It follows, therefore, that the sequence of reflection coefficients in (1) has a predictor polynomial with all of its roots on a circle of radius $|\rho|$.

IV. Conclusions

Decaying geometric sequences of reflection coefficients arise from Gaussian-shaped autocorrelation functions. It has now been shown that such reflection coefficient sequences have predictor polynomials with all roots on a circle centered at the origin of radius $|\rho|$ where ρ is the ratio of two consecutive terms in the reflection coefficient sequence.

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Chebyshev Nonuniform Sampling Cascaded with the Discrete Cosine Transform for Optimum Interpolation

VICTOR-EMIL NEAGOE

Abstract—**This correspondence presents a new method for discrete representation of signals $\{g(t), t \in [0, L], g \in \mathcal{L}^2(0, L)\}$ consisting of a cascade having two stages: a) nonuniform sampling according to Chebyshev polynomial roots; and b) discrete cosine transform applied on the nonuniformly taken samples. We have proved that the considered signal samples and the coefficients of the corresponding Chebyshev polynomial finite series are essentially a discrete cosine transform pair. It provides a method for fast computation of the coefficients of the optimum interpolation formula (which minimizes the maximum instantaneous error). If the signal $g(t)$ is band limited and has a finite energy, we deduce the condition of convergence for interpolation.**

Introduction

The Shannon sampling theorem and its variants [2], [8], [14] are well known as performing the reconstruction of a band-limited signal from the knowledge of its uniformly taken samples. There are also a few approaches to signal reconstruction from nonuniformly spaced samples [6], [8], [10]-[12]. The classical results of function interpolation theory in computational mathematics [4], [7] show that the best choice of interpolation points to minimize the maximum modulus of the instantaneous error corresponds to the roots of the Chebyshev polynomials of the first kind.

Recently, considerable attention has been paid to the use of orthogonal transforms applied to the uniformly spaced samples. These transforms concentrate the signal energy in the "low generalized frequency" spectrum and have applications for data compression and feature extraction in pattern recognition. If we consider data compression ability as a criterion, one of the best orthogonal transforms having a fast algorithm available was proved to be the discrete cosine transform introduced by Ahmed *et al.* [1], [2].

Based on the classical theory of interpolation and quadrature formulas, we have built a model of discrete representation of signals, consisting of a cascade of Chebyshev nonuniform sampling (CNS) followed by the discrete cosine transform (DCT). It provides a method for fast computation of the coefficients of the optimum interpolation formula for a given signal $\{g(t), t \in [0, L], g \in \mathcal{L}^2(0, L)\}$.

Manuscript received June 29, 1987; revised September 27, 1989.

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IEEE Log Number 9037810.

Chebyshev Nonuniform Sampling and Discrete Cosine Transform

Theorem: Consider a real-valued signal $\{g(t), t \in [0, L], g \in \mathcal{L}^2(0, L)\}$. Choose the nonuniform sampling grid vector $t_N = (t_j)_{j=0}^{N-1}$ given by

$$t_j = (L/2)(1 + x_j), \quad 0 \le j \le N - 1 \tag{1}$$

where x_j represent the roots of the Nth degree Chebyshev polynomial of the first kind, i.e.,

$$x_j = -\cos\frac{(2j+1)}{2N}\pi = \cos\frac{2(N-j)-1}{2N}\pi$$
$$j = 0, 1, \cdots, N-1 \tag{2}$$

$(-1 < x_0 < x_1 \cdots < x_{N-1} < 1)$.

Consider the matrix $\boldsymbol{\psi}^{\mathrm{DCT}} = (\psi_{hj}) 0 \le h \le N-1; 0 \le j \le N-1$ characterizing the discrete cosine transform as

$$\psi_{hj} = \sqrt{\frac{2}{N}}\, k(h) \cos\left[\frac{(2j+1)h\pi}{2N}\right] = \sqrt{\frac{2}{N}}\, \tilde{T}_h(x_j) \tag{3}$$

where

$$k(h) = \begin{cases} \dfrac{1}{\sqrt{2}}, & \text{for } h = 0 \\ (-1)^h, & \text{for } h = 1, \cdots, N-1 \end{cases}$$

the values x_j are given by (2) and $\tilde{T}_h(x)$ is the hth degree normalized Chebyshev polynomial

$$\tilde{T}_0(x) = \frac{1}{\sqrt{2}}; \quad \tilde{T}_h(x) = T_h(x) = \cos(h \arccos x);$$
$$h = 1, \cdots, N-1. \tag{4}$$

In (3), h is the row index and j is the column index.

Denote by $g_N = (g(t_j))_{j=0}^{N-1}$ the vector of nonuniformly spaced samples according to the Chebyshev sampling grid vector t_N. Denote by $C_N = (C_0 C_1 \cdots C_{N-1})^T$ the direct discrete cosine transformation (DDCT) of g_N, defined as

$$C_N = \boldsymbol{\psi}_N^{\mathrm{DDCT}} \cdot g_N \tag{5}$$

where

$$\boldsymbol{\psi}_N^{\mathrm{DDCT}} = (\sqrt{2/N})\boldsymbol{\psi}_N^{\mathrm{DCT}}. \tag{6}$$

The inverse discrete cosine transform (IDCT) is expressed by

$$g_N = \boldsymbol{\psi}_N^{\mathrm{IDCT}} \cdot C_N \tag{7}$$

where

$$\boldsymbol{\psi}_N^{\mathrm{IDCT}} = (\boldsymbol{\psi}_N^{\mathrm{DDCT}})^{-1} = \sqrt{\frac{N}{2}}\,(\boldsymbol{\psi}_N^{\mathrm{DCT}})^T. \tag{8}$$

Then:

i) For any $t \in [0, L]$, the optimum reconstruction formula which minimizes the maximum instantaneous modulus error is

$$\hat{g}_N(t) = \sum_{h=0}^{N-1} C_h \tilde{T}_h \left(2\frac{t}{L} - 1\right). \tag{9}$$

ii) If the signal $g(t)$ is assumed to be band limited to W, and having a finite energy

$$E = \int_{-W}^{+W} |G(f)|^2 \, df \tag{10}$$

where $G(f)$ is the Fourier transform of $g(t)$

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-i2\pi f t) \, dt, \qquad i^2 = -1$$

which satisfies $|G(f)| = 0$, for $|f| > W$, then the maximum interpolation error is

$$\epsilon_N = \max_{t \in [0, L]} |g(t) - \hat{g}_N(t)| = \frac{2\sqrt{EW}}{\sqrt{N + \frac{1}{2}}} \cdot \frac{\left(\frac{\pi r}{4} \cdot N\right)^N}{N!} \tag{11}$$

where r is a coefficient defining the average sampling interval

$$\bar{\tau} = \frac{L}{N} = r \cdot \frac{1}{2W}. \tag{12}$$

iii) In order that interpolation converges ($\lim_{N \to \infty} \epsilon_N = 0$), it is sufficient to have $r < (4/\pi e) \cong 0.468$. It means to choose an average sampling rate 2.135 times faster than the Shannon sampling rate.

Proof: The Chebyshev polynomials of the first kind, $T_h(x) = \cos(h \arccos x)$ $h = 1, \cdots, N$ satisfy the following recurrence relation:

$$T_{h+1}(x) = 2xT_h(x) - T_{h-1}(x), \qquad x \in [-1, 1]$$
$$h = 1, 2, \cdots, N - 1 \tag{13}$$

where

$$T_0(x) = 1, \qquad T_1(x) = x.$$

The orthogonal and normalized Chebyshev polynomials given by (4) fulfill the condition

$$\int_{-1}^{1} \rho(x) \tilde{T}_n(x) \tilde{T}_m(x) \, dx = \delta_{n,m} \tag{14}$$

where

$$\rho(x) = 2/(\pi\sqrt{1 - x^2}); \qquad \delta_{n,m} = 0$$
$$\text{for } n \neq m; \qquad \delta_{n,n} = 1.$$

Approximate $g(x)$, for $x \in [-1, 1]$, by a finite Chebyshev polynomial series

$$\hat{g}_N(x) = \sum_{h=0}^{N-1} C_h \tilde{T}_h(x). \tag{15}$$

We used the Hermite quadrature formula [4], [7]

$$\int_{-1}^{1} \frac{h(x)}{\sqrt{1 - x^2}} \, dx = \frac{\pi}{N} \sum_{j=0}^{N-1} h(x_j). \tag{16}$$

Relations (14)–(16) lead to

$$C_h = \int_{-1}^{1} \frac{2}{\pi\sqrt{1 - x^2}} g(x) \tilde{T}_h(x) \, dx = \frac{2}{N} \sum_{j=0}^{N-1} \tilde{T}_h(x_j) g(x_j). \tag{17}$$

Ahmed *et al.* [1], [2], proved that the matrix φ_N having its general element

$$\phi_{h,j} = \sqrt{\frac{2}{N}} k(h) \cos\left[\frac{(2j+1)h\pi}{2N}\right] = (-1)^h \sqrt{\frac{2}{N}} \tilde{T}_h(x_j) \tag{18}$$

(where $k(0) = 1/\sqrt{2}$, $k(h) = 1$, for $h = 1, \cdots, N - 1$) is an orthogonal and normalized matrix. Observe that the general term ψ_{hj} of the matrix ψ_N^{DCT} given by (3), differs by the factor $(-1)^h$ only from the general term ϕ_{hj} of the matrix φ_N. Hence, it results that ψ_N^{DCT} is also an orthogonal and normalized matrix. For $t \in [0, L]$, we use the changing of variables

$$t = (L/2)(1 + x) \tag{19}$$

and from (15) and (17) obtain relation (9). It is easy to prove that $\hat{g}_N(t_j) = g(t_j)$ and to observe that the finite series Chebyshev polynomial (9) is identical with the Lagrangian polynomial of degree $N - 1$ corresponding to the optimum approximation characterized by the grid t_N given by (1) and (2), which minimizes the maximum absolute instantaneous error.

It is well known that [4]

$$\epsilon_N = \max_{t \in [0, L]} |g(t) - \hat{g}_N(t)| \leq \frac{N^N}{2^{2N-1} \cdot N!} \max_{t \in [0, L]} |g^{(N)}(t)|. \tag{20}$$

Taking into account the bound on $g^{(N)}(t)$ given in [13], we obtain

$$|g^{(N)}(t)| \leq \sqrt{E}\left[\pi(2N + 1)\right]^{-1/2} (2\pi W)^{N + (1/2)} \tag{21}$$

where E is given by relation (10).

Using the notation (12) (where $r = (L/N)/[1/(2W)]$ represents the ratio of the average sampling interval to the classical Shannon interval), we obtain relation (11).

According to Stirling's formula [3]

$$N! \cong N^N e^{-N} \sqrt{2\pi N} \left(1 + \frac{1}{12N}\right). \tag{22}$$

Hence

$$\epsilon_N = \sqrt{\frac{2EW}{\pi}} \cdot \frac{1}{\left(1 + \frac{1}{12N}\right)\sqrt{1 + \frac{1}{2N}}} \cdot \frac{\left(\frac{\pi r e}{4}\right)^N}{N}. \tag{23}$$

In order that $\lim_{N \to \infty} \epsilon_N = 0$, it is sufficient to have $r < (4/\pi e) = 0.468$, i.e., the average sampling rate N/L to be approximately two times faster than the Shannon sampling rate.

Normalized Truncation Error Bound Evaluation for Band-Limited Signals

The normalized instantaneous truncation error bounds obtained for sampling reconstruction by our method as well as by Shannon interpolation are given in Table I. The signal $g(t)$ is assumed to be band-limited to the frequency W, and having a finite energy E (relation (10)), where $N' = L$ (even) is the number of nonuniformly taken samples over the interval $[-L/2, L/2]$, according to the proposed method; for the uniform sampling, assume the number of samples is $N' = L + 1$. We deduce the important advantage of our method over the Shannon interpolation for $r = 2W \leq 0.468$. Note that for our method the error bound is considered on the whole definition interval $[-L/2, L/2]$, while for the Shannon interpolation the error bound is taken on the central zone only.

Numerical Examples

Example: Consider the signal $\{g(t) = e^{-t}, t \in [0, 4]\}$.

Assume $N = 8$.

The Chebyshev sampling grid is

$$t_8 = (0.0384294 \quad 0.3370607 \quad 0.8888595 \quad 1.6098194$$
$$2.3901806 \quad 3.1111405 \quad 3.6629392 \quad 3.9615706)^T.$$

TABLE I
NORMALIZED INSTANTANEOUS ERROR BOUND FOR UNIFORM AND NONUNIFORM SAMPLING RECONSTRUCTION

TYPE OF INTERPOLATION BOUND		L \ r	0.436	0.468	0.500
SHANNON UNIFORM SAMPLING RECONSTRUCTION ERROR BOUND IN THE CENTRAL ZONE $\max_{\lvert t\rvert < \frac{1}{2}} \frac{\lvert g(t)-\hat{g}(t)\rvert}{\sqrt{E}}$ (for a signal, g(t), uniformly sampled at the points t=-L/2, -L/2+1,...,-1, 0,1,...,L/2-1,L/2; the number of samples is N'=L+1; L=even; the sampling interval=1; r=1/(1/2W)=2W)	BALAKRISHNAN-PIPER BOUND	L=16	0.159154	0.159154	0.159154
		L=32	0.112539	0.112539	0.112539
		L=64	0.079577	0.079577	0.079577
	PIPER BOUND	L=16	0.097956	0.103068	0.108390
		L=32	0.048978	0.051534	0.054195
		L=64	0.024489	0.025767	0.027097
	YAO-THOMAS BOUND	L=16	0.059310	0.065145	0.071644
		L=32	0.029650	0.032572	0.035820
		L=64	0.014827	0.016286	0.017911
	BROWN BOUND	L=16	0.114763	0.120751	0.126957
		L=32	0.057381	0.060375	0.063493
		L=64	0.028690	0.030187	0.031746
ERROR BOUND OVER THE INTERVAL [-L/2,L/2] OBTAINED BY THE PROPOSED INTERPOLATION CASCADE OF CHEBYSHEV NONUNIFORM SAMPLING FOLLOWED BY DISCRETE COSINE TRANSFORM $\max_{\lvert t\rvert < \frac{L}{2}} \frac{\lvert g(t)-\hat{g}(t)\rvert}{\sqrt{E}}$ (for a signal g(t) nonuniformly sampled in N'=L points over the interval [-L/2,L/2]; the average sampling interval $\bar{\tau}$=L/N=1; r=$\bar{\tau}$/(1/2W) = 2W; the convengence condition is: r < 0.468)		L=16	0.007395	0.023796	0.084072
		L=32	0.001174	0.011737	0.070860
		L=64	0.000059	0.005710	0.100716

$(g: [-L/2, L/2] \rightarrow \mathbb{R}, G(f) = \int_{-\infty}^{\infty} g(t) \exp(-i \cdot 2\pi f t)\, dt, i^2 = -1, E = \int_{-W}^{W} |G(f)|^2\, df)$

The vector of Chebyshev nonuniformly spaced samples is

$$g_8 = (0.9622996 \quad 0.7138654 \quad 0.4111243 \quad 0.1999237 \quad 0.0916131 \quad 0.0445501 \quad 0.1256569 \quad 0.0190331)^T.$$

The DDCT of g_8 is

$$C_8 = \psi_8^{\text{DDCT}} \cdot g_8 = \sqrt{\tfrac{8}{2}}\, \psi_8^{\text{DCT}} \cdot g_8 = (0.4362965 \quad -0.4305386 \quad 0.186478 \quad -0.0575823 \quad 0.0137306 \quad -2.65935 \cdot 10^{-3} \quad 4.32901 \cdot 10^{-4} \quad -5.995 \cdot 10^{-5})^T.$$

The interpolation formula (9), using the expressions of Chebyshev polynomials given by (4) and (13), leads to

$$\hat{g}_8(t) = \sum_{k=0}^{7} C_k \tilde{T}_k[2t/4 - 1] = 0.9999899 - 0.9996933t + 0.498378t^2 - 0.1633072t^3 + 0.381451t^4 - 6.23515 \cdot 10^{-3}t^5 + 6.361 \cdot 10^{-4}t^6 - 2.9975 \cdot 10^{-5}t^7.$$

A. Interpolation Performance Evaluation

We further evaluate by computer simulation the following interpolation performances:

i) The maximum modulus of the instantaneous error

$$\epsilon_N = \max_{t \in [0, L]} \left| g(t) - \hat{g}_N(t) \right|.$$

ii) The root mean squared interpolation error (continuous variant)

$$(\epsilon_{\text{rms}})_c = \sqrt{\frac{1}{L} \int_0^L \left[g(t) - \hat{g}_N(t) \right]^2 dt}.$$

For the proposed cascade CNS-DCT, the above considered numerical example leads to $(\epsilon_8)_{\text{CNS-DCT}} = 0.932 \cdot 10^{-5}$ and $(\epsilon_{\text{rms}})_{c\text{-CNS-DCT}} = 0.538 \cdot 10^{-5}$.

For comparison, we considered the truncated uniform Shannon interpolation (for the same number of samples $N = 8$). It leads to $(\epsilon_8)_{\text{Shannon}} = 0.579$ and $(\epsilon_{\text{rms}})_{c\text{-Shannon}} = 0.106$. The important advantage of our method over Shannon interpolation is obvious for the considered signal (which is not band limited).

B. Data Compression Performance Evaluation

If we retain only $M = N/2 = 4$ DCT coefficients, the interpolation after compression is given by

$$\hat{g}_{N,M}(t) = \hat{g}_{8,4} = \sum_{k=0}^{M-1} C_k \tilde{T}_k(2t/L - 1) = \sum_{k=0}^{3} C_k \tilde{T}_k(2t/4 - 1) = 0.9831071 - 0.8473456t + 0.2659859t^2 - 0.0287911t^3.$$

iii) To evaluate the fidelity of reconstruction, we define the discrete variant of the root mean-squared reconstruction error

$$(\epsilon_{\text{rms}})_d = \sqrt{\frac{1}{N} \sum_{k=0}^{N-1} \left[g\left(k\frac{L}{N}\right) - \hat{g}_{N,M}\left(k\frac{L}{N}\right) \right]^2}.$$

We obtain $(\epsilon_{\text{rms}})_{d\text{-CNS-DCT}} = 0.0114$.

For comparison, consider the case of the following cascade for the same signal: uniform sampling (US) at the moments $t_k = (kL)/N$, $(k = 0, 1, \cdots, 7)$, the direct discrete cosine transformation (DDCT), compression by retaining the first $M = N/2 = 4$ coefficients and the inverse discrete cosine transform (IDCT) to reconstruct $g(kL/N)$. We obtain $(\epsilon_{\text{rms}})_{d\text{-US-DCT}} = 0.0433$.

The advantage of the cascade CNS-DDCT-compression-interpolation over the classical cascade of US-DDCT-compression-IDCT (for the same average sampling rate) is obvious.

CONCLUDING REMARKS

This correspondence provides a method to rapidly compute the coefficients of the optimum interpolation formula starting from the Chebyshev nonuniformly spaced samples of the signal $g(t)$, $t \in [0, L]$, and then applying the DCT. For fast computation of the DCT, there are a lot of available algorithms [5], [9].

In order that interpolation converges, a sufficient theoretical condition is to have an average sampling rate approximately two-times faster than that one required by the Shannon theorem. The advantage of our method over the Shannon interpolation is obvious from Table I (for $r \leq 0.468$).

If we retain only the first M C_k's (DCT coefficients), where $M < N$, relation (9) remains valid, having M instead of N. Thus, we have an efficient formula for signal reconstruction in the case of data compression.

Within a future paper, we intend to present the extension of the proposed cascade CNS-DCT for two-dimensional signals.

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On the Maximum Entropy Method for Interval Covariance Sequences

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***Abstract*—Given an interval covariance sequence, we consider the existence of a maximum entropy spectral estimate. It is shown that a maximum entropy spectral estimate does exist, and, moreover, it is unique.**

I. Introduction

Consider a real and scalar stationary zero-mean stochastic process y_t, $t \in \mathbf{Z}_+$. The estimation of the power spectral density of y_t, based on observed samples, usually proceeds in two steps. First, an estimate for the first $n + 1$ covariance lags $c_k = E[y_t y_{t+k}]$, $k = 0, 1, \cdots, n$ is obtained. Second, a spectral density function consistent with the partial covariance sequence $C_n := (c_0, c_1, \cdots, c_n)$ is postulated.

Manuscript received May 5, 1989; revised November 2, 1989. This work was supported in part by the NSF under Grant ECS-8705291, MIP-8708811, ECS-8996307, and MIP-8996305.

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IEEE Log Number 9037949.

In the first step errors of a statistical nature are induced on the terms c_k, and the second step involves a nonunique choice among spectra which are consistent with the data. To resolve the nonuniqueness in the second step, and to obtain a "canonical" spectrum, the maximum entropy principle is usually invoked. This requires a spectral estimate that is consistent with the available data and is maximally noncommittal with respect to the unavailable data (see Burg [4] and Jaynes [6]).

To address the issue of the uncertainty associated with the first step, several different formulations have been proposed. For instance, in the work by Schott and McClellan [12], a covariance estimate contaminated by noise is being considered. In that work, the covariance matching constraint of the maximum entropy method is replaced by a weighted inequality, where the weight is based on knowledge of the corrupting noise. Lang and Marzetta [8], [9] use linear programming to provide an upper and lower bound of the power spectral estimate. The inverse Fourier transform relating the covariance at lag k, c_k, to the spectral density function provides the constraints. The approach also provides bounds in the case of "fuzzy" covariance estimates.

In this correspondence, we consider a partial interval covariance sequence defined as follows:

$$\begin{aligned}\mathcal{C}_n := \Big\{ &(c_0, c_1, \cdots, c_n) : c_0 = 1,\\ &c_1 \in [c_{1l}, c_{1u}], c_2 \in [c_{2l}, c_{2u}], \cdots, c_n \in [c_{nl}, c_{nu}],\\ &\text{and such that } (c_0, c_1, \cdots, c_n)\\ &\text{is an admissible covariance sequence.} \Big\}\end{aligned}$$

This is a set of possible partial covariance sequences. Any sequence $C_n \in \mathcal{C}_n$ has a unique covariance extension c_k, $k = n + 1, \cdots$, which maximizes the entropy integral

$$H = \int_{-\pi}^{\pi} \log f(\theta)\, d\theta$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ki\theta}\, d\sigma(\theta), \qquad k = 0, \pm 1, \pm 2, \cdots,$$

and $f(\theta) = \sigma'(\theta)$ a.e. is the respective spectral density function. This is the maximum entropy extension of C_n and the respective extremal value of the entropy integral we denote by $H_{ME}(C_n)$. In this paper we show that there exists a unique element $C_n \in \mathcal{C}_n$ maximizing $H_{ME}(C_n)$ over all $C_n \in \mathcal{C}_n$.

II. Notation and Preliminaries

The set of square matrices of dimension n with real elements is denoted by M_n. The determinant of a matrix M is denoted det (M) and the determinant of M with the ith row and ith column deleted is denoted det $(M(i|i))$.

Consider a sequence $C_n = (c_0, c_1, \cdots, c_n)$, $c_i \in \mathbf{R}$, for $i = 0, 1, \cdots, n$, and the associated Toeplitz matrix

$$\mathbf{T}_{C_n} := [c_{m-k}]_{k,m=0}^{n}$$

where $c_{-n} = c_n$. The sequence C_n is said to be positive (respectively, nonnegative) if $\mathbf{T}_{C_n}$ is positive definite (respectively, positive semidefinite). For a stationary zero-mean stochastic process y_t, $t \in \mathbf{Z}_+$, the covariance sequence $(c_0, c_1, \cdots, c_n)$, $c_i = E[y_t y_{t+i}]$, $i = 0, 1, \cdots, n$ is nonnegative. Conversely, C_n qualifies as a partial covariance sequence of a stationary zero-mean stochastic process if it is nonnegative. Without a loss of generality, we normalize the sequence C_n so that $c_0 = 1$.

Define $\mathcal{P}_n := \{C_n : \mathbf{T}_{C_n} \geq 0\}$. This set is bounded and convex in the Euclidian space $\mathbf{R}^n$. A partial interval covariance sequence