On Pointwise Binomial Approximation for Independent Beta Binomial Random Variables

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Abstract

Stein's method and the beta binomial w -functions are used to derive a non-uniform bound for the point metric between the distribution of a sum of n independent beta binomial random variables, each with parameters n_i, α_i and β_i , and a binomial distribution with parameters $m = \sum_{i=1}^n n_i$ and $p = \frac{1}{m} \sum_{i=1}^n \frac{n_i \alpha_i}{\alpha_i + \beta_i}$. The bound is an appropriate measurement when all β_i are large, with respect to all n_i and α_i .

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1 Introduction

Let $X_1, \dots, X_n$ be independently distributed beta binomial random variables, each with the probability function $p_{X_i}(k) = \binom{n_i}{k} \frac{B(k+\alpha_i, n_i-k+\beta_i)}{B(\alpha_i, \beta_i)}$ for $k = 0, \dots, n_i$, where B is the complete beta function, and mean $\mu_i = \frac{n_i \alpha_i}{\alpha_i + \beta_i}$ and variance $\sigma_i^2 = \frac{n_i \alpha_i \beta_i (n_i + \alpha_i + \beta_i)}{(\alpha_i + \beta_i)^2 (1 + \alpha_i + \beta_i)}$. Let $\mathcal{S}_n = \sum_{i=1}^n X_i$ and $\mathcal{B}_{m,p}$ denote the binomial random variable with parameters $m = \sum_{i=1}^n n_i$ and $p = \frac{1}{m} \sum_{i=1}^n \mu_i$. For $A \subseteq \{0, \dots, m\}$, [4] gave a uniform bound for the distance between a beta binomial distribution with parameter n, α and β and a binomial distribution with parameters n and

$p = \frac{\alpha}{\alpha+\beta}$, $d_A(\mathcal{S}_n, \mathcal{B}_{m,p}) = |P(\mathcal{S}_n \in A) - P(\mathcal{B}_{m,p} \in A)|$, as follows:

$$d_A(\mathcal{S}_n, \mathcal{B}_{m,p}) \leq \frac{1 - p^{m+1} + q^{m+1}}{(m+1)pq} \sum_{i=1}^n \left\{ |p\beta_i - q\alpha_i| + \frac{(n_i - 1)\beta_i}{\alpha_i + \beta_i + 1} \right\} \frac{n_i\alpha_i}{(\alpha_i + \beta_i)^2}. \quad (1.1)$$

However, for $A = \{x_0; x_0 \in \{0, \dots, m\}\}$ and $d_{x_0}(\mathcal{S}_n, \mathcal{B}_{m,p}) = |P(\mathcal{S}_n = x_0) - P(\mathcal{B}_{m,p} = x_0)|$, the result in (1.1) becomes

$$d_{x_0}(\mathcal{S}_n, \mathcal{B}_{m,p}) \leq \frac{1 - p^{m+1} + q^{m+1}}{(m+1)pq} \sum_{i=1}^n \left\{ |p\beta_i - q\alpha_i| + \frac{(n_i - 1)\beta_i}{\alpha_i + \beta_i + 1} \right\} \frac{n_i\alpha_i}{(\alpha_i + \beta_i)^2} \quad (1.2)$$

for every x_0 . It is observed that the bound is a uniform constant for the point metric $d_{x_0}(\mathcal{S}_n, \mathcal{B}_{m,p})$. With this situation, a non-uniform bound with respect to x_0 is required. In this paper, we focus on deriving a non-uniform bound for this point metric.

2 Method

The following lemma gives the beta binomial w -functions, which are directly obtained from [3].

Lemma 2.1. *For $1 \leq i \leq n$, let w_i be the w -function associated with the beta binomial random variable X_i , then we have the following:*

$$w_i(k) = \frac{(n_i - k)(\alpha_i + k)}{(\alpha_i + \beta_i)\sigma_i^2}, \quad k = 0, \dots, n_i. \quad (2.1)$$

The following relation is an important property for proving the result, which was stated by [2].

$$\begin{aligned} Cov(\mathcal{S}_n, f(\mathcal{S}_n)) &= \sum_{i=1}^n Cov \left(X_i, f \left(X_i + \sum_{j \neq i} X_j \right) \right) \\ &= \sum_{i=1}^n \sigma_i^2 E[w_i(X_i) \Delta f(\mathcal{S}_n)], \end{aligned} \quad (2.2)$$

for any function $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ for which $E|w_i(X_i) \Delta f(\mathcal{S}_n)| < \infty$, where $\Delta f(x) = f(x+1) - f(x)$.

For Stein's method in the binomial approximation, it can be applied for every $m \in \mathbb{N}$ and $0 < p = 1 - q < 1$, for every $x_0 \in \{0, \dots, m\}$ and bounded real-valued function $f = f_{\{x_0\}} : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ defined as in [1], where $f(0) = f(1)$

and $f(x) = f(m)$ for $x \geq m$. So, Stein's equation for these conditions is as follows:

$$P(\mathcal{S}_n = x_0) - P(\mathcal{B}_{m,p} = x_0) = E[(m - \mathcal{S}_n)pf(\mathcal{S}_n + 1) - q\mathcal{S}_nf(\mathcal{S}_n)]. \quad (2.3)$$

For $x_0, x \in \mathbb{N} \cup \{0\}$, [5] showed that

$$\sup_{x \geq 0} |\Delta f(x)| \leq \delta(x_0) = \begin{cases} \frac{1-q^m}{np} & \text{if } x_0 = 0, \\ \min \left\{ \frac{1-p^m}{x_0 q}, \frac{1-p^{m+1}q^{m+1}}{(m+1)pq} \right\} & \text{if } x_0 > 0. \end{cases} \quad (2.4)$$

3 Result

The following theorem presents the main result of this study.

Theorem 3.1. *For $x_0 \in \{0, \dots, m\}$, then we have the following:*

$$d_{x_0}(\mathcal{S}_n, \mathcal{B}_{m,p}) \leq \delta(x_0) \sum_{i=1}^n \left\{ |p\beta_i - q\alpha_i| + \frac{(n_i - 1)\beta_i}{\alpha_i + \beta_i + 1} \right\} \frac{n_i \alpha_i}{(\alpha_i + \beta_i)^2}, \quad (3.1)$$

where $\delta(x_0)$ is defined in (2.4).

Proof. From (2.3), it follows that

$$\begin{aligned} d_{TV}(\mathcal{S}_n, \mathcal{B}_{m,p}) &= |E[(m - \mathcal{S}_n)pf(\mathcal{S}_n + 1) - q\mathcal{S}_nf(\mathcal{S}_n)]| \\ &= |E[mpf(\mathcal{S}_n + 1) - p\mathcal{S}_n\Delta f(\mathcal{S}_n) - \mathcal{S}_nf(\mathcal{S}_n)]| \\ &= |E[mp\Delta f(\mathcal{S}_n)] - pE[\mathcal{S}_n\Delta f(\mathcal{S}_n)] - Cov(\mathcal{S}_n, f(\mathcal{S}_n))| \\ &= \left| \sum_{i=1}^n \{E[\mu_i\Delta f(\mathcal{S}_n)] - pE[X_i\Delta f(\mathcal{S}_n)] - Cov(X_i, f(\mathcal{S}_n))\} \right|. \end{aligned}$$

Using (2.2) and Lemma 2.1, we have

$$\begin{aligned}
 d_{TV}(\mathcal{S}_n, \mathcal{B}_{m,p}) &= \left| \sum_{i=1}^n \left\{ E[(\mu_i - pX_i)\Delta f(\mathcal{S}_n)] - \sigma_i^2 E[w_i(X_i)\Delta f(\mathcal{S}_n)] \right\} \right| \\
 &\leq \sum_{i=1}^n E\{|\mu_i - pX_i - \sigma_i^2 w_i(X_i)| |\Delta f(\mathcal{S}_n)|\} \\
 &\leq \sup_{x \geq 0} |\Delta f(x)| \sum_{i=1}^n E \left\{ \left| \frac{n_i \alpha_i}{\alpha_i + \beta_i} - pX_i - \frac{(n_i - X_i)(\alpha_i + X_i)}{\alpha_i + \beta_i} \right| \right\} \\
 &= \sup_{x \geq 0} |\Delta f(x)| \sum_{i=1}^n E \left\{ \left| -pX_i - \frac{(n_i - X_i)X_i}{\alpha_i + \beta_i} + \frac{\alpha_i X_i}{\alpha_i + \beta_i} \right| \right\} \\
 &\leq \sup_{x \geq 0} |\Delta f(x)| \sum_{i=1}^n \left\{ |p\beta_i - p\alpha_i| E(X_i) + \frac{E(n_i X_i - X_i^2)}{1 + \alpha_i + \beta_i} \right\} \frac{1}{\alpha_i + \beta_i} \\
 &= \sup_{x \geq 0} |\Delta f(x)| \sum_{i=1}^n \left\{ |p\beta_i - p\alpha_i| + \frac{(n_i - 1)\beta_i}{1 + \alpha_i + \beta_i} \right\} \frac{n_i \alpha_i}{(\alpha_i + \beta_i)^2}.
 \end{aligned}$$

Hence, by (2.4), (3.1) is obtained. $\square$

4 Conclusion

In this study, a non-uniform bound for the point metric between the distribution of a sum of n independent beta binomial random variables and a binomial distribution with parameters $m = \sum_{i=1}^n n_i$ and $p = \frac{1}{m} \sum_{i=1}^n \frac{n_i \alpha_i}{\alpha_i + \beta_i}$ was derived by Stein's method and the beta binomial w -functions. By comparing, the bound in this study is sharper than that presented in (1.2).

References

- [1] A.D. Barbour, L. Holst, S. Janson, Poisson approximation, Oxford Studies in Probability 2, Clarendon Press, Oxford, 1992.
- [2] T. Cacoullos, V. Papathanasiou, Characterization of distributions by variance bounds, Statist. Probab. Lett., **7** (1989), 351–356. [http://dx.doi.org/10.1016/0167-7152\(89\)90050-3](http://dx.doi.org/10.1016/0167-7152(89)90050-3)
- [3] K. Teerapabolarn, A bound on the binomial approximation to the beta binomial distribution, Int. Math. Forum, **3** (2008), 1355–1358.
- [4] K. Teerapabolarn, Binomial approximation for a sum of independent beta binomial random variables, Appl. Math. Sci., **8**(2014), 8929–8932. <http://dx.doi.org/10.12988/ams.2014.410837>

- [5] K. Teerapabolarn, P. Wongkasem, On pointwise binomial approximation by w -functions, *Int. J. Pure Appl. Math.*, **71** (2011), 57–66.

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