\hat{F}_p(x_1, w_{\text{a}}) = -NN_a([x_1], w_{\text{a}})
\]
where \( NN_a([x_1], w_{\text{a}}) \) is a periodic RBF network
\[
NN_a([x_1], w_{\text{a}}) = \sum_{n=1}^{N_a} w_{\text{a}n} r([x_1] - p_{\text{a}n}) = R^T_a([x_1]) w_{\text{a}}
\]
where \( N_a = 10 \), and \( r([x_1] - p_{\text{a}n}) = \exp\left[-[(x_1 - p_{\text{a}n}^2)/(2\sigma_{\text{a}}^2)]\right] \) are Gaussian basis functions whose centers \( p_{\text{a}n} \) are equidistantly located in \( 0 \leq [x_1] < P \). The essential width of the basis functions is chosen as \( \sigma_{\text{a}} = (\sqrt{2}/\pi)(p_{\text{a}n} - p_{(n-1)a}) \). The nonlinear function \( F(x) \) is thus modeled as
\[
\hat{F}(x) = -NN_a([x_1], w_{\text{a}}) - w_{a_{\text{fa}}/2, -w_{a_{\text{fa}}} \text{sign}(x_2)}.
\]

To summarize, since the nominal function is chosen as \( F_0(x) = 0 \), we can model \( \hat{\Delta}_F(x) \) in (5) as
\[
\Delta_F(x, w_F) = \phi^T_F(x) w_F
\]
\[
\phi^T_F(x) = [-R^T_a([x_1]), -x_2, -\text{sign}(x_2)]
\]
\[
w_F = [w^T_{a_{\text{fa}}}, w_{a_{\text{fe}}}, w_{f_c}]^T.
\]

The bounds of the unknown parameters are given as follows:
\[
0 = w_{a_{\text{fa}}} \leq w_{a_{\text{fa}}} \leq w_{a_{\text{fe}}} = 300
\]
\[
0 = w_{f_c} \leq w_{f_c} \leq w_{f_c} = 30
\]
\[
0 = w_{\tilde{G}} \leq w_{\tilde{G}} \leq w_{\tilde{G}} = 30.
\]

The time-constant of \( Q(s) \), control gains and adaptive gains are given as
\[
c_{1p} = 200, \quad c_{1i} = 12000, \quad c_{2} = 90, \quad \lambda = 0.01, 0.03, 0.1
\]
\[
\kappa_{21} = 3, \quad \kappa_{22} = 3, \quad \kappa_{23} = 3, \quad \kappa_{24} = 3, \quad \kappa_{25} = 3
\]
\[
\gamma_{\text{F}} = 1000, \quad \gamma_{\text{G}} = 20.
\]

Notice that different values of \( \lambda \) are used for comparative investigations of the cooperative performance of the DOB and adaptive laws.

The results are shown in Figs. 3–5, where from the top to the bottom are respectively the position error \( x_1 \), velocity error \( x_2 \), control input \( u \) and the estimate of \( w_{\tilde{G}} \). Due to the space limitation, the behaviour of the other parameter estimates is omitted. We can find that when \( \lambda \) is sufficiently small, the proposed controller brings significant improvement to suppress the error amplitudes. Also, we can find that due to the cooperative effects of the DOB and adaptive laws, the control performance is not sensitive to the value of \( \lambda \).

Furthermore, when the value of \( \lambda \) is smaller, the adaptive parameter \( \delta \tilde{G} \) converges more slowly. This is because that when \( \lambda \) is smaller and thus the pass-band of \( Q(s) \) is broader, the high-pass filter \( \tilde{Q}(s) = 1 - Q(s) \) cuts off more signal components so that the adaptive law (27) works with less signal information. We have also verified that if either of the DOB or the adaptive mechanism is removed, the results are not acceptable. The results, however, are omitted due to the limitation of paper length as a brief.

Also, comparing the results of Figs. 3–5 with those of Fig. 2, we can conclude that at least for the example under study, the
The proposed controller does not suffer from the chattering problem and thus achieves smaller position-tracking error. However, due to the use of the DOB, we have to calculate some signals that are filtered by the low-pass filter $Q(s)$. Therefore, our method requires more computing burden.

Finally, we can conclude that all of these results exactly reflect the theoretic analyses.

VII. CONCLUSION

In this brief, an adaptive robust nonlinear motion controller combined with the DOB for positioning control of a nonlinear SISO mechanical system has been proposed. Stability analysis was performed as well. Extensive simulation studies were carried out to support the theoretical analysis. Our major contribution is to incorporate the DOB technique and adaptive technique which have been considered as two contrastively different approaches in the literature. Moreover, it has been shown theoretically how the DOB and adaptive mechanism play in a cooperative way so that the proposed controller is able to handle broader classes of uncertainties than the individual adaptive controllers or DOB-based controllers.

REFERENCES


