On constructing regular filter banks from domain bounded polynomials

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Abstract—The design of regular two channel bi-orthogonal filter banks is shown to be reducible to the design of pairs of real polynomials with domain bounded to the interval [-1,1]. Techniques for designing polynomials satisfying various constraints are outlined. Transformation of polynomials to multidimensional bi-orthogonal filter banks is presented.

Keywords— Polynomials, digital filter, filter banks, perfect reconstruction, bi-orthogonal.

EDICS Nr—Filter Bank Theory 2.4.2

I . Introduction

In this correspondence we will be concerned with the design of critically sampled, 2-channel, zero phase bi-orthogonal (BO) filter banks (FB) [3] [17]. The approach we follow is to reduce the design of 1-D zero phase BO FB to the design of appropriate real polynomials on the interval [-1,1]. These polynomials are then transformed to yield 1-D and also multidimensional (m-D) zero phase BO FB. A 1-D bi-orthogonal filter bank of the above class is characterized by the pair of filters $\{G_0, H_0\}$ in the low pass channel. The filters in the high pass channel are up to an even delay given by $\{z^{-1}G_1(z), zH_1(z)\} = \{z^{-1}H_0(-z), zG_0(-z)\}$. With this restriction, the perfect reconstruction (PR) property of the filter bank is expressed by the formula [6]

$$(G_0(z)G_1(-z))_e = 1, \tag{1}$$

where we use the notation f_e (f_o) to denote the even (odd) part of a polynomial f.

The question is how to design 'good' filters $\{G_0, G_1\}$? The goodness of filters here is measured against two criteria. First of all, the filters are required to have a good low pass/high pass characteristics in order not to cause too much alias distorsion after subsampling. Secondly, the filter bank is required to be "well-behaved" when applied in a wavelet decomposition tree [17]. This is popularly known as the *regularity* property of wavelets. Our main aim here is *not* to design wavelets in the strict sense, however we use the concept of regularity loosely in the design of our filters since it has shown to have merits for image coding [2]. In [9] it is shown that the order of the zero that G_0 and H_0 have at $e^{j\pi}$ is a good measure for regularity. We use this necessary (though not sufficient) condition to impose a loose regularity on our filters. Finally, we also require the low pass filter to be "flat" (the derivatives should be zero) around DC.

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As mentioned earlier, this correspondence aims at showing that the design problem mentioned above can be translated into a problem on polynomials over the reals. The key ingredient is the following theorem (see [11]).

Theorem 1 *(polynomial and filters)*
Let $G(z)$ be a (real) zero-phase filter. Then there exists a polynomial $P_G(x)$ with real coefficients such that $G(z) = P_G((z+z^{-1})/2)$

This theorem says that there is a one-to-one relationship between zero-phase filters and polynomials. The origin of this theorem goes back to McClellan [8]. Substituting the expression $(z+z^{-1})$ by some (multi-dimensional) kernel filter $K(z)$ is known as the McClellan transform. This transform has successfully been used to design (non-separable) multi-dimensional filters and filter banks [12] [11] [14] [7] [4] [16]. Most authors take the original 1-D filter as the starting point for their designs. This however will sometimes hide the simplicity of the underlying polynomial, thereby preventing insight into the design process. A typical example in this respect is given by the class of Lagrange half-band filter banks [15] [1]. This correspondence tries to convey the message that the polynomials are a more natural starting point.

We start by showing that the design criteria for our filter banks are directly expressible in terms of polynomials. The low pass character of a filter G can be stated by requiring that $P_G(x)$ satisfies

$$P_G(x) \approx \begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 < x \le 1 \end{cases} \tag{2}$$

Let f and g be the polynomials corresponding to G_0 and G_1. The perfect reconstruction condition is easily seen to translate to [6] $(f(x)g(-x))_e = 1$. The condition on the number of zeros also carries over: it can be checked that if a zero-phase filter G has n zeros at $e^{j\pi}$, then n is even, and the corresponding $P_G(x)$ has a zero of order $n/2$ at $x = -1$.

Section II presents the analytical construction of pairs of polynomials with a good low-pass/high-pass character having a prescribed number of zero derivatives ("flatness") at ± 1. The Lagrange half-band filters are also discussed as seen from the polynomial viewpoint. We continue in Section III with the formulation of a condition which ensures that a polynomial f has a companion polynomial g, i.e. $(f(x)g(-x))_e = 1$. Moreover, we generalize the above to find *all* companion polynomials to f. The design of polynomials with special care to get *both* the low pass and high pass polynomials of good characteristics is discussed in Section IV. Finally, the transformation of the polynomials to 1-D and m-D filter banks is presented in Section V.

II . Analytical construction of polynomials

In this section, we will analytically construct polynomials p satisfying Eq. 2. The constraint that p be flat for $x = \pm 1$ means that its higher order derivatives are zero in $x = \pm 1$; in other words, the derivative of p should look like $(1-x^2)^n r$ for some (large) integer n and polynomial r. For simplicity we consider the case that $r = 1$. Hence, we are interested in the the polynomial q_n defined by $q_n(x) = \int_0^x (1-t^2)^n \, dt$. Clearly, q_n is an odd polynomial with derivative $(1-x^2)^n$. Consequently, q_n is monotonically increasing on the interval $[-1, 1]$ and its j^{th} derivative in $x = \pm 1$ equals zero ($j = 1, 2, \ldots, n$). Now we lift and scale q_n such that we obtain a function p_n of the same form that satisfies $p_n(-1) = 0$ and $p_n(1) = 1$. We find

$$p_n(x) = \frac{1}{2}\left(1 + \frac{q_n(x)}{q_n(1)}\right). \tag{3}$$

It is not difficult to obtain the coefficients of p_n explicitly. Indeed, by expanding $(1-t^2)^n$ with Newton's formula and integrating each of the terms separately, we obtain $q_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{2j+1} x^{2j+1}$. Also the scaling factor $q_n(1)$ can be computed explicitly. Indeed, integrating by parts we obtain that $q_n(1) = 2n q_{n-1}(1) - 2n q_n(1)$. With this recurrence and the starting value $q_0(1) = 1$, we obtain $q_n(1) = \prod_{i=1}^{n} \left(\frac{2i}{2i+1}\right) = \frac{4^n}{(2n+1)\binom{2n}{n}}$.

The polynomial q_n happens to belong to the class of *Lagrangian interpolation* polynomials [10], and has been used in the construction of Lagrangian half-band filters and filter banks [15] [1]. Our method of constructing polynomials on the interval $[-1, 1]$ and then transforming them into filter banks, rather than considering them on the unit circle in the complex plane, is a much simpler approach.

III . Constructing companion polynomials

In section I we have seen that a pair of two-band filters allows perfect reconstruction if and only if their defining polynomials f and g satisfy

$$((f(x)g(-x))_e = 1. \tag{4}$$

In this section we address the questions of determining if a given polynomial f has a *companion polynomial* g (i.e. a polynomial g satisfying Eq. 4) and of finding all such polynomials. This topic is also extensively treated in [17]; our *companion* polynomials correspond to the *complementary* filters in [17]. The main difference in approach is that we are considering polynomials whereas expressions in the complex z domain are considered in [17].

From the definition, g is a companion polynomial to f if and only if

$$1 = (f(x)g(-x))_e = f_e g_e - f_o g_o = \begin{vmatrix} f_e & f_o \\ g_e & g_o \end{vmatrix}.^1 \tag{5}$$

[1] The expression $\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$ denotes the determinant of a matrix.

It follows from Eq. 5 that if f has a companion polynomial, then f_o and f_e have no common zeros (as that would imply a zero of 1 which is clearly impossible). Conversely, suppose that f_o and f_e have no common zeros. The Euclidean algorithm [5] for finding the greatest common divisor of f_o and f_e can be used to find polynomials s and t such that $f_e s - f_o t = 1$. Taking even parts, we see that we in fact have $f_e s_e - f_o t_o = 1$; so $g \stackrel{\text{def}}{=} s_e + t_o$ is a companion polynomial to f.

So if a polynomial f has a companion polynomial, then we can find *one* companion polynomial with the Euclidean algorithm. There are, however, more companion polynomials to f. The following lemma describes the freedom we have in the choice of a companion polynomial to f.

Lemma 1 *Let g be a companion polynomial to f. The polynomial h is a companion polynomial to f if and only if $h = g + \lambda f$ for some odd polynomial λ (see also [17], Proposition 4.5).*

Proof: Let λ be an odd polynomial and $h = g + \lambda f$, then $f_e h_e - f_o h_o = f_e(g_e + \lambda f_o) - f_o(g_o + \lambda f_e) = f_e g_e - f_o g_o = 1$, so h is a companion polynomial to f.
On the other hand, suppose that h is a companion polynomial to f. Let $\Delta \stackrel{\text{def}}{=} h - g$, then we have $f_e \Delta_e - f_o \Delta_o = f_e(h_e - g_e) - f_o(h_o - g_o) = (f_e h_e - f_o h_o) - (f_e g_e - f_o g_o) = 1 - 1 = 0$. That is, we have $f_e \Delta_e = f_o \Delta_o$. Using this equality and the fact that g is a companion polynomial to f, we find that $\Delta_e = \Delta_e(f_e g_e - f_o g_o) = \Delta_o f_o g_e - \Delta_e f_o g_o = f_o \lambda$, where $\lambda = \Delta_o g_e - \Delta_e g_o$. In the same way, we find that $\Delta_o = f_e \lambda$. Consequently, as λ is obviously odd, $\Delta = \lambda f$. □

IV . Design of Polynomials

In this section we will use the class of polynomials given by Eq. 3 to design a low-pass polynomial, $f(x)$. We then proceed to find companion polynomials, $g_n(x)$, having "good" high-pass characteristics and satisfying Eq. 4 for PR. Flatness of the polynomials around ± 1 and monotonicity are important factors in determining goodness of the filters.

Example 1 (basic polynomial) We start with the basic polynomial $f_n(x) = p_n(x)$ of Eq. 3. The minimal companion polynomial $g = 2$ as can be easily seen from the following equation: $\begin{vmatrix} \frac{1}{2} & \frac{q_n(x)}{2} \\ 0 & 2 \end{vmatrix} = 1$, where we assume $q_n(x)$ to be normalized. Following Lemma 1, all the companion polynomials of f_n can be found, and are of the form:

$$\hat{g}_n(x) = 2 + \lambda(x) f_n(x) \tag{6}$$

for any odd λ. We need the chosen λ to satisfy some shape constraints, such as $\hat{g}(1) = 0$ and for flatness, $\hat{g}'(1) \stackrel{def}{=} \frac{\partial \hat{g}(x)}{\partial x}|_{x=1} = 0$. We use the prime to denote derivative. The simplest odd function that can satisfy the two constraints is of the form $\lambda(x) = ax + bx^3$. We solve for a and b resulting in $\lambda(x) = -3x + x^3$. This λ is used in Eq. 6 yielding a simple companion to f_n. Fig. 1 and Fig. 2 show f_n and $\hat{g_n}$ respectively for $n = 3$. Notice that f is an extremely flat (around $x = \pm 1$) and monotonically increasing (from -1 to +1) function, but $\hat{g}$ is not such a nice function. It is very flat at -1 and also has one zero at +1, however it is not monotonic. □

The function $\hat{g}$ in Example 1 has a gain factor of 2, which can be taken care of by scaling, and more zeros can be added at +1. However, it is insightful to see if there are any fundamental constraints on the values of the polynomials at certain points.

Lemma 2 *Let g be a companion polynomial to f. If f and g vanish at -1 and 1 respectively, then $f(0)g(0) = 1$ and $f(1)g(-1) = 2$.*

Proof: As g is a companion polynomial to f, we have $f_e(x)g_e(x) - f_o(x)g_o(x) = 1$. As $f_o(0) = g_o(0) = 0$ (we are dealing with odd functions), it follows that $f(0)g(0) = f_e(0)g_e(0) - f_o(0)g_0(0) = 1$, proving the first part of the theorem.
For the second part, clearly, we have $0 = f(-1) = f_e(-1) + f_o(-1) = f_e(1) - f_o(1)$; that is $f_e(1) = f_o(1)$. In the same way, we have $g_e(-1) = g_o(-1)$. Plugging in $x = 1$ in Eq. 5, we find $1 = f_e(1)g_e(1) - f_0(1)g_o(1) = f_e(1)(g_e(-1) + g_o(-1)) = f_e(1)g(-1) = \frac{1}{2}f(1)g(-1)$. □

Now let g be a companion polynomial to f such that $f(-1) = g(1) = 0$, and suppose that f and g have a low-pass and high-pass characteristic respectively. Then f and g cannot simultaneously have attenuated very much in $x = 0$. Indeed, we have $\frac{f(0)}{f(1)} \times \frac{g(0)}{g(-1)} = \frac{1}{2}$, and hence, $\max(\frac{f(0)}{f(1)}, \frac{g(0)}{g(-1)})$ exceeds $\frac{1}{2}\sqrt{2}$. Following Lemma 2 above, it is easy to understand the result of Example 1. Since $f(1) = 1$ it follows that $\hat{g}(-1) = 2$. Similarly, $f(0) = 1/2$ forces $\hat{g}(0) = 2$. The fact $\hat{g}(-1) = \hat{g}(0) = 2$ implies that $\hat{g}$ can be constant, but *never* be monotonically decreasing over the interval [-1,1]. It is obvious, that to improve the shape of the companion polynomial $\hat{g}$, f must be altered. If f is modified such that $f(1) = \sqrt{2}$ and $f(0) = 1$, then this forces a companion polynomial $g(-1) = \sqrt{2}$ and $g(0) = 1$. With these values, the monotonicity of the companion may be achieved.

We now design two new polynomials f and g which obey the following additional constraints:

$$f(0) = g(0) = 1,\ f(1) = g(-1) = \sqrt{2},\ f(-1) = g(1) = 0 \tag{7}$$

Compared to the f_n of Example 1, the new f must be raised a bit, but still be 0 at $x = -1$. We construct such a polynomial and its companion in the next example:

Example 2 (Raised polynomial) To meet the additional constraints, we modify the f_n of Example 1 by adding an even term to it. A simple companion can be immediately found, both are given by:

$$\begin{aligned} f_n(x) &= 1 - (1 - \frac{1}{2}\sqrt{2})q_n(x)^2 + \frac{1}{2}\sqrt{2}\ q_n(x) \\ g_n(x) &= (1 - \sqrt{2})q_n(x) + 1 \end{aligned} \tag{8}$$

It is easy to verify that f_n and g_n above satisfy Eq. 4. Using Lemma 1, we can construct a more general companion given by:

$$\hat{g}_n(x) = g_n(x) + \lambda(x)f_n(x). \tag{9}$$

We choose the simplest odd λ such that $\hat{g}_n(1) = 0$, which is $\lambda(x) = (1 - \sqrt{2})x$. The resulting companion is now given by: $\hat{g}_n(x) = g_n(x) + (1 - \sqrt{2})x\, f_n(x)$ Fig. 3 and Fig. 4 show $f_n(x)$ and $\hat{g}_n(x)$ designed above for $n = 3$. It can be seen that the constraints of Eq. 7 do indeed have the desired effect. Both $f_n(x)$ and $\hat{g}_n(x)$ are monotonic, and cross each other at $x = 0$. □

In the Example 2 above, $\hat{g}_n(x)$ is not very flat around +1. This can be easily solved by a better choice of λ as is shown in the following example:

Example 3 (Placing zeros at $\hat{g}_n(1)$) To obtain a flatter $\hat{g}_n(x)$ around 1, we require its derivative to be 0 there (placing a zero there), or $\hat{g}'_n(1) = 0$. This can be shown to yield the equivalent requirement $\lambda'(1) = 0$. Choosing $\lambda(x) = ax + bx^3$ and solving for the condition above, along with the requirement $\hat{g}_n(1) = 0$ we find $a = \frac{1}{2}\sqrt{2} - \frac{1}{2}$, $b = \frac{3}{2} - \frac{3}{2}\sqrt{2}$; and so we obtain:

$$\hat{g}_n(x) = g_n(x) + (\frac{1}{2}\sqrt{2} - \frac{1}{2})(x^3 - 3x)\ \ f_n(x) \tag{10}$$

Fig. 5 shows $\hat{g}_n(x)$ designed above for $n = 3$, which is indeed flat around +1 due to the placing of a zero there. □

Following the method outlined above, $\hat{g}_n(x)$ can be made to have more zeros at +1 by choosing λ with higher order derivatives to be zero at +1. One such choice for λ in Eq. 9 is: $\lambda_n(x) = (1 - \sqrt{2})q_n(x)$, which yields a highly flat $\hat{g}_n(x)$, shown for $n = 3$ in Fig. 6. The above examples are just an indication of how various constraints can be placed on the polynomials. Moreover, using simple algebraic manipulations, both the low pass and companion high pass polynomials can be designed. In the next section we address the generation of filters from the polynomials.

V . Generating filters from polynomials

Since our interest is in generation of both 1-D and m-D filters, in the rest of the paper, we deal with m-D signals, denoted by a vector symbol. 1-D signals are treated as a special case of the general m-D case. A zero phase m-D BO FB is also characterized by a pair of filters $\{G_0(\vec{z}), H_0(\vec{z})\}$ in the low pass channel. Correspondingly, the filters in the high pass channel are given by $m^{-1}(\vec{z})G_1(\vec{z}) = m^{-1}(\vec{z})H_0(\vec{w}\vec{z})$ and $m(\vec{z})H_1(\vec{z}) = m(\vec{z})G_0(\vec{w}\vec{z})$, where $m(\vec{z})$ denotes a odd monomial[2], and w corresponds to the aliasing frequency of the m-D sampling lattice. The PR property of the FB is represented, in a manner similar to Eq. 1, as [6] $(G_0(\vec{z})G_1(\vec{w}\vec{z}))_e = 1$

We now transform the polynomial pairs $\{f(x), g(x)\}$ designed above into a pair of bi-orthogonal filters $\{F(\vec{z}), G(\vec{z})\}$, including the 1-D case. The type of transformation we consider here is the substitution $x \rightarrow K(\vec{z})$, where $K(\vec{z})$

[2] In this context the even (odd) part of a filter is the restriction of the impulse response to sampling lattice (complement of the lattice).

is some (one)multi-dimensional kernel. The resulting filters are $\{F(\vec{z}) = f(K(\vec{z})),\ G(\vec{z}) = g(K(\vec{z}))\}$. There are three main objectives of this transformation: preservation of PR property of the polynomials, control over the shape of the filters and preservation of "flatness" at DC and zeros at alias frequencies. For preservation of PR we have the following theorem, proved in [11] [6] [13].

Theorem 2 *(Preservation of PR)*
Given two polynomials $\{f(x), g(x)\}$ such that $(f(x)g(-x))_e = 1$ then transformation of the polynomials by a kernel $K(\vec{z})$ will result in filters obeying $(f(K(\vec{z}))g(-K(\vec{z})))_e = 1$ (the PR condition) if and only if $K(\vec{z})$ is odd.

The requirement on $K(\vec{z})$ for the 1-D and the m-D quincunx lattices can be expressed as $K(-\vec{z}) = -K(\vec{z})$. Such kernels will be called *valid kernels*. For the general definition of valid kernels, see [13]. For control over the shape of the transformed filter, we require $K(\vec{z})$ to be zero-phase and bounded in the interval [-1,1] as z ranges over the unit (multi)-circle [13]. Preservation of flatness at DC and zeros at alias frequencies means appropriate transfer of the higher order derivatives of the polynomials at $x = \pm 1$ to the DC and alias frequencies of the filters respectively. The following theorem can be proved.

Theorem 3 *(Preservation of zeros)*
Let $f(x)$ be a polynomial and let $K(z)$ be an N-D kernel. Assume that the following conditions are satisfied: $f(x)$ has a zero of order n at $x = -1$, $f(x)$ is m-flat at $x = +1$, i.e. $\frac{\partial^l f}{\partial x^l}(1) = 0, \quad l = 1, \ldots, m-1$, $K(z)$ is valid for a 2-band N-D sampling lattice with aliasing frequency w, and $K(\vec{1}) = 1$ (to be referred to as normality *of K). Let $F(z) = f(K(z))$. Then the following three assertions hold: $F(\vec{1}) = f(1)$, $F(z)$ has a zero of order $2n$ at w, and $F(z)$ is $(2m)$-flat at $\vec{1}$.*

Proof:

1. $F(\vec{1}) = f(K(\vec{1})) = f(1)$.
2. Write $K(z) = \sum_{r \in S} a_r (z^r + z^{-r})$, where $S \cup -S$ is the support of $K(z)$. Note that normality implies that $\sum_{r\in S} a_r = 1/2$, and that validity implies that $w^{\pm r} = -1$ for all $r \in S$. Also, as we are dealing with 2-band filter banks, all the entries of w are equal to ± 1.

   Let $P \in \mathbf{Z}^N$ be such that $\tilde{K}(z) \stackrel{def}{=} \sum_{r\in S} a_r (z^{P+r} + z^{P-r})$ is a polynomial in z, i.e. z has only positive exponents. Then clearly $K(z) = z^{-P}\tilde{K}(z)$.

   As f has a zero of order n at -1 we can write $f(x) = (x+1)^n g(x)$. Therefore we can express $F(z)$ as

$$\begin{aligned} F(z) &= f(K(z)) \\ &= (K(z)+1)^n g(K(z)) \\ &= z^{-nP}(\tilde{K}(z) + z^P)^n g(K(z)) \end{aligned}$$

   We are done if we can prove that $(\tilde{K}(z) + z^P)$ has only terms of degree 2 and higher when expanded at w. Writing $z = w + y$ this will follow from

$$\begin{aligned} &\tilde{K}(w+y) + z^P = \\ &\quad \{\sum_{r\in S} a_r((w+y)^{P+r} + (w+y)^{P-r})\} + (w+y)^P \\ &= -w^P\{\sum_{r\in S} a_r(1 + (P+r)wy + 1 + (P-r)wy)\} + \\ &\quad w^P(1 + Pwy) + \{\text{terms of degree} \geq 2\} \\ &= -w^P\{\sum_{r\in S} 2a_r(1+Pwy)\} + \\ &\quad w^P(1+Pwy) + \{\text{terms of degree} \geq 2\} \\ &= \{\text{terms of degree} \geq 2\} \end{aligned} \qquad (11)$$

   Here we have used the properties $w^r = 1$, $w^{-1} = 1$ and $\sum_{r\in S} a_r = 1/2$.
3. Define $\hat{f}(x)$ to be $\hat{f}(x) = f(-x) - 1$. $\hat{f}(x)$ has a zero of order m at -1. By assertion 2 of this theorem (proved above) $\hat{f}(K(z))$ has a zero of order $2m$ at w. As $F(z) - 1 = f(K(z)) - 1 = \hat{f}(-K(z)) = \hat{f}(K(wz))$ it follows that $F(z) - 1$ has a zero of order $2m$ at $w^2 = \vec{1}$. Therefore $F(z)$ is $(2m)$-flat at 1. □

To summarize, if the kernel $K(z)$ is odd and bounded zero-phase, then the resulting filters will be a bi-orthogonal PR pair, with control over the shape given by the polynomials, and the kernel. Readers may recognize this as the insertion step of generating filter banks using the McClellan transform [12] [11] [14] [4] [7] [16]. In our case, the starting point is not filters, but polynomials over the reals. Moreover, if the kernels are also normal, then a zero of order n at -1 will be transformed to $2n$ zeros at the aliasing frequency of the filter.

Example 4 (1-D and 2-D filter bank from polynomials) The function $\tilde{f}_3(x)$ of Eq. 8 and $\hat{g}_3(x)$ of Eq. 10, shown in Fig. 3 and Fig. 5 respectively, are transformed using kernels $K(z) = \frac{1}{2}(z + z^{-1})$ and $K(z_1, z_2) = (z_1 + z_1^{-1} + z_2 + z_2^{-1})/4$, to yield 1-D and 2-D filters respectively. It is easy to check that these are normal kernels as defined in Thm. 3. The resulting zero-phase BO filters $\tilde{f}_3(K(z))$ and $\hat{g}_3(K(z))$ are shown in Fig. 7 and Fig. 8 for the 1-D case, and in Fig. 9 and Fig. 10 for the 2-D case respectively. □

VI . Conclusions

We have shown that the design of *all* two channel zero phase 1-D bi-orthogonal filter banks can be reduced to the design of pairs of real polynomials with domain bounded to [-1,1]. A particular class of regular polynomials is designed, along with their companion polynomials, and it has been shown that various constraints can be imposed on the polynomials using simple manipulations. The polynomials are then transformed to yield multidimensional bi-orthogonal filter banks.

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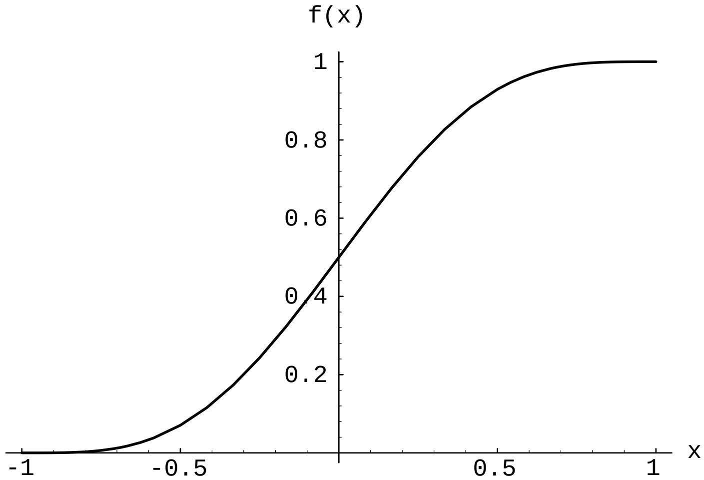

Fig. 1. The basic polynomial $f_3(x)$

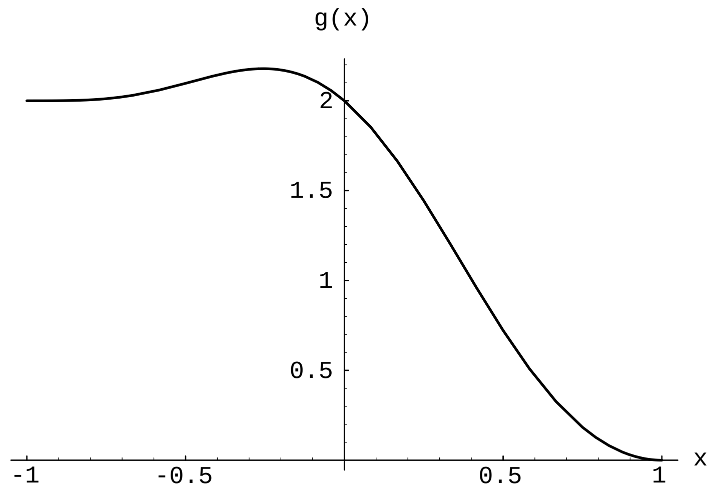

Fig. 2. The companion polynomial $\hat{g}_3(x)$

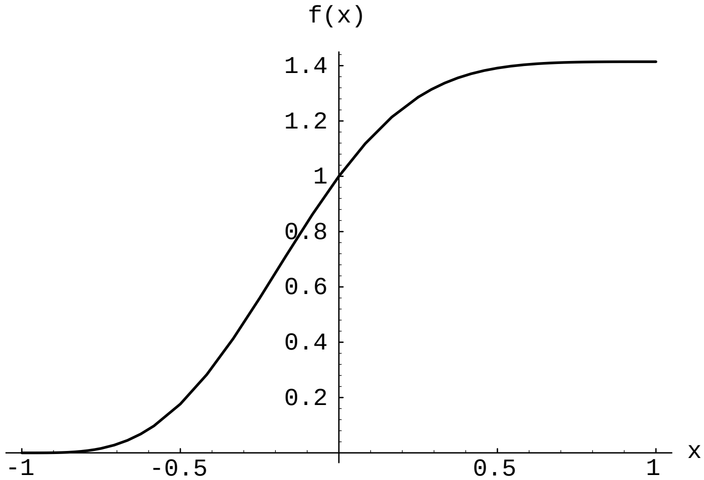

Fig. 3. The “raised” polynomial $f_3(x)$

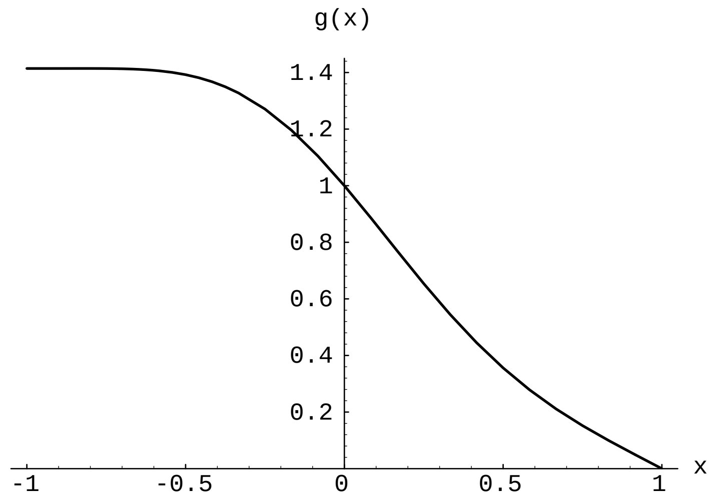

Fig. 4. A companion $\hat{g}_3(x)$ to $f_3(x)$

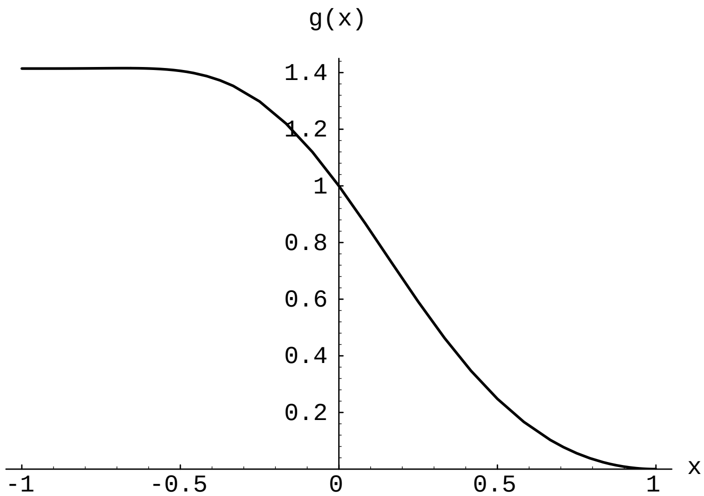

Fig. 5. A companion $\hat{g}_3(x)$ with a zero at 1

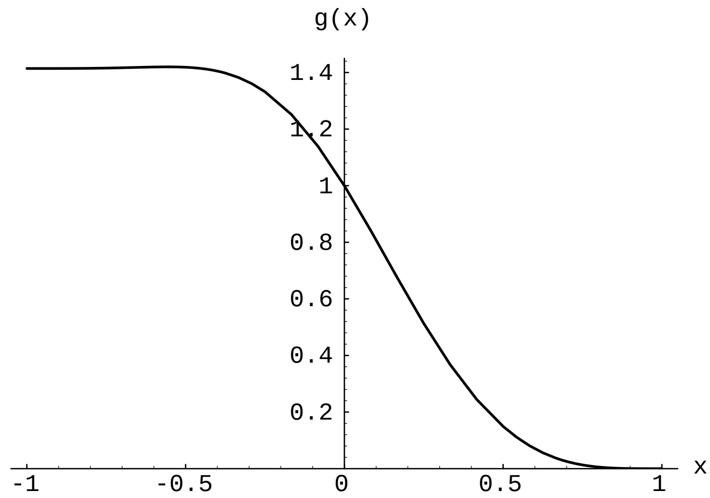

Fig. 6. Another companion $\hat{g}_3(x)$ with many zeros at 1

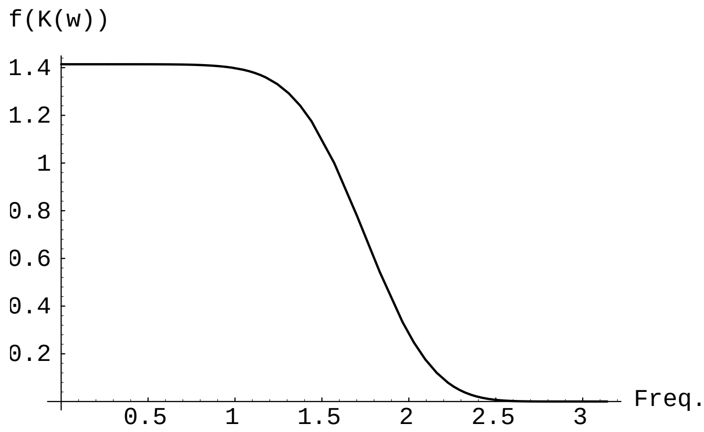

Fig. 7. The low pass filter $f_3(K(z))$.

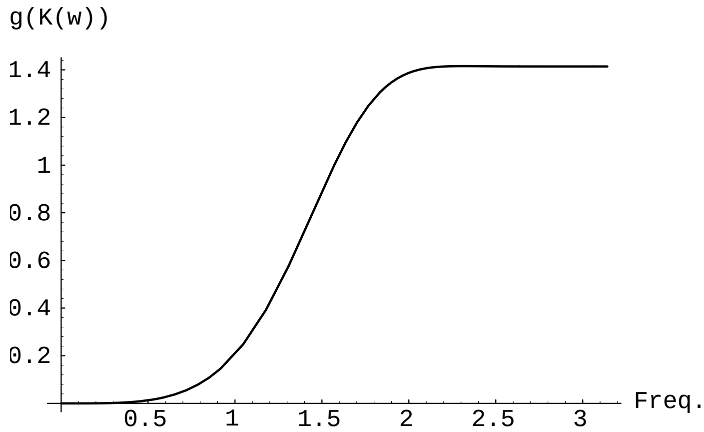

Fig. 8. The high pass filter $\hat{g}_3(K(z))$.

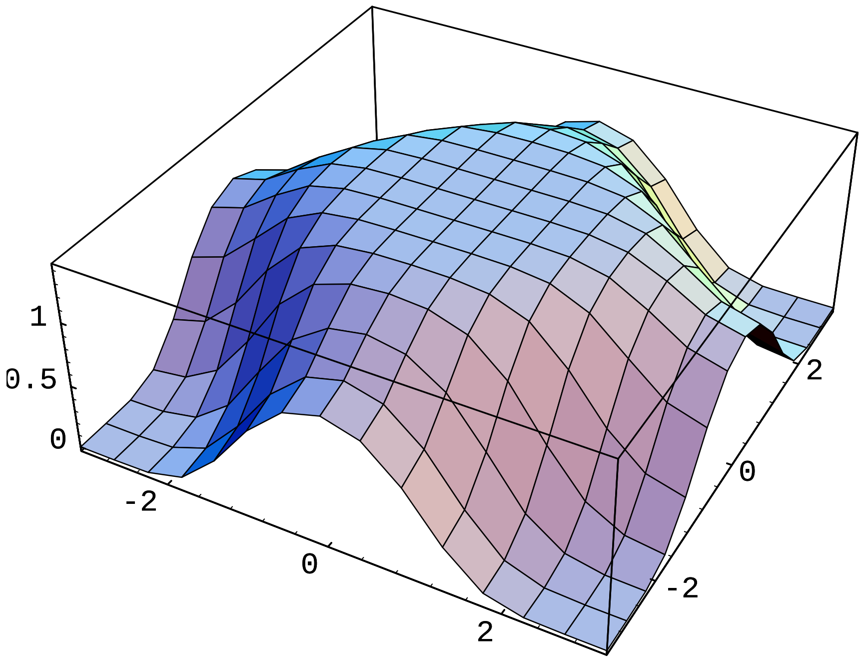

Fig. 9. The diamond shape low pass filter $f_3(K(z))$.

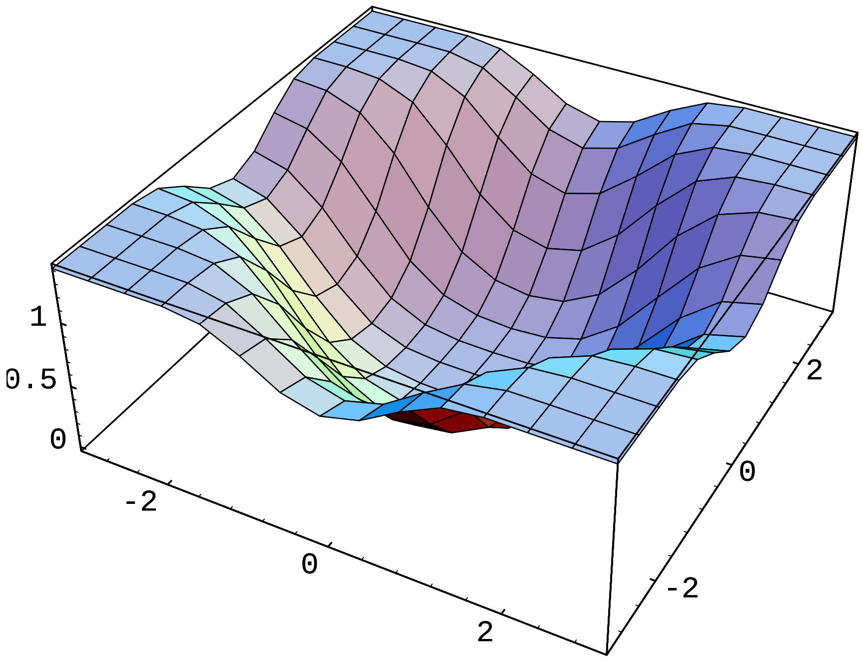

Fig. 10. The diamond shape high pass filter $\hat{g}_3(K(z))$