

# On Dynamic Decision Making to Meet Consumption Targets

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## Abstract

We investigate a dynamic decision model that facilitates a target-oriented decision maker in regulating her risky consumption based on her desired target consumption level in every period in a finite planning horizon. We focus on dynamic operational decision problems of a firm where risky cash flows are being resolved over time. The firm can finance consumption by borrowing or saving to attain prescribed consumption targets over time. To evaluate the ability of the consumption in meeting respective targets, we propose the Consumption Shortfall Risk (CSR) criterion, which has salient properties of attainment content, starvation aversion, subadditivity and positive homogeneity. We show that if borrowing and saving are unrestricted and their interest rates are common, the optimal policy that minimizes the CSR criterion is to finance consumption at the target level for all periods except the last. [For general convex dynamic decision problems, the optimal policies correspond to those that maximize an additive expected utility, in which the underlying utility functions are concave and increasing.](#) Despite the interesting properties, this approach violates the principle of normative utility theory and we discuss the limitations of [our](#) target-oriented decision model.

**Keywords:** dynamic programming, targets, riskiness index

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# 1 Introduction

Firms plan their operational decisions such as procurement, production, and inventory replenishment to generate cash flow for day-to-day expenditures (e.g., wages, dividend payments, etc) and more importantly, to grow the company. Due to multiple sources of randomness (e.g., demand volatility) embedded in the business environment, cash flows arising from operational decisions are risky and this complicates the decision making process. In principle, an analyst could construct a dynamic model to evaluate a decision that will reveal the corresponding cash flow profile, i.e., the probability distributions of the present values of the risky cash flow over time. [The decision criterion serves as a key factor in this process as it translates the preference of a decision maker.](#) Among the others, the risk neutral framework is widely adopted as it can often be analyzed and solved via Bellman's (1957) dynamic programming when the state space is of moderate size.

Despite the technical attractiveness of risk neutral decision criteria, they neglect the risks involved in the operational cash flows and may not appeal to managers who are averse to potential losses along the planning horizon. They also ignore the fact that decision makers can be sensitive to the timing of the resolution of uncertainties. For example, suppose a firm will have a risky cash flow 10 years from now, knowing it today can be preferred than knowing it 10 years later because by having the knowledge of the cash flow today, managers can better plan corporate activities such as expansion and investment in new technology. Markowitz (1959) and Matheson and Howard (1989), among others, recognize this as the problem of "temporal risk." In order to resolve this problem, decision makers should be allowed to borrow or save to smooth out their *consumption* over the planning horizon. In the corporate world, consumption can refer to expenditures such as wages and dividend payments. More details on "temporal risk problem" can be found in Smith (1998).

Recognizing the risks in the operational cash flow as well as the temporal risk, we study a firm's operational and financing decisions concurrently by modelling a finite horizon dynamic decision problem. We refer to *operational decisions* as those (e.g., production quantity and inventory planning) that would directly affect the operational cash flow in response to underlying uncertainties, and *financing decisions* as the amount of money the firm borrows or saves that

will impact the status of wealth of the firm. In every period except for the last, the total cash flow arising from the operational and financing decisions is the firm's consumption. To illustrate, if the firm needs to consume more than the operational cash flow, she'll borrow at a cost and if she consumes less, she will save and earn the interest. The consumption in the final period is defined as the total wealth of the company in the consideration that a firm's ultimate goal is to grow the company. We define *consumption profile* as the probability distributions of the present values of the risky consumption over time.

One widely adopted criterion for evaluating the consumption profile is expected utility, which captures a decision maker's risk awareness and also has a strong normative basis. Dynamic programming under the expected utility criterion has been proposed and studied by Howard and Matheson (1972), Porteus (1975) and Jaquette (1976) among others.

In this paper, we explore a different decision criterion for dynamic optimization to cater for a subset of decision makers who are target-oriented instead of utility maximizers. [The practical prevalence of targets has been well documented in literature on descriptive decision making \(see, for instance, Simon \(1955\); Lanzillotti \(1958\) and Mao \(1970\)\).](#) From the normative perspective, research interests in target-oriented utility can be traced to Borch (1968) and have rekindled in recent years (Bordley and LiCalzi, 2000; Bordley and Kirkwood, 2004; Castagnoli and Calzi, 1996 and Tsetlin and Winkler, 2007). Abbas and Matheson (2005, 2009) also attempt to incorporate the concept of target in the utility theory and they study the conditions under which maximizing the probability of target attainment would be equivalent to the preference that is based on expected utility. Nevertheless, Diecidue and van de Ven (2008) argue that the criterion based on the probability of target attainment is "too crude to be normatively or descriptively relevant" .

We develop a new dynamic decision criterion, *Consumption Shortfall Risk* (CSR) criterion, that evaluates the prospect of a consumption profile in achieving consumption targets over time. This decision criterion is especially of practical meaning, as an important aspect of many managers' decision making is attainment of predetermined targets (of consumption levels, profits, or company stock prices), which is a direct reflection of their performance. Therefore, we investigate the proposed CSR criterion to capture the effect of targets. In the case of single period, CSR criterion is related to the satisficing measure axiomatized by Brown and Sim (2009) and Brown et al. (2012).

In managing operations and financing decisions to meet consumption targets, we investigate the approach to optimize over a set of feasible policies to yield the solution with the lowest CSR criterion. We show that if the borrowing and saving are unconstrained and their interest rates are common, all consumption targets, except the terminal one, are met with certainty. Moreover, for convex dynamic decision problems, the optimal policies correspond to those that optimize an expected additive utility. Through the choice of targets, our approach determines the weights and parameters of the utility functions by solving an optimization problem. Despite these interesting properties, it is important to note that our approach deviates from the principles of normative utility theory, and we caution the use of this approach and discuss its limitations in the paper.

The rest of the paper is organized as follows. Section 2 discusses a general framework for joint operational and financing decisions. The decision criterion CSR is also introduced. Section 3 provides the optimal policies for both the scenarios of unrestricted and restricted financing. We discuss the limitations of the target-oriented decision criterion in Section 4 and conclude the paper in Section 5.

**Notations:** A vector such as  $\mathbf{x}$  is represented as a boldface character and  $x_i$  denotes the  $i^{th}$  element of the vector. To simplify the notation, for any given vector  $\mathbf{x}$ , we define  $(y, \mathbf{x}_{-i})$  as the vector with all elements identical to those in  $\mathbf{x}$ , except the  $i$ th element being  $y$ ; i.e.,  $(y, \mathbf{x}_{-i}) = (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_T)$ . We denote a random variable by a character with the tilde sign such as  $\tilde{z}$ , and  $z$  is its realization. A random variable is defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where  $\Omega$  is the set of possible outcomes,  $\mathcal{F}$  is a  $\sigma$ -algebra that describes the set of all possible events, and  $\mathbb{P}$  is the probability measure function. We also use the shorthand  $\tilde{\mathbf{u}} \geq \tilde{\mathbf{v}}$  to denote  $\mathbb{P}(\tilde{\mathbf{u}} \geq \tilde{\mathbf{v}}) = 1$ , and use  $\tilde{\mathbf{u}} > \tilde{\mathbf{v}}$  to represent that there exists  $\epsilon > 0$  such that  $\tilde{\mathbf{u}} \geq \tilde{\mathbf{v}} + \epsilon \mathbf{1}$ . Table 1 summarizes the notations used throughout the paper.

## 2 Influence of operational and financing decisions on consumption

We consider a firm making operational decisions (e.g., inventory planning, procurement) in the presence of uncertainties such as demand variability and supply volatility. The resulting cash

Miscellaneous		Parameters	
$\mathbf{1}$	vector of ones	$\tilde{\mathbf{z}}_t$	vector of uncertainties in period $t \in \mathcal{T}$
$\mathbf{0}$	vector of zeros	$\mathbf{z}_t$	realizations of $\tilde{\mathbf{z}}_t$ , $t \in \mathcal{T}$
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space	$\mathbf{z}_{[t]}$	realizations of $(\tilde{\mathbf{z}}_1, \dots, \tilde{\mathbf{z}}_t)$ ( $\mathbf{z}_{[0]} = \{\}$ )
Sets and Indices		$w_t$	wealth at the beginning of period $t \in \mathcal{T}$
$T$	length of planning horizon	$\mathbf{x}_t$	operational state at the beginning of period $t \in \mathcal{T}$
$\mathcal{T}$	set defined as $\{1, \dots, T\}$	$\tau_t$	consumption target at the end of period $t \in \mathcal{T}$
$\mathcal{T}^-$	set defined as $\{1, \dots, T-1\}$	$r_t$	operational cash flow in period $t \in \mathcal{T}$
$t$	time period $t \in \mathcal{T}$	$c_t$	consumption level at period $t \in \mathcal{T}$
$\mathcal{Q}$	set of operational policies $\mathbf{\Pi}$	$\beta_t^S$	saving interest rate at period $t \in \mathcal{T}$
$\mathcal{P}_H$	set of policies $\mathbf{\Psi} = (\mathbf{\Pi}, \mathbf{\Phi})$	$\beta_t^B$	borrowing interest rate at period $t \in \mathcal{T}$
$\mathcal{V}$	set of all random variables	$\beta$	common interest rate when $\beta_t^S = \beta_t^B \forall t \in \mathcal{T}$
		$\tilde{v}_t$	random net present value of the consumption excess at the end of period $t \in \mathcal{T}$ , it depends on $\mathbf{\Psi}$
Decisions		Functions	
$\mathbf{a}_t$	operational control in period $t \in \mathcal{T}$	$\mathcal{A}_t$	mapping from $\mathbf{x}_t$ to the feasible set of $\mathbf{a}_t$
$b_t$	financing level at period $t \in \mathcal{T}$	$\theta_t$	mapping from $\mathbf{x}_t$ to $\tau_t$
$\boldsymbol{\pi}_t$	mapping from $\mathbf{z}_{[t-1]}$ into $\mathbf{a}_t$	$g_t$	mapping from $(\mathbf{x}_t, \mathbf{a}_t, \mathbf{z}_t)$ to $r_t$
$\mathbf{\Pi}$	history dependent operational policy	$\mathbf{f}_t$	mapping from $(\mathbf{x}_t, \mathbf{a}_t, \mathbf{z}_t)$ to $\mathbf{x}_{t+1}$
$\phi_t$	mapping from $\mathbf{z}_{[t]}$ into $b_t$	$F_t$	mapping from $(\mathbf{x}_{t+1}, w_t, r_t)$ to the feasible set of $b_t$
$\mathbf{\Phi}$	history dependent financing policy	$m_t$	function for wealth updating : $w_{t+1} = m_t(w_t) - b_t$
$\mathbf{\Psi}$	admissible history dependent policy	$\psi$	CSR criterion
		$\mu$	convex risk measure
		$\mu_U$	shortfall risk measure
		$\psi_U$	utility based CSR
		$u$	increasing concave normalized utility function

Table 1: Notations

flow from operational decisions are used for consumption as well as increasing firm's wealth (or value in other words). The firm makes financing decisions via saving and borrowing that would impact her wealth status to smooth out her consumption over the planning horizon  $T$ .

We have  $\tilde{\mathbf{z}}_t : \Omega \mapsto \mathfrak{R}^n$ ,  $t \in \{1, \dots, T\}$  represent the vector of uncertainties in period  $t$ , which are independently distributed and resolved overtime. We further define the  $n \times t$  vector  $\mathbf{z}_{[t]} = (\mathbf{z}_1, \dots, \mathbf{z}_t)$ ,  $t \in \{1, \dots, T\}$  as the realizations of the uncertainties from the first period to the end of period  $t$  and that  $\mathbf{z}_{[0]} = \{\}$ . For convenience, we define the index sets,  $\mathcal{T} = \{1, \dots, T\}$  and  $\mathcal{T}^- = \{1, \dots, T-1\}$ .

The sequence of events in any period  $t$ ,  $t \in \mathcal{T}^-$  is as follows: I) At the beginning of the period, the wealth of the firm is represented by  $w_t \in \mathfrak{R}$ . The firm observes her operational state of the system,  $\mathbf{x}_t \in \mathfrak{R}^{l_t}$ , which is Markovian. We assume that the system operational state contains sufficient statistics for the decision maker to determine her desired consumption target at the end of the period, which is given by  $\tau_t = \theta_t(\mathbf{x}_t)$ <sup>1</sup>. This approach provides the flexibility for the decision maker to have fixed targets for the entire horizon or endogenous targets that are influenced by, for instance, the states of the economy. II) The firm then administers an operational control  $\mathbf{a}_t$ , which takes values in a nonempty set  $\mathcal{A}_t(\mathbf{x}_t) \subseteq \mathfrak{R}^{o_t}$ , i.e.,  $\mathbf{a}_t \in \mathcal{A}_t(\mathbf{x}_t)$ . III) Near the end of the period, the uncertainty  $\tilde{\mathbf{z}}_t$  is resolved and takes a value of  $\mathbf{z}_t$ , which results in an operational cash flow of  $r_t = g_t(\mathbf{x}_t, \mathbf{a}_t, \mathbf{z}_t)$ . The operational state of the system is updated as  $\mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t, \mathbf{z}_t)$ . IV) The firm then determines the financing level,  $b_t$ , to fulfill the level of consumption  $c_t$  at the end of the period so that  $c_t = r_t + b_t$ . Whenever the consumption exceeds the operational income, i.e.,  $c_t > r_t$ , cash  $b_t$  is utilized from the firm's wealth to support consumption, which may require the firm to borrow if there is a shortfall in the wealth level. Likewise, whenever  $c_t < r_t$ , the surplus of operational cash flow over consumption,  $-b_t$  is pumped back to the firm's wealth. Note that here "financing" is defined in a broad sense since borrowing from an external institute occurs only if there is no sufficient wealth. We study both the cases when the financing level,  $b_t$  is constrained and when it is unlimited. In the former case, we note that the financing level,  $b_t$  can depend on the firm's wealth level,  $w_t$ , the system's updated

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<sup>1</sup>Note that in general, we can extend the results so that the target update function,  $\theta_t$  depends on both the system operational state  $\mathbf{x}_t$  and the state of wealth,  $w_t$ . This would [necessarily](#) increase the state space of the dynamic optimization problem and affect the structure of its optimal policy

operational state,  $\mathbf{x}_{t+1}$  as well as the realized operational cash flow  $r_t$ , i.e.,  $b_t \in F_t(\mathbf{x}_{t+1}, w_t, r_t)$ . Further, by having  $F_t(\mathbf{x}_{t+1}, w_t, r_t) \subseteq \{b : r_t + b \geq 0\}$ , we can enforce nonnegative consumption, which ensures that the operational expenditures for generating the cash flow  $r_t$  can be fully paid from financing decisions, before the excess cash flow is used for consumption. For the case of unrestricted financing, we have  $F_t(\cdot) = \mathfrak{R}$ . V) Finally, the state of wealth is updated as

$$w_{t+1} = m_t(w_t) - b_t,$$

where

$$m_t(w_t) = (1 + \beta_t^S) \max\{w_t, 0\} + (1 + \beta_t^B) \min\{w_t, 0\} \quad \forall t \in \mathcal{T}.$$

Note that  $\beta_t^S$  and  $\beta_t^B$  are the saving and borrowing interest rates, respectively, and that we assume  $\beta_t^S \leq \beta_t^B$  for all  $t \in \mathcal{T}$ .<sup>2</sup> Hence,  $m_t$  is a concave increasing function. In particular, we refer to *common interest rate financing* if  $\beta_t^B = \beta_t^S = \beta$  for all  $t \in \mathcal{T}^-$ , where  $\beta$  is the discount rate for evaluating net present values.

The sequence of events in the terminal period  $T$  is the same as that for earlier periods except that there is no financing decision involved, i.e.,  $b_T = 0$ . This is because we require all outstanding borrowing and interest earning to be settled by the end of period  $T$  so that the firm does not have outstanding balances. Given the different events in the terminal period and other periods, the consumption over the entire horizon is given as follows. For  $t \in \mathcal{T}^-$ ,  $c_t = r_t + b_t$  and in the terminal period,  $c_T = r_T + m_T(w_T)$ . Taking a closer look at  $c_T$ , we can see that it represents the accumulated wealth gained from the operational cash flow subtracting the consumption and financing costs incurred in earlier periods.

Note that in the event where the firm cannot pay off the debt in the last period, i.e.,  $c_T < 0$ , it can go bankrupt. The bankruptcy risk can be captured by the borrowing rate  $\beta_t^B$ , which can include a bankruptcy-risk associated premium. Instead of modeling the borrowing rate endogenously, which can depend on the operational cash flow, we assume the bankruptcy cost is exogenously given. We also acknowledge that we do not cover the broader context of financing decisions such as those associated with bankruptcy processes. As such, the treatment of the financial dynamics in our model (e.g., no bankruptcy process, common interest rate financing) is

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<sup>2</sup>It is not uncommon to see a higher borrowing rate than a saving rate.

somewhat idealistic. We will demonstrate, however, that by trading off a more realistic financial system, we are able to have tractable analysis, obtain interesting insights on the policy structure, and gain computational benefits for our proposed target-oriented dynamic optimization model. Hence, this approach can serve as a practical approximation for more complex financial dynamics.

**Definition 1** *A history dependent operational policy is a sequence of  $T$  measurable functions given by  $\mathbf{\Pi} = \{\pi_1, \dots, \pi_T\}$  where  $\pi_t : \mathbb{R}^{n \times (t-1)} \mapsto \mathbb{R}^{o_t}$  maps from a history of realizations of uncertainties  $\mathbf{z}_{[t-1]}$  into operational control  $\mathbf{a}_t = \pi_t(\mathbf{z}_{[t-1]})$ . Likewise, a history dependent financing policy is a sequence of  $T - 1$  measurable functions given by  $\mathbf{\Phi} = \{\phi_1, \dots, \phi_{T-1}\}$  where  $\phi_t : \mathbb{R}^{n \times t} \mapsto \mathbb{R}$  maps from a history of realizations of uncertainties  $\mathbf{z}_{[t]}$  into financing decision  $b_t = \phi_t(\mathbf{z}_{[t]})$ .*

Note that the function  $\pi_1$  is a constant. In addition,  $\pi_{t+1}$  and  $\phi_t$  map from the same space due to the sequence of events described earlier – there is a time period lag between the operational control and the financing decision.

Starting with an initial operational state  $\mathbf{x}_1$ , an initial wealth  $w_1$ , we focus on the admissible history dependent policies, which are defined as follows.

**Definition 2** *An admissible history dependent policy  $\Psi = (\mathbf{\Pi}, \mathbf{\Phi})$  is a pair of a history dependent operational policy  $\mathbf{\Pi}$  and a history dependent financing policy  $\mathbf{\Phi}$ , such that  $\pi_t(\mathbf{z}_{[t-1]}) \in \mathcal{A}_t(\mathbf{x}_t)$  and  $\phi_t(\mathbf{z}_{[t]}) \in F_t(\mathbf{x}_{t+1}, w_t, r_t)$  for all possible histories  $\mathbf{z}_{[t]}$ . We define  $\mathcal{P}_H$  as the set of all admissible history dependent policies.*

Given an admissible history dependent policy,  $\Psi = (\mathbf{\Pi}, \mathbf{\Phi}) \in \mathcal{P}_H$ , the operational states  $\tilde{\mathbf{x}}_t$  and the wealth of the firm  $\tilde{w}_t$  are random variables with distributions defined through the following system equations:

$$\tilde{\mathbf{x}}_{t+1} = \mathbf{f}_t(\tilde{\mathbf{x}}_t, \pi_t(\tilde{\mathbf{z}}_{[t-1]}), \tilde{\mathbf{z}}_t)$$

and

$$\tilde{w}_{t+1} = m_t(\tilde{w}_t) - \phi_t(\tilde{\mathbf{z}}_{[t]})$$



for all  $t \in \mathcal{T}^-$ . The net present value of the consumption excess at the end of period  $t$  is a random variable given by

$$\tilde{v}_t(\Psi) = \begin{cases} \frac{1}{(1+\beta)^t} (g_t(\tilde{\mathbf{x}}_t, \boldsymbol{\pi}_t(\tilde{\mathbf{z}}_{[t-1]}), \tilde{\mathbf{z}}_t) + \phi_t(\tilde{\mathbf{z}}_{[t]}) - \theta_t(\tilde{\mathbf{x}}_t)) & \text{if } t \in \mathcal{T}^-, \\ \frac{1}{(1+\beta)^T} (g_T(\tilde{\mathbf{x}}_T, \boldsymbol{\pi}_T(\tilde{\mathbf{z}}_{[T-1]}), \tilde{\mathbf{z}}_T) + m_T(\tilde{w}_T) - \theta_T(\tilde{\mathbf{x}}_T)) & \text{if } t = T. \end{cases}$$

We define  $\tilde{\mathbf{v}}(\Psi) = (\tilde{v}_1(\Psi), \dots, \tilde{v}_T(\Psi))$  to represent the consumption excess profile as a function of the admissible history dependent policies,  $\Psi \in \mathcal{P}_H$ .

Note that although we do not directly model bankruptcy processes, our model is still potentially useful for a firm to mitigate her risk of bankruptcy. For instance, if liability is disallowed at all periods, we could impose the constraint on financing decision  $b_t$  so that  $b_t \in F_t(\mathbf{x}_{t+1}, w_t, r_t) \subseteq \{b : m_t(w_t) - b_t \geq 0\}$ , which may lead to rather conservative or even infeasible solution. Alternatively, we can mitigate the bankruptcy risk by minimizing the riskiness of the debt not fully paid at the end of the planning horizon, which could be modeled by imposing a target of zero at the the end of the horizon, i.e.,  $\tau_T = 0$ .

## Consumption Shortfall Risk (CSR) criterion

Although the joint probability of achieving targets is a natural candidate for a target-oriented decision criterion, Diecidue and van de Ven (2008) argue against success probability as it tacitly assumes that the decision maker is indifferent to the magnitude of the losses when they occur. Our criterion is built upon the *satisficing measure* recently axiomatized by Brown and Sim (2009) and the riskiness index of Aumann and Serrano (2008). Let  $\mathcal{V}$  be a set of random variables on  $\Omega$  in which  $\tilde{v}_t \in \mathcal{V}$  denotes the present value of the uncertain consumption excess that will be realized at the end of period  $t$ .

**Definition 3** *A function  $\psi : \mathcal{V}^T \mapsto [0, \infty]$  is a Consumption Shortfall Risk (CSR) criterion if for all  $\tilde{\mathbf{v}}, \tilde{\mathbf{v}}^o \in \mathcal{V}^T$ , it satisfies the following properties:*

(P1) *Monotonicity: if  $\tilde{\mathbf{v}} \geq \tilde{\mathbf{v}}^o$ , then  $\psi(\tilde{\mathbf{v}}) \leq \psi(\tilde{\mathbf{v}}^o)$ .*

(P2) *Attainment content: if  $\tilde{\mathbf{v}} \geq \mathbf{0}$ , then  $\psi(\tilde{\mathbf{v}}) = 0$ . Moreover, if  $\exists t$  such that  $\tilde{v}_t \geq 0$ , then  $\psi(\tilde{\mathbf{v}}) = \psi((0, \tilde{\mathbf{v}}_{-t}))$ .*

(P3) *Starvation aversion: if  $\exists t$  such that  $\tilde{v}_t < 0$ , then  $\psi(\tilde{\mathbf{v}}) = \infty$ .*

(P4) *Period-wise additive*:  $\psi(\tilde{\mathbf{v}}) = \sum_{t \in \mathcal{T}} \psi(\tilde{v}_t \mathbf{e}_t)$ .

(P5) *Order invariance*:  $\psi(\tilde{\mathbf{v}}) = \psi(\mathbf{P}\tilde{\mathbf{v}})$  for all permutation matrix  $\mathbf{P}$ .

(P6) *Subadditivity*:  $\psi(\tilde{\mathbf{v}} + \tilde{\mathbf{v}}^o) \leq \psi(\tilde{\mathbf{v}}) + \psi(\tilde{\mathbf{v}}^o)$ .

(P7) *Positive homogeneity*:  $\psi(\lambda\tilde{\mathbf{v}}) = \lambda\psi(\tilde{\mathbf{v}})$ ,  $\forall \lambda > 0$ .

(P8) *Right continuity*:  $\lim_{a \downarrow 0} \psi(\tilde{\mathbf{v}} + a\mathbf{1}) = \psi(\tilde{\mathbf{v}})$ .

We define the CSR criterion so that a lower value is associated with a consumption excess profile that has a lower risk of shortfalls. Monotonicity implies that if a consumption profile beats another almost surely, then it would not be less preferred.

Attainment content reflects that a consumption excess profile without shortfall risks will attain the lowest CSR value of zero. Furthermore, if there exists a time period in which the target consumption can be met almost surely, then the CSR criterion would be insensitive to the magnitude of its excess over the target.

Starvation aversion states that the CSR criterion is absolutely intolerant to any consumption profile in which the targets of at least one of the periods cannot be attained almost surely. Period-wise additive implies that the overall consumption shortfall risk is an aggregation of the risks in individual periods. Hence, if more periods are considered, then the overall consumption shortfall risk would be added accordingly.

Order-invariance implies that the sequence of the consumption excesses over the time periods does not matter. Recall that  $\tilde{\mathbf{v}}$  denotes the present values of the uncertain consumption excesses that will be realized in future. Hence, two consumption excesses at different periods with the same marginal distributions will be equally ranked under order invariance. However, with order invariance, consumption preference at different periods could still be specified by having different targets over time.

Subadditivity is a property that is associated with the preference for diversification, which has motivation in financial risk management (see for instance coherent risk measures of Artzner et al. (1999) ). It implies that when two consumption profiles are combined, the collective risk in meeting the overall targets would not be more than the sum of the risks from individual consumption profiles in meeting their own targets.

Positive homogeneity embodies the cardinal nature, such that  $2\tilde{\mathbf{v}}$  is “twice” as risky as  $\tilde{\mathbf{v}}$  for all  $\mathbf{v} \in \mathcal{V}^T$ . Motivations for positive homogeneity can be also be found in Aztner et al. (1999) and Aumann and Serrano (2008).

Finally, the right continuity implies that if the consumption is augmented with an infinitesimally small but positive amount, the risk level remains the same; i.e., the CSR criterion is insensitive to small increase in the consumption. Moreover, it also enables us to characterize an alternate representation of the CSR criterion via a convex risk measure as follows:

**Theorem 1**  $\psi : \mathcal{V}^T \mapsto [0, \infty]$  is a CSR criterion if and only if it has the representation

$$\psi(\tilde{\mathbf{v}}) = \inf \left\{ \sum_{t \in \mathcal{T}} \alpha_t : \mu \left( \frac{\tilde{v}_t}{\alpha_t} \right) \leq 0 \text{ and } \alpha_t > 0, \forall t \in \mathcal{T} \right\} \quad (1)$$

where we define  $\inf \emptyset = \infty$  by convention and  $\mu : \mathcal{V} \mapsto \mathfrak{R}$  is a convex risk measure, i.e., for all  $\tilde{v}, \tilde{v}^\circ \in \mathcal{V}$  it satisfies the following properties:

1. *Monotonicity:* If  $\tilde{v} \geq \tilde{v}^\circ$ , then  $\mu(\tilde{v}) \leq \mu(\tilde{v}^\circ)$ .
2. *Cash invariance:*  $\mu(\tilde{v} + w) = \mu(\tilde{v}) - w$  for all  $w \in \mathfrak{R}$ .
3. *Convexity:*  $\mu(\lambda\tilde{v} + (1 - \lambda)\tilde{v}^\circ) \leq \lambda\mu(\tilde{v}) + (1 - \lambda)\mu(\tilde{v}^\circ)$  for all  $\lambda \in [0, 1]$ .
4. *Normalization:*  $\mu(0) = 0$ .

Conversely, given a CSR criterion  $\psi$ , the underlying convex risk measure is represented by

$$\mu(\tilde{v}) = \min \{a : \psi((\tilde{v} + a)\mathbf{e}_1) \leq 1\}. \quad (2)$$

**Proof :** We first prove that any  $\psi$  is a CSR criterion if it has representation (1) with  $\mu$  being a convex risk measure.

*Monotonicity:* It is an obvious result from the monotonicity of  $\mu$ .

*Attainment content:* If  $\tilde{\mathbf{v}} \geq \mathbf{0}$ , we have  $\mu(\tilde{v}_t/k) \leq \mu(0) = 0$  for any  $t \in \mathcal{T}$ ,  $k > 0$ ; hence,  $\psi(\tilde{\mathbf{v}}) = 0$ .

If  $\exists t \in \mathcal{T}$  with  $\tilde{v}_t \geq 0$ , we observe that both  $\psi(\tilde{\mathbf{v}})$  and  $\psi((0, \tilde{\mathbf{v}}_{-t}))$  have the value of

$$\inf \left\{ \sum_{j \in \mathcal{T}, j \neq t} k_j : \mu \left( \frac{\tilde{v}_j}{k_j} \right) \leq 0 \text{ and } k_j > 0, \forall j \in \mathcal{T} \text{ and } j \neq t \right\}.$$

*Starvation aversion:* If  $\exists t \in \mathcal{T}$  with  $\tilde{v}_t < 0$ , then  $\forall \mathbf{k} > \mathbf{0}$  we have  $\tilde{v}_t/k_t < 0$  and hence  $\exists \epsilon > 0$  such that  $\tilde{v}_t/k_t + \epsilon \leq 0$ . Therefore,  $\mu(\tilde{v}_t/k_t) = \mu(\tilde{v}_t/k_t + \epsilon) + \epsilon \geq \epsilon > 0$ ,  $\psi(\tilde{\mathbf{v}}) = \infty$ .

*Period-wise additive:* This trivially follows from representation (1).

*Order invariance:* This trivially follows from representation (1).

*Subadditivity:* Let  $S_v = \{\mathbf{k} > \mathbf{0} : \mu(\tilde{v}_t/k_t) \leq 0, \forall t \in \mathcal{T}\}$ ,  $S_{v^o} = \{\mathbf{k} > \mathbf{0} : \mu(\tilde{v}_t^o/k_t) \leq 0, \forall t \in \mathcal{T}\}$ , and  $S_{vv^o} = \{\mathbf{k} > \mathbf{0} : \mu((\tilde{v}_t + \tilde{v}_t^o)/k_t) \leq 0, \forall t \in \mathcal{T}\}$ . For any  $\mathbf{k}^v \in S_v, \mathbf{k}^{v^o} \in S_{v^o}$ , and  $t \in \mathcal{T}$ , by the convexity of  $\mu$  we have

$$\mu\left(\frac{\tilde{v}_t + \tilde{v}_t^o}{k_t^v + k_t^{v^o}}\right) = \mu\left(o\frac{\tilde{v}_t}{k_t^v} + (1-o)\frac{\tilde{v}_t^o}{k_t^{v^o}}\right) \leq o\mu\left(\frac{\tilde{v}_t}{k_t^v}\right) + (1-o)\mu\left(\frac{\tilde{v}_t^o}{k_t^{v^o}}\right) \leq 0,$$

where  $o = k_t^v/(k_t^v + k_t^{v^o}) \in (0, 1)$ . Therefore,  $\mathbf{k}^v + \mathbf{k}^{v^o} \in S_{vv^o}$  and hence  $S_v + S_{v^o} \subseteq S_{vv^o}$ ,

$$\begin{aligned} \psi(\tilde{\mathbf{v}}) + \psi(\tilde{\mathbf{v}}^o) &= \inf\left\{\sum_{t \in \mathcal{T}} k_t : \mathbf{k} \in S_v\right\} + \inf\left\{\sum_{t \in \mathcal{T}} k_t : \mathbf{k} \in S_{v^o}\right\} \\ &= \inf\left\{\sum_{t \in \mathcal{T}} (k_t^v + k_t^{v^o}) : \mathbf{k}^v \in S_v, \mathbf{k}^{v^o} \in S_{v^o}\right\} \\ &\leq \inf\left\{\sum_{t \in \mathcal{T}} k_t : \mathbf{k} \in S_{vv^o}\right\} \\ &= \psi(\tilde{\mathbf{v}} + \tilde{\mathbf{v}}^o). \end{aligned}$$

*Positive homogeneity:* We observe that  $\forall \lambda > 0$ ,

$$\begin{aligned} \psi(\lambda\tilde{\mathbf{v}}) &= \inf\left\{\sum_{t \in \mathcal{T}} k_t : \mu(\lambda\tilde{v}_t/k_t) \leq 0 \text{ and } k_t > 0, \forall t \in \mathcal{T}\right\} \\ &= \inf\left\{\sum_{t \in \mathcal{T}} \lambda k_t : \mu(\lambda\tilde{v}_t/\lambda k_t) \leq 0 \text{ and } k_t > 0, \forall t \in \mathcal{T}\right\} \\ &= \lambda \times \inf\left\{\sum_{t \in \mathcal{T}} k_t : \mu(\tilde{v}_t/k_t) \leq 0 \text{ and } k_t > 0, \forall t \in \mathcal{T}\right\} \\ &= \lambda\psi(\tilde{\mathbf{v}}). \end{aligned}$$

*Right continuity:* Denote  $k^* = \psi(\tilde{\mathbf{v}})$ ,  $k_t^* = \psi(\tilde{v}_t \mathbf{e}_t)$ ,  $t \in \mathcal{T}$ . By Period-wise additive, which has been proved, we have  $k^* = \sum_{t \in \mathcal{T}} k_t^*$ . We now prove the right continuity in two scenarios.

- Consider the case that  $k_t^* \in \mathfrak{R}_+ \forall t \in \mathcal{T}$ , then  $k^* \in \mathfrak{R}_+$ . We need to show that  $\forall \epsilon > 0$ , there exists  $\bar{a} > 0$  such that  $\forall a \in (0, \bar{a})$ ,  $\psi(\tilde{\mathbf{v}} + a\mathbf{1}) \geq \psi(\tilde{\mathbf{v}}) - \epsilon = k^* - \epsilon$ . The case of  $\epsilon \geq k^*$  is

trivial. If  $\epsilon \in (0, k^*)$ , we define  $\mathcal{I} = \{t \in \mathcal{T} : k_t^* = 0\}$ , and  $\mathcal{N} = \mathcal{T} \setminus \mathcal{I} = \{t \in \mathcal{T} : k_t^* > 0\}$ , then we have  $\epsilon < k^* = \sum_{t \in \mathcal{N}} k_t^*$ . Hence, we can find a vector  $\{\epsilon_t\}_{t \in \mathcal{N}}$  such that  $\sum_{t \in \mathcal{N}} \epsilon_t = \epsilon$ , and  $\epsilon_t \in (0, k_t^*) \forall t \in \mathcal{N}$ . Choose  $\bar{a} = \min_{t \in \mathcal{N}} (k_t^* - \epsilon_t) \mu\left(\frac{\tilde{v}_t}{k_t^* - \epsilon_t}\right)$ , which is strictly positive since  $k_t^* = \psi(\tilde{v}_t \mathbf{e}_t) = \inf\{k > 0 : \mu(\tilde{v}_t/k) \leq 0\} \forall t \in \mathcal{N}$ . Consider any  $a \in (0, \bar{a})$ . For  $t \in \mathcal{N}$ , we have

$$\mu\left(\frac{\tilde{v}_t + a}{k_t^* - \epsilon_t}\right) = \mu\left(\frac{\tilde{v}_t}{k_t^* - \epsilon_t}\right) - \frac{a}{k_t^* - \epsilon_t} > \mu\left(\frac{\tilde{v}_t}{k_t^* - \epsilon_t}\right) - \frac{\bar{a}}{k_t^* - \epsilon_t} \geq 0,$$

and hence by Lemma 6 we know  $\psi((\tilde{v}_t + a)\mathbf{e}_t) \geq k_t^* - \epsilon_t$ . For  $t \in \mathcal{I}$ , obviously we have  $\psi((\tilde{v}_t + a)\mathbf{e}_t) = 0$ . Therefore,

$$\psi(\tilde{\mathbf{v}}) = \sum_{t \in \mathcal{N}} \psi(\tilde{v}_t \mathbf{e}_t) \geq \sum_{t \in \mathcal{N}} (k_t^* - \epsilon_t) = k^* - \epsilon.$$

- Consider the case that  $\exists t \in \mathcal{T}$  such that  $k_t^* = \infty$ , then  $k^* = \infty$ . Since  $\psi(\tilde{\mathbf{v}} + a\mathbf{1})$  is nonincreasing in  $a \in \mathfrak{R}_+$ ,  $\lim_{a \downarrow 0} \psi(\tilde{\mathbf{v}} + a\mathbf{1})$  exists, as real number or  $\infty$ . Suppose  $\lim_{a \downarrow 0} \psi(\tilde{\mathbf{v}} + a\mathbf{1}) = k^o \in \mathfrak{R}^+$ . It implies that  $\forall \epsilon > 0, \exists \bar{a} > 0$  such that  $\forall a \in (0, \bar{a})$  we have  $\psi(\tilde{\mathbf{v}} + a\mathbf{1}) \in [k^o - \epsilon, k^o + \epsilon]$ . As  $k_t^* = \psi(\tilde{v}_t \mathbf{e}_t) = \infty, \mu(\tilde{v}_t/k) > 0 \forall k > 0$ . Hence, we are able to choose

$$a = \min \left\{ \frac{\bar{a}}{2}, \hat{k} \left( \mu\left(\frac{\tilde{v}_t}{\hat{k}}\right) - \delta \right) \right\} \in (0, \bar{a}),$$

where  $\hat{k} = k^o + 2\epsilon$ , and  $\delta$  is any real number in  $(0, \mu(\tilde{v}_t/\hat{k}))$ . Since  $a \in (0, \bar{a})$ , we still have  $\psi((\tilde{v}_t + a)\mathbf{e}_t) \leq \psi(\tilde{\mathbf{v}} + a\mathbf{1}) \leq k^o + \epsilon$ . Therefore, by Lemma 6, we know  $\mu((\tilde{v}_t + a)/\hat{k}) \leq 0$  since  $\hat{k} > k^o + \epsilon$ . We also observe that

$$\mu\left(\frac{\tilde{v}_t + a}{\hat{k}}\right) = \mu\left(\frac{\tilde{v}_t}{\hat{k}}\right) - \frac{a}{\hat{k}} \geq \mu\left(\frac{\tilde{v}_t}{\hat{k}}\right) - \left(\mu\left(\frac{\tilde{v}_t}{\hat{k}}\right) - \delta\right) = \delta > 0,$$

which contradicts with  $\mu((\tilde{v}_t + a)/\hat{k}) \leq 0$ . So the assumption is false,  $\lim_{a \downarrow 0} \psi(\tilde{\mathbf{v}} + a\mathbf{1}) = \infty$ .

We now prove that given a CSR criterion  $\psi$ , the  $\mu$  defined by (2) is a convex risk measure.

*Monotonicity.* It is an obvious result from the monotonicity of  $\psi$ .

*Cash invariance* Notice that

$$\mu(\tilde{v} + w) = \min \{a : \psi((\tilde{v} + w + a)\mathbf{e}_1) \leq 1\} = \min \{a - w : \psi((\tilde{v} + a)\mathbf{e}_1) \leq 1\} = \mu(\tilde{v}) - w.$$

*Convexity.* Notice that

$$\begin{aligned} \psi((\lambda\tilde{v} + (1-\lambda)\tilde{v}^o + \lambda\mu(\tilde{v}) + (1-\lambda)\mu(\tilde{v}^o))\mathbf{e}_1) &\leq \lambda\psi((\tilde{v} + \mu(\tilde{v}))\mathbf{e}_1) + (1-\lambda)\psi((\tilde{v}^o + \mu(\tilde{v}^o))\mathbf{e}_1) \\ &\leq \lambda + (1-\lambda) \\ &= 1, \end{aligned}$$

where the first inequality follows from the subadditivity and positive homogeneity of  $\psi$ , and the second inequality is due to the definition of  $\mu$ . Hence, we know

$$\mu(\lambda\tilde{v} + (1-\lambda)\tilde{v}^o) = \min\{a : \psi((\lambda\tilde{v} + (1-\lambda)\tilde{v}^o + a)\mathbf{e}_1) \leq 1\} \leq \lambda\mu(\tilde{v}) + (1-\lambda)\mu(\tilde{v}^o).$$

*Normalization.* Obviously  $\psi(a\mathbf{e}_1)$  is 0 if  $a \geq 0$  and it is  $\infty$  if  $a < 0$ . Therefore,

$$\mu(0) = \min\{a : \psi(a\mathbf{e}_1) \leq 1\} = 0.$$

Finally, we need to show given any convex risk measure  $\mu$  defined as (2), the equation (1) holds, which can be done as follows. By the right continuity of CSR, the minimum in (2) is achievable. Therefore,

$$\begin{aligned} \inf \left\{ k > 0 : \mu \left( \frac{\tilde{v}_t}{k} \right) \leq 0 \right\} &= \inf \left\{ k > 0 : \exists a \leq 0, \psi \left( \left( \frac{\tilde{v}_t}{k} + a \right) \mathbf{e}_1 \right) \leq 1 \right\} \\ &= \inf \left\{ k > 0 : \psi \left( \frac{\tilde{v}_t}{k} \mathbf{e}_1 \right) \leq 1 \right\} \\ &= \inf \{ k > 0 : \psi(\tilde{v}_t \mathbf{e}_1) \leq k \} \\ &= \psi(\tilde{v}_t \mathbf{e}_1), \end{aligned}$$

where the second equality holds since  $\psi((\tilde{v}_t/k + a)\mathbf{e}_1)$  is nonincreasing in  $a$ , the third equality follows from the monotonicity of CSR. By the period-wise additive, (1) holds.  $\blacksquare$

The concept of convex risk measure is popularized in the past decade (see for instance Föllmer and Schied (2002, 2004)). In particular, an important class of convex risk measures termed as *shortfall risk measures* (Föllmer and Schied, 2002) has the following relation with expected utility preferences.

**Definition 4** A *shortfall risk measure* is any function  $\mu_U : \mathcal{V} \rightarrow \mathfrak{R}$  defined by

$$\mu_U(\tilde{v}) = \inf_{\eta \in \mathfrak{R}} \{ \eta : \mathbb{E}[u(\tilde{v} + \eta)] \geq 0 \}, \quad (3)$$

where  $u : \mathfrak{R} \rightarrow \mathfrak{R}$  is an increasing concave normalized utility function such that  $u(0) = 0$ . Correspondingly, through the dual representation in Theorem 1, we define the utility based CSR criterion,  $\psi_U : \mathcal{V}^T \mapsto [0, \infty]$  as one that is constructed from a shortfall risk measure as follows:

$$\psi_U(\tilde{\mathbf{v}}) = \inf \left\{ \sum_{t \in \mathcal{T}} \alpha_t : \boldsymbol{\alpha} > \mathbf{0}, \mathbb{E} \left[ u \left( \frac{\tilde{v}_t}{\alpha_t} \right) \right] \geq 0, \forall t \in \mathcal{T} \right\}. \quad (4)$$

A shortfall risk measure,  $\mu_U(\tilde{v})$  can be interpreted as the minimum amount cash,  $\eta$ , to be augmented to the consumption excess so that their overall expected utility exceeds zero. We can interpret the utility based CSR criterion,  $\psi_U(\tilde{\mathbf{v}})$  as the aggregation over  $t \in \mathcal{T}$  of the reciprocal of the largest positive scale factor  $\kappa_t$  such that the corresponding expected utility of  $\kappa_t \tilde{v}_t$  remains nonnegative. Under the utility based CSR criterion, the consumption target in every period is refrained from exceeding the risk neutral or expected consumption level in that period such that  $\mathbb{E}[\tilde{v}_t] < 0$  for some  $t \in \mathcal{T}$ .

**Proposition 1** *The utility based CSR criterion has the property of risk-seeking avoidance, i.e., if  $\exists t \in \mathcal{T}$  such that  $\mathbb{E}[\tilde{v}_t] < 0$ , then  $\psi_U(\tilde{\mathbf{v}}) = \infty$ .*

**Proof :** Suppose  $\mathbb{E}[\tilde{v}_t] < 0$ , then by Jensen's inequality

$$\mathbb{E} [u(\tilde{v}_t/\alpha_t)] \leq u(\mathbb{E}[\tilde{v}_t]/\alpha_t) < u(0) = 0$$

for all  $\alpha_t > 0$ . Hence, the optimization problem in (3) will be infeasible and by definition,  $\psi_U(\tilde{\mathbf{v}}) = \infty$ . ■

### 3 Optimizing CSR criterion in dynamic decision problems

We assume that the preference of the decision maker follows the CSR criterion and she establishes a “pre-committed” optimal policy at the beginning of the first period. Thereafter, the policy is committed and she would not change the policy for the rest of the planning horizon. In other words, we only evaluate the optimality from the viewpoint of the beginning of the time horizon.

In this study, we call a policy optimal if it yields a  $T$ -period consumption excess profile with the lowest CSR criterion, which can be obtained by solving the following optimization problem:

$$\psi^* = \min_{\Psi \in \mathcal{P}_H} \psi(\tilde{\mathbf{v}}(\Psi)), \quad (5)$$

where  $\psi$  is a CSR criterion.

By the expressions of  $\tilde{c}_t$  introduced in Section 2,  $\tilde{c}_t$ ,  $t \in \mathcal{T}^-$  refers to the consumption and  $\tilde{c}_T$  refers to the firm's wealth increase after the consumption in the previous periods. The CSR framework is consistent with the situation where apart from meeting day-to-day corporate consumption such as labor cost, it is also crucial to achieve a pre-determined target profit at the end of the horizon, which often plays a significant role in managers' performance evaluation as well as in affecting firms' stock prices.

In this model, we say a policy  $\Psi \in \mathcal{P}_H$  is feasible if  $\psi(\tilde{\mathbf{v}}(\Psi)) < \infty$ . Hence, we assume that the decision maker can aptly set her targets so that  $\psi^* \in (0, \infty)$ .

## Policy under unrestricted and common interest rate financing

We first analyze the case of unrestricted and common interest rate financing, i.e.,  $F_t(\cdot) = \Re$  and  $m_t(w_t) = (1 + \beta)w_t$  for all  $t \in \mathcal{T}^-$ . These assumptions are similar to those made in Smith (1998) and Chen et al. (2007) so that a tractable analysis can be made. We will show that under the CSR criterion, there exists an optimal financing policy in which the consumption in each period  $t$ ,  $t \in \mathcal{T}^-$  is exactly at its target. We call this a *financing-at-target (FAT)* policy.

**Definition 5** *Given an admissible operational policy  $\mathbf{\Pi}$ , the financing-at-target (FAT) policy is a financing policy  $\Phi = \{\phi_1, \dots, \phi_{T-1}\}$ ,  $\phi_t : \Re \mapsto \Re$  such that for all  $t \in \mathcal{T}^-$ ,*

$$\phi_t(r_t) = \tau_t - r_t,$$

where  $r_t = g_t(\mathbf{x}_t, \mathbf{a}_t, \mathbf{z}_t)$  and  $\tau_t = \theta_t(\mathbf{x}_t)$  are the realized operational cash flow and the desired consumption target, respectively, in period  $t$  under policy  $\mathbf{\Pi}$ .

**Theorem 2** *Consider a CSR objective function and suppose the problem is feasible and the objective is finite. There exists an optimal FAT policy that minimizes the criterion under unrestricted and common interest rate financing.*



**Proof :** Let  $(\mathbf{\Pi}^*, \mathbf{\Phi}^*) \in \mathcal{P}_H$  be an optimal admissible history dependent policy to Model (5) in which  $\mathbf{\Phi}^*$  may not be a FAT policy. For given operational policy  $\mathbf{\Pi}^*$ , we show that the corresponding FAT policy,  $\hat{\mathbf{\Phi}}$  would not yield a consumption excesses profile with worse CSR value. With unrestricted and common interest rate financing, the net present value of consumption excesses under the FAT policy is

$$\tilde{v}_t = \begin{cases} 0 & \text{for } t \in \mathcal{T}^-, \\ w_1 + \sum_{k \in \mathcal{T}} \frac{\tilde{r}_k - \tilde{\tau}_k}{(1 + \beta)^k} & \text{for } t = T. \end{cases}$$

where  $\tilde{r}_t$  and  $\tilde{\tau}_t$ ,  $t \in \mathcal{T}$  are respectively the uncertain operational cash flow and targets under the operational policy  $\mathbf{\Pi}^*$ . Let  $\tilde{\mathbf{v}}^* = (\tilde{v}_1^*, \dots, \tilde{v}_T^*)$  denote the net present value of consumption excesses under the optimal policy  $(\mathbf{\Pi}^*, \mathbf{\Phi}^*)$  and  $\tilde{b}_t$ ,  $t \in \mathcal{T}^-$  denotes the corresponding financing cash flow. Therefore,

$$\tilde{v}_t^* = \begin{cases} \frac{\tilde{r}_t + \tilde{b}_t - \tilde{\tau}_t}{(1 + \beta)^t} & \text{for } t \in \mathcal{T}^-, \\ w_1 + \frac{\tilde{r}_T - \tilde{\tau}_T}{(1 + \beta)^T} - \sum_{k \in \mathcal{T}^-} \frac{\tilde{b}_k}{(1 + \beta)^k} & \text{for } t = T. \end{cases}$$

Hence, we have

$$\begin{aligned} \psi(\tilde{\mathbf{v}}^*) &= \sum_{t \in \mathcal{T}} \psi(\tilde{v}_t^* \mathbf{e}_t) = \sum_{t \in \mathcal{T}} \psi(\tilde{v}_t^* \mathbf{e}_T) \\ &\geq \psi\left(\sum_{t \in \mathcal{T}} \tilde{v}_t^* \times \mathbf{e}_T\right) = \psi(\tilde{v}_T \mathbf{e}_T) = \sum_{t \in \mathcal{T}} \psi(\tilde{v}_t \mathbf{e}_t) \\ &= \psi(\tilde{\mathbf{v}}), \end{aligned}$$

where the first and last equalities are due to the period-wise additive property, the second equality follows from the order invariance, the inequality follows from the subadditivity, the second last equality follows from  $\psi((\tilde{c}_t - \tilde{\tau}_t) \mathbf{e}_t) = \psi(\mathbf{0}) = 0 \forall t \in \mathcal{T}^-$ .  $\blacksquare$

The optimality of the FAT policy implies that as long as Model (5) is feasible, one can always attain the desired consumption targets through financing, with the exception of the last target. Hence, we can perfectly regulate the consumption in period  $t \in \mathcal{T}^-$  by minimizing the CSR criterion and in doing so, relegate consumption uncertainty to the last period. [Note that this policy does not necessarily increase the likelihood of bankruptcy at the end of the horizon. In](#)

the case of the utility based CRI criterion, the *risk-seeking avoidance* property implies that the targets set must be realistic and do not promote risk seeking behavior. In particular, if the expected consumption in the last period is less than its target, then the policy would not yield a finite objective value.

Observe that if the target functions  $\theta_t(\mathbf{x}_t)$  are nonnegative, the FAT policy would also imply nonnegative consumption and hence ensure that operational expenses can be fully paid from finance decisions before the excess cash flow is used for consumption. Note that the optimality of the FAT policy is driven by the assumption of unrestricted and common interest rate financing, which may be over idealistic. Nevertheless, this can be viewed as an approximation to the more realistic financing scenarios.

We now demonstrate the procedure for obtaining the optimal operational policy. To derive explicit results for dynamic optimization and illustrate the insights on the connection to the classical expected utility model, we will focus on the utility based CSR criterion. Henceforth, we refer to a CSR criterion,  $\psi$  as one that takes the form of (4). Therefore, Problem (5) can be formulated equivalently as

$$\begin{aligned} \psi^* &= \inf_{\Psi \in \mathcal{P}_H} \sum_{t \in \mathcal{T}} \alpha_t \\ \text{s.t.} \quad & \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi)}{\alpha_t} \right) \right] \geq 0 \quad \forall t \in \mathcal{T}, \\ & \boldsymbol{\alpha} > \mathbf{0}, \end{aligned} \tag{6}$$

where  $u$  is the underlying increasing concave normalized utility function.

We observe that the corresponding CSR criterion of the consumption profile under the FAT policy can be expressed as follows:

$$\psi \left( \left( w_1 + \sum_{t \in \mathcal{T}} \frac{\tilde{r}_t - \tilde{r}_t}{(1 + \beta)^t} \right) \mathbf{e}_T \right) = \inf \left\{ \alpha > 0 \mid \mathbb{E} \left[ u \left( \frac{w_1 + \sum_{t \in \mathcal{T}} \frac{\tilde{r}_t - \tilde{r}_t}{(1 + \beta)^t}}{\alpha} \right) \right] \geq 0 \right\},$$

where  $\tilde{r}_t$ ,  $t \in \mathcal{T}$  is the uncertain operational cash flow. Therefore, we can formulate the optimization problem to minimize the CSR criterion as follows:

$$\begin{aligned} \psi^* &= \min \alpha \\ \text{s.t.} \quad & \max_{\mathbf{H} \in \mathcal{Q}} \mathbb{E} \left[ u \left( \frac{w_1 + \sum_{t \in \mathcal{T}} \frac{\tilde{r}_t - \tilde{r}_t}{(1 + \beta)^t}}{\alpha} \right) \right] \geq 0 \\ & \alpha > 0, \end{aligned} \tag{7}$$

where  $\mathcal{Q}$  is the set of all admissible operational policies,  $\tilde{r}_t$  and  $\tilde{\tau}_t$ ,  $t \in \mathcal{T}$  denote the uncertain operational cash flow and targets, respectively, under policy  $\mathbf{\Pi} \in \mathcal{Q}$ . Note that the minimum in (7) is attainable since we assume  $\psi^* \in (0, \infty)$ . We next present the optimal operational policy.

**Theorem 3** *Consider a utility based CSR objective function and suppose the problem is feasible and the objective is finite. Under unrestricted and common interest rate financing, there exists an optimal operational policy that minimizes the criterion, which can be obtained by solving the dynamic programming problem given by*

$$\boldsymbol{\pi}_t(\mathbf{x}, w) = \arg \max_{\mathbf{a}_t \in \mathcal{A}_t(\mathbf{x})} \mathbb{E} \left[ U_{t+1}^{\psi^*}(\mathbf{f}_t(\mathbf{x}, \mathbf{a}_t, \tilde{\mathbf{z}}_t), (1 + \beta)w + g_t(\mathbf{x}, \mathbf{a}_t, \tilde{\mathbf{z}}_t) - \theta_t(\mathbf{x})) \right] \quad t \in \mathcal{T},$$

where

$$U_t^\alpha(\mathbf{x}, w) = \begin{cases} u\left(\frac{w}{\alpha(1 + \beta)^T}\right) & \text{for } t = T + 1, \\ \mathbb{E} \left[ U_{t+1}^\alpha(\mathbf{f}_t(\mathbf{x}, \boldsymbol{\pi}_t(\mathbf{x}, w), \tilde{\mathbf{z}}_t), (1 + \beta)w + g_t(\mathbf{x}, \boldsymbol{\pi}_t(\mathbf{x}, w), \tilde{\mathbf{z}}_t) - \theta_t(\mathbf{x})) \right] & \text{for } t \in \mathcal{T}, \end{cases}$$

defined for  $\alpha > 0$ ,  $u$  is the utility function associated with the CSR criterion, and

$$\psi^* = \min \left\{ \alpha > 0 \mid U_1^\alpha(\mathbf{x}_1, w_1) \geq 0 \right\}.$$

Moreover,  $U_1^\alpha(\mathbf{x}_1, w_1) \geq 0$  if and only if  $\alpha \geq \psi^*$ ; hence,  $\psi^*$  can be found by a standard bisection search on  $\alpha$ .

**Proof :** See Appendix. ■

**Remark :** Before using the bisection search of Theorem 3, we need to find a range  $[\underline{\alpha}, \bar{\alpha}]$  satisfying  $\psi^* \in [\underline{\alpha}, \bar{\alpha}]$ . For simplicity, we can take  $\underline{\alpha} = 0$ . An upper bound can be found by starting with any  $\bar{\alpha} > 0$ , then doubling its value repeatedly whenever  $U_1^{\bar{\alpha}}(\mathbf{x}_1, w_1) < 0$ .

## Optimization for convex dynamic decision problems

We now consider the financing decision in a general setting where a FAT policy may not necessarily be admissible, which is a harder problem to solve. We will focus on a special class of convex dynamic decision problems in which the structure of the optimal policies under the CSR criterion can be analyzed.

Analogous to a convex maximization problem, the feasible policies of a convex dynamic decision problem are convex and the consumptions are concave functions with respect to the policies. The first step is to ensure that the feasible set of the optimization problem is closed, which is necessary for our results to hold. Hence, we analyze the policy of an  $\epsilon$ -closure of Model (6) as follows.

**Proposition 2** *For any  $\epsilon > 0$ , we have  $\varphi^* \leq \varphi_\epsilon^* \leq \varphi^* + T\epsilon$ , where*

$$\begin{aligned} \psi_\epsilon^* = \min_{\Psi \in \mathcal{P}_H} \quad & \sum_{t \in \mathcal{T}} \alpha_t \\ \text{s.t.} \quad & \mathbb{E} \left[ u \left( \frac{\bar{v}_t(\Psi)}{\alpha_t} \right) \right] \geq 0, t \in \mathcal{T} \\ & \alpha \geq \epsilon \mathbf{1}. \end{aligned} \tag{8}$$

**Proof :** See Appendix. ■

By Proposition 2, the optimal policy of Model (8) can be made arbitrarily close to that of Model (6). Therefore, with an abuse of terminology, we refer to an optimal policy of Model (8) as one that also minimizes the CSR criterion in the general financing case.

From here forward, we assume finite discrete distributions, i.e.,  $\Omega = \{\omega_1, \dots, \omega_K\}$  and that  $\mathbb{P}\{\omega_k\} > 0$ . While this assumption aims to simplify the analysis, it is not practically limiting in most applications of dynamic decision problems. Under the assumption of a finite sample space, at any point in time, there are only finitely many possible sample paths that will influence control decisions. Therefore, the history dependent policies can be perceived as vectors representing concatenation of controls corresponding to all the possible sample paths. Hence, Model (8) can be expressed as a finite dimensional optimization problem. In spite of this, there could be exponentially large number of decision variables and it is computationally prohibitive to solve Model (8) directly as a mathematical optimization problem. Instead, we propose to address the problem by solving a sequence of dynamic optimization problems, which may enable us to exploit the structures of their optimal policies for efficient computations. We make further assumptions on the problem.

**Assumption 1** *The set of admissible policies  $\mathcal{P}_H$  is closed, bounded, convex, i.e. for all  $\Psi^1, \Psi^2 \in \mathcal{P}_H$ ,*

$$\lambda \Psi^1 + (1 - \lambda) \Psi^2 \in \mathcal{P}_H, \quad \forall \lambda \in [0, 1],$$

and is strictly feasible, i.e.  $\exists \Psi \in \text{int}\mathcal{P}_H$ , where  $\text{int}\mathcal{P}_H$  refers to the interior of  $\mathcal{P}_H$ , such that  $\varphi(\tilde{v}_t(\Psi)\mathbf{e}_t) > 0$ . The consumption excesses are concave with respect to the policy, i.e, for all  $t \in \mathcal{T}$ ,

$$\tilde{v}_t(\lambda\Psi^1 + (1-\lambda)\Psi^2) \geq \lambda\tilde{v}_t(\Psi^1) + (1-\lambda)\tilde{v}_t(\Psi^2) \quad \forall \lambda \in [0, 1].$$

Note that since the wealth update function,  $m_t(w)$  is concave and increasing, we can establish that the consumption excess  $\tilde{v}_t(\mathbf{\Pi}, \mathbf{\Phi})$  is concave on  $\mathbf{\Phi}$ .

**Theorem 4** *Consider a utility based CSR objective function and suppose the problem is feasible and the objective is finite. Under Assumption 1, the optimal policy under the criterion is one that maximizes an additive expected utility over the  $T$  periods as follows:*

$$\max_{\Psi \in \mathcal{P}_H} \mathbb{E} \left[ \sum_{t \in \mathcal{T}} \delta_t u \left( \frac{\tilde{v}_t(\Psi)}{\alpha_t} \right) \right], \quad (9)$$

for some  $\boldsymbol{\delta} \geq \mathbf{0}$ ,  $\boldsymbol{\alpha} \geq \epsilon \mathbf{1}$ , and  $u$  is the utility function associated with the CSR criterion.

**Proof :** Note for any  $\tilde{v} \in \mathcal{V}$ ,  $\alpha > 0$ ,  $E[u(\tilde{v}/\alpha)] \geq 0$  if and only if  $\alpha \mathbb{E}[u(\tilde{v}/\alpha)] \geq 0$ . We can therefore formulate Model (8) equivalently as follows:

$$\begin{aligned} \min_{\Psi \in \mathcal{P}_H} \quad & \sum_{t \in \mathcal{T}} \alpha_t \\ \text{s.t.} \quad & \alpha_t \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi)}{\alpha_t} \right) \right] \geq 0, \quad t \in \mathcal{T} \\ & \boldsymbol{\alpha} \geq \epsilon \mathbf{1}. \end{aligned} \quad (10)$$

We claim that for any given  $\tau \in \mathfrak{R}$ , the function  $\alpha \mathbb{E}[u(\tilde{v}_t(\Psi)/\alpha)]$  is jointly concave in  $(\alpha, \Psi)$  on  $\alpha > 0$ . Indeed, given  $\alpha^1, \alpha^2 > 0$ ,  $\Psi^1, \Psi^2 \in \mathcal{P}_H$ , let  $\alpha^\lambda = \lambda\alpha^1 + (1-\lambda)\alpha^2$  and  $\Psi^\lambda = \lambda\Psi^1 + (1-\lambda)\Psi^2$  for any  $\lambda \in (0, 1)$ . Observe that

$$\begin{aligned} \alpha^\lambda \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi^\lambda)}{\alpha^\lambda} \right) \right] & \geq \alpha^\lambda \mathbb{E} \left[ u \left( \frac{\lambda\tilde{v}_t(\Psi^1) + (1-\lambda)\tilde{v}_t(\Psi^2)}{\alpha^\lambda} \right) \right] \\ & = \alpha^\lambda \mathbb{E} \left[ u \left( \frac{\lambda\alpha^1}{\alpha^\lambda} \cdot \frac{\tilde{v}_t(\Psi^1)}{\alpha^1} + \frac{(1-\lambda)\alpha^2}{\alpha^\lambda} \cdot \frac{\tilde{v}_t(\Psi^2)}{\alpha^2} \right) \right] \\ & \geq \lambda\alpha^1 \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi^1)}{\alpha^1} \right) \right] + (1-\lambda)\alpha^2 \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi^2)}{\alpha^2} \right) \right] \end{aligned}$$

where the first inequality holds for the concavity of the dynamic decision problem, the second inequality follows from the concavity of  $u$ . Hence, Problem (10) is a concave optimization problem

with a finite number of decision variables and it is strictly feasible by assumption. We let this be the primal problem, and the Lagrange dual problem follows:

$$\begin{aligned} \max \quad & g(\boldsymbol{\lambda}) \\ \text{s.t.} \quad & \boldsymbol{\lambda} \geq \mathbf{0}, \end{aligned} \tag{11}$$

where

$$g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\alpha} \geq \epsilon \mathbf{1}, \boldsymbol{\Psi} \in \mathcal{P}_H} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}),$$

$$L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}) = \sum_{t \in \mathcal{T}} \left( \alpha_t + \lambda_t \left( -\alpha_t \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\boldsymbol{\Psi})}{\alpha_t} \right) \right] \right) \right).$$

Since the primal problem has a finite objective, is convex and strictly feasible, strong duality holds and the dual variables  $\boldsymbol{\lambda}$  are attainable. (See for instance, Boyd and Vandenberghe 2004, section 5.2.3). Let  $(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*)$  be any optimal solution to the primal problem (10), and  $\boldsymbol{\lambda}^*$  be any optimal solution to the dual problem (11). By complementary slackness we have

$$L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*, \boldsymbol{\lambda}^*) = \sum_{t \in \mathcal{T}} \left( \alpha_t^* + \lambda_t^* \left( -\alpha_t^* \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\boldsymbol{\Psi}^*)}{\alpha_t^*} \right) \right] \right) \right) = \sum_{t \in \mathcal{T}} \alpha_t^* = g(\boldsymbol{\lambda}^*).$$

Therefore,  $L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*, \boldsymbol{\lambda}^*) = \min_{\boldsymbol{\Psi} \in \mathcal{P}_H} L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}, \boldsymbol{\lambda}^*)$ , and  $\boldsymbol{\Psi}^*$  is an optimal solution to the problem given by

$$\max_{\boldsymbol{\Psi} \in \mathcal{P}_H} \sum_{t \in \mathcal{T}} \left( \delta_t^* \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\boldsymbol{\Psi}^*)}{\alpha_t^*} \right) \right] \right),$$

where  $\delta_t^* = \lambda_t^* \alpha_t^*$ . ■

**Proposition 3** *The optimal policy that maximizes an expected additive utility  $\mathbb{E} \left[ \sum_{t \in \mathcal{T}} u_t(\tilde{v}_t(\boldsymbol{\Psi})) \right]$ , where*

$$u_t(x) = \delta_t u \left( \frac{x}{\alpha_t} \right)$$

for some  $\boldsymbol{\delta} \geq \mathbf{0}$ ,  $\boldsymbol{\alpha} \geq \epsilon \mathbf{1}$  and  $u$  is the utility function associated with the CSR criterion, can be obtained by solving the dynamic programming algorithm given by

$$\boldsymbol{\pi}_t(\mathbf{x}, w) = \begin{cases} \arg \max_{\mathbf{a}_T \in \mathcal{A}_T(\mathbf{x})} \mathbb{E}_{\tilde{\mathbf{z}}_T} \left[ u_T \left( \frac{m_T(w) + g_T(\mathbf{x}, \mathbf{a}_T, \tilde{\mathbf{z}}_T) - \theta_T(\mathbf{x})}{(1 + \beta)^T} \right) \right] & t = T, \\ \arg \max_{\mathbf{a}_t \in \mathcal{A}_t(\mathbf{x})} \mathbb{E}_{\tilde{\mathbf{z}}_t} \left[ V_t^f(\mathbf{f}_t(\mathbf{x}, \mathbf{a}_t, \tilde{\mathbf{z}}_t), w, g_t(\mathbf{x}, \mathbf{a}_t, \tilde{\mathbf{z}}_t), \theta_t(\mathbf{x})) \right] & t \in \mathcal{T}^-, \end{cases}$$

$$\phi_t(\mathbf{x}, w, r, \tau) = \arg \max_{b_t \in F_t(\mathbf{x}, w, r)} \left\{ u_t \left( \frac{r + b_t - \tau}{(1 + \beta)^t} \right) + V_{t+1}^o(\mathbf{x}, m_t(w) - b_t) \right\} \quad t \in \mathcal{T}^-,$$

where

$$V_t^o(\mathbf{x}, w) = \begin{cases} \mathbb{E}_{\tilde{\mathbf{z}}_T} \left[ u_T \left( \frac{m_T(w) + g_T(\mathbf{x}, \boldsymbol{\pi}_T(\mathbf{x}, w), \tilde{\mathbf{z}}_T) - \theta_T(\mathbf{x})}{(1 + \beta)^T} \right) \right] & t = T \\ \mathbb{E}_{\tilde{\mathbf{z}}_t} \left[ V_t^f(\mathbf{f}_t(\mathbf{x}, \boldsymbol{\pi}_t(\mathbf{x}, w), \tilde{\mathbf{z}}_t), w, g_t(\mathbf{x}, \boldsymbol{\pi}_t(\mathbf{x}, w), \tilde{\mathbf{z}}_t), \theta_t(\mathbf{x})) \right] & t \in \mathcal{T}^- \end{cases}$$

$$V_t^f(\mathbf{x}, w, r, \tau) = u_t \left( \frac{r + \phi_t(\mathbf{x}, w, r, \tau) - \tau}{(1 + \beta)^t} \right) + V_{t+1}^o(\mathbf{x}, m_t(w) - \phi_t(\mathbf{x}, w, r, \tau)) \quad t \in \mathcal{T}^-.$$

**Proof :** See Appendix. ■

To obtain the optimal policy of Model (8), it remains to find the appropriate parameters  $\boldsymbol{\delta}$  and  $\boldsymbol{\alpha}$ . We next present an algorithm for obtaining the optimal policy as follows.

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### Algorithm Main

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**Input:** Initial multipliers  $\boldsymbol{\lambda} \geq \mathbf{0}$  and parameters  $\boldsymbol{\alpha} \geq \epsilon \mathbf{1}$ ,  $i = 1$ .

**repeat**

    Run **Algorithm Coordinate Descent** with input  $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$  and output  $(\mathbf{h}, \boldsymbol{\alpha}, \boldsymbol{\Psi})$ .

    Update multipliers,  $\boldsymbol{\lambda} := \max(\boldsymbol{\lambda} + d_i \mathbf{h}, \mathbf{0})$

    Update index  $i := i + 1$ .

**until** stopping criterion is met.

**Output:**  $\boldsymbol{\Psi}$

---

Here  $d_i > 0$  is a decreasing sequence of step sizes satisfying  $\lim_{i \rightarrow \infty} d_i = 0$  and  $\sum_{i=1}^{\infty} d_i = \infty$ .

---

### Algorithm Coordinate Descent

---

**Input:** Multipliers  $\boldsymbol{\lambda} \geq \mathbf{0}$  and parameters  $\boldsymbol{\alpha} \geq \epsilon \mathbf{1}$ .

**repeat**

    Update  $\boldsymbol{\Psi} := \arg \min_{\boldsymbol{\Psi} \in \mathcal{P}_H} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ .

    Update  $\boldsymbol{\alpha} := \arg \min_{\boldsymbol{\alpha} \geq \epsilon \mathbf{1}} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ .

**until** stopping criterion is met.

**Output:**  $(\mathbf{h}, \boldsymbol{\alpha}, \boldsymbol{\Psi})$ , where  $h_t := -\alpha_t \mathbb{E} [u(\tilde{v}_t(\boldsymbol{\Psi})/\alpha_t)]$ ,  $t \in \mathcal{T}$ .

---

Here the Lagrangian  $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$  is defined as follows:

$$L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}) = \sum_{t \in \mathcal{T}} \left( \alpha_t - \lambda_t \alpha_t \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\boldsymbol{\Psi})}{\alpha_t} \right) \right] \right).$$

Observe that from Proposition 3, we can use dynamic programming to obtain the optimal policy  $\Psi$  that minimizes the Lagrangian for given  $(\alpha, \lambda)$ . To obtain the optimal solution  $\alpha$  that minimizes the Lagrangian for given  $(\lambda, \Psi)$ , we can do so by solving the univariate convex optimization problem,

$$\alpha_t = \arg \min \left\{ \alpha - \lambda_t \alpha \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi)}{\alpha} \right) \right] \mid \alpha \geq \epsilon \right\},$$

via bisection search techniques such as the Golden search methods (Kiefer 1953). To ensure that the algorithm converges to the optimal policy, we require a differentiability assumption on the Lagrangian, which is implied in our next assumption.

**Assumption 2** *Given  $\alpha > 0$  and  $\lambda \geq 0$ , the Lagrangian  $L(\alpha, \Psi, \lambda)$  is differentiable with respect to  $\Psi$  for all  $\Psi \in \mathcal{P}_H$ .*

**Theorem 5** *Under Assumptions 1 and 2, Algorithm Main returns an optimal policy that minimizes the utility based CSR criterion [if the problem is feasible and the objective is finite](#).*

**Proof :** See Appendix. ■

**Remark :** [We note that the parameters of the additive utility are obtained by solving a sequence of convex optimization problems involving the feasible set of policies and the targets. Hence, these parameters do not depend on the targets alone and there is no one-to-one correspondence between these parameters and the given targets.](#)

## 4 Limitations and discussions

It is important to note that as our approach is not in the same vein of expected utility theory, which is the bedrock for rational decision making, it would necessarily violate some salient properties of a prescriptive model. We discuss the limitations of this approach as follows.

### Difficulties of setting targets

The target-oriented decision criterion requires the decision maker to specify her targets as functions of the state of the environment, which is difficult to determine in practice. We are not aware of how this can be done in the literature. The simplest model would be to assume fixed targets, which may not reflect the actual preference of a target-oriented decision maker in general.



## Violation of normative utility theory

The normative utility theory postulates the *Independence* or *Substitution Axiom* such that if a rational decision-maker is indifferent between two possible outcomes, then these outcomes can be substituted within a compound lottery without having to affect the overall preference. Indeed, the CSR criterion would violate this property. As a result, under unconstrained borrowings and savings, the target-oriented approach would favor consumption exactly at target levels in all periods except the last, and recede the accumulated risk to the last period. In contrast, a rational utility maximizing agent would likely prefer allocating her consumption in a manner to spread the risky consumption across different time periods.

## Time inconsistency

Our target-oriented decision criterion does not generally ensure time consistency, and that an optimal policy perceived in one time period may not be recognized as optimal in another. On the descriptive front, it is well-known that human subjects often violate time consistency in behavioral experiments even in the absence of uncertainty; see for instance Thaler (1981), Frederick et al. (2002) and Loch and Wu (2007). To ensure time consistency, decision makers should adopt the pre-committed policy instead of dabbling with different policies at different periods (see, for instance, Basak and Chabakauri, 2010, Kydland and Prescott, 1977, Richardson, 1989 and Bajeux-Besnainou and Portait, 1998).

## Limitations of time-additive expected utility in resolving inter-temporal risks

As a consequence of Theorem 4, when the problem is convex, the optimal policy would correspond to one of time-additive expected utility. Although the time-additive expected utility criterion remains popular due to its simplicity and computational tractability, it does not adequately capture the desired preferences associated with inter-temporal risks. For instance, individuals under such preference would be indifferent to the correlations of risky positions across different periods (see, for instance, Skiadis 2007, Sobel 2013). As an illustration, let  $\tilde{r}_t$ ,  $t \in \{1, \dots, T\}$  represent  $T$  IID uncertain incomes. We consider two income streams  $\tilde{\mathbf{r}}_A = (\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_T)$  and  $\tilde{\mathbf{r}}_B = (\tilde{r}_1, \tilde{r}_1, \dots, \tilde{r}_1)$ . Note that in  $\tilde{\mathbf{r}}_B$ , the incomes in all periods are perfectly correlated and

hence the risks are not diversifiable across periods. Therefore,  $\tilde{\mathbf{r}}_B$  would be a riskier income stream compared to  $\tilde{\mathbf{r}}_A$ . Observe that under the time additive utility model,  $\tilde{\mathbf{r}}_A$  and  $\tilde{\mathbf{r}}_B$  are equally preferred since  $\tilde{r}_t, t \in \{1, \dots, T\}$  are IID. Likewise, under the CSR criterion  $\psi(\tilde{\mathbf{r}}_A - \boldsymbol{\tau}) = \psi(\tilde{\mathbf{r}}_B - \boldsymbol{\tau})$  for any fixed target vector  $\boldsymbol{\tau} \in \mathfrak{R}^T$ .

One approach to resolve the inter-temporal risk preferences is to consider endogenous targets as in Constantinides (1990), where the target in each period is a weighted sum of previous returns given by

$$\tilde{r}_t = \tau_t^0 + \sum_{s=\max\{1, t-L\}}^{t-1} \gamma_{t-s} \tilde{r}_s,$$

for  $t \geq 2$  and some  $L \geq 1$ . While this approach may address the issue of inter-temporal risk preferences, it would necessarily increase the system state dimensions and can impede efficient computations in practice.

Another approach to address this issue without the need for endogenous targets, which is what our paper does, is to adopt the approach of Smith (1988) and consider financing decisions as part of the decision model. Optimizing cash flows via financing allows us to move cash across different time periods and capture inter-temporal risk preferences. For simplicity, we assume a common interest rate with  $\beta = 0$ , and let  $\mathcal{B}$  denote the set of all feasible financing decisions under unrestricted financing. We observe that

$$\begin{aligned} & \min_{\tilde{\mathbf{b}} \in \mathcal{B}} \psi(\tilde{\mathbf{r}}_A + \tilde{\mathbf{b}} - \boldsymbol{\tau}) \\ &= \psi\left(0, 0, \dots, 0, \sum_{t=1}^n \tilde{r}_t - \sum_{t=1}^T \tau_t\right) && : \text{FAT policy of Theorem 2} \\ &\leq \sum_{t=1}^T \psi\left(0, 0, \dots, 0, \tilde{r}_t - \frac{1}{T} \sum_{t=1}^T \tau_t\right) && : \text{Subadditivity} \\ &= \sum_{t=1}^T \psi\left(0, 0, \dots, 0, \tilde{r}_1 - \frac{1}{T} \sum_{t=1}^T \tau_t\right) \\ &= T\psi\left(0, 0, \dots, 0, \tilde{r}_1 - \frac{1}{T} \sum_{t=1}^T \tau_t\right) \\ &= \psi\left(0, 0, \dots, 0, T\tilde{r}_1 - \sum_{t=1}^T \tau_t\right) && : \text{Positive homogeneity} \\ &= \min_{\tilde{\mathbf{b}} \in \mathcal{B}} \psi(\tilde{\mathbf{r}}_B + \tilde{\mathbf{b}} - \boldsymbol{\tau}) && : \text{FAT policy of Theorem 2} \end{aligned}$$

Therefore, when considering optimized cash flows via financing, the CSR criterion identifies  $\tilde{\mathbf{r}}_A$

as being less risky and hence captures the dynamic preferences for this situation. In the absence of financing decisions, however, our CSR criterion would inherit the weakness of time-separable utilities and is incapable of resolving the correlation effect associated with inter-temporal risks.

## 5 Conclusions

We caution that although there are a reasonable set of axioms to motivate the target-oriented decision criterion, it would still violate the principle of normative utility theory and lead to inferior choices with respect to a rational decision maker. Likewise, we also would like to caution the use of time-additive expected utility, which despite its ubiquity, has limitations in addressing inter-temporal risks.

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# Appendix

## Proof of Theorem 3

Given any operational state  $\mathbf{x}$  and state of wealth  $w$  at period  $t$ , we define

$$U_t^\alpha(\mathbf{x}, w) = \begin{cases} u\left(\frac{w}{\alpha(1+\beta)^T}\right) & \text{for } t = T + 1, \\ \max_{\mathbf{\Pi} \in \mathcal{Q}} \mathbb{E} \left[ u\left(\frac{w}{(1+\beta)^{t-1} + \sum_{i=t}^T \frac{\tilde{r}_i - \tilde{\tau}_i}{(1+\beta)^i}}\right) \right] & \text{for } t \in \mathcal{T}, \end{cases}$$

where  $\tilde{r}_t$  and  $\tilde{\tau}_t$ ,  $t \in \mathcal{T}$  denote respectively the uncertain operational cash flow and targets under the operational policy  $\mathbf{\Pi} \in \mathcal{Q}$  while we have operational state  $\mathbf{x}$  and state of wealth  $w$  at the beginning of period  $t$ . Subsequently, we have

$$U_t^\alpha(\mathbf{x}, w) = \max_{\mathbf{a}_t \in \mathcal{A}_t(\mathbf{x})} \mathbb{E} [U_{t+1}^\alpha(\mathbf{f}_t(\mathbf{x}, \mathbf{a}_t, \tilde{\mathbf{z}}_t), (1+\beta)w + g_t(\mathbf{x}, \mathbf{a}_t, \tilde{\mathbf{z}}_t) - \theta_t(\mathbf{x}))] \quad t \in \mathcal{T},$$

and specifically,

$$U_1^\alpha(\mathbf{x}, w) = \max_{\mathbf{\Pi} \in \mathcal{Q}} \mathbb{E} \left[ u\left(\frac{w + \sum_{t=1}^T \frac{\tilde{r}_t - \tilde{\tau}_t}{(1+\beta)^t}}{\alpha}\right) \right].$$

Therefore,  $\psi^* = \min \left\{ \alpha > 0 \mid U_1^\alpha(\mathbf{x}_1, w_1) \geq 0 \right\}$ , and the policy can be obtained in deriving  $U_t^{\psi^*}$ . Finally,  $U_1^\alpha(\mathbf{x}_1, w_1) \geq 0$  if and only if  $\alpha \geq \psi^*$  is the result from Lemma 6 in which the shortfall risk measure is a special case of the general convex risk measure.  $\blacksquare$

## Proof of Proposition 2

Observe that the concavity of  $u : \mathfrak{R} \rightarrow \mathfrak{R}$  implies that it is continuous, and hence  $\mathbb{E}[u(\frac{\tilde{v}_t(\Psi)}{\alpha_t})]$  is continuous in  $\alpha_t \geq \epsilon \forall t \in \mathcal{T}$ . Therefore, the feasible set of Problem (8) is a closed set and hence  $\psi_\epsilon^*$  is well defined.

Note that  $\psi^* \leq \psi_\epsilon^*$  follows from the feasible set of Problem (8) being a subset of that from Problem (6). To see  $\psi_\epsilon^* \leq \psi^* + T\epsilon$ , we first observe that for any  $\alpha$  feasible in Problem (6),  $\alpha + \epsilon \mathbf{1}$

is feasible in Problem (8) since for all  $t \in \mathcal{T}$ ,

$$\begin{aligned}
\mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi)}{\alpha_t + \epsilon} \right) \right] &= \mathbb{E} \left[ u \left( \frac{\alpha_t}{\alpha_t + \epsilon} \frac{\tilde{v}_t(\Psi)}{\alpha_t} + \frac{\epsilon}{\alpha_t + \epsilon} \cdot 0 \right) \right] \\
&\geq \mathbb{E} \left[ \frac{\alpha_t}{\alpha_t + \epsilon} u \left( \frac{\tilde{v}_t(\Psi)}{\alpha_t} \right) + \frac{\epsilon}{\alpha_t + \epsilon} u(0) \right] \\
&= \frac{\alpha_t}{\alpha_t + \epsilon} \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\Psi)}{\alpha_t} \right) \right] \\
&\geq 0,
\end{aligned}$$

where the first inequality and the second equality follow from the concavity and the normalization of  $u$ , respectively, and the second inequality is due to the assumption that  $\alpha$  is feasible in Problem (6). Therefore,

$$\begin{aligned}
\psi_\epsilon^* &= \inf \left\{ \sum_{t \in \mathcal{T}} \alpha_t : \alpha \text{ feasible in Problem (8)} \right\} \\
&\leq \inf \left\{ \sum_{t \in \mathcal{T}} (\alpha_t + \epsilon) : \alpha \text{ feasible in Problem (6)} \right\} \\
&= \inf \left\{ \sum_{t \in \mathcal{T}} \alpha_t : \alpha \text{ feasible in Problem (6)} \right\} + T\epsilon \\
&= \psi^* + T\epsilon.
\end{aligned}$$

■

### Proof of Proposition 3

Following standard dynamic programming procedure,  $V_t^o(\mathbf{x}, w)$  represents the optimal expected utility of consumption excess from the  $t^{\text{th}}$  period to the end of the horizon (i.e.,  $\sum_{i=t}^T \mathbb{E}[u_i(\tilde{v}_i)]$ ) given the system operational state  $\mathbf{x}$  and wealth state  $w$  at the beginning of the period. Right after the uncertainty is realized in the  $t^{\text{th}}$  period, the realized income is  $r$ , the target is updated as  $\tau$  and the system operational state becomes  $\bar{\mathbf{x}}$ . We denote  $V_i^f(\bar{\mathbf{x}}, w, r, \tau)$  as the optimal expected utility of consumption excess from the  $t^{\text{th}}$  period to the end of the horizon for given  $(\bar{\mathbf{x}}, w, r, \tau)$ . Observe that

$$\begin{aligned}
V_T^o(\mathbf{x}, w) &= \max_{\mathbf{a}_T \in \mathcal{A}_T(\mathbf{x})} \mathbb{E}_{\tilde{\mathbf{z}}_T} \left[ u_T \left( \frac{m_T(w) + g_T(\mathbf{x}, \mathbf{a}_T, \tilde{\mathbf{z}}_T) - \theta_T(\mathbf{x})}{(1 + \beta)^T} \right) \right] \\
&= \mathbb{E}_{\tilde{\mathbf{z}}_T} \left[ u_T \left( \frac{m_T(w) + g_T(\mathbf{x}, \boldsymbol{\pi}_T(\mathbf{x}, w), \tilde{\mathbf{z}}_T) - \theta_T(\mathbf{x})}{(1 + \beta)^T} \right) \right],
\end{aligned}$$



and we can obtain  $V_t^f(\bar{\mathbf{x}}, w, r, \tau)$  and  $V_t^o(\mathbf{x}, w)$ ,  $t \in \mathcal{T}^-$  by standard backward induction.  $\blacksquare$

## Proof of Theorem 5

Algorithm Main is a standard subgradient optimization routine for obtaining the optimal multiplier solution,  $\boldsymbol{\lambda}$  in the nondifferential dual function

$$g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\alpha} \geq \epsilon \mathbf{1}, \boldsymbol{\Psi} \in \mathcal{P}_H} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$$

(see for instance, Bertsekas 1999, section 6.3). It calls upon Algorithm Coordinate Descent to obtain the subgradient  $\mathbf{h}$ . We first show for any input  $\boldsymbol{\lambda}$ , the limit point of the sequence  $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$ , which is generated by Algorithm Coordinate Descent, minimizes  $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ . We write  $f(\boldsymbol{\alpha}, \boldsymbol{\Psi}) = L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$  for ease of notation.

We first observe that  $\boldsymbol{\alpha}^{(i)}$  is bounded below by  $\epsilon \mathbf{1}$  and that  $\boldsymbol{\Psi}^{(i)}$  is bounded, since we assume that the policy set is bounded. We will next show that the sequence  $\{\boldsymbol{\alpha}^{(i)}\}$  is bounded above by  $\max\{\mathbf{1}, \hat{\boldsymbol{\alpha}}\}$ , where  $\hat{\boldsymbol{\alpha}}$  is defined by

$$\hat{\alpha}_t = \begin{cases} \epsilon + 2 - 2\lambda_t \inf_{\boldsymbol{\Psi}^o \in \mathcal{P}_H} \mathbb{E}[u(\tilde{v}_t(\boldsymbol{\Psi}^o))] & \text{if } \lambda_t \leq \frac{0.5}{\bar{u}}, \\ \epsilon + \max \left\{ 2 - 2\lambda_t \inf_{\boldsymbol{\Psi}^o \in \mathcal{P}_H} \mathbb{E}[u(\tilde{v}_t(\boldsymbol{\Psi}^o))], \frac{\bar{v}_t}{u^{-1}(0.5/\lambda_t)} \right\} & \text{if } \lambda_t > \frac{0.5}{\bar{u}}, \end{cases}$$

and  $\bar{u} = \sup_{x \in \mathfrak{R}} u(x)$  represents the maximal utility value,  $\bar{v}_t = \sup_{\boldsymbol{\Psi}^o \in \mathcal{P}_H} \inf\{x \mid \mathbb{P}(\tilde{v}_t(\boldsymbol{\Psi}^o) \leq x) = 1\}$  represents the maximal consumption excess in period  $t$ . Let us assume  $\epsilon \in (0, 1)$ . Suppose  $\boldsymbol{\alpha} \geq \mathbf{1}$  such that  $\alpha_t > \hat{\alpha}_t$  for some  $t \in \mathcal{T}$ . We will show that  $\boldsymbol{\alpha}$  is not the optimal solution that minimizes the Lagrangian for any given  $(\boldsymbol{\lambda}, \boldsymbol{\Psi})$ . Indeed, we have:

$$\alpha_t - \lambda_t \alpha_t \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\boldsymbol{\Psi})}{\alpha_t} \right) \right] \geq \alpha_t - \lambda_t \alpha_t u \left( \frac{\bar{v}_t}{\alpha_t} \right) = \alpha_t \left( 1 - \lambda_t u \left( \frac{\bar{v}_t}{\alpha_t} \right) \right),$$

and we define  $\varsigma_t = \alpha_t(1 - \lambda_t u(\bar{v}_t/\alpha_t))$  to represent the right hand side of the above inequality. If  $\lambda_t \leq 0.5/\bar{u}$ , then  $\lambda_t u(\bar{v}_t/\alpha_t) \leq 0.5$  and hence  $\varsigma_t \geq 0.5\alpha_t$ . If  $\lambda_t > 0.5/\bar{u}$ , as  $\alpha_t > \hat{\alpha}_t > \bar{v}_t/u^{-1}(0.5/\lambda_t)$ , we have  $u(\bar{v}_t/\alpha_t) < 0.5/\lambda_t$  and hence  $\lambda_t u(\bar{v}_t/\alpha_t) < 0.5$ ,  $\varsigma_t \geq 0.5\alpha_t$ . Therefore,

$$\begin{aligned} & \alpha_t - \lambda_t \alpha_t \mathbb{E} \left[ u \left( \frac{\tilde{v}_t(\boldsymbol{\Psi})}{\alpha_t} \right) \right] \\ & \geq \varsigma_t \geq 0.5\alpha_t > 0.5\hat{\alpha}_t > 1 - \lambda_t \inf_{\boldsymbol{\Psi}^o \in \mathcal{P}_H} \mathbb{E}[u(\tilde{v}_t(\boldsymbol{\Psi}^o))] \geq 1 - \lambda_t \mathbb{E}[u(\tilde{v}_t(\boldsymbol{\Psi}))]. \end{aligned}$$

In other words, we can lower the value of the Lagrangian,  $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$  by changing  $\alpha_t$  to 1 and hence,  $\boldsymbol{\alpha}^{(i+1)}$  must be bounded above by  $\max\{\mathbf{1}, \hat{\boldsymbol{\alpha}}\}$ . Since,  $(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})$  is a bounded sequence and there must exist at least one limit point.

Let  $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$  be a limit point of the sequence  $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$ . Algorithm Coordinate Descent ensures that the following inequalities holds:

$$f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)}) \geq f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i+1)}) \geq f(\boldsymbol{\alpha}^{(i+1)}, \boldsymbol{\Psi}^{(i+1)}), \quad (12)$$

for all  $i$ . Therefore, since  $(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})$  is bounded, which implies  $f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)}) > -\infty$ , the sequence  $\{f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$  is nonincreasing and converges to the limit point,  $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ . It remains to prove that  $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$  minimizes  $f$ , which we will show by contradiction. Let  $\{(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) \mid j = 0, 1, \dots\}$  be a subsequence of  $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$  that converges to  $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ . Suppose there exists  $\boldsymbol{\Psi}^o \in \mathcal{P}_H$  such that  $\Delta = f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) > 0$ . Under Assumption 1, the function  $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$  is differentiable with respect to  $\boldsymbol{\alpha} > \mathbf{0}$  and  $\boldsymbol{\Psi} \in \mathcal{P}_H$ . Since continuity is implied by differentiability, we can find  $\delta > 0$  such that  $|f(\boldsymbol{\alpha}, \boldsymbol{\Psi}^o) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o)| < \Delta/2$  for all  $\boldsymbol{\alpha}$  with  $\|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}\| \leq \delta$ . Since  $\{\boldsymbol{\alpha}^{(i_j)}\}$  converges to  $\bar{\boldsymbol{\alpha}}$ , we can find  $M_1$  such that for all  $j > M_1$ ,  $\|\boldsymbol{\alpha}^{(i_j)} - \bar{\boldsymbol{\alpha}}\| \leq \delta$  and therefore  $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^o) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o)| < \Delta/2$ . Moreover, since  $\{f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)})\}$  converges to  $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ , there exists  $M_2$  such that for all  $j > M_2$ ,  $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)}) - f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})| < \Delta/2$ . Therefore, for  $j > \max\{M_1, M_2\}$ , we have:

$$f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) < f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)}) + \Delta/2 \leq f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^o) + \Delta/2 < f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) + \Delta,$$

which contradicts the assumption that  $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) = f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) + \Delta$ . Hence, we conclude that  $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) \leq f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}), \forall \boldsymbol{\Psi} \in \mathcal{P}_H$ . Therefore,  $\bar{\boldsymbol{\Psi}}$  minimizes  $f(\bar{\boldsymbol{\alpha}}, \cdot)$ . Under the assumption of differentiability and convexity, the optimality condition is equivalent to  $\nabla_{\boldsymbol{\Psi}} f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})'(\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}) \geq 0$  for all  $\boldsymbol{\Psi} \in \mathcal{P}_H$  (see for instance, Bertsekas 1999, Proposition 2.1.2).

Similarly, suppose there exists  $\boldsymbol{\alpha}^o \geq \epsilon \mathbf{1}$  such that  $\Delta = f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) > 0$ . By continuity argument, we can find  $\delta > 0$  such that  $|f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}})| < \Delta/2$  for all  $\boldsymbol{\Psi}$  with  $\|\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}\| \leq \delta$ . Since  $\{\boldsymbol{\Psi}^{(i_j)}\}$  converges to  $\bar{\boldsymbol{\Psi}}$ , we can find  $M_1$  such that for all  $j > M_1$ ,  $\|\boldsymbol{\Psi}^{(i_j)} - \bar{\boldsymbol{\Psi}}\| \leq \delta$  and therefore  $|f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}^{(i_j)}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}})| < \Delta/2$ . Moreover, since  $\{f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)})\}$  converges to  $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ , there exists  $M_2$  such that for all  $j > M_2$ ,  $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) - f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})| < \Delta/2$ . Therefore,

for  $j > \max\{M_1, M_2\}$ , we have:

$$f(\bar{\alpha}, \bar{\Psi}) < f(\alpha^{(i_j)}, \Psi^{(i_j)}) + \Delta/2 \leq f(\alpha^o, \Psi^{(i_j)}) + \Delta/2 < f(\alpha^o, \bar{\Psi}) + \Delta,$$

which contradicts the assumption that  $f(\bar{\alpha}, \bar{\Psi}) = f(\alpha^o, \bar{\Psi}) + \Delta$ . Hence, we conclude that  $f(\bar{\alpha}, \bar{\Psi}) \leq f(\alpha, \bar{\Psi})$ ,  $\forall \alpha \geq \epsilon \mathbf{1}$ . Since  $f(\alpha, \bar{\Psi})$  is differentiable and convex in  $\alpha$ , it implies by the optimality condition that  $\nabla_{\alpha} f(\bar{\alpha}, \bar{\Psi})'(\alpha - \bar{\alpha}) \geq 0$  for all  $\alpha \geq \epsilon \mathbf{1}$ .

Combining the two above results, we have:

$$\nabla f(\bar{\alpha}, \bar{\Psi})'((\alpha, \Psi) - (\bar{\alpha}, \bar{\Psi})) \geq 0, \quad \forall \alpha \geq \epsilon \mathbf{1}, \Psi \in \mathcal{P}_H.$$

Since  $f(\alpha, \Psi)$  is also jointly convex in  $(\alpha, \Psi)$ , this implies that  $(\bar{\alpha}, \bar{\Psi})$  minimizes  $f(\cdot, \cdot)$ .

Finally we show that the output of Algorithm Coordinate Descent,  $\mathbf{h}$  is indeed a subgradient of  $g(\cdot)$  at the input  $\lambda$ . We observe  $g(\cdot)$  is a pointwise minimum of a family of affine functions and hence is concave. Moreover, for any  $\lambda^o \geq \mathbf{0}$ , we have:

$$\begin{aligned} g(\lambda^o) - g(\lambda) &= \min_{\alpha \geq \epsilon \mathbf{1}, \Psi \in \mathcal{P}_H} L(\alpha, \Psi, \lambda^o) - L(\bar{\alpha}, \bar{\Psi}, \lambda) \\ &\leq L(\bar{\alpha}, \bar{\Psi}, \lambda^o) - L(\bar{\alpha}, \bar{\Psi}, \lambda) \\ &= (\lambda^o - \lambda)' \mathbf{h}. \end{aligned}$$

Therefore,  $\mathbf{h}$  is a subgradient of  $g(\cdot)$  at  $\lambda$ . ■

**Lemma 6** *If  $\mu : \mathcal{V} \mapsto \mathfrak{R}$  is a convex risk measure and  $k^* = \inf \{k > 0 : \mu(\frac{\tilde{v}}{k}) \leq 0\} \in \mathfrak{R}_+$ , then we have  $\mu(\frac{\tilde{v}}{k}) \leq 0$ ,  $\forall k > k^*$ .*

**Proof :** We prove the result by contradiction. Suppose  $\exists k^o > k^*$  such that  $\mu(\tilde{v}/k^o) > 0$ . Given any  $k \in (0, k^o]$ , by the convexity and normalization of  $\mu$  we have

$$\frac{k}{k^o} \mu\left(\frac{\tilde{v}}{k}\right) = \frac{k}{k^o} \mu\left(\frac{\tilde{v}}{k}\right) + \left(1 - \frac{k}{k^o}\right) \mu(0) \geq \mu\left(\frac{k}{k^o} \times \frac{\tilde{v}}{k} + 0\right) = \mu\left(\frac{\tilde{v}}{k^o}\right) > 0.$$

It implies that  $\mu(\tilde{v}/k) > 0$  and hence  $\inf \{k > 0 : \mu(\tilde{v}/k) \leq 0\} \geq k^o$ , which contradicts with  $k^o > k^*$ . ■