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A Constructor-Based Approach to Positive/Negative-Conditional Equational Specifications

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We study algebraic specifications given by finite sets R of positive/negative-conditional equations (i. e. universally quantified first-order implications with a single equation in the succedent and a conjunction of positive and negative (i. e. negated) equations in the antecedent). The class of models of such a specification R does not contain in general a minimum model in the sense that it can be mapped to any other model by some homomorphism. We present a constructor-based approach for assigning appropriate semantics to such specifications. We introduce two restrictions: firstly, for a condition to be fulfilled we require the evaluation values of the terms of the negative equations to be in the constructor sub-universe which contains the evaluation values of all constructor ground terms; secondly, we restrict the constructor equations to have "Horn"-form and to be "constructor-preserving". A reduction relation for R is defined, which allows to generalize the fundamental results for positive-conditional rewrite systems. This reduction relation is monotonic w. r. t. consistent extension of the specification, which is of practical importance as it allows for an incremental construction process of complex specifications without destroying reduction steps which were possible before. Under the assumption of confluence, the factor algebra of the term algebra modulo the congruence of the reduction relation is a minimal model which is (beyond that) the minimum of all models that do not identify more objects of the constructor sub-universe than necessary. We define several kinds of compatibility of R with a reduction ordering for achieving decidability of reducibility, and present several criteria for the confluence of our reduction relation.

Contents

1 Introduction and Overview
2 Basic Notions and Notations
3 Motivation
4 Syntax and Semantics of Specifications
5 The Reduction Relation
6 Testing for Confluence
7 Compatible Rule Systems
8 Conclusion
A The Proofs
References

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1. Introduction and Overview

We present a constructor-based approach for assigning semantics to algebraic specifications with finite sets R of positive/negative-conditional equations. In this approach, the non-constructor function symbols can be used for (possibly partially) specifying functions on a domain of discourse supplied by the constructor ground terms and called the *constructor sub-universe*. For such partial specifications of functions, variables ranging over the constructor terms (or the constructor sub-universe) are likely to be more convenient than variables ranging over all terms (including "junk" terms) (or the whole universe), because the specifier usually (unless he wants to specify error-recovery or non-strict functions) does not intend to tell how the functions behave on objects that are "*undefined*" in the sense that they do not belong to the domain of discourse. Therefore we generalize unconditional equations not only by adding positive and negative conditions but also by allowing *constructor variables* in addition to the usual *general variables*.

In general, specifications with positive/negative-conditional equations lack an initial model. This becomes relevant when a unique "computational model" (abstract data type) or appropriate notions of inductive validity are to be chosen. The most promising attempt in literature to overcome this problem has been that in Kaplan (1988). There, one of the quasi-initial models is distinguished by means of control information extracted from the rules, which must be compatible with a noetherian ordering. In addition, Kaplan gives a straightforward ground term reduction relation. However, the distinction of his quasi-initial model cannot be expressed without the control part of the specification. Furthermore, his reduction relation is not monotonic w. r. t. consistent extension of the specification. For these reasons, we choose a new different approach. Instead of using control information we introduce two syntactically expressible restrictions:

(A) For a condition to be true, the terms of its negative equations must be "*defined*" in the sense that their evaluations fall into the constructor sub-universe. This requirement is achieved by adding condition literals expressing this property and goes well with our intention of taking the constructor sub-universe as the domain of discourse.

(B) We restrict the constructor rules (which express equalities among the constructor terms) to have "Horn"-form and to be "constructor-preserving".

We can then define our reduction relation, which does not need to be noetherian, without using noetherian orderings anymore. Contrary to Kaplan, we can show the monotonicity of this reduction relation w. r. t. consistent extension of the specification. As in Kaplan's approach, assuming confluence of our reduction relation, the factor algebra of the ground term algebra modulo the congruence of our reduction relation is a quasi-initial model for our specification. Unlike Kaplan, however, it is also initial in the class of all models which do not identify more objects of the constructor sub-universe than necessary. Thus, the distinction of our intended computational model is not based on control information, but on homomorphisms between the models of the first-order logical part of the specification.

Finally, to achieve decidability of reducibility and to enhance our means of testing for confluence, we define several kinds of compatibility of R with a reduction ordering, which enable us to present several confluence criteria for our reduction relation.

The more difficult proofs of presented results can be found in appendix A.

2. Basic Notions and Notations

Since our approach is based on the consequent distinction of constructors, we have to be quite explicit about terms, substitutions, and algebras.

We use '$\uplus$' for the union of disjoint classes and 'id' for the identity function. For classes A, B we define: $\mathrm{dom}(A) := \{\, a \mid \exists b\colon (a,b) \in A \,\}$; $B[A] := \{\, b \mid \exists a \in A\colon (a,b) \in B \,\}$.

2.1. Terms and Substitutions

We will consider terms of fixed arity over many-sorted signatures. A *signature*

$$\mathrm{sig} = (\mathrm{F}, \mathrm{S}, \alpha)$$

consists of an enumerable set of function symbols F, a finite set of sorts S (disjoint from F), and a computable arity-function $\alpha : \mathrm{F} \to \mathrm{S}^{+}$. For $f \in \mathrm{F} : \alpha(f)$ is the list of argument sorts augmented by the sort of the result of f; to ease reading we will sometimes insert a '$\longrightarrow$' between a nonempty list of argument sorts and the result sort.

A *constructor sub-signature of the signature* sig is a signature

$$\mathrm{cons} = (\mathrm{C}, \mathrm{S}, \alpha|_{\mathrm{C}})$$

such that the set C is a decidable subset of F. C is called the set of *constructor symbols*; the complement $\mathrm{N} = \mathrm{F} \setminus \mathrm{C}$ is called the set of *non-constructor symbols*.

Example 2.1. (Signature with Constructor Sub-signature)

$$\begin{array}{rcl|rcl}
\mathrm{C} & = & \{\mathsf{0}, \mathsf{s}, \mathsf{false}, \mathsf{true}, \mathsf{nil}, \mathsf{cons}\} & \alpha(\mathsf{false}) & = & \mathsf{bool} \\
\mathrm{N} & = & \{\mathsf{-}, \mathsf{memberp}\} & \alpha(\mathsf{true}) & = & \mathsf{bool} \\
\mathrm{S} & = & \{\mathsf{nat}, \mathsf{bool}, \mathsf{list}\} & \alpha(\mathsf{nil}) & = & \mathsf{list} \\
 & & & \alpha(\mathsf{cons}) & = & \mathsf{nat\ list} \longrightarrow \mathsf{list} \\
\alpha(\mathsf{0}) & = & \mathsf{nat} & \alpha(\mathsf{-}) & = & \mathsf{nat\ nat} \longrightarrow \mathsf{nat} \\
\alpha(\mathsf{s}) & = & \mathsf{nat} \longrightarrow \mathsf{nat} & \alpha(\mathsf{memberp}) & = & \mathsf{nat\ list} \longrightarrow \mathsf{bool}
\end{array}$$

A *variable-system for a signature* sig is an S-sorted family of decidable sets of variable symbols which are mutually disjoint and disjoint from F. As the basis for our terms throughout the whole paper we assume two fixed disjoint variable-systems $\mathrm{V_{SIG}}$ of *general variables* and $\mathrm{V_{CONS}}$ of *constructor variables* such that for each $s \in \mathrm{S}$ we have $|\mathrm{V}_{\mathrm{SIG},s}|$, $|\mathrm{V}_{\mathrm{CONS},s}| \notin \mathbb{N}$. By abuse of notation we will use the symbol 'X' for an S-sorted family to denote not only the family $X = (X_s)_{s \in \mathrm{S}}$ itself, but also the union of its *ranges*: $\bigcup_{s \in \mathrm{S}} X_s$. $\mathcal{T}(\mathrm{sig}, \mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}})$ denotes the S-sorted family of all well-sorted *(variable-mixed) terms* over $\mathrm{sig}/\mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}}$, while $\mathcal{GT}(\mathrm{sig})$ denotes the S-sorted family of all well-sorted *ground terms* over sig. Similarly, $\mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}})$ denotes the S-sorted family of all *(variable-mixed) constructor terms*, $\mathcal{T}(\mathrm{cons}, \mathrm{V_{CONS}})$ denotes the S-sorted family of all *pure constructor terms*, while $\mathcal{GT}(\mathrm{cons})$ denotes the S-sorted family of all *constructor ground terms*. To avoid problems with empty sorts, we assume $\mathcal{GT}(\mathrm{cons})$ to have nonempty ranges only.

As exhibited in Avenhaus & Becker (1992), it is adequate to describe our terms, substitutions, and algebras within the order-sorted framework in the style of Gogolla (1983) or Smolka & al. (1989): Take $\{\mathrm{SIG}, \mathrm{CONS}\} \times \mathrm{S}$ for the sorts with the sort declaration that for each $s \in \mathrm{S}$ the sort (CONS, s) is a sub-sort of the sort (SIG, s); and replace each arity declaration of the form $\alpha(f) = s_0 \ldots s_{n-1} \longrightarrow s_n$ with the arity declaration

$$\alpha(f) \ni (\mathrm{SIG}, s_0) \ldots (\mathrm{SIG}, s_{n-1}) \longrightarrow (\mathrm{SIG}, s_n);$$

moreover, for $f \in \mathrm{C}$ add the arity declaration

$$\alpha(f) \ni (\mathrm{CONS}, s_0) \ldots (\mathrm{CONS}, s_{n-1}) \longrightarrow (\mathrm{CONS}, s_n).$$

A *variable-system for a signature* sig *with sub-signature* cons is defined to be a $\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}$-sorted family $\mathrm{V} = (\mathrm{V}_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}$ of decidable sets which are mutually disjoint and disjoint from F. We use $\mathcal{V}(A)$ to denote the $\{\mathrm{SIG},\mathrm{CONS}\}\times$S-sorted family of variables occurring in a structure A (e. g. a term or a list of terms).

Now the order-sorted notation for our sets of terms are the $\{\mathrm{SIG},\mathrm{CONS}\}\times$S-sorted families $\mathcal{T} = (\mathcal{T}_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}$ and $\mathcal{GT} = (\mathcal{GT}_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}$ given by $\mathcal{T}_{\mathrm{SIG},s} := \mathcal{T}(\mathrm{sig},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathrm{CONS}})_s$, $\mathcal{T}_{\mathrm{CONS},s} := \mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{CONS}})_s$, $\mathcal{GT}_{\mathrm{SIG},s} := \mathcal{GT}(\mathrm{sig})_s$, and $\mathcal{GT}_{\mathrm{CONS},s} := \mathcal{GT}(\mathrm{cons})_s$. To avoid confusion: Note that $\mathcal{T}_{\mathrm{CONS},s} \subseteq \mathcal{T}_{\mathrm{SIG},s}$ for $s\in\mathrm{S}$, whereas $\mathrm{V}_{\mathrm{CONS},s}\cap\mathrm{V}_{\mathrm{SIG},s} = \emptyset$. Our custom of reusing the symbol of a family for the union of its ranges now allows to write $\mathcal{T}$ as a shorthand for $\mathcal{T}(\mathrm{sig},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathrm{CONS}})$.

For a term $t\in\mathcal{T}$ we denote by $\mathcal{POS}(t)$ the *set of its positions* (which are lists of positive natural numbers), by t/p the subterm of t at position p, and by $t[\,p\leftarrow t'\,]$ the result of replacing t/p with t' at position p in t. We write $p|q$ to express that neither p is a prefix of q, nor q a prefix of p. For $P\subseteq\mathcal{POS}(t)$; $\forall p,q\in P: (p{\neq}q \Rightarrow p|q)$; we denote by $t[\,p\leftarrow t'_p \mid p\in P\,]$ the result of replacing for each $p\in P$ the subterm at position p in the term t with the term t'_p. t is *linear* :iff $\forall p,q\in\mathcal{POS}(t) : (t/p{=}t/q{\in}\mathrm{V} \Rightarrow p{=}q)$.

The set of *substitutions* from a variable-system $\mathrm{X} = (\mathrm{X}_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}$ to a $\{\mathrm{SIG},\mathrm{CONS}\}\times$S-sorted family of sets $T = (T_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}$ is defined to be

$$\mathcal{SUB}(\mathrm{X},T) := \{\ \sigma:\mathrm{X}\to T \mid \forall(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S} : \forall x\in\mathrm{X}_{\varsigma,s} : \sigma(x)\in T_{\varsigma,s}\ \}\ .$$

Important sets of substitutions are $\mathcal{SUB}(\mathrm{V},\mathcal{T})$ and $\mathcal{SUB}(\mathrm{V},\mathcal{GT})$. The definition is consistent with the notion of substitutions in our order-sorted framework from above. A fortiori we get $\forall\sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}) : \forall(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S} : \forall t\in\mathcal{T}_{\varsigma,s}:\ t\sigma\in\mathcal{T}_{\varsigma,s}$.

Let E be a finite set of equations and X a finite set of variables. A substitution $\sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T})$ is called *a unifier for* E :iff $E\sigma\subseteq\mathrm{id}$. Such a unifier is called *most general on* X :iff for all unifiers μ for E there is some $\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T})$ such that $(\sigma\tau)|_{\mathrm{X}} = \mu|_{\mathrm{X}}$. If E has a unifier, then it also has a most general unifier† on X, denoted by $\mathrm{mgu}(E,\mathrm{X})$.

2.2. Algebras

We define a sig/cons-*algebra* $\mathcal{A}$ *over the signature* $\mathrm{sig} = (\mathrm{F},\mathrm{S},\alpha)$ *with constructor sub-signature* $\mathrm{cons} = (\mathrm{C},\mathrm{S},\alpha|_{\mathrm{C}})$ to be a function defined on $\mathrm{F}\uplus(\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S})$ with $\forall s\in\mathrm{S} : (\ \emptyset\ \neq\ \mathcal{A}(\mathrm{CONS},s)\ \subseteq\ \mathcal{A}(\mathrm{SIG},s)\ )$ and

$f^{\mathcal{A}} : \mathcal{A}(\mathrm{SIG},s_0)\ \times\cdots\times\mathcal{A}(\mathrm{SIG},s_{n-1})\ \to\ \mathcal{A}(\mathrm{SIG},s_n)$ for $f\in\mathrm{F}$ with $\alpha(f){=}s_0\ldots s_n$,

$c^{\mathcal{A}}[\mathcal{A}(\mathrm{CONS},s_0)\times\cdots\times\mathcal{A}(\mathrm{CONS},s_{n-1})]\subseteq\mathcal{A}(\mathrm{CONS},s_n)$ for $c\in\mathrm{C}$ with $\alpha(c){=}s_0\ldots s_n$.

We write $f^{\mathcal{A}}$ instead of $\mathcal{A}(f)$ for $f\in\mathrm{F}$. $\mathcal{A}(\varsigma,s)$ is called the *universe of* $\mathcal{A}$ *for* $(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}$. The sig/cons-algebras of this definition are nothing but the order-sorted algebras over the order-sorted signature exhibited in sect. 2.1. A sig/cons-algebra $\mathcal{A}$ is called *trivial* :iff $\forall(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S} :\ |\mathcal{A}(\varsigma,s)|\ = 1$.

A (total) sig/cons-*homomorphism* $h{::}\mathcal{A}{\to}\mathcal{B}$ from a sig/cons-algebra $\mathcal{A}$ to a sig/cons-algebra $\mathcal{B}$ is an S-sorted family $h = (h_s)_{s\in\mathrm{S}}$ of functions $h_s : \mathcal{A}(\mathrm{SIG},s)\to\mathcal{B}(\mathrm{SIG},s)$ which are compatible with sig and cons: For $f\in\mathrm{F}$; $\alpha(f){=}s_0\ldots s_{n-1}s_n$; $\forall i{<}n$: $a_i\in\mathcal{A}(\mathrm{SIG},s_i)$:

$$h_{s_n}(f^{\mathcal{A}}(a_0,\ldots,a_{n-1})) = f^{\mathcal{B}}(h_{s_0}(a_0),\ldots,h_{s_{n-1}}(a_{n-1}))$$

and for all $s\in\mathrm{S}$: $\quad h_s[\mathcal{A}(\mathrm{CONS},s)]\ \subseteq\ \mathcal{B}(\mathrm{CONS},s)$.

† For this most general unifier σ we could, as usual, even require $\sigma\sigma = \sigma$ but (unless we restrict the variables in our terms either to be from $\mathrm{V}_{\mathrm{SIG}}$ only (as in Wirth & Gramlich (1993)) or from $\mathrm{V}_{\mathrm{CONS}}$ only (as in Avenhaus & Becker (1992))) *not* $\mathcal{V}(\sigma[\mathcal{V}(E)])\ \subseteq\ \mathcal{V}(E)$. To see this, consider $x,y\in\mathrm{V}_{\mathrm{CONS},\mathsf{nat}}$; $Y\in\mathrm{V}_{\mathrm{SIG},\mathsf{nat}}$; a mgu for $\{(x,\ \mathsf{s}(Y))\}$ must be something like $\{\ x\mapsto\mathsf{s}(y),\ Y\mapsto y\ \}$.

Taking the class of sig/cons-algebras for the class of objects and the class of sig/cons-homomorphisms for the class of arrows, we get the sig/cons-*homomorphism category of* sig/cons-*algebras*. The composition $hk{::}\mathcal{A}{\to}\mathcal{C}$ of $h{::}\mathcal{A}{\to}\mathcal{B}$ and $k{::}\mathcal{B}{\to}\mathcal{C}$ is defined by $hk := (h_s \circ k_s)_{s\in\mathrm{S}}$ and the identity homomorphism for $\mathcal{A}$ is $(\mathrm{id}|_{\mathcal{A}(\mathrm{SIG},s)})_{s\in\mathrm{S}}{::}\mathcal{A}{\to}\mathcal{A}$.

Let $\mathrm{X}\subseteq\mathrm{V}$. We use $\mathcal{T}(\mathrm{X})$ to denote the *term algebra over* X and sig/cons/V. This term algebra has $\mathcal{T}_{\varsigma,s} \cap \mathcal{T}(\mathrm{sig},\mathrm{X})$ as the universe for each $(\varsigma,s) \in \{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}$ and for $f \in \mathrm{F}$: $f^{\mathcal{T}(\mathrm{X})}$ is given by $f^{\mathcal{T}(\mathrm{X})}(t_0,\ldots,t_{n-1}) = f(t_0,\ldots,t_{n-1})$. Similarly, we sometimes use $\mathcal{GT}$ for the *ground term algebra* $\mathcal{T}(\emptyset)$ over sig/cons instead of the family of ground terms. An $\mathcal{A}$-*valuation* κ *of* X is an element of

$$\mathcal{SUB}(\mathrm{X},\mathcal{A}) = \mathcal{SUB}((\mathrm{V}_{\varsigma,s}\cap\mathrm{X})_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}, (\mathcal{A}(\varsigma,s))_{(\varsigma,s)\in\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}}) \ .$$

The evaluation homomorphism $\mathcal{A}_\kappa{::}\mathcal{T}(\mathrm{X}){\to}\mathcal{A}$ is recursively defined by $\mathcal{A}_\kappa(x) = \kappa(x)$ for $(x \in \mathrm{X})$; and $\mathcal{A}_\kappa(f(t_0,\ldots,t_{n-1})) = f^{\mathcal{A}}(\mathcal{A}_\kappa(t_0),\ldots,\mathcal{A}_\kappa(t_{n-1}))$.

Lemma 2.2. (Substitution-Lemma)
Let $\mathcal{A}$ *be a* sig/cons-*algebra and* κ *an* $\mathcal{A}$-*valuation of* X. *For* $t\in\mathcal{T}$ *and* $\sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$:

$$\mathcal{A}_\kappa(t\sigma) = \mathcal{A}_{\sigma\circ\mathcal{A}_\kappa}(t)$$

For dunno $\in \{\mathrm{sig},\mathrm{cons}\}$, a sig/cons-algebra $\mathcal{A}$ is called dunno-*term-generated* :iff

$$\forall s\in\mathrm{S} : \forall a \in \mathcal{A}(\mathrm{SIG},s) : \exists t \in \mathcal{GT}(\mathrm{dunno})_s : \ \mathcal{A}(t) = a \ .$$

A sig/cons-*congruence* $\sim$ *on* $\mathcal{A}$ is an S-sorted family $\sim\, = (\sim_s)_{s\in\mathrm{S}}$ of equivalences $\sim_s$ on $\mathcal{A}(\mathrm{SIG},s)$ being compatible with sig, i. e. satisfying for $f \in \mathrm{F}$; $\alpha(f) = s_0\ldots s_n s_{n+1}$; $\forall i \leq n : a_i \in \mathcal{A}(\mathrm{SIG},s_i)$:

If $a_j \sim_{s_j} b$, then $f^{\mathcal{A}}(a_0,\ldots,a_n) \sim_{s_{n+1}} f^{\mathcal{A}}(a_0,\ldots,a_{j-1},b,a_{j+1},\ldots,a_n)$.

The *factor algebra of* $\mathcal{A}$ *modulo* $\sim$ is the sig/cons-algebra $\mathcal{B}$ (denoted by $\mathcal{A}/{\sim}$) given by:

$$\mathcal{B}(\varsigma,s) := \{\ \sim_s[\{a\}] \mid a \in \mathcal{A}(\varsigma,s)\ \} \qquad (\ (\varsigma,s) \in \{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}\ )$$
$$f^{\mathcal{B}}(\sim_{s_0}[\{a_0\}],\ldots,\sim_{s_{n-1}}[\{a_{n-1}\}]) := \ \sim_{s_n}[\{f^{\mathcal{A}}(a_0,\ldots,a_{n-1})\}]$$
$$(f\in\mathrm{F};\ \alpha(f) = s_0\ldots s_{n-1}s_n;\ \forall i<n : a_i \in \mathcal{A}(\mathrm{SIG},s_i))$$

The *canonical* sig/cons-*epimorphism of* $\mathcal{A}$ *modulo* $\sim$ is the sig/cons-homomorphism $k{::}\mathcal{A}{\to}\mathcal{A}/{\sim}$ given by $(s\in\mathrm{S};\ a\in\mathcal{A}(\mathrm{SIG},s))$: $k_s(a) := \ \sim_s[\{a\}]$.

For a sig/cons-homomorphism $h{::}\mathcal{A}{\to}\mathcal{B}$ we define its *kernel* to be the sig/cons-congruence $\ker(h)$ given by $(s\in\mathrm{S};\ a,b\in\mathcal{A}(\mathrm{SIG},s))$: $(a,b)\in\ker(h)_s$:iff $h_s(a) = h_s(b)$.

Theorem 2.3. (Homomorphism-Theorem)
Let $h{::}\mathcal{A}{\to}\mathcal{C}$ *be a* sig/cons-*homomorphism. Let* $\sim$ *be a* sig/cons-*congruence on* $\mathcal{A}$ *with* $\forall s\in\mathrm{S}{:} \sim_s \subseteq \ker(h)_s$. *Define* $\mathcal{B} := \mathcal{A}/{\sim}$. *Let* k *be the canonical* sig/cons-*epimorphism of* $\mathcal{A}$ *modulo* $\sim$. *Now* $h{=}kl$ *uniquely defines an* S-*sorted family of functions* $l = (l_s)_{s\in\mathrm{S}}$ *with* $l_s : \mathcal{B}(\mathrm{SIG},s) \to \mathcal{C}(\mathrm{SIG},s)$ *for* $s \in \mathrm{S}$. *Furthermore, this* l *is a* sig/cons-*homomorphism* $l{::}\mathcal{B}{\to}\mathcal{C}$. *Moreover, if* $\sim\, = \ker(h)$ *holds, then* l_s *is injective for each* $s \in \mathrm{S}$, *i. e.* $l{::}\mathcal{B}{\to}\mathcal{C}$ *is monic in the* sig/cons-*homomorphism category of* sig/cons-*algebras.*

By specialization of notions of category theory to full sub-categories of the sig/cons-homomorphism category of sig/cons-algebras and to the forgetful functor we define for a class K of sig/cons-algebras; a sig/cons-algebra $\mathcal{A}$; $\mathrm{X}\subseteq\mathrm{V}$; and $\kappa \in \mathcal{SUB}(\mathrm{X},\mathcal{A})$: $\mathcal{A}$ is *initial in* K :iff $\mathcal{A}\in\mathrm{K}$ and for all $\mathcal{B}\in\mathrm{K}$ there is a unique $h{::}\mathcal{A}{\to}\mathcal{B}$. $\mathcal{A}$ is *free for* K *over* X *w. r. t.* κ :iff for all $\mathcal{B}\in\mathrm{K}$ and $\mu\in\mathcal{SUB}(\mathrm{X},\mathcal{B})$ there is a unique $h{::}\mathcal{A}{\to}\mathcal{B}$ with $\mu = \kappa h$. $\mathcal{A}$ is *free in* K *over* X *w. r. t.* κ :iff $\mathcal{A}\in\mathrm{K}$ and $\mathcal{A}$ is free for K over X w. r. t. κ.

2.3. Relations

Let $X \subseteq V$. A relation R on $\mathcal{T}$ is called:

X-*stable (w. r. t. substitution)* :iff $\forall (t_0, \ldots t_{n-1}) \in R \colon \forall \sigma \in \mathcal{SUB}(V, \mathcal{T}(X))\colon$
$(t_0\sigma, \ldots, t_{n-1}\sigma) \in R$
X-*monotonic* :iff $\forall (t', t'') \in R : \forall t \in \mathcal{T}(\mathrm{sig}, X) : \forall p \in \mathcal{POS}(t) : \forall s \in S :$
$(t/p, t', t'' \in \mathcal{T}_{\mathrm{SIG},s} \;\Rightarrow\; (t[p \leftarrow t'], t[p \leftarrow t'']) \in R)$
sort-invariant :iff $\forall (t, t') \in R : \exists s \in S : \; t, t' \in \mathcal{T}_{\mathrm{SIG},s}$
sufficiently complete (w. r. t. $\mathcal{GT}(\mathrm{cons})$*)* :iff $\forall t \in \mathcal{GT}(\mathrm{sig}) : \exists t' \in \mathcal{GT}(\mathrm{cons}) : (t, t') \in R$

A *reduction ordering on* $\mathcal{T}$ is a V-monotonic, V-stable, and well-founded ordering. The *subterm ordering* $\lhd_{\mathrm{ST}}$ *on* $\mathcal{T}$ is the V-stable and well-founded ordering defined by: $t \unlhd_{\mathrm{ST}} t'$:iff $\exists p \in \mathcal{POS}(t') : t = t'/p$. A *simplification ordering on* $\mathcal{T}$ is a reduction ordering on $\mathcal{T}$ containing $\lhd_{\mathrm{ST}}$. For further information on orderings cf. Dershowitz (1987).

The symmetric, transitive, and reflexive & transitive closure of a relation $\longrightarrow$ will be denoted by $\longleftrightarrow$, $\stackrel{+}{\longrightarrow}$, and $\stackrel{*}{\longrightarrow}$, resp.. Two terms v, w are called *joinable w. r. t.* $\longrightarrow$:iff $v \downarrow w$:iff $v \stackrel{*}{\longrightarrow} \circ \stackrel{*}{\longleftarrow} w$. $\longrightarrow$ is called *confluent below* u :iff $\forall v, w : ((v \stackrel{*}{\longleftarrow} u \stackrel{*}{\longrightarrow} w) \Rightarrow (v \downarrow w))$. $\longrightarrow$ is called *locally confluent below* u :iff $\forall v, w : ((v \longleftarrow u \longrightarrow w) \Rightarrow (v \downarrow w))$; it is called *[locally] confluent* :iff it is [locally] confluent below all u.

3. Motivation

Kaplan (1988) defines a [negated] equation in the condition of a positive/negative-conditional equation to hold :iff its terms [do not] have a common reduct (w. r. t. $\stackrel{*}{\longrightarrow}$). If the resulting reduction relation is confluent and the rules are *decreasing* w. r. t. some ordering $\rhd$ (cf. Dershowitz & al. (1988a)), then its congruence closure is minimal (but not a minimum!) w. r. t. set-inclusion among the congruence relations whose factor algebras (w. r. t. $\mathcal{GT}$) are models of R. Despite of the lack of an initial model even in this restricted case, positive/negative-conditional equations are necessary for convenient specification, as illustrated by the following example, where '$\longleftarrow$' precedes the condition of an equation.

Example 3.1. (continuing Example 2.1)

R: $x - 0 = x$ | $\mathsf{memberp}(x,\mathsf{nil}) = \mathsf{false}$
$\mathsf{s}(x) - \mathsf{s}(y) = x - y$ | $\mathsf{memberp}(x,\mathsf{cons}(y,l)) = \mathsf{true} \longleftarrow x = y$
| $\mathsf{memberp}(x,\mathsf{cons}(y,l)) = \mathsf{memberp}(x,l) \longleftarrow x \neq y$

The Importance of Confluence

Why is confluence essential for reduction with positive/negative-conditional rules? Firstly (even without negative conditions), confluence is needed for the completeness of testing semantic equality of two condition terms by looking for a common reduct. This means: We need confluence for the congruence defined in Kaplan (1988) to yield a model of R. Secondly, it is needed for guaranteeing the congruence to be minimal:

Example 3.2. *Let* $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}$ *be constants of the same sort,* $\mathsf{a} \rhd \mathsf{b} \rhd \mathsf{c} \rhd \mathsf{d} \rhd \mathsf{e}$. *Let* R*:* $\mathsf{c}{=}\mathsf{d}$ *;* $\mathsf{c}{=}\mathsf{e}$ *;* $\mathsf{a}{=}\mathsf{b} {\longleftarrow} \mathsf{e}{\neq}\mathsf{d}$ *. In this case, the congruence closure* $\stackrel{*}{\longleftrightarrow}$ *of the reduction relation* $\longrightarrow = \{ (\mathsf{c}, \mathsf{d}), (\mathsf{c}, \mathsf{e}), (\mathsf{a}, \mathsf{b}) \}$ *of Kaplan (1988) is not minimal among the congruences satisfying* R *since it properly contains the congruence closure of* $\{ (\mathsf{c}, \mathsf{d}), (\mathsf{c}, \mathsf{e}) \}$ *.*

While confluence can be dropped for merely positive conditional equations by testing for congruence instead of testing for the existence of a common reduct of two condition terms, the situation is worse for positive/negative-conditional equations: It does not suffice to test non-congruence for inequality of two condition terms if confluence is not provided:

EXAMPLE 3.3. *Let the signature and the ordering be as in the previous example. Let* R*:* $\mathsf{a}{=}\mathsf{d}$ *;* $\mathsf{a}{=}\mathsf{e}{\longleftarrow}\mathsf{b}{\neq}\mathsf{c}$ *;* $\mathsf{b}{=}\mathsf{c}{\longleftarrow}\mathsf{d}{=}\mathsf{e}$ *. Any congruence yielding a model of* R *must contain* (b,c)*: If it did not, it would contain* (a,e) *by the second rule, then by the first rule* (d,e)*, and hence by the last rule* (b,c)*. Therefore, no matter which congruence we actually use for condition-testing, the test of* $\mathsf{b}{\neq}\mathsf{c}$ *with such a (model-yielding) congruence will always fail, such that we cannot establish* $\mathsf{a}\overset{\oplus}{\longleftrightarrow}\mathsf{e}$ *by testing the condition of the second rule, and hence cannot establish* $\mathsf{b}\overset{\oplus}{\longleftrightarrow}\mathsf{c}$ *by testing the condition of the last rule. But* R *has the minimum model "*$\mathsf{a}{=}\mathsf{d}$*;* $\mathsf{b}{=}\mathsf{c}$*", which cannot be obtained by the simple method of condition-testing anymore, but only by paramodulation and factoring instead, which in our opinion are too complicated for establishing just a simple reduction step.*

By this we conclude that in case of negative equations in the condition, confluence is required for computing a correct reduct by the method of condition-testing.

Problematic Aspects of Kaplan (1988)

The major shortcoming of the reduction relation in Kaplan (1988), however, is (as noted above) that its congruence closure is not a minimum (i. e. being smaller than anything else) but only minimal (i. e. there is nothing smaller) among the congruences yielding a model of R. Thus, contrary to the case of merely positive conditional specifications, there might be reductions $s{\longrightarrow}t$ with $s{=}t$ not holding in all models logically specified by R.

Kaplan correctly argues as follows: By writing "$\mathsf{c}{=}\mathsf{d} \longleftarrow \mathsf{d}{\neq}\mathsf{e}$" instead of the logically equivalent "$\mathsf{c}{=}\mathsf{d} \vee \mathsf{d}{=}\mathsf{e}$" the specifier adds some "operational" information to the logical part of the specification. This "operational" information may therefore be used to control the choice of the intended minimal congruence "$\mathsf{c}{=}\mathsf{d}$" of the congruences yielding a model of R ("$\mathsf{c}{=}\mathsf{d}$", "$\mathsf{d}{=}\mathsf{e}$", and "$\mathsf{c}{=}\mathsf{d}{=}\mathsf{e}$").

However, if the ordering context given by other rules does not allow "$\mathsf{c} \rhd \mathsf{e}$" without extending '$\rhd$' to a non-noetherian relation, then the specifier is not at all allowed to write "$\mathsf{c}{=}\mathsf{d} \vee \mathsf{d}{=}\mathsf{e}$" in the form of "$\mathsf{c}{=}\mathsf{d} \longleftarrow \mathsf{d}{\neq}\mathsf{e}$". Even if he actually is allowed to specify his intended control information, he is likely to be unable to keep track of the consequences of all his pieces of "operational" information, especially because he is forced to include some operational information into each rule he writes.

All this would not be crucial, if the operational information were used only for admissibility of a specification, as is the case with our approach where the operational information given by writing first-order clauses in the form of positive/negative-conditional rules is used for our reduction relation only, which again must be confluent for the specification to be admissible. The distinction of our computational model for an admissible specification, however, does not depend on the rules' operational information anymore, but only on homomorphisms between the models of the logical part of the specification. Therefore our computational semantics (of a specification which has passed the admissibility test depending on its operational information) can be grasped on a more abstract level in terms of models and homomorphisms without any knowledge of rewriting, confluence, orderings on terms, termination, etc.. Contrariwise, in Kaplan's approach not only admissibility of a specification but also its computational model semantics itself depends on the rules' operational information and is not expressible without.

This loss of the pure logic view on an admissible specification goes with the loss of a property which is very important in practice (cf. Theorem 5.16 and the discussion which precedes it): The monotonicity of logic is lost:

EXAMPLE 3.4. (continuing Example 3.1)
$\mathsf{memberp}(\mathsf{0},\mathsf{cons}(\mathsf{0}-\mathsf{s}(\mathsf{0}),\mathsf{nil})) \overset{\circledast}{\longrightarrow} \mathsf{false}$ *no longer holds after adding the rule* $\mathsf{0}-\mathsf{s}(x)=\mathsf{0}$. *This shows that completing the definition of a partially specified function (here: '−') (even in a way that does not confuse different constructor terms) might destroy some reductions and congruences which were possible before.*

Similarly, reduction of non-ground terms is of no use because the reduction relation is not stable:

EXAMPLE 3.5. *As* $X \in \mathrm{V_{SIG}}$ *does not reduce to* $\mathsf{0}$ *, one might say*

$$\mathsf{memberp}(\mathsf{0},\mathsf{cons}(X,\mathsf{nil})) \overset{\circledast}{\longrightarrow} \mathsf{false}.$$

But for $X \mapsto \mathsf{0}$ *this does not make sense.*

Looking for Remedy

One could think that in practice the problem of a minimal congruence not being a minimum hardly arises or can be avoided by convenient purely syntactic restrictions on the defining rules. Using the specification of Example 3.1 above (which is not a sophisticated but a really standard specification and therefore essential in practice), the example below will on the contrary exhibit that the problem is relevant in practice and that purely syntactic restrictions on the defining rules cannot be reasonable because they would have to forbid such a very restricted use of negative conditions as in Example 3.1.

EXAMPLE 3.6. (continuing Example 3.1)
We exclude the function symbols $\mathsf{0}$, s, *and* $-$ *(together with their respective rules), and enrich the signature with two constants* a, b *with* $\alpha(\mathsf{a})=\alpha(\mathsf{b})=\mathsf{nat}$. *The reduction relation* $\longrightarrow$ *of Kaplan (1988) is confluent (since there are no feasible critical pairs) and noetherian (lexicographic path ordering given by* $\mathsf{memberp} \succ \mathsf{true},\mathsf{false}$*) in this case.*

Consider the following two congruence relations on ground terms, given by their congruence classes for the sorts nat *and* bool*:*

$\underline{\overset{\circledast}{\longleftrightarrow}:}$ $\{\,\mathsf{a}\,\}$
$\{\,\mathsf{b}\,\}$
$\{\,\mathsf{false}\,\} \cup \{\,\mathsf{memberp}(x,l) \mid (x \in \{\mathsf{a},\mathsf{b}\} \;\wedge\; (x \textit{ does not occur in } l))\,\}$
$\{\,\mathsf{true}\,\} \cup \{\,\mathsf{memberp}(x,l) \mid (x \in \{\mathsf{a},\mathsf{b}\} \;\wedge\; (x \textit{ does occur in } l))\,\}$

$\underline{\sim:}$ $\{\,\mathsf{a},\ \mathsf{b}\,\}$
$\{\,\mathsf{false},\ \mathsf{memberp}(\mathsf{a},\mathsf{nil}),\ \mathsf{memberp}(\mathsf{b},\mathsf{nil})\,\}$
$\{\,\mathsf{true}\,\} \cup \{\,\mathsf{memberp}(x,l) \mid (x \in \{\mathsf{a},\mathsf{b}\} \;\wedge\; l{\neq}\mathsf{nil})\,\}$

Now, both $\overset{\circledast}{\longleftrightarrow}$ *and* $\sim$ *yield a model of* R. *By* $a \sim b$ *and* $a \overset{\circledast}{\not\longleftrightarrow} b$ *we know that* $\sim$ *is no minimum. By* $\mathsf{memberp}(\mathsf{a},\mathsf{cons}(\mathsf{b},\mathsf{nil})) \overset{\circledast}{\longleftrightarrow} \mathsf{false}$ *and* $\mathsf{memberp}(\mathsf{a},\mathsf{cons}(\mathsf{b},\mathsf{nil})) \not\sim \mathsf{false}$ *we know that* $\overset{\circledast}{\longleftrightarrow}$ *is no minimum either. But both* $\overset{\circledast}{\longleftrightarrow}$ *and* $\sim$ *are minimal among the congruences that yield a model of* R. *Hence their intersection does not yield a model of* R.

Thus, we have to choose between $\mathsf{a} \neq \mathsf{b}$ and $\mathsf{memberp(a,cons(b,nil))} \neq \mathsf{false}$. As $\overset{\oplus}{\longleftrightarrow}$ is somehow more appealing than $\sim$, one may argue that $\mathsf{a} \neq \mathsf{b}$ is somewhat more important than $\mathsf{memberp(a,cons(b,nil))} \neq \mathsf{false}$ by stating a, b to be constructors and thinking freeness of constructors to be more important than that of non-constructors. But this treatment does not solve the problem in general: If

(1) a or b is changed into a non-constructor term,

or (less likely in our special case here)

(2) memberp is stated to be a constructor symbol too,

then the very same problem arises again.

Our Solution

Now, while the simple attempt above fails, the intended bias towards freeness of constructor terms can be achieved with the help of a new† unary predicate 'Def' (in addition to the binary predicate '=') (cf. sect. 4) in the following way:

(A) Adding condition literals expressing *definedness* for all terms of negative equations in the condition. In essence (cf. Definition 4.2), a term t is *defined* :iff 'Def t' holds :iff t has a congruent constructor ground term. For our example above this means that the last memberp-rule is not applicable if a or b is undefined, thereby avoiding the problem of (1) above.

(B) Forcing each *constructor rule* (which is a rule whose left-hand side is a constructor term) to have no negative equations in its condition and to be *constructor-preserving*, which means that all its terms are constructor terms and all its variables occur in its left-hand side. For our example above, this means that 'memberp' cannot be a constructor symbol, thereby avoiding the problem of (2) above.

(B) is purely syntactic and not very restrictive in practice as it only limits congruences between constructor terms (and this even less restrictively than usual). (A) is not a usage of control information. It just means that '$\neq$' is restricted to defined terms. Since this restriction is made syntactically explicit, the semantics of '$\neq$' remains unchanged.

Undefined terms are due to some *partially specified* function, by which we mean a function with symbol say 'f' for which the application to some constructor ground terms $t_0, \ldots, t_{n-1}$ is not congruent to any constructor ground term, i. e. for which "$f(t_0, \ldots, t_{n-1})$" is an undefined term. In the context of our specifications, functions are partially specified not because the specifier has explicitly stated their partiality as a property of importance, but because he has partially left open their definition, maybe due to partial information, due to irrelevance of the functions' further behaviour for the specification in the current state of development, or even due to partiality being actually intended. Thus, partiality and undefinedness are not part of the specification but a result from its incompleteness. For this reason, the undefined terms are often thought to be equal to some unknown constructor ground terms:

Kapur & Musser (1987&1986) consider those congruences which are maximally enlarged by random identification of undefined terms with constructor ground terms, as long as this identification does not identify two distinct constructor ground terms. Their

† which is also useful for sufficient expressibility for inductive theorem proving: Lemmas of the form "Def $f(x_0, \ldots, x_{n-1})$" (the x_i being different constructor variables), expressing that the symbol 'f' denotes a *totally specified* function, are important for inductive theorem proving. Cf. Wirth (1991).

intended congruence is then the intersection of all those maximally enlarged congruences. In Kapur & Musser (1987) the maximal congruences are allowed to have some undefined terms left; this causes the problem that one cannot describe the intended congruence by monotonic model semantics[‡]. Therefore in Kapur & Musser (1986) the intersection is done only over those congruences that have no undefined terms left: These congruences can easily be described in terms of model semantics: A model $\mathcal{A}$ is required to satisfy the following (besides making true the universally quantified equations of R): Let $\overset{\oplus}{\longleftrightarrow}$ denote the initial congruence of R (which exists because they consider unconditional equations only) and k its canonical cons-epimorphism from $\mathcal{GT}(\mathrm{cons})$ to $\mathcal{GT}(\mathrm{cons})/\overset{\oplus}{\longleftrightarrow}$. Now the unique cons-homomorphism h from $\mathcal{GT}(\mathrm{cons})/\overset{\oplus}{\longleftrightarrow}$ to $\mathcal{A}$ given by $kh = (\mathcal{A}|_{\mathcal{GT}(\mathrm{cons})_s})_{s \in \mathrm{S}}$ (cf. the Homomorphism-Theorem) is required to be an isomorphism. A third way of removing the undefined terms is to require h to be epimorphic instead of isomorphic, i. e. $\mathcal{A}$ is required to be cons-term-generated. While the theory of the last two approaches is beautiful, the resulting congruences may be very difficult to understand: One needs a sophisticated way of argumentation for showing two terms equal — even for some very simple examples.

Based on this tradition of thinking undefined terms to be possibly equal to constructor ground terms, the above item (A) of our approach can be justified the following way:

Considering dynamic extension of specifications: If two terms can be shown equal by $\overset{\oplus}{\longleftrightarrow}$, they will keep being equal even if an undefined term will be identified with a defined term later on (cf. Theorem 5.16). On the other hand might an undefined term become equal to a previously unequal term when identifying an undefined term with a defined term. Thus, we had better be cautious: We should not pretend to be able to distinguish something undefined from anything else (as the former might in the sequel be defined to be the latter).

From a static point of view on the specification: Two distinct terms may be equal or unequal, no matter whether they are defined or undefined. In particular may an undefined term be both unequal to some distinct undefined term and equal to some other. This inequality between undefined terms, however, differs from the inequality between defined terms in that it is not considered sufficient for the fulfilledness of an inequality literal in the condition of an equation. This means that we have a "*closed world assumption*" which is restricted to the constructor ground terms, saying that two constructor ground terms are meant to be unequal unless their equality is specified by the constructor rules. According to this, we use "*negation as failure*" on the defined terms only, and not on the undefined terms where the specification is allowed to be incomplete and open.

[‡] Of course, this is tried to be done in Kapur & Musser (1987). But their "inductive model" (which is defined to be a model with free constructors whose proper epimorphic images are no models with free constructors) is rather peculiar: Normally, a model uses to keep being a model when one throws away some equations of the specification, thereby establishing the monotonicity of logic. The "inductive models" do not have this property. To see this take $\mathrm{C} = \{\mathsf{false}, \mathsf{true}, \mathsf{0}\}$; $\mathrm{N} = \{\mathsf{s}, \mathsf{zerop}\}$; $\mathrm{R} = \{\ \mathsf{zerop}(\mathsf{0}) = \mathsf{true}\ ,\ \mathsf{zerop}(\mathsf{s}(x)) = \mathsf{false}\ \}$. Now the following $\mathcal{A}$ is an "inductive model" for R but not for $\emptyset$ (where we need $|\mathcal{A}(\mathsf{nat})| = 1$): $\mathcal{A}(\mathsf{bool}) = \{\mathsf{FALSE}, \mathsf{TRUE}\}$; $\mathcal{A}(\mathsf{nat}) = \{0, 1\}$; $\mathsf{true}^{\mathcal{A}} = \mathsf{TRUE}$; $\mathsf{false}^{\mathcal{A}} = \mathsf{FALSE}$; $\mathsf{0}^{\mathcal{A}} = 0$; $\mathsf{s}^{\mathcal{A}}(x) = 1$; $\mathsf{zerop}^{\mathcal{A}}(0) = \mathsf{TRUE}$; $\mathsf{zerop}^{\mathcal{A}}(1) = \mathsf{FALSE}$.

We can also see by this that we indeed have no monotonic logic here: $\emptyset \models \mathsf{0} = \mathsf{s}(\mathsf{0})$; but (as seen by $\mathcal{A}$): $\mathrm{R} \not\models \mathsf{0} = \mathsf{s}(\mathsf{0})$.

Concluding Comparison with Other Approaches

In the following sections we will show that by the requirements (A) and (B) we get a straightforward reduction relation $\longrightarrow$ that has the following advantages (compared to the one of Kaplan (1988)) (cf. sect. 5):

(1) Its congruence closure $\overset{\circledast}{\longleftrightarrow}$ yields a model that is not only minimal but also the (up to isomorphism) uniquely determined minimum among those sig-term-generated models of R that do not identify more constructor ground terms than necessary (provided (as also required for Kaplan's $\overset{\circledast}{\longleftrightarrow}$ to be minimal) that $\longrightarrow$ is confluent).

(2) It is monotonic w. r. t. the addition of new rules that do not have old constructor terms as left-hand sides.

(3) It is stable when defined also on non-ground terms.

As shown in the examples above, the reduction relation of Kaplan (1988) has none of these properties. We will now revisit these examples to illustrate how our restrictions solve the problems mentioned.

(1) (Example 3.6). If a and b are defined terms, then $\overset{\circledast}{\longleftrightarrow}$ becomes the *minimum* among those congruences which do not identify more constructor ground terms than necessary. Contrariwise, if a or b is undefined, then the intersection of $\overset{\circledast}{\longleftrightarrow}$ and $\sim$ becomes a model of R because the last memberp-rule now reads

$$\mathsf{memberp}(x,\mathsf{cons}(y,l)) = \mathsf{memberp}(x,l) \longleftarrow x \neq y \ , \ \mathrm{Def}\, x \ , \ \mathrm{Def}\, y \ ;$$

thus, $\mathsf{memberp}(\mathsf{a},\mathsf{cons}(\mathsf{b},\mathsf{nil}))$ is neither true nor false now, but undefined instead.

(2) (Example 3.4). We do not have $\mathsf{memberp}(0,\mathsf{cons}(0-\mathsf{s}(0),\mathsf{nil})) \overset{\circledast}{\longrightarrow} \mathsf{false}$ anymore: $\mathsf{memberp}(0,\mathsf{cons}(0-\mathsf{s}(0),\mathsf{nil}))$ is irreducible because $0-\mathsf{s}(0)$ is undefined.

(3) (Example 3.5). As X is undefined: $\mathsf{memberp}(0,\mathsf{cons}(X,\mathsf{nil})) \overset{\circledast}{\not\longrightarrow} \mathsf{false}$.

Moreover, we are not only able to give control independent semantics for admissible specifications, but are also able to remove the control aspect of requiring the rules to be decreasing for admissibility. Thus, in principle, our reduction relation does not need to be noetherian. For practical purposes, however, in particular for verifying confluence, termination of (at least some sub-relation of) the reduction relation is often indispensable.

For a final comparison between the reduction relation of Kaplan (1988) and the ground term restriction of our reduction relation, suppose that R satisfies (B) from above and is decreasing w. r. t. Kaplan's reduction relation and some ordering $\rhd$. By induction over $\rhd$ one can easily show the following: The two relations do not differ on constructor terms. If the reflexive & transitive closure of Kaplan's relation is sufficiently complete, then ours contains Kaplan's. If Kaplan's relation is confluent, then it contains ours. If the reflexive & transitive closure of one of the relations is sufficiently complete and one of the relations is confluent, then there is no difference between our ground reduction relation and that of Kaplan (1988). Therefore, in this important case, where all functions are totally specified and no undefined ground terms exist, we offer control independent semantics for the reduction relation of Kaplan.

The *perfect model* semantics approach of Bachmair & Ganzinger (1991), which also includes a completion procedure, generalizes Kaplan's approach by abstracting the control information hidden in the syntactic form of rules into a reduction ordering which must be total on ground terms and which determines the construction process of perfect models. The perfect model semantics is very similar to Kaplan's in that it still does not provide control independent semantics and in that it is still not monotonic w. r. t. consistent extensions of the specification. Cf. also Becker (1993) for the interrelation between the three approaches of Kaplan (1988), Bachmair & Ganzinger (1991), and ours.

Two types of variables

An additional feature of our presentation is our distinction between two kinds of variables. While the distinction between constructor terms and general terms is commonly accepted and considered fruitful, our distinction between constructor variables and general variables may require some explanation: General variables may be substituted by any term of the whole signature. Constructor variables, however, may only be substituted by pure constructor terms consisting of constructor function and constructor variable symbols. In the field of model semantics, this distinction is mirrored by the possible valuations: While a general variable can take the value of any object in the universe of its sort, a constructor variable can take the value of an object of the constructor sub-universe only.

General variables are the common ones in the field of term rewriting. They allow to express semantic properties that cannot be expressed by constructor variables. (Consider equations for error recovery or for non-strict functions whose meaning does not depend on the definedness of all its variables, e. g. “ $\mathsf{or}(\mathsf{true},Y) = \mathsf{true}$ ”.) Furthermore, general variables allow a higher abstraction from evaluation strategies than constructor variables which result in an innermost rewriting strategy in case of free constructors.

Constructor variables are convenient in the field of inductive theorem proving for expressing important lemmas that do not hold for undefined terms. (E. g., one certainly should be able to express a commutativity lemma for addition of rational numbers, but one cannot expect it hold for “ $\mathsf{1/0}$ ” or other undefined terms.) Semantically we could remove the whole order-sorted frame by considering ‘Def’ to be an interpreted first-order predicate, then by stating for each $c \in \mathrm{C}$ with $\alpha(c) = s_0 \ldots s_n$ that $\mathrm{Def}\, c(x_0, \ldots, x_{n-1})$ holds for $\forall i < n : x_i \in \mathrm{V}_{\mathrm{CONS},s_i}$, and finally by replacing each formula A containing a variable $x \in \mathrm{V}_{\mathrm{CONS}}$ with the formula “ $A\{x{\mapsto}X\} \longleftarrow \mathrm{Def}\, X$ ” for a new variable $X \in \mathrm{V}_{\mathrm{SIG}}$. While the order-sorted frame can therefore be considered to be syntactic, it is not just syntactic sugar, since it deeply influences termination and confluence of reduction relations. E. g., the means to automatically show termination of the functions of classic inductive theorem proving (cf. e. g. Boyer & Moore (1979), Walther (1988)) depend on the variables in the function definitions being bound to constructor terms only. This dependence, however, and the intended meaning of the variables at all, are usually hidden in the formalism and not made as explicit as in Avenhaus & Becker (1992) where it is shown that the restriction to constructor variables only, is beneficial to confluence (cf. also our Theorem 7.18) and termination of rewriting systems.

All in all, both kinds of variables have their benefits for specification with positive/negative-conditional equations and for expressing (inductive) properties with first-order clauses, as well as for rewriting and (inductive) theorem proving. Since the technical treatment of both kinds of variables can be achieved by simple means, we have decided to include both of them in our constructor-based approach for positive/negative-conditional equations here. Together with the generalization to positive- and negative-conditional equations, the addition of constructor variables to classic term rewriting provides us with a unifying approach to the function specification style of classic inductive theorem proving on the one hand and to term rewriting on the other.

4. Syntax and Semantics of Specifications

Definition 4.1. (Syntax of CRS) *A (positive/negative-)*conditional rule system *(*CRS*)* R *over* sig/cons/V *is a finite subset of the* set of rules $\mathcal{RUL}(\mathrm{sig},\mathrm{cons},\mathrm{V})$ *over* sig/cons/V*, which will be defined in Definition 5.1. The only thing we have to know about it now is:* $\mathcal{RUL}(\mathrm{sig},\mathrm{cons},\mathrm{V}) \subseteq \mathrm{DEq}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}) \times (\mathcal{LIT}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}))^*$ *, where* $\mathrm{DEq}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}})$ *is the set of directed equations and* $\mathcal{LIT}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}})$ *is the set of* condition literals *over the following predicate symbols on terms from* $\mathcal{T}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}})$*: '=', '≠' (binary, symmetric, sort-invariant), and 'Def' (unary). A rule* $((l,r),\emptyset)$ *with an empty condition will be written* $l{=}r$*. Note that* $l{=}r$ *differs from* $r{=}l$ *whenever the equation is used as a reduction rule. A rule* $((l,r),C)$ *with condition* C *will be written* $l{=}r{\longleftarrow}C$*. We call* l *the* left-hand side *and* r *the* right-hand side *of the rule* $l{=}r{\longleftarrow}C$*; the terms*[†] *of the condition literals in* C *are called* condition terms *and their set is denoted by* $\mathcal{TERMS}(C)$*. A rule is said to be* left-linear *:iff its left-hand side is a linear term. A rule* $l{=}r{\longleftarrow}C$ *is said to be* extra-variable free *:iff* $\mathcal{V}(r,\mathcal{TERMS}(C)) \subseteq \mathcal{V}(l)$*. The whole CRS* R *is said to have one of these properties :iff each of its rules has it.*

A rule $l{=}r{\longleftarrow}C$ expresses a universally quantified implication with the conjunction of the literals in C as the condition and with "$l{=}r$" as the conclusion. The meaning of the predicate symbols '=' and '≠' is not open to interpretation. As usual, the fixed meaning of '=' is the equality in a sig/cons-algebra $\mathcal{A}$; '≠' is its negation. 'Def', however, is the "definedness" predicate which states that the evaluation of its argument belongs (with sort invariant) to the *constructor sub-universe* of $\mathcal{A}$ which contains the set of evaluation values of constructor ground terms and which is intended to supply a domain for (possibly partially) specifying functions on it. We speak of our new kind of model just as a "sig/cons-model" (without any further attributes), because if we removed the new predicate symbol 'Def' and the constructor sub-universes, we would just get the usual model concept of algebra; i. e., our sig/cons-model is an upward-compatible extension.

Definition 4.2. (Semantics of CRS) *Let* R *be a CRS over* sig/cons/V*; let* $\mathcal{A}$ *be a* sig/cons-*algebra. Now* $\mathcal{A}$ *is a* sig/cons-model of R *:iff*

$$\forall((l,r),C)\in \mathrm{R} : \forall\kappa\in\mathcal{SUB}(\mathrm{V},\mathcal{A}) : \big((C \textit{ is true w. r. t. } \mathcal{A}_\kappa) \;\Rightarrow\; \mathcal{A}_\kappa(l)=\mathcal{A}_\kappa(r)\big) \;,$$

where C *is* true w. r. t. $\mathcal{A}_\kappa$ *:iff*

$$\forall s\in\mathrm{S} : \forall u,v\in\mathcal{T}_{\mathrm{SIG},s} : \begin{pmatrix} ((\;(u{=}v\;)\textit{ in } C) \;\Rightarrow\; \mathcal{A}_\kappa(u)=\mathcal{A}_\kappa(v)\;) & \wedge \\ ((\;(u{\neq}v\;)\textit{ in } C) \;\Rightarrow\; \mathcal{A}_\kappa(u)\neq\mathcal{A}_\kappa(v)\;) & \wedge \\ ((\;(\mathrm{Def}\,u)\textit{ in } C) \;\Rightarrow\; \mathcal{A}_\kappa(u)\in\mathcal{A}(\mathrm{CONS},s)\;) & \end{pmatrix}$$

As we have negative equations in our conditions, we cannot hope to get a minimum model because we can express things like "a=b ∨ b=c", which has the incomparable minimal models "a=b≠c" and "a≠b=c". What we will get instead is a model which is the (up to isomorphism) uniquely determined minimum of all sig-term-generated models which are minimal w. r. t. the identification of constructor ground terms (cf. Corollary 5.15). For formally expressing these minimality-properties, we need the following definition.

[†] To avoid misunderstanding: For a condition list, say " $s{=}t,\; u{\neq}v,\; \mathrm{Def}\,w$ ", we mean the top level terms $s,t,u,v,w \in \mathcal{T}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}})$, but neither their proper subterms nor the literals "$s{=}t$", "$u{\neq}v$", "$\mathrm{Def}\,w$" themselves.

Definition 4.3. *Define* $\cdot_{\mathrm{H}}$ *and* $\cdot_{\mathrm{CONS}}$ *as (proper class) relations on* sig/cons-*algebras by:* $\mathcal{A}\cdot_{\mathrm{H}}\mathcal{B}$ *:iff there is a* sig/cons-*homomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$*.* $\mathcal{A}\cdot_{\mathrm{CONS}}\mathcal{B}$ *:iff there is a* cons-*homomorphism from the* cons-*algebra* $\mathcal{A}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}$ *to* $\mathcal{B}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}$ *.*

We trivially get $\cdot_{\mathrm{H}} \subseteq \cdot_{\mathrm{CONS}}$ *(by restriction of the homomorphism); and* $\cdot_{\mathrm{H}}$*,* $\cdot_{\mathrm{CONS}}$ *are quasi-orderings. The corresponding equivalences, orderings, and reflexive orderings will be denoted by* $\approx$*,* $<$*,* $\leq$*, resp., with the corresponding subscript.*

A sig/cons-*algebra* $\mathcal{A}$ *will be called a* minimum model *(or else a* constructor-minimum model*) of a CRS* R *over* sig/cons/V *:iff* $\mathcal{A}$ *is a* $\cdot_{\mathrm{H}}$*-minimum (or else* $\cdot_{\mathrm{CONS}}$*-minimum) of the class of all* sig/cons-*models of* R*.*

Similarly, a sig/cons-*algebra* $\mathcal{A}$ *will be called a* minimal model *(or else a* constructor-minimal model*) of a CRS* R *over* sig/cons/V *:iff* $\mathcal{A}$ *is a* sig/cons-*model of* R *and there is no* sig/cons-*model* $\mathcal{B}$ *of* R *with* $\mathcal{B}<_{\mathrm{H}}\mathcal{A}$ *(or else* $\mathcal{B}<_{\mathrm{CONS}}\mathcal{A}$ *).*

The following lemma tells us that, considering minimum models, we can think in terms of sig/cons-congruences on $\mathcal{GT}$ instead of algebras:

Lemma 4.4. *Let* $\mathcal{B}$ *be a* sig/cons-*model of the CRS* R *over* sig/cons/V*. Define the factor algebra* $\mathcal{A} := \mathcal{GT}/\mathrm{ker}(\mathcal{B})$ *. Now:*

(1) $\mathcal{A}$ *is a* sig/cons-*model of* R*.*

(2) $\mathcal{A}\cdot_{\mathrm{H}}\mathcal{B}$ *. Moreover, there is a unique* sig/cons-*homomorphism* $l{::}\mathcal{A}\rightarrow\mathcal{B}$ *(, which is monic in the* sig/cons-*homomorphism category of* sig/cons-*algebras).*

(3) $\mathcal{A}\cdot_{\mathrm{CONS}}\mathcal{B}$ *. (However, we do not have* $\mathcal{A}\approx_{\mathrm{CONS}}\mathcal{B}$ *in general, because* $\mathcal{B}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}$ *need not be* cons-*term-generated.)*

The following lemma of theoretical nature ensures the existence of minimal models. It resembles Theorem 2.1 in Kaplan (1988). Note however, that our $\cdot_{\mathrm{H}}$ and $\cdot_{\mathrm{CONS}}$ are reflexive and therefore different from the relation $\leq$ in Kaplan (1988), where the homomorphism is additionally required to be unique.

Lemma 4.5. *Let* R *be a CRS over* sig/cons/V*.*

(1) The trivial sig/cons-*algebra is a* sig/cons-*model of* R*.*

(2) If $\mathcal{B}$ *is a* sig/cons-*model of* R*, then there is a minimal model* $\mathcal{A}$ *of* R *with* $\mathcal{A}\leq_{\mathrm{H}}\mathcal{B}$*.*

(3) R *has a minimal model.*

5. The Reduction Relation

In this section we are going to define a reduction relation $\longrightarrow$ which is convenient for the semantics defined in the previous section. The overall idea is to reduce a left-hand side of a rule to its right-hand side only if the condition of this rule can somehow be shown valid by means of the same reduction relation again.

Many authors impose rather strong restrictions on constructor equations, such as "no equations between constructors" ("free constructors") or "unconditional equations between constructors only". Compared to these, our restrictions are very weak. They serve to guarantee a constructor-minimum model for the constructor equations that is unique modulo $\approx_{\mathrm{CONS}}$, by requiring the constructor equations to have "Horn"-form and to be constructor-preserving[†].

Definition 5.1. (Set of Rules)
(continuing Definition 4.1 by adding the restrictions on constructor equations)
The set of rules *over* sig/cons/V *is defined to be:* $\mathcal{RUL}(\mathrm{sig},\mathrm{cons},\mathrm{V}) :=$

$$\left\{ \; ((l,r),C) \in (\mathrm{DEq}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}) \times (\mathcal{LIT}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}))^*) \;\middle|\; \left(l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}) \Rightarrow \left(\begin{array}{ll} \forall L \;in\; C : \forall u,v:\; L \neq (u{\neq}v) & \wedge \\ \mathcal{V}(r,\mathcal{TERMS}(C)) \subseteq \mathcal{V}(l) & \wedge \\ r \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}) & \wedge \\ \mathcal{TERMS}(C) \subseteq \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}}) & \end{array} \right) \right) \right\}$$

We are now going to define our reduction relation, having in mind to require it to be confluent in the sequel, whereas we do not require confluence for the definition because we cannot prove confluence criteria if the non-confluent case is undefined. Therefore, we have to be explicit about how we test the condition literals — even if this testing is not straightforward when confluence is not provided. Our "operational" semantics for testing condition literals is the following: "$u{=}v$" is fulfilled if u, v have reducts $\hat{u}, \hat{v}$, resp., which are syntactically equal. "Def u" is fulfilled if u has a constructor ground reduct, which means that our reduction relation depends on the constructor sub-signature 'cons' beyond the signature 'sig' — just as our notion of "sig/cons-model" does. Finally, "$u{\neq}v$" is fulfilled if u, v have constructor ground reducts $\hat{u}, \hat{v}$, resp., which are not joinable. Thus, two terms in a condition literal are operationally equal if they are joinable, whereas they are unequal if they are not joinable after some reduction to constructor ground terms. The non-joinability alone of two terms is not sufficient for regarding them as unequal because we are never sure about the inequality of "undefined" terms (cf. sect. 3). Note that our operational logic is four-valued, i. e. '=' and '$\neq$' can independently be fulfilled or not. In case of confluence, however, it is impossible that both "$u{=}v$" and "$u{\neq}v$" are fulfilled simultaneously; in case of free or confluent constructors, such a simultaneous fulfilledness occurs only if we have something like an ambiguous function definition.

[†] The constructor-preservation is really necessary here for guaranteeing the existence of a a minimal constructor-minimum model as in Theorem 5.14: Let 0, 1, true, false be constructor constants, let weirdp be a non-constructor constant, and take R: 1=0 ⟵ weirdp=true ; weirdp=true ⟵ true≠false . Now there are sig/cons-models of R with 0≠1 and models with true≠false but no models with "0≠1 ∧ true≠false". Also notice, that the constructor-preservation has some additional advantages, e. g.:
(1) The rules become sort-decreasing w. r. t. to the order-sorted signature exhibited in sect. 2.1, i. e. the right-hand side and the condition terms are from $\mathcal{T}(\mathrm{cons},\mathrm{V_{CONS}})$ if the left-hand side is.
(2) For $u \in \mathcal{GT}$ with computable and unique normal form $\mathrm{NF}(u)$ we can test

$$\text{“ } \exists \hat{u} \in \mathcal{GT}(\mathrm{cons}) : u \xrightarrow{\circledast} \hat{u} \text{ ” by “ } \mathrm{NF}(u) \in \mathcal{GT}(\mathrm{cons}) \text{ ”.}$$

(3) Theorem 5.16 has no reasonable analogue for CRSs which are not constructor-preserving.

Definition 5.2. (Fulfilledness) *A list* $D \in \mathcal{LIT}(\mathrm{sig},\mathrm{X})^*$ *of condition literals is said to be* fulfilled w. r. t. some relation $\longrightarrow$ *:iff*

$$\forall u,v \in \mathcal{T}: \begin{pmatrix} (\ (\ (u{=}v\ )\ \mathit{in}\ D) & \Rightarrow & & u \downarrow v\ \ ) & \wedge \\ (\ (\ (\mathrm{Def}\, u)\ \mathit{in}\ D) & \Rightarrow & \exists \hat{u} \in \mathcal{GT}(\mathrm{cons}): & u \overset{\oplus}{\longrightarrow} \hat{u}\ ) & \wedge \\ (\ (\ (u{\neq}v\ )\ \mathit{in}\ D) & \Rightarrow & \exists \hat{u},\hat{v} \in \mathcal{GT}(\mathrm{cons}): & u \overset{\oplus}{\longrightarrow} \hat{u} \not\downarrow \hat{v} \overset{\oplus}{\longleftarrow} v\ ) & \end{pmatrix}$$

Usually one gets a minimal reduction relation by taking the closure over a finitary generating relation. This is not possible here, because we have a negative condition ($\hat{u} \not\downarrow \hat{v}$). By the "Horn"-form of our constructor equations (and the constructor-preservation), however, this negative condition does not influence the reduction of constructor terms; and (in Definition 5.2) '$\not\downarrow$' is applied to constructor (ground) terms only. Thus, we can get our intended minimal reduction relation by a double closure: first for constructor rules only; second for general rules, knowing the constructor reduction to remain unchanged.

Definition 5.3. ($\longrightarrow_{\mathrm{R,X}}$) *Let* R *be a CRS over* sig/cons/V*. Let* X$\subseteq$V*. Let* $\prec$ *denote the ordering on the ordinal numbers. For* $\beta \preceq \omega{+}\omega$ *the reduction relations* $\longrightarrow_{\mathrm{R,X},\beta}$ *on* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *are inductively defined as follows:* $\longrightarrow_{\mathrm{R,X},0} := \emptyset$ *. For* $i \in \mathbb{N}$*:*

$s \longrightarrow_{\mathrm{R,X},i+1} t$ *:iff* $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})\ \wedge$

$$\exists ((l,r),C) \in \mathrm{R}: \exists \sigma \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})): \exists p \in \mathcal{POS}(s): \begin{pmatrix} l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}}) & \wedge \\ s/p = l\sigma\ \wedge\ t = s[\,p \leftarrow r\sigma\,] & \wedge \\ C\sigma \textit{ is fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X},i} & \end{pmatrix}.$$

$\longrightarrow_{\mathrm{R,X},\omega} := \bigcup_{i \in \mathbb{N}} \longrightarrow_{\mathrm{R,X},i}$. *For* $i \in \mathbb{N}$*:*

$s \longrightarrow_{\mathrm{R,X},\omega+i+1} t$ *:iff* $s \longrightarrow_{\mathrm{R,X},\omega} t$ *or* $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})\ \wedge$

$$\exists ((l,r),C) \in \mathrm{R}: \exists \sigma \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})): \exists p \in \mathcal{POS}(s): \begin{pmatrix} s/p = l\sigma\ \wedge\ t = s[\,p \leftarrow r\sigma\,] & \wedge \\ C\sigma \textit{ is fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X},\omega+i} & \end{pmatrix}.$$

$\longrightarrow_{\mathrm{R,X}} := \longrightarrow_{\mathrm{R,X},\omega+\omega} := \bigcup_{i \in \mathbb{N}} \longrightarrow_{\mathrm{R,X},\omega+i}$.

We will drop "R, X" in $\longrightarrow_{\mathrm{R,X}}$ and $\longrightarrow_{\mathrm{R,X},\beta}$ when referring to some fixed R, X. Instantiations of X which are important in theory and practice are at least $\emptyset$, $\mathrm{V_{SIG}}$, and V. We have introduced the parameter X since it is more convenient than triplicating statements about properties (e. g. confluence) for "X=$\emptyset$" (ground confluence), "X=$\mathrm{V_{SIG}}$", and "X=V".

Lemma 5.4. *Let* $S_{\mathrm{R,X}}$ *be the set of all relations* $\rightharpoonup$ *satisfying*

(1) $(\rightharpoonup \cap (\mathcal{GT}(\mathrm{cons}) \times \mathcal{T})) \subseteq \longrightarrow_{\mathrm{R,X},\omega}$ *as well as*

(2) $s \rightharpoonup t$ *if* $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})\ \wedge$

$$\exists ((l,r),C) \in \mathrm{R}: \exists \sigma \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})): \exists p \in \mathcal{POS}(s): \begin{pmatrix} s/p{=}l\sigma\ \wedge\ t{=}s[\,p \leftarrow r\sigma\,]\ \wedge \\ C\sigma \textit{ is fulfilled w. r. t. } \rightharpoonup \end{pmatrix}.$$

Now $\longrightarrow_{\mathrm{R,X}}$ *is the minimum (w. r. t. set-inclusion) in* $S_{\mathrm{R,X}}$*, and* $S_{\mathrm{R,X}}$ *is closed under non-empty intersection.*

Note that the first requirement of Lemma 5.4 is really necessary: To see this, consider Example 3.6 with the additional declaration that a and b are *constructor* symbols, i. e. that $\mathsf{a},\mathsf{b} \in \mathrm{C}$. If we now take $\rightharpoonup$ to be the relation $\sim$ of Example 3.6 (which does not satisfy the first requirement of Lemma 5.4 since $\mathsf{a} \sim \mathsf{b}$ but not $\mathsf{a} \longrightarrow_{\mathrm{R},\emptyset,\omega} \mathsf{b}$), then $\rightharpoonup$ satisfies the second requirement of Lemma 5.4, but we do not have $\longrightarrow_{\mathrm{R},\emptyset} \subseteq \rightharpoonup$ since $\mathsf{memberp(a,cons(b,nil))} \overset{\oplus}{\longrightarrow}_{\mathrm{R},\emptyset} \mathsf{false}$ but not $\mathsf{memberp(a,cons(b,nil))} \overset{\oplus}{\rightharpoonup} \mathsf{false}$.

Before we go on, we want to spend some more words on the way we test fulfilledness of negative equations in the condition of a rule. While other formulations (e. g. a universal instead of the existential quantification) might seem to be more satisfactory, ours is the one required for a correct definition. One might have expected "$u \not\downarrow v$" instead of "$\exists \hat{u}, \hat{v} \in \mathcal{GT}(\mathrm{cons}) : \; u \stackrel{\oplus}{\longrightarrow} \hat{u} \not\downarrow \hat{v} \stackrel{\oplus}{\longleftarrow} v$" for "$u \neq v$", but this modification would not allow the conclusion that $\longrightarrow_{\mathrm{R,X}}$ is minimal in the sense of Lemma 5.4, as can be seen from:

EXAMPLE 5.5. *Let* c, d *be constructor and* a, b, e *be non-constructor constants and take* R*:* a=c⟵b≠d ; b=d⟵e≠c ; e=a *. Now with the modified definition of fulfilledness we would get* $\longrightarrow_{\mathrm{modified,R},\emptyset} = \{\, (\mathsf{a},\mathsf{c}),\ (\mathsf{b},\mathsf{d}),\ (\mathsf{e},\mathsf{a}) \,\}$ *. Furthermore,* $\{\, (\mathsf{a},\mathsf{c}),\ (\mathsf{e},\mathsf{a}) \,\}$ *and* $\{\, (\mathsf{b},\mathsf{d}),\ (\mathsf{e},\mathsf{a}) \,\}$ *would be* $\subseteq$*-incomparable minimal relations satisfying requirement of Lemma 5.4 with modified fulfilledness. Their intersection* $\{\, (\mathsf{e},\mathsf{a}) \,\}$ *, however, satisfies the requirement for our original, non-modified fulfilledness only, and is equal to* $\longrightarrow_{\mathrm{R},\emptyset}$.

By induction over the construction of $\longrightarrow_{\mathrm{R,X},\omega+\omega}$ one can easily show:

COROLLARY 5.6. (MONOTONICITY OF $\longrightarrow$ W. R. T. REPLACEMENT)
$\longrightarrow_{\mathrm{R,X},\beta}$ *(for* $\beta \preceq \omega{+}\omega$*) and* $\longrightarrow_{\mathrm{R,X}}$ *are* X*-monotonic.*

COROLLARY 5.7. (STABILITY OF $\longrightarrow$)
$\longrightarrow_{\mathrm{R,X},\beta}$ *(for* $\beta \preceq \omega{+}\omega$*),* $\longrightarrow_{\mathrm{R,X}}$*, and their respective fulfilledness-predicates are* X*-stable.*

The following two technical lemmas state constructor-preservation and that there is no need for a second closure for reduction of constructor terms.

LEMMA 5.8. *For* $\mathrm{X} \subseteq \mathrm{Y} \subseteq \mathrm{V}$*:*

$$\forall n \in \mathbb{N}\colon \forall s \in \mathcal{T}(\mathrm{cons},\mathrm{X}) : \forall t\colon \left(s \stackrel{n}{\longrightarrow}_{\mathrm{R,Y}} t \;\Rightarrow\; (s \stackrel{n}{\longrightarrow}_{\mathrm{R,Y},\omega} t \in \mathcal{T}(\mathrm{cons},\mathrm{X})) \right)$$

LEMMA 5.9. $\downarrow \cap \; (\, \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}}) \times \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}})\,) \;\subseteq\; \downarrow_{\omega}$

LEMMA 5.10. (MONOTONICITY OF $\longrightarrow_{\beta}$ AND OF FULFILLEDNESS W. R. T. $\longrightarrow_{\beta}$ IN β)
For $\beta \preceq \gamma \preceq \omega{+}\omega$*:* $\longrightarrow_{\beta} \;\subseteq\; \longrightarrow_{\gamma} \;\subseteq\; \longrightarrow$ *; and if* C *is fulfilled w. r. t.* $\longrightarrow_{\beta}$ *and* $\omega \preceq \beta \;\vee\; \forall u,v : ((u{\neq}v)$ *is not in* $C)$ *, then* C *is fulfilled w. r. t.* $\longrightarrow_{\gamma}$ *and w. r. t.* $\longrightarrow$.

LEMMA 5.11. (FULFILLEDNESS TEST MAY BE SIMPLE)
Let $C \in (\mathcal{LIT}(\mathrm{sig},\mathrm{X}))^{*}$*. If for each element* $u \in \mathcal{TERMS}(C)$*:*

(1) u *has a normal form* $\mathrm{NF}(u)$ *(i. e.* $u \stackrel{\oplus}{\longrightarrow} \mathrm{NF}(u) \notin \mathrm{dom}(\longrightarrow)$ *) and*
(2) $\longrightarrow$ *is confluent below* u*,*

then C *is fulfilled w. r. t.* $\longrightarrow$ *iff*

$$\forall u,v \in \mathcal{T}\colon \begin{pmatrix} (\;(\,(u{=}v\,) \text{ in } C) \;\Rightarrow\; \mathrm{NF}(u) = \mathrm{NF}(v) \;) & \wedge \\ (\;(\,(\mathrm{Def}\,u) \text{ in } C) \;\Rightarrow\; \mathrm{NF}(u) \in \mathcal{GT}(\mathrm{cons}) \;) & \wedge \\ (\;(\,(u{\neq}v\,) \text{ in } C) \;\Rightarrow\; (\mathrm{NF}(u), \mathrm{NF}(v) \in \mathcal{GT}(\mathrm{cons}) \,\wedge\, \mathrm{NF}(u) \neq \mathrm{NF}(v)) \;) & \end{pmatrix}$$

LEMMA 5.12. *Let* $\mathrm{X} \subseteq \mathrm{Y} \subseteq \mathrm{V}$ *. Now: For all* $\beta \preceq \omega{+}\omega$ *and* $n \in \mathbb{N}$ *with* $n{\neq}0$*:* $\stackrel{n}{\longrightarrow}_{\mathrm{R,X},\beta} = \stackrel{n}{\longrightarrow}_{\mathrm{R,Y},\beta} \cap (\mathcal{T}(\mathrm{sig},\mathrm{X}) \times \mathcal{T}(\mathrm{sig},\mathrm{X}))$*, and for* $C \in \mathcal{LIT}(\mathrm{sig},\mathrm{X})$*:* C *is fulfilled w. r. t.* $\longrightarrow_{\mathrm{R,X},\beta}$ *iff* C *is fulfilled w. r. t.* $\longrightarrow_{\mathrm{R,Y},\beta}$*. Furthermore,* $\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) = \mathrm{dom}(\longrightarrow_{\mathrm{R,Y}}) \cap \mathcal{T}(\mathrm{sig},\mathrm{X})$*. Finally, if* $\longrightarrow_{\mathrm{R,Y}}$ *is confluent, then* $\longrightarrow_{\mathrm{R,X}}$ *is confluent, too.*

By Example 5.5 we know that (for guaranteeing a minimal model via a minimal reduction relation) we have to restrict the terms of the negated equations to be "defined". This semantic restriction is made syntactically explicit in the following definition which specifies a "well-behaved" subclass of the class of CRSs, in which inequalities are founded on constructor objects. For a detailed motivation cf. item (A) in sect. 3 ("Our Solution").

DEFINITION 5.13. (DEF-MODERATE CONDITIONAL RULE SYSTEMS (Def-MCRS))
A CRS R *is a* Def-moderate conditional rule system *(*Def-MCRS*) :iff*

$$\forall((l,r),C)\in \mathrm{R} : \forall(u{\neq}v) \text{ in } C : \big(\ \mathrm{Def}\,u,\ \mathrm{Def}\,v \text{ are in } C\ \big)\ .$$

Now we are able to state the fundamental theorem about $\longrightarrow$, which is the first main result of this paper. While the theorem in its general form is indeed necessary for establishing appropriate notions of inductive validity (cf. Wirth & al. (1993)), its meaning is easier to grasp from its corollary below, saying that (for Def-moderate CRSs R with confluent $\longrightarrow_{\mathrm{R},\emptyset}$) the factor algebra $\mathcal{GT}/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\emptyset}$ is an (up to isomorphism) uniquely determined sig/cons-model being initial in a class of models which, in our opinion, captures the intuition behind constructor-based specifications. Furthermore, this unique model $\mathcal{GT}/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\emptyset}$ can be constructed by means of the congruence induced by our reduction relation. Thus $\mathcal{GT}/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\emptyset}$ provides a computational model for positive/negative-conditional specifications in a fashion very similar to the initial model (or abstract data type) for positive- or un-conditional specifications.

THEOREM 5.14.
(MINIMAL MODEL BEING FREE IN THE CONSTRUCTOR-MINIMAL MODELS)
Let R *be a* Def-*MCRS over* sig/cons/V*. Let* X⊆V*. Let* K *be the class of all constructor-minimal models of* R*. Let* κ *be given by (*$x \in$X*):* $x \mapsto \overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{X}}[\{x\}]$ *.*
Now, if $\longrightarrow_{\mathrm{R},\emptyset}$ *is confluent*†*, then* $\mathcal{T}(\mathrm{X})/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{X}}$ *is free for* K *over* X *w. r. t.* κ*.*
Furthermore, if we assume $\longrightarrow_{\mathrm{R},\mathrm{X}}$ *to be confluent*‡ *and* $\mathrm{X}\subseteq\mathrm{V}_{\mathrm{SIG}}$ *, then:*

(1) $\mathcal{T}(\mathrm{X})/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{X}}$ *is a constructor-minimum model of* R*.*
(2) $\mathcal{T}(\mathrm{X})/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{X}}$ *is free in* K *over* X *w. r. t.* κ*.*
(3) $\mathcal{T}(\mathrm{X})/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{X}}$ *is a minimal model of* R*.*

COROLLARY 5.15. *Let* R *be a* Def-*MCRS over* sig/cons/V*. Furthermore, assume* $\longrightarrow_{\mathrm{R},\emptyset}$ *to be confluent. Now:* $\mathcal{GT}/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\emptyset}$ *is a minimal model of* R*, initial in the class of all constructor-minimal models of* R*, and the (up to isomorphism) unique (*$\cdot_{\mathrm{H}}$*-) minimum of the* sig*-term-generated constructor-minimal models of* R*.*

† The remark of footnote ‡ with $\mathrm{X} := \emptyset$ is applicable here.
‡ The following allows to apply the confluence criterion of Theorem 6.5: If we additionally require $\forall((l,r),C)\in\mathrm{R} : \forall(u{=}v) \text{ in } C : \big(\ \mathrm{Def}\,u,\ \mathrm{Def}\,v \text{ are in } C\ \big)$, then we can weaken the confluence requirement to confluence of $\longrightarrow_{\mathrm{R},\mathrm{X}} \cap (D_{\mathrm{X}} \times D_{\mathrm{X}})$ for $D_{\mathrm{X}} := \{\ u \in \mathcal{T}(\mathrm{sig},\mathrm{X}) \mid \exists \hat{u} \in \mathcal{GT}(\mathrm{cons}) : u \overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{X}} \hat{u}\ \}$.

Finally in this section, we present the second fundamental theorem for our approach, which states that our reduction relation is monotonic w. r. t. consistent extension of the specification. Consistent extensions play an important role for incremental refinement and modular construction of specifications. For inductive theorem proving it is of major[†] importance not to lose the already shown theorems when extending the specification in a consistent manner. The following theorem can be used to establish monotonicity of inductive validity (of first-order clauses) defined to be validity in $\mathcal{T}(\mathrm{V_{SIG}})/\overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\mathrm{V_{SIG}}}$, cf. Wirth & al. (1993).

Theorem 5.16.
(Monotonicity of $\longrightarrow_{\mathrm{R,X}}$ w. r. t. Consistent Extension of the Specification)
Let R *be a CRS over* sig/cons/V. *Let* X⊆V. *Let* R′ *be another CRS, but over* sig′/cons′/V′; *and* X′⊆V′ *with*

$$\left|\begin{array}{rl} \mathrm{sig}' &= (\mathrm{F}',\mathrm{S}',\alpha') \\ \mathrm{cons}' &= (\mathrm{C}',\mathrm{S}',\alpha'|_{\mathrm{C}}) \\ \mathrm{V}'|_{\{\mathrm{SIG},\mathrm{CONS}\}\times\mathrm{S}} &= \mathrm{V} \end{array}\right| \begin{array}{l} \mathrm{F}\subseteq\mathrm{F}' \\ \mathrm{C}\subseteq\mathrm{C}'\ \subseteq\ \mathrm{F}' \\ \mathrm{S}\subseteq\mathrm{S}' \\ \alpha\subseteq\alpha' \end{array}\left|\begin{array}{l} \mathrm{R}\subseteq\mathrm{R}' \\ \mathrm{X}\subseteq\mathrm{X}' \end{array}\right|$$

Thus, sig′/cons′/V′ *is an enrichment of* sig/cons/V *in the most general*[‡] *sense we can think of.*
Moreover, assume[§]*:* $\forall((l,r),C)\in(\mathrm{R}'\setminus\mathrm{R}):\ l\notin\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V_{CONS}})$ *(:$)*
Now we have:

(1) $\forall s\in\mathcal{T}(\mathrm{cons},\mathrm{X}):\forall t:\Big((s\overset{\oplus}{\longrightarrow}_{\mathrm{R,X}}t)\ \Leftrightarrow\ (s\overset{\oplus}{\longrightarrow}_{\mathrm{R',X'}}t)\Big)$

"no change on old constructor terms"

(2) $\longrightarrow_{\mathrm{R,X}}\ \subseteq\ \longrightarrow_{\mathrm{R',X'}}$ *"monotonicity"*

(3) $\forall\beta\preceq\omega{+}\omega:\ \longrightarrow_{\mathrm{R,X},\beta}\ \subseteq\ \longrightarrow_{\mathrm{R',X'},\beta}$ *"monotonicity"*

When proving theorems dealing with signature enrichments, one usually has to be very careful because notations like '$\longrightarrow_{\mathrm{R,X}}$' do not indicate whether $\longrightarrow_{\mathrm{R,X}}$ is defined on sig/cons/V or sig′/cons′/V′, which may be important under several aspects. E. g., it is important for Theorem 5.16 that $\longrightarrow_{\mathrm{R,X}}$ tests "$u{\neq}v$" in a condition of an equation by " $\exists\hat{u},\hat{v}\in\mathcal{GT}(\underline{\underline{\mathrm{cons}}}):\ u\overset{\oplus}{\longrightarrow}_{\mathrm{R,X}}\hat{u}\not\downarrow_{\mathrm{R,X}}\hat{v}\overset{\oplus}{\longleftarrow}_{\mathrm{R,X}}v$ " instead of " $\exists\hat{u},\hat{v}\in\mathcal{GT}(\underline{\underline{\mathrm{cons}'}}):\ u\overset{\oplus}{\longrightarrow}_{\mathrm{R,X}}\hat{u}\not\downarrow_{\mathrm{R,X}}\hat{v}\overset{\oplus}{\longleftarrow}_{\mathrm{R,X}}v$ ". With this exception, however, for the validity of the theorem it does not matter whether $\longrightarrow_{\mathrm{R,X}}$ is defined on $\mathcal{T}(\mathrm{sig},\mathrm{X})$ or $\mathcal{T}(\mathrm{sig}',\mathrm{X}')$.

[†] Contrary to deductive first-order theorem proving, inductive theorem proving often is only successful when one tries to show stronger theorems than one initially intended to show. This is because induction hypotheses are not only a task but also a tool for the inductive argumentation. Sometimes the required induction hypotheses or lemmas are not expressible by first-order clauses unless we extend the specification in a consistent manner.

[‡] One may even introduce new constructor symbols for the old sorts and take them from the old non-constructor symbols. Since all $\mathrm{V}_{\varsigma,s}$ are infinite, the restriction on V′ is not severe.

[§] This has to be required for keeping the negative conditions fulfilled: Having founded our inequalities on old constructor ground terms, all we have to take care of now is not to confuse these terms.

6. Testing for Confluence

The following notions and notations are standard, with the exception of "quasi overlay joinable" which is a slight weakening of "overlay joinable" of Dershowitz & al. (1988a) in that it allows an identical non-overlay part in the critical pair.

If the left-hand side of a rule $l_0{=}r_0{\longleftarrow}C_0$ and the subterm at non-variable (i. e. $l_1/p \notin \mathrm{V}$) position $p \in \mathcal{POS}(l_1)$ of the left-hand side of a rule $l_1{=}r_1{\longleftarrow}C_1$ (assuming $\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0) \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$ w. l. o. g.) are unifiable by $\sigma = \mathrm{mgu}(\{(l_0, l_1/p)\}, \mathcal{V}(l_0{=}r_0{\longleftarrow}C_0, l_1{=}r_1{\longleftarrow}C_1))$ and if the resulting critical peak is non-trivial (i. e. $l_1[p \leftarrow r_0]\sigma \neq r_1\sigma$), then $(((l_1[p \leftarrow r_0], r_1), C_0C_1)\sigma,\ \ l_1\sigma,\ \ p)$ is a (non-trivial) *critical peak* consisting of of the conditional critical pair, its peak $l_1\sigma$, and the overlap position p. The set of all critical peaks of a CRS R is denoted by CP(R). R is said to be *overlapping* :iff $\mathrm{CP}(\mathrm{R}) \neq \emptyset$. A critical peak $(((t_0, t_1), D), \hat{t}, p)$ is *joinable w. r. t.* R, X (for X⊆V) :iff $\forall\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})) : ((D\varphi \text{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}}) \Rightarrow t_0\varphi{\downarrow_{\mathrm{R,X}}}t_1\varphi)$. A critical peak $(((t_0, t_1), D), \hat{t}, p)$ is *overlay joinable w. r. t.* R, X :iff it is joinable w. r. t. R, X and $p{=}\emptyset$. It is *quasi overlay joinable w. r. t.* R, X :iff $\forall\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$:

$$\left(\left(\begin{array}{l} D\varphi \text{ fulfilled} \\ \text{w. r. t. } \longrightarrow_{\mathrm{R,X}} \end{array}\right) \Rightarrow \left(\begin{array}{ll} t_1\varphi = t_0\varphi[\,p \leftarrow t_1\varphi/p\,] & \wedge \\ (t_0/p)\varphi \downarrow_{\mathrm{R,X}} t_1\varphi/p\ \ (\longleftarrow_{\mathrm{R,X}} \cup \lhd_{\mathrm{ST}})^{\oplus}\ \ (\hat{t}/p)\varphi & \end{array}\right)\right).$$

Lemma 6.1. (Joinability of Critical Peaks is Necessary for Confluence)
If $\longrightarrow_{\mathrm{R,X}}$ *is confluent, then all critical peaks in* CP(R) *are joinable w. r. t.* R, X.

Lemma 6.2. (Overlay Joinable ⇒ Quasi Overlay Joinable ⇒ Joinable)
Let $(((t_0, t_1), D), \hat{t}, p) \in \mathrm{CP}(\mathrm{R})$. *Now w. r. t.* R, X *the following holds:*

(1) If $(((t_0, t_1), D), \hat{t}, p)$ *is overlay joinable, then it is quasi overlay joinable.*

(2) If $(((t_0, t_1), D), \hat{t}, p)$ *is quasi overlay joinable, then it is joinable.*

Sufficient criteria for confluence of reduction relations for merely positive conditional rule systems are studied in Dershowitz & al. (1988a). As counterexamples for suggested sufficient confluence criteria for merely positive conditional rule systems are counterexamples for Def-MCRSs too, we repeat the results of Dershowitz & al. (1988a) here: There are (left-linear) non-overlapping positive-conditional rule systems whose reduction relations are not (locally) confluent[†] (but necessarily non-noetherian then, cf. Dershowitz & al. (1988a), Theorem 4, p. 39). Therefore, syntactic confluence criteria for non-noetherian conditional rule systems must be difficult to develop. Semantic confluence criteria (in the style of Plaisted (1985)) seem to require noetherian (or at least normalizing) reduction relations because they rely on the irreducible reducts of the terms; furthermore irreducibility is not (semi-) decidable. Thus, for our confluence criteria we require (at least a sub-relation of) $\longrightarrow_{\mathrm{R,X}}$ to be noetherian. Even then the situation is not very encouraging, because there are noetherian and non-confluent reduction relations of (left-linear, normal, and) merely positive conditional rule systems whose critical peaks are all joinable, cf. Dershowitz & al. (1988a), Example B, p. 36. Moreover, semantic confluence criteria remain difficult because irreducibility is still not (semi-) decidable. However, for merely positive conditional rule systems there are two known syntactic solutions of major interest:[‡] One requires either more than joinability for the critical peaks (as, e. g., in Theorem 4 in Dershowitz & al. (1988a)) or the condition terms of a rule to be somehow smaller

[†] Cf. Dershowitz & al. (1988a), Example A, p. 36, taken from Bergstra & Klop (1986)

than the left-hand side (as, e. g., in Theorem 3 in Dershowitz & al. (1988a)). We will study the latter approach (which is the more important one in practice, cf. Example 6.4) later, cf. theorems 7.17 and 7.18. The following result is a generalization of Theorem 4 in Dershowitz & al. (1988a) from positive-conditional to positive/negative-conditional rule systems and, moreover, from overlay joinability to quasi overlay joinability.

Theorem 6.3. (Syntactic Confluence Criterion)
Let R *be a CRS over* sig/cons/V *and* $\mathrm{X}\subseteq\mathrm{V}$. *If* $\longrightarrow_{\mathrm{R,X}}$ *is noetherian and all critical peaks in* CP(R) *are quasi overlay joinable w. r. t.* R, X, *then* $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

While this theorem is nice (theoretically) and has a pretty complicated proof, it may be difficult to apply, even for merely positive conditional equations:

Example 6.4. *Let* R: $\mathsf{f}(\mathsf{s}(x))=\mathsf{0} \longleftarrow \mathsf{f}(x)=\mathsf{0}\ ;\ \mathsf{f}(\mathsf{s}(x))=\mathsf{1} \longleftarrow \mathsf{f}(x)=\mathsf{1}\ ;\ \mathsf{f}(\mathsf{0})=\ldots$ *Assume* 0 *and* 1 *to be irreducible. Now for showing the critical peak between the first two rules to be (quasi) overlay joinable, one has to show that it is impossible that both conditions hold simultaneously for a substitution* $\{x\mapsto t\}$. *However, in order to prove this, we need the confluence below '*$\mathsf{f}(t)$*', which we are not allowed to assume for the joinability test here, contrary to the theorems 7.17 and 7.18 (regarding Definition 7.16).*

Finally in this section, we present a semantic confluence criterion. Note that its second part is interesting because the reduction relation is required to be noetherian on the defined terms only, i. e. infinite reduction sequences on terms which do not return a result (in the sense that they are congruent to some constructor ground term) are in principle no obstacle for applying the criterion.

Theorem 6.5. (Semantic Confluence Criterion)
Let R *be a CRS over* sig/cons/V *and* $\mathrm{X}\subseteq\mathrm{V}$. *Let* $\mathcal{A}$ *be a* sig/cons*-model of* R *and* κ *an* $\mathcal{A}$*-valuation of* X. *Now:*

(1) If $\forall s\in\mathrm{S}:\forall\hat{u},\hat{v}\in\mathcal{T}(\mathrm{sig},\mathrm{X})_s\backslash\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}):(\ \mathcal{A}_\kappa(\hat{u})=\mathcal{A}_\kappa(\hat{v})\ \Rightarrow\ \hat{u}=\hat{v}\ )$ *and* $\longrightarrow_{\mathrm{R,X}}$ *is noetherian, then* $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

(2) Define† $D_{\mathrm{X}}\ :=\ \{\ u\in\mathcal{T}(\mathrm{sig},\mathrm{X})\mid\exists\hat{u}\in\mathcal{GT}(\mathrm{cons}):u\overset{\circledast}{\longleftrightarrow}_{\mathrm{R,X}}\hat{u}\ \}$. *If*§
$$\forall s\in\mathrm{S}:\forall\hat{u}\in\mathcal{GT}(\mathrm{cons})_s\backslash\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}):\forall\hat{v}\in\mathcal{T}(\mathrm{sig},\mathrm{X})_s\backslash\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}):$$
$$(\ \mathcal{A}_\kappa(\hat{u})=\mathcal{A}_\kappa(\hat{v})\ \Rightarrow\ \hat{u}=\hat{v}\ )$$
and $\longrightarrow_{\mathrm{R,X}}\cap(D_{\mathrm{X}}\times D_{\mathrm{X}})$ *is noetherian, then* $\longrightarrow_{\mathrm{R,X}}\cap(D_{\mathrm{X}}\times D_{\mathrm{X}})$ *is confluent.*

‡ We do not discuss Theorem 1 (which is taken from Bergstra & Klop (1986) and interesting in so far as confluence can be guaranteed without requiring $\longrightarrow$ to be noetherian) and Theorem 2 of Dershowitz & al. (1988a) here, which state that left-linear and normal rule systems are confluent if they are non-overlapping or both shallow-joinable and noetherian. Each of these conditions is really necessary for guaranteeing confluence: Cf. Example C (on p. 36 of Dershowitz & al. (1988a)) for normality and Example D for left-linearity. The combination of left-linearity and normality, however, is rather restrictive, because left-linearity forbids the positive part of the specification of an equality predicate by ($s\in\mathrm{S}$) " $\mathsf{eq}_s(X,X)=-$ " ($X\in\mathrm{V}_{\mathrm{SIG},s}$), which is the common trick for achieving normality by transformation of "$v=v$" in a condition of a rule into "$\mathsf{eq}_s(v,u)=-$". For sufficiently complete $\overset{\circledast}{\longrightarrow}$ and free constructors, however, the two theorems are important, since then it is indeed possible to specify this positive part by equations of the form ($c\in\mathrm{C}$) " $\mathsf{eq}_{s_n}(c(x_0,\ldots,x_{n-1}),c(y_0,\ldots,y_{n-1}))=-\ \longleftarrow\ \mathsf{eq}_{s_0}(x_0,y_0)=-,\ldots,\mathsf{eq}_{s_{n-1}}(x_{n-1},y_{n-1})=-$ ".

† Cf. footnote ‡ of Theorem 5.14

§ The following differs from the similar condition of part 1: Irreducible, syntactically different, non-constructor terms (i. e. in case of confluence: non-congruent "undefined" terms) may be identified by $\mathcal{A}_\kappa$.

7. Compatible Rule Systems

Compatibility restrictions on rule systems w. r. t. well-founded orderings (saying, in essence, that the left-hand side must be bigger than the condition terms and the right-hand side of the rule) enhance our means of deciding reducibility and confluence. It is well-known that such restrictions are necessary (cf. e. g. Theorem 3.3 of Kaplan (1984)):

LEMMA 7.1. (REDUCIBILITY OF GROUND TERMS IS NOT CO-SEMI-DECIDABLE)
There is a left-linear, non-overlapping, extra-variable free, merely positive conditional rule system R *with noetherian and confluent reduction relation* $\longrightarrow_{\mathrm{R,V}}$ *for which reducibility of ground terms is not co-semi-decidable.*

LEMMA 7.2. (REDUCIBILITY OF GROUND TERMS IS NOT SEMI-DECIDABLE)
There is a left-linear, non-overlapping, extra-variable free, Def-*moderate CRS* R *with noetherian and confluent reduction relation* $\longrightarrow_{\mathrm{R,V}}$ *for which reducibility of ground terms is not semi-decidable.*

The following theorem generalizes Theorem 3.4 of Kaplan (1984) to negative conditions and to non-confluent and non-noetherian $\longrightarrow$.

THEOREM 7.3. *Let* R *be a CRS over* sig/cons/V. *Let* X *be an enumerable subset of* V.

(1) $\longrightarrow_{\mathrm{R,X}}$*-reducibility of terms from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is co-semi-decidable if a* $\longrightarrow_{\mathrm{R,X}}$*-normal form for each term from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is computable (i. e. there is a computable (partial) function* f *with* $\mathrm{dom}(f) = \{\ s \in \mathcal{T}(\mathrm{sig},\mathrm{X}) \mid \exists t : (s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X}}))\ \}$ *such that* $\forall s \in \mathrm{dom}(f) : s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} f(s) \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$ *).*

(2) A $\longrightarrow_{\mathrm{R,X}}$*-normal form for each term from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is computable (cf. above) if* $\longrightarrow_{\mathrm{R,X}}$*-reducibility of terms from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is co-semi-decidable and*
$\forall s \in \mathcal{GT}(\mathrm{cons}) : \exists t : s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$.

COROLLARY 7.4. *Let* R *be a CRS over* sig/cons/V. *Let* X *be an enumerable subset of* V. *Assume* $\longrightarrow_{\mathrm{R,X}}$ *to be noetherian. Now, co-semi-decidability of* $\longrightarrow_{\mathrm{R,X}}$*-reducibility of terms from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is logically equivalent to computability of a* $\longrightarrow_{\mathrm{R,X}}$*-normal form for each term from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$.

7.1. THE USE OF ORDERINGS

In this and the following section we want to give minimal reasonable compatibility requirements for achieving additional properties for our reduction relation. We start with a discussion on how to use orderings for reduction with conditional rules. This discussion mainly depends on the method of testing the conditions uniformly by the same reduction relation again, where well-founded orderings are needed for guaranteeing termination of condition-testing and reduction. Since this method does not depend on the concrete form of our rules, the situation under discussion does not differ from the case of merely positive conditional equations.

As we test our conditions by reduction we must be allowed to switch from reduction to condition-testing, and then to reduction of the condition terms, and so on. Hence we want our compatibility requirement to imply that $(\longrightarrow_{\mathrm{R,X}} \cup \imath_{\mathrm{R,X}})$ is noetherian, where $s \imath_{\mathrm{R,X}}\ t$:iff $s \in \mathcal{T}(\mathrm{sig},\mathrm{X}) \ \wedge\ \exists((l,r),C) \in \mathrm{R} : \exists \sigma \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})) : \exists p \in \mathcal{POS}(s)$: $(\ s/p = l\sigma \ \wedge\ \exists u \in \mathcal{TERMS}(C) : t = u\sigma \ \wedge\ (C\sigma$ is fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X}})\ )$.

We are now going to find out how to formulate a compatibility requirement on a set of rules R in such a way that it is guaranteed to be satisfiable for appropriate well-founded orderings iff $(\longrightarrow_{\mathrm{R,X}} \cup \imath_{\mathrm{R,X}}\ )$ is noetherian. By Lemma 5.12 and the following lemma, this condition is logically equivalent to $(\longrightarrow_{\mathrm{R,V}}\ \cup\ (\trianglerighteq_{\mathrm{ST}} \circ \imath_{\mathrm{R,V}}\ ))$ being noetherian.

LEMMA 7.5.
If $(\longrightarrow_{\mathrm{R,X}} \cup \imath_{\mathrm{R,X}}\ )$ *is noetherian, then* $(\longrightarrow_{\mathrm{R,V}}\ \cup\ (\trianglerighteq_{\mathrm{ST}} \circ \imath_{\mathrm{R,V}}\ ))$ *is noetherian, too.*

For the sake of practical applicability, we allow to replace '$\longrightarrow_{\mathrm{R,V}}$' and '$\imath_{\mathrm{R,V}}$' with some abstract relations '´' and '$\hookrightarrow$': By Lemma 5.12 and the corollaries 5.6 and 5.7, the termination of $(\longrightarrow_{\mathrm{R,V}}\ \cup\ (\trianglerighteq_{\mathrm{ST}} \circ \imath_{\mathrm{R,V}}\ ))$ is logically equivalent to the existence of two relations '´' and '$\hookrightarrow$' satisfying the following:
$\longrightarrow_{\mathrm{R,X}}\ \subseteq\ ´$; $\imath_{\mathrm{R,X}}\ \subseteq \hookrightarrow$; ´ is sort-invariant, V-monotonic, and V-stable; $\hookrightarrow$ is V-stable; and $(´\ \cup (\trianglerighteq_{\mathrm{ST}} \circ \hookrightarrow))$ is noetherian. The following two lemmas show that the last requirement is equivalent to even $(´ \cup \triangleright_{\mathrm{ST}} \cup \hookrightarrow)$ being noetherian:

LEMMA 7.6. *Let ´ be a sort-invariant (This can always be achieved by identifying all sorts.) and* X*-monotonic relation on* $\mathcal{T}(\mathrm{sig},\mathrm{X})$. *Define* $\succ\ := (´ \cup \triangleright_{\mathrm{ST}})^{\oplus}$. *Now:*

(1) *If ´ is noetherian [and* Y*-stable], then* $\succ$ *is a well-founded [and* Y*-stable] ordering, which does not need to be sort-invariant or* $\emptyset$*-monotonic.*
(2) *(1) does not hold in general if ´ is not sort-invariant or not* X*-monotonic.*
(3) $\mathrm{id}|_{\mathcal{T}(\mathrm{sig},\mathrm{X})} \circ \triangleright_{\mathrm{ST}} \circ\ ´\ \subseteq\ ´ \circ \triangleright_{\mathrm{ST}}$. *Moreover, for* X=V*:* $\triangleright_{\mathrm{ST}} \circ\ ´\ \subseteq\ ´ \circ \triangleright_{\mathrm{ST}}$.
(4) $\succ\ =\ \triangleright_{\mathrm{ST}} \cup (\trianglerighteq_{\mathrm{ST}} \circ\ ´^{\oplus} \circ \trianglerighteq_{\mathrm{ST}})$. *Moreover, for* X=V*:* $\succ\ =\ \triangleright_{\mathrm{ST}} \cup (´^{\oplus} \circ \trianglerighteq_{\mathrm{ST}})$.

LEMMA 7.7. *Let* $\triangleright_{\mathrm{ST}} \circ\ ´ \subseteq\ ´ \circ \triangleright_{\mathrm{ST}}$ *(cf. Lemma 7.6(3)). Assume* $(´\ \cup (\trianglerighteq_{\mathrm{ST}} \circ \hookrightarrow))$ *to be noetherian (cf. requirement above 7.6). Assume* $(´ \cup \triangleright_{\mathrm{ST}})$ *to be noetherian (cf. 7.6(1)). [Assume ´ and* $\hookrightarrow$ *to be* V*-stable.] Now:*
$\triangleright := (´ \cup \hookrightarrow \cup \triangleright_{\mathrm{ST}})^{\oplus}$ *is a well-founded [and* V*-stable] ordering.*

Finally, writing '>' for '$´^{\oplus}$', this motivates the following definition:

DEFINITION 7.8. (TERMINATION-PAIR)
A termination-pair *over* sig/V *is a pair* $(>, \triangleright)$ *of binary relations on* $\mathcal{T}$ *with:*

(1) $>$ *is a* V*-monotonic and* V*-stable*[†] *ordering*[‡].
(2) $\triangleright$ *is a* V*-stable*[†] *and well-founded ordering.*
(3) $> \subseteq \triangleright$
(4) $\triangleright_{\mathrm{ST}} \subseteq \triangleright$ [§]

[†] V-stability is included because it can always be achieved (for ´ and $\hookrightarrow$; and thereby for $>$ and $\triangleright$, too) by restriction to ground terms — and the non-ground part of an ordering $\triangleright$ whose V-stable closure is not noetherian anymore is of no use for showing termination anyway because then its $\emptyset$-stable closure is not noetherian, either.

[‡] As discussed above, sort-invariance can be required here; but it is of no use for us and omitted for convenience. For the benefit from this cf. Example 7.9(3).

[§] Notice that for proving alignment (cf. Definition 7.10 below) in practice we only have to show $\triangleright_{\mathrm{ST}} \circ \triangleright\ \subseteq\ \triangleright$ and then take $(\triangleright \cup \triangleright_{\mathrm{ST}})^{\oplus}$ instead of $\triangleright$, because then we know by Lemma 7.6 (applied to the sort-invariant restriction of $>$) and then by Lemma 7.7 that $(\triangleright \cup \triangleright_{\mathrm{ST}})^{\oplus}$ will do the job of $\triangleright$.

EXAMPLE 7.9. *The standard examples for a termination-pair* $(>, \rhd)$ *are:*

(1) $>$ *some sort-invariant reduction ordering;* $\rhd := \rhd_{\mathrm{ST}} \cup (> \circ \trianglerighteq_{\mathrm{ST}})$, *cf. Lemma 7.6(4).*

(2) $\rhd$ *some* V*-stable and well-founded ordering containing* $\rhd_{\mathrm{ST}}$*;* $> := \bigcup_{s \in \mathrm{S}} \{\ (t', t'')\ \mid$

$$t', t'' \in \mathcal{T}_{\mathrm{SIG},s} \ \wedge\ \forall t \in \mathcal{T}\colon \forall p \in \mathcal{POS}(t)\colon (t/p \in \mathcal{T}_{\mathrm{SIG},s} \Rightarrow t[p \leftarrow t'] \rhd t[p \leftarrow t'']) \ \}.^{\dagger}$$

(3) $>$ *some simplification ordering;* $\rhd := >$.

To avoid misunderstandings we will use "aligned" for the local ordering restriction on a single rule and "compatible" (with different prefixes) for ordering restrictions that involve the reduction relation of the whole rule system.

DEFINITION 7.10. (ALIGNMENT OF A RULE W. R. T. A TERMINATION-PAIR)
Let $(>, \rhd)$ *be a termination-pair over* sig/V*.*
A rule $((l, r), C) \in \mathcal{RUL}(\mathrm{sig}, \mathrm{cons}, \mathrm{V})$ *is called* aligned with $(>, \rhd)$ *:iff*

$$l > r \quad \wedge \quad \forall u \in \mathcal{TERMS}(C) : \ l \rhd u$$

The following kind of compatibility is a generalization to negative conditions and also a slight weakening of the notion of "decreasingness" of Dershowitz & al. (1988a), cf. below.

DEFINITION 7.11. (COMPATIBILITY WITH A TERMINATION-PAIR) *A CRS* R *over* sig/cons/V *is* X-compatible *with a termination-pair* $\mathrm{T} = (>, \rhd)$ *over* sig/V *:iff*
$\forall ((l, r), C) \in \mathrm{R} : \forall \tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$:

$$\big((C\tau \textit{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}}) \ \Rightarrow\ \big(\ ((l, r), C)\tau \textit{ is aligned with } \mathrm{T}\ \big)\big)$$

Compatibility of a CRS R guarantees alignment of an instantiated rule of R when its condition is fulfilled. But, while this kind of compatibility is convenient for obtaining further theoretical properties of the reduction relation, we have a problem when using this kind of compatibility of R in practice of reduction: The terms in $C\tau$ must be smaller than $l\tau$ only if $C\tau$ is fulfilled; but for easily deciding whether $C\tau$ is fulfilled we need its terms to be smaller than $l\tau$ and the analogous property for the other rules. That this need not be a vicious circle is shown by the following definition, which allows us to test the literals in the condition from left to right. Notice that the difference to Definition 7.11 is in the quantified variable i occurring as an index which allows us to step inductively from $(< i)$ to i.

DEFINITION 7.12. (LEFT-RIGHT-COMPATIBILITY) *A CRS* R *over* sig/cons/V *is* X-left-right-compatible *with a termination-pair* $\mathrm{T} = (>, \rhd)$ *over* sig/V *:iff*
$\forall ((l, r), L_0 \ldots L_{n-1}) \in \mathrm{R} : \forall \tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$:

$$\left(\begin{array}{ll} \forall i < n : (& ((L_0 \ldots L_{i-1})\tau \textit{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}}) \Rightarrow \forall u \in \mathcal{TERMS}(L_i) : l\tau \rhd u\tau\ ) \\ \wedge \qquad (& ((L_0 \ldots L_{n-1})\tau \textit{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}}) \Rightarrow l\tau > r\tau \qquad\qquad) \end{array}\right)$$

† While irreflexivity, transitivity, sort-invariance and V-monotonicity of $>$ are trivial, V-stability for $t' > t''$; $t', t'' \in \mathcal{T}_{\mathrm{SIG},s}$ w. r. t. a substitution $\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T})$ can be seen the following way: For arbitrary $t \in \mathcal{T}$ and $p \in \mathcal{POS}(t)$ with $t/p \in \mathcal{T}_{\mathrm{SIG},s}$ let $\xi \in \mathcal{SUB}(\mathrm{V}, \mathrm{V})$ be a bijection with $\xi[\mathcal{V}(t)] \cap \mathcal{V}(t', t'') = \emptyset$ and then define $\varrho := \sigma|_{\mathcal{V}(t',t'')} \cup \xi^{-1}|_{\mathrm{V} \setminus \mathcal{V}(t',t'')}$. Now by $t\xi[p \leftarrow t'] \rhd t\xi[p \leftarrow t'']$ we get $t[p \leftarrow t'\sigma] = t\xi[p \leftarrow t']\varrho \rhd t\xi[p \leftarrow t'']\varrho = t[p \leftarrow t''\sigma]$.

Definition 7.13. (Don't-Care-Compatibility) *A CRS* R *over* sig/cons/V *is* X-don't-care-compatible *with a termination-pair* $\mathrm{T}=(>,\,\rhd)$ *over* sig/V *:iff*
$\forall((l,r),L_0\ldots L_{n-1})\in\mathrm{R}:\forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})):$

$$\left(\begin{array}{llr}\forall i<n: & & \forall u\in\mathcal{TERMS}(L_i):l\tau\rhd u\tau \\ \wedge & (\ ((L_0\ldots L_{n-1})\tau \text{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}})\Rightarrow l\tau>r\tau &)\end{array}\right)$$

Having a left-right-compatible CRS, we do not have to test the instantiated rules for alignment anymore, provided that we test the literals of the instantiated conditions from left to right until one of them fails. Having a don't-care-compatible CRS, we can even test the literals of the instantiated conditions in parallel and don't have to care for the position of these equations in the condition list.

The compatibility of Definition 7.11 (which seems to be the least restrictive one tractable in theory) is intended to be an interface for generating logically stronger kinds of compatibility that are useful in practice (cf. definitions 7.12, 7.13), where the don't-care-compatibility seems to be the most important one. For restrictions of $\longrightarrow_{\mathrm{R,X}}$, however, even weaker kinds of compatibility than the one of Definition 7.11 may be sufficient.

Finally in this section, we will have a brief look on similar ordering restrictions in literature. In Kaplan (1987) we find a notion of alignment: There, an instantiated rule $l{=}r{\longleftarrow}C$ is called "simplifying" if $l>r$ and $\forall u\in\mathcal{TERMS}(C)\colon l>u$ for some simplification ordering $>$, cf. also our Example 7.9(3). Well-known are also the following compatibility notions (i. e. notions involving $\longrightarrow_{\mathrm{R,X}}$): In Jouannaud & Waldmann (1986) an instantiated rule $l{=}r{\longleftarrow}C$ is called "reductive" if fulfilledness of C w. r. t. $\longrightarrow_{\mathrm{R,X}}$ implies $l>r$ and if $\forall u\in\mathcal{TERMS}(C)\colon l>u$ for some sort-invariant reduction ordering $>$, cf. Example 7.9(1). A less restrictive compatibility requirement which is conceptually the same as our don't-care compatibility can be found in Dershowitz & al. (1988a&b): R is called "decreasing" if there exists a well-founded ordering $\rhd$, containing $\rhd_{\mathrm{ST}}$, such that $s\rhd t$ whenever $s\longrightarrow_{\mathrm{R,X}}t$ and, for each instantiated rule $l{=}r{\longleftarrow}C$ of R, $\forall u\in\mathcal{TERMS}(C)$: $l\rhd u$. While our additional $>$ does not occur in this definition, it is useful both in theory and practice, since the infinite requirement of $s\rhd t$ above can be reduced to $l>r$, but not to $l\rhd r$. Furthermore, our discussion preceding Definition 7.8 reveals the interference of $>$ and $\rhd$ with rules in practice and how to establish the properties required.

While we require $>$ to be a reduction ordering, we avoid the superfluous commonplace restriction along Example 7.9(1), restricting $\rhd$ to be

$$\rhd := \rhd_{\mathrm{ST}}\cup\ ((>\cap\textstyle\bigcup_{s\in\mathrm{S}}(\mathcal{T}_{\mathrm{SIG},s}\times\mathcal{T}_{\mathrm{SIG},s}))\circ\trianglerighteq_{\mathrm{ST}})$$

(which is also implicit in the notion of "reductive" above) because this may not be sufficient for alignment of given rules as in the following example of Dershowitz & al. (1988b):

Example 7.14. ($\rhd := \rhd_{\mathrm{ST}}\cup(>\circ\trianglerighteq_{\mathrm{ST}})$ is Too Restrictive (cf. Ex. 7.9(1)))
R*:* b=c *;* a=c⟵b=c *;* f(b)=f(a) *.*
Alignment of these rules requires $\mathsf{a}\rhd\mathsf{b}$ *which we cannot achieve by the above construction of* $\rhd$*:* $\mathsf{a}>\mathsf{b}$ *is impossible since alignment of the third rule requires* $\mathsf{f(b)}>\mathsf{f(a)}$ *, which also forbids* $\mathsf{a}>\mathsf{f}^{n+1}(\mathsf{b})$ *, since then we get* $\mathsf{a}>\mathsf{f}^{(n+1)}(\mathsf{a})>\mathsf{f}^{2(n+1)}(\mathsf{a})>\cdots$*.*

Thus, for theoretical treatment, the procedure of (2) of Example 7.9 is to be preferred to that of (1) of Example 7.9, whereas (for practically guaranteeing alignment of rules) (2) of Example 7.9 lacks any hints on how to semi-decide $>$ (even for decidable $\rhd$).

All in all, there seems to be no proper reason for preferring one of $>$, $\rhd$ to the other and we thus have introduced the notion of a termination-pair $(>,\,\rhd)$.

7.2. Results for Compatible CRSs

Lemma 7.15. *Let* R *be a CRS over* sig/cons/V*;* X⊆Y⊆V *; and* T = (>, ⊳) *a termination-pair over* sig/V*. Assume that* R *is* Y*-compatible with* T*. Now we have* $\longrightarrow_{\mathrm{R,Y}} \subseteq >$ *and* $\longrightarrow_{\mathrm{R,Y}} \cup \mathbf{1}_{\mathrm{R,Y}} \ \cup \rhd_{\mathrm{ST}} \subseteq \rhd$*, which is noetherian.*
Furthermore, R *is* X*-compatible with* T *and* V*-compatible with the termination-pair* $(\overset{\oplus}{\longrightarrow}_{\mathrm{R,V}},\ (\longrightarrow_{\mathrm{R,V}} \cup \mathbf{1}_{\mathrm{R,V}} \ \cup \rhd_{\mathrm{ST}})^{\oplus} \ )$ *over* sig/V*.*

The following notion of "weakly joinable" weakens "joinable" by adding a confluence requirement to the premise.

Definition 7.16.
A critical peak $(((t_0, t_1), D), \hat{t}, p)$ *is* ⊳-weakly joinable w. r. t. R, X *:iff*
$\forall\tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$:

$$\left(\begin{pmatrix} D\tau \text{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}} & \wedge \\ \forall u : (\ \ u \lhd \hat{t}\tau \ \Rightarrow (\ \longrightarrow_{\mathrm{R,X}} \text{ is confluent below } u \ ) \ ) \end{pmatrix} \Rightarrow \ t_0\tau{\downarrow_{\mathrm{R,X}}}t_1\tau \right).$$

For compatible CRSs we can now give a complete confluence test à la Knuth-Bendix:

Theorem 7.17. (Syntactic Confluence Test)
Let R *be a CRS over* sig/cons/V *and* X⊆V *. Assume that* R *is* X*-compatible with a termination-pair* T = (>, ⊳) *over* sig/V*. The following two are logically equivalent:*

(1) $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

(2) All critical peaks in CP(R) *are (*⊳*-weakly) joinable w. r. t.* R, X*.*

The following theorem, which is similar to Theorem 5.1 in Avenhaus & Becker (1992), drops the compatibility restriction of Theorem 7.17 for those condition literals which do not contain general variables, while it does not require (quasi) overlay joinability as Theorem 6.3.

Theorem 7.18. (Syntactic Confluence Test)
Let R *be a CRS over* sig/cons/V*;* X⊆V *; and* T = (>, ⊳) *a termination-pair over* sig/V*. Assume the constructors to be free, i. e. each left-hand side of* R *contains a non-constructor function symbol. Furthermore, we require the following compatibility-property:*
$\forall((l, r), C) \in \mathrm{R} : \forall\tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$:

$$\left((C\tau \text{ fulfilled w. r. t. } \longrightarrow_{\mathrm{R,X}}) \ \Rightarrow \ \begin{pmatrix} l\tau > r\tau \ \ \wedge \\ \forall L \text{ in } C : \begin{pmatrix} \forall u \in \mathcal{TERMS}(L) : l\tau \rhd u\tau \\ \vee \ \ \mathcal{V}(L) \subseteq \mathrm{V_{CONS}} \end{pmatrix} \end{pmatrix} \right).$$

Now, the following two are logically equivalent:

(1) $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

(2) All critical peaks in CP(R) *are (*⊳*-weakly) joinable w. r. t.* R, X*.*

LEMMA 7.19.
Let R *be a CRS over* sig/cons/V *and* T = (>, ⊳) *a termination-pair over* sig/V*. Let* X *be an enumerable subset of* V*. Now, if (This condition is essential: Cf. Lemma 7.2.)*

(1) R *is* X*-left-right-compatible with* T*,*

or

(2) R *is* X*-compatible with* T*,* $\rhd \cap (\mathcal{T}(\mathrm{sig},\mathrm{X}) \times \mathcal{T}(\mathrm{sig},\mathrm{X}))$ *is semi-decidable, and* $\rhd \cap (\mathcal{GT}(\mathrm{cons}) \times \mathcal{GT}(\mathrm{cons}))$ *is decidable,*

then the following statements hold:

(1) $\longrightarrow_{\mathrm{R,X}}$*-reducibility of terms from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is semi-decidable.*

(2) $\{\ t \mid s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t\ \}$ *is (universally*[†]*) enumerable for all* $s \in \mathcal{T}(\mathrm{sig},\mathrm{X})$*.*

(3) $\{\ t \mid s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t\ \}$ *is a finite computable set for all* $s \in \mathcal{GT}(\mathrm{cons})$*.*[‡]

The following lemma, however, shows that compatibility does not imply decidability of reducibility as long as extra-variables are permitted.

LEMMA 7.20. (REDUCIBILITY OF GROUND TERMS STILL NOT CO-SEMI-DECIDABLE)
There is a left-linear, non-overlapping, merely positive conditional rule system R *with noetherian and confluent reduction relation* $\longrightarrow_{\mathrm{R,V}}$*, which is* V*-don't-care-compatible with a termination-pair* (⊳, ⊳) *with decidable* ⊳*, and for which reducibility of ground terms is not co-semi-decidable.*

If we do not allow extra-variables, however, we get the following decidability result, which is important because (in combination with Corollary 7.4) it says that normal forms become computable.

LEMMA 7.21.
Let R *be a CRS over* sig/cons/V *and* T = (>, ⊳) *a termination-pair over* sig/V*. Let* X *be an enumerable subset of* V*. Now, if* R *is extra-variable free (This condition is essential: Cf. Lemma 7.20.) and if (This condition is essential: Cf. lemmas 7.1, 7.2.)*

(1) R *is* X*-left-right-compatible with* T*,*

or

(2) R *is* X*-compatible with* T *and* $\rhd \cap (\mathcal{T}(\mathrm{sig},\mathrm{X}) \times \mathcal{T}(\mathrm{sig},\mathrm{X}))$ *is decidable*

then the following statements hold:

(1) $\longrightarrow_{\mathrm{R,X}}$*-reducibility of terms from* $\mathcal{T}(\mathrm{sig},\mathrm{X})$ *is decidable.*

(2) $\{\ t \mid s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t\ \}$ *is a finite computable set for all* $s \in \mathcal{T}(\mathrm{sig},\mathrm{X})$*.*

(3) Confluence of $\longrightarrow_{\mathrm{R,X}}$ *is co-semi-decidable.*

[†] By this we want to express that there is not only for each $s \in \mathcal{T}(\mathrm{sig},\mathrm{X})$ some computable function which enumerates $\{\ t \mid s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t\ \}$ but even one single computable universal function which enumerates $\{\ t \mid s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t\ \}$ when its first argument (or index) is s.

[‡] I. e. there is some computable function f such that, for each $s \in \mathcal{GT}(\mathrm{cons})$, $f(s)$ is a list of exactly the elements of $\{\ t \mid s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} t\ \}$.

Confluence of $\longrightarrow_{\mathrm{R},\emptyset}$ (i. e. ground confluence) cannot be semi-decidable for extra-variable free, V-don't-care-compatible Def-MCRSs R because it is not semi-decidable even for extra-variable free, noetherian, left-linear, monadic, unconditional rule systems, cf. Kapur & al. (1990). While confluence of $\longrightarrow_{\mathrm{R,V}}$, however, is decidable for extra-variable free, noetherian, unconditional rule systems R (Note that each condition is essential here.), the following lemma does not give us a chance in general to decide confluence of $\longrightarrow_{\mathrm{R,V}}$ for extra-variable free, don't-care-compatible Def-MCRSs R.

LEMMA 7.22. (CONFLUENCE OF $\longrightarrow_{\mathrm{R,V}}$ IS NOT SEMI-DECIDABLE)
There is a signature sig *with sub-signature* cons *and a termination-pair* $(\rhd, \rhd)$ *over* sig/V *with decidable* $\rhd$*, such that confluence of* $\longrightarrow_{\mathrm{R,V}}$ *is not semi-decidable in general for left-linear, extra-variable free, merely positive conditional rule systems* R *over* sig/cons/V *which are* V*-don't-care-compatible with* $(\rhd, \rhd)$*.*

8. Conclusion

We have presented a novel constructor-based approach to positive/negative-conditional equational specifications, which was heavily inspired by previous work of Kapur & Musser (1987&1986) (for the case of unconditional equations only) and Zhang (1988). Under some reasonable restrictions on the syntactic form of positive/negative-conditional rules it turns out that the combination of these ideas with the approach of Kaplan (1988) becomes very fruitful and also relevant for practical purposes since many natural specifications involve both conditional equations with positive and negative conditions and partially specified functions. For such specifications we have been able to define semantics admitting a unique model, being initial in the class of constructor-minimal models, if (ground) confluence of our reduction relation is provided. The lack of an initial model is one of the main disadvantages of the approach of Kaplan (1988). The addition of constructor variables conceptually completes the constructor-based approach and (together with the positive and negative conditions of our equations) provides us with a unifying framework for the function specification style of classic inductive theorem proving on the one hand and for classic term rewriting on the other. Furthermore, a thorough and precise analysis of termination and decidability issues has led to some useful and slightly weakened "decreasingness"-notions for positive/negative-conditional rule systems. Finally, we have also been able to provide some interesting confluence criteria. Since (under reasonable assumptions) our reduction relation is monotonic w. r. t. consistent extension of the specification, the whole approach may be considered to be a firm theoretical basis for inductive validity and inductive theorem proving in theories specified by positive/negative-conditional equations.

Dedication:
To my father, Friedhelm Wirth, in gratefulness.

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A. The Proofs

Proof of Lemma 5.4

We claim the following:

(1) $\longrightarrow_{\mathrm{R,X},\omega}$ is the minimum among all relations satisfying the requirement of Lemma 5.4 with " $l \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}})$ " added into the second requirement.

(2) $\forall s \in \mathcal{GT}(\mathrm{cons}) : \forall t : (s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\omega} t \;\Rightarrow\; t \in \mathcal{GT}(\mathrm{cons}))$

(3) $\forall i \in \mathbb{N} : \forall n \in \mathbb{N} :\; (\overset{n}{\longrightarrow}_{\mathrm{R,X},\omega+i} \cap (\mathcal{GT}(\mathrm{cons}) \times \mathcal{T})) \;\subseteq\; \overset{n}{\longrightarrow}_{\mathrm{R,X},\omega}$

(4) $\forall i \in \mathbb{N} :\; \longrightarrow_{\mathrm{R,X},\omega+i} \;\subseteq\; \longrightarrow_{\mathrm{R,X},\omega+i+1}$

(5) $\longrightarrow_{\mathrm{R,X}}$ satisfies the requirement of Lemma 5.4; i. e. $\longrightarrow_{\mathrm{R,X}} \in S$.

(6) $\longrightarrow_{\mathrm{R,X},\omega} \;\subseteq\; \bigcap S$

(7) $\forall \rightsquigarrow \in S : \forall n \in \mathbb{N} : (\;\rightsquigarrow^{n} \;\cap\; (\mathcal{GT}(\mathrm{cons}) \times \mathcal{T})\;) \;\subseteq\; \overset{n}{\longrightarrow}_{\mathrm{R,X},\omega}$

(8) $\forall i \in \mathbb{N} :\; \longrightarrow_{\mathrm{R,X},\omega+i} \;\subseteq\; \bigcap S$

(9) $\longrightarrow_{\mathrm{R,X}}$ is the minimum $\bigcap S \in S$.

(10) For $\emptyset \neq S' \subseteq S :\; \bigcap S' \in S$.

To the proofs of these claims:

(1) By the restriction on the rules of Definition 5.1, $\longrightarrow_{\mathrm{R,X},\omega}$ is just the standard closure over a finitary relation.

(2) By the restriction on the rules of Definition 5.1.

(3) By induction on i using (1), (2), and the restriction on the rules of Definition 5.1.

(4) By induction on i using (3) and the restriction on the rules of Definition 5.1.

(5) The first requirement follows directly from (3). The second follows from (4), taking $\longrightarrow_{\mathrm{R,X},\omega+i+1}$ on the left-hand side of the second requirement for the maximum i of all $\longrightarrow_{\mathrm{R,X},\omega+i}$ occurring positively (i. e. not in a $\not\downarrow$-statement) in the fulfilledness condition (expanded via 5.2) of the right-hand side of the second requirement.

(6) By (1), $\longrightarrow_{\mathrm{R,X},\omega}$ is the intersection of a superset of S.

(7) By $(\;\rightsquigarrow \;\cap\; (\mathcal{GT}(\mathrm{cons}) \times \mathcal{T})\;) \;\subseteq\; \longrightarrow_{\mathrm{R,X},\omega}$ and (2).

(8) <u>$i = 0$:</u> By (6). <u>$i \Rightarrow (i+1)$:</u> By (6) and (7).

(9) By (5) and (8).

(10) By (7) and (6).

<u>Q. e. d. (**Proof of Lemma 5.4**)</u>

Proof of Theorem 5.14

Let $\mathcal{A} := \mathcal{T}(\mathrm{X}) / \overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}$. Let $\mathcal{I} := \mathcal{GT} / \overset{\oplus}{\longleftrightarrow}_{\mathrm{R},\emptyset}$.

<u>Claim 1</u>: If $\mathcal{C}$ is a sig/cons-model of R; $\mu \in \mathcal{SUB}(\mathrm{X}, \mathcal{C})$; then

$$\forall \beta \preceq \omega : \forall s \in \mathrm{S} :\; \longrightarrow_{\mathrm{R,X},\beta} \cap\, (\mathcal{T}_{\mathrm{SIG},s} \times \mathcal{T}_{\mathrm{SIG},s}) \;\subseteq\; \mathrm{ker}(\mathcal{C}_{\mu})_{s} \;.$$

The proof of Claim 1 is omitted because it reads just like the proof of Claim 2 until it comes to $\beta \succ \omega$.

<u>Proof of *(1)*</u>: By the Axiom of Choice, each element of $\mathcal{SUB}(\mathrm{V}, \mathcal{A})$ can be written $\sigma \circ \mathcal{A}_{\kappa}$ for some $\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$. Thus (by the Substitution-Lemma(2.2) and $\mathrm{X} \subseteq \mathrm{V_{SIG}}$), for $\mathcal{A}$ being a sig/cons-model of R it is sufficient to note that (by confluence of $\longrightarrow_{\mathrm{R,X}}[\cap(D_{\mathrm{X}} \times D_{\mathrm{X}})]$, the fact that R is a Def-**M**CRS, and Lemma 5.8) for $((l,r),C) \in \mathrm{R}$; $\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$:

$C\sigma$ is fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X}}$ iff

$$\forall u,v\in\mathcal{T}:\left(\begin{array}{llr}(\,(\,(u{=}v\,)\text{ in }C\sigma) & \Rightarrow & u\overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}v\,)\quad\wedge\\ (\,(\,(\mathrm{Def}\,u)\text{ in }C\sigma) & \Rightarrow\ \exists\hat{u}\in\mathcal{GT}(\mathrm{cons}): & u\overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}\hat{u}\,)\quad\wedge\\ (\,(\,(u{\neq}v\,)\text{ in }C\sigma) & \Rightarrow & u\overset{\oplus}{\not\longleftrightarrow}_{\mathrm{R,X}}v\,)\end{array}\right).$$

For the proof of $\mathcal{A}$ being a *constructor-minimum* model, suppose $\mathcal{C}$ to be a sig/cons-model of R. We have to find a cons-homomorphism from $\mathcal{A}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}$ to $\mathcal{C}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}$. Let $\mathcal{B}$ be the ground term algebra over cons. There is a cons-homomorphism $h{::}\mathcal{A}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}\to(\mathcal{B}/(\overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}\cap(\mathcal{GT}(\mathrm{cons})\times\mathcal{GT}(\mathrm{cons}))))$ given by $(s\in\mathrm{S};\,A\in\mathcal{A}(\mathrm{CONS},s))$: $A\ \mapsto\ \mathcal{GT}(\mathrm{cons})\cap A$. Thus we only have to find a cons-homomorphism from $\mathcal{B}/(\overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}\cap(\mathcal{GT}(\mathrm{cons})\times\mathcal{GT}(\mathrm{cons})))$ to $\mathcal{C}|_{\mathrm{C}\uplus(\{\mathrm{CONS}\}\times\mathrm{S})}$. Using the Homomorphism-Theorem (the usual one for cons-homomorphisms, not ours) all we have to show is $\forall s\in\mathrm{S}:\ \overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}\cap(\mathcal{GT}_{\mathrm{CONS},s}\times\mathcal{GT}_{\mathrm{CONS},s})\ \subseteq\ \ker(\mathcal{C})_s$, which by confluence of $\longrightarrow_{\mathrm{R,X}}[\cap(D_{\mathrm{X}}\times D_{\mathrm{X}})]$ is the same as $\forall s\in\mathrm{S}:\ \downarrow_{\mathrm{R,X}}\cap(\mathcal{GT}_{\mathrm{CONS},s}\times\mathcal{GT}_{\mathrm{CONS},s})\ \subseteq\ \ker(\mathcal{C})_s$. Because of Lemma 5.8, $\forall s\in\mathrm{S}:\ \longrightarrow_{\mathrm{R,X},\omega}\cap(\mathcal{GT}_{\mathrm{CONS},s}\times\mathcal{GT}_{\mathrm{CONS},s})\ \subseteq\ \ker(\mathcal{C})_s$ is sufficient for this. But this is implied by Claim 1. Q. e. d. *((1))*

Claim 2: If $\mathcal{C}\in\mathrm{K}$; $\mu\in\mathcal{SUB}(\mathrm{X},\mathcal{C})$; $\longrightarrow_{\mathrm{R},\emptyset}[\cap(D_\emptyset\times D_\emptyset)]$ is confluent; then

$$\forall\beta\preceq\omega{+}\omega:\forall s\in\mathrm{S}:\ \longrightarrow_{\mathrm{R,X},\beta}\cap(\mathcal{T}_{\mathrm{SIG},s}\times\mathcal{T}_{\mathrm{SIG},s})\ \subseteq\ \ker(\mathcal{C}_\mu)_s\ .$$

Proof of Claim 2: For the limit ordinals 0, ω, $\omega{+}\omega$ the induction step is trivial. For a non-limit ordinal $\beta{+}1$ the induction step is as follows: Suppose $s\longrightarrow_{\mathrm{R,X},\beta+1}t$. If $\omega\preceq\beta$ and $s\longrightarrow_{\mathrm{R,X},\omega}t$, then we already have $\mathcal{C}_\mu(s)=\mathcal{C}_\mu(t)$. Otherwise there are $((l,r),C)\in\mathrm{R}$; $\sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; $p\in\mathcal{POS}(s)$; with $s/p=l\sigma$; $t=s[p\leftarrow r\sigma]$; and $C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X},\beta}$. As $\mathcal{C}$ is a sig/cons-model of R, for inferring $\mathcal{C}_\mu(s)=\mathcal{C}_\mu(t)$ (by the Substitution-Lemma(2.2)) we only have to show that the condition $C\sigma$ is true w. r. t. $\mathcal{C}_\mu$. Three cases for L in $C\sigma$: If $L=(u{=}v)$, by $u\downarrow_{\mathrm{R,X},\beta}v$ the induction hypothesis implies $\mathcal{C}_\mu(u)=\mathcal{C}_\mu(v)$. If $L=(\mathrm{Def}\,u)$ with $u\in\mathcal{T}_{\mathrm{SIG},s'}$, by the existence of some $\hat{u}\in\mathcal{GT}_{\mathrm{CONS},s'}$ with $u\overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\beta}\hat{u}$ the induction hypothesis implies $\mathcal{C}_\mu(u)=\mathcal{C}_\mu(\hat{u})\in\mathcal{C}(\mathrm{CONS},s')$. If $L=(u{\neq}v)$, by the existence of some $\hat{u},\hat{v}\in\mathcal{GT}(\mathrm{cons})$ with $u\overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\beta}\hat{u}\not\downarrow_{\mathrm{R,X},\beta}\hat{v}\overset{\oplus}{\longleftarrow}_{\mathrm{R,X},\beta}v$ the induction hypothesis implies $\mathcal{C}_\mu(u)=\mathcal{C}_\mu(\hat{u})=\mathcal{C}(\hat{u})$ and $\mathcal{C}(\hat{v})=\mathcal{C}_\mu(\hat{v})=\mathcal{C}_\mu(v)$. Thus, for $\mathcal{C}_\mu(u)\neq\mathcal{C}_\mu(v)$ it is sufficient to show $\mathcal{C}(\hat{u})\neq\mathcal{C}(\hat{v})$. By $\omega\preceq\beta$ and Lemma 5.9 we know that $\hat{u}\not\downarrow_{\mathrm{R,X}}\hat{v}$; then by Lemma 5.12 $\hat{u}\not\downarrow_{\mathrm{R},\emptyset}\hat{v}$; and therefore (by confluence of $\longrightarrow_{\mathrm{R},\emptyset}[\cap(D_\emptyset\times D_\emptyset)]$) $\hat{u}\overset{\oplus}{\not\longleftrightarrow}_{\mathrm{R},\emptyset}\hat{v}$. Because of *(1)* (for $\mathrm{X}:=\emptyset$), $\mathcal{C}$ must be not only a constructor-minimal model but also a constructor-minimum model of R. Hence $\mathcal{C}(\hat{u})\neq\mathcal{C}(\hat{v})$ simply because $\hat{u},\hat{v}\in\mathcal{GT}(\mathrm{cons})$; $\mathcal{I}(\hat{u})\neq\mathcal{I}(\hat{v})$; and $\mathcal{I}$ is (by *(1)*) a sig/cons-model of R. Q. e. d. (Claim 2)

Claim 3: If $\longrightarrow_{\mathrm{R},\emptyset}[\cap(D_\emptyset\times D_\emptyset)]$ is confluent, then $\mathcal{A}$ is free for K over X w. r. t. κ.

Proof of Claim 3: Suppose $\mathcal{C}\in\mathrm{K}$ and μ to be a $\mathcal{C}$-valuation of X. The uniqueness of the required sig/cons-homomorphism $h{::}\mathcal{A}{\to}\mathcal{C}$ with $\mu=\kappa h$ is trivial. For its existence (by the Homomorphism-Theorem(2.3)) we only have to show

$$\forall s\in\mathrm{S}:\ \overset{\oplus}{\longleftrightarrow}_{\mathrm{R,X}}\cap(\mathcal{T}_{\mathrm{SIG},s}\times\mathcal{T}_{\mathrm{SIG},s})\ \subseteq\ \ker(\mathcal{C}_\mu)_s\ ,$$

which is implied by Claim 2. Q. e. d. (Claim 3)

Claim 4: If $\longrightarrow_{\mathrm{R,X}}[\cap(D_{\mathrm{X}}\times D_{\mathrm{X}})]$ is confluent, then $\longrightarrow_{\mathrm{R},\emptyset}[\cap(D_\emptyset\times D_\emptyset)]$ is confluent, too.

Proof of Claim 4: Trivial by Lemma 5.12. Q. e. d. (Claim 4)

Proof of *(2)*: Since $\mathcal{A}\in\mathrm{K}$ by *(1)*, this is implied by the claims 4 and 3.

Proof of *(3)*: Suppose $\mathcal{C}$ to be a sig/cons-model of R with $\mathcal{C}\cdot_{\mathrm{H}}\mathcal{A}$. By *(1)* we get $\mathcal{C}\in\mathrm{K}$, and then by *(2)* $\mathcal{A}\cdot_{\mathrm{H}}\mathcal{C}$. Q. e. d. (**Proof of Theorem 5.14**)

Proof of Theorem 5.16
First note that the remark below the theorem is respected during the whole proof.
Claim 1: $\forall i \in \mathbb{N}\colon \forall n \in \mathbb{N}\colon \forall s \in \mathcal{T}(\mathrm{cons},\mathrm{X})\colon \forall t\colon$

$$\left(s \xrightarrow{n}_{\mathrm{R}',\mathrm{X}',i} t \;\Rightarrow\; (s \xrightarrow{n}_{\mathrm{R},\mathrm{X},i} t \in \mathcal{T}(\mathrm{cons},\mathrm{X}))\right)$$

Claim 2: $\forall\beta \preceq \omega{+}\omega : \;\longrightarrow_{\mathrm{R},\mathrm{X},\beta} \;\subseteq\; \longrightarrow_{\mathrm{R}',\mathrm{X}',\beta}$
Claim 2 and Claim 1 (using Lemma 5.8 and $\mathcal{T}(\mathrm{cons},\mathrm{X}) \subseteq \mathcal{T}(\mathrm{cons}',\mathrm{X}')$) imply *(1)*; and Claim 2 implies *(2)* and *(3)*.
Proof of Claim 1: $i = 0$: $\longrightarrow_{\mathrm{R}',\mathrm{X}',0} = \emptyset$. $i \Rightarrow (i+1)$: $n = 0$: Trivial.
$n \Rightarrow (n+1)$: Suppose $s \xrightarrow{n}_{\mathrm{R}',\mathrm{X}',i+1} s' \longrightarrow_{\mathrm{R}',\mathrm{X}',i+1} t$. By induction hypothesis in n we know $s \xrightarrow{n}_{\mathrm{R},\mathrm{X},i+1} s' \in \mathcal{T}(\mathrm{cons},\mathrm{X})$. By (\$) there must be some $l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}} \uplus \mathrm{V_{CONS}})$; $((l,r),C) \in \mathrm{R}$; $\sigma \in \mathcal{SUB}(\mathcal{V}(l),\mathcal{T})$; $p \in \mathcal{POS}(s')$ with $s'/p = l\sigma$; $t = s'[p \leftarrow r\sigma]$; $C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R}',\mathrm{X}',i}$; $\forall x \in \mathcal{V}(l) : x\sigma \in \mathcal{T}(\mathrm{cons},\mathrm{X})$. By the structure of constructor equations we have no inequality literal in $C\sigma$. For $(u{=}v)$ in $C\sigma$ we have $u, v \in \mathcal{T}(\mathrm{cons},\mathrm{X})$ and $u{\downarrow_{\mathrm{R}',\mathrm{X}',i}}v$ and therefore by induction hypothesis in i: $u{\downarrow_{\mathrm{R},\mathrm{X},i}}v$. For $(\mathrm{Def}\,u)$ in $C\sigma$ we have $u \in \mathcal{T}(\mathrm{cons},\mathrm{X})$ and $u \xrightarrow{\oplus}_{\mathrm{R}',\mathrm{X}',i} \hat{u}$ for some $\hat{u} \in \mathcal{GT}(\mathrm{cons}')$ and hence by induction hypothesis in i: $u \xrightarrow{\oplus}_{\mathrm{R},\mathrm{X},i} \hat{u} \in (\mathcal{T}(\mathrm{cons},\mathrm{X}) \cap \mathcal{GT}(\mathrm{cons}')) = \mathcal{GT}(\mathrm{cons})$. Thus, finally we conclude that $C\sigma$ is fulfilled w. r. t. $\longrightarrow_{\mathrm{R},\mathrm{X},i}$, i. e. $s' \longrightarrow_{\mathrm{R},\mathrm{X},i+1} t \in \mathcal{T}(\mathrm{cons},\mathrm{X})$.
Q. e. d. (Claim 1)
Proof of Claim 2: The induction step for the limit ordinals 0, ω, and $\omega+\omega$ is trivial. For a non-limit ordinal $\beta+1$ the induction step is as follows: Suppose $s \longrightarrow_{\mathrm{R},\mathrm{X},\beta+1} t$. In the case of $\omega \preceq \beta$ this may be due to $s \longrightarrow_{\mathrm{R},\mathrm{X},\omega} t$; but then by induction hypothesis we succeed by $\longrightarrow_{\mathrm{R},\mathrm{X},\omega} \;\subseteq\; \longrightarrow_{\mathrm{R}',\mathrm{X}',\omega} \;\subseteq\; \longrightarrow_{\mathrm{R}',\mathrm{X}',\beta+1}$. Otherwise there must be some $((l,r),C) \in \mathrm{R}$; $\sigma \in \mathcal{SUB}(\mathrm{V},\mathcal{T})$; $p \in \mathcal{POS}(s)$ with $s/p = l\sigma$; $t = s[p \leftarrow r\sigma]$; $C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R},\mathrm{X},\beta}$. The only thing to be shown is: $C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R}',\mathrm{X}',\beta}$. For $(u{=}v)$ in $C\sigma$ we have $u{\downarrow_{\mathrm{R},\mathrm{X},\beta}}v$ and therefore by induction hypothesis $u{\downarrow_{\mathrm{R}',\mathrm{X}',\beta}}v$. For $(\mathrm{Def}\,u)$ in $C\sigma$ we have $u \xrightarrow{\oplus}_{\mathrm{R},\mathrm{X},\beta} \hat{u}$ for some $\hat{u} \in \mathcal{GT}(\mathrm{cons})$ and therefore by induction hypothesis $u \xrightarrow{\oplus}_{\mathrm{R}',\mathrm{X}',\beta} \hat{u}$. For $(u{\neq}v)$ in $C\sigma$ we have $u \xrightarrow{\oplus}_{\mathrm{R},\mathrm{X},\beta} \hat{u} \not\downarrow_{\mathrm{R},\mathrm{X},\beta} \hat{v} \xleftarrow{\oplus}_{\mathrm{R},\mathrm{X},\beta} v$ for some $\hat{u}, \hat{v} \in \mathcal{GT}(\underline{\mathrm{cons}})$ and $\omega \preceq \beta$. We get $\hat{u} \not\downarrow_{\mathrm{R},\mathrm{X},\omega} \hat{v}$, and then (by Claim 1) $\hat{u} \not\downarrow_{\mathrm{R}',\mathrm{X}',\omega} \hat{v}$ and (by Lemma 5.9) $\hat{u} \not\downarrow_{\mathrm{R}',\mathrm{X}'} \hat{v}$. Finally, by induction hypothesis we get $u \xrightarrow{\oplus}_{\mathrm{R}',\mathrm{X}',\beta} \hat{u} \not\downarrow_{\mathrm{R}',\mathrm{X}',\beta} \hat{v} \xleftarrow{\oplus}_{\mathrm{R}',\mathrm{X}',\beta} v$.

Proof of Theorem 6.3
For the proof of Sub-claim 3 below, we enrich the signatures by a new sort s_{new} and new constructor symbols $\mathsf{eq}_{\bar{s}}$ for each old sort $\bar{s} \in \mathrm{S}$ with arity $\bar{s}\bar{s} \longrightarrow s_{\mathrm{new}}$ and $-$ with arity s_{new}. We take (in addition to R) the following set of new rules (with $X_{\bar{s}} \in \mathrm{V}_{\mathrm{SIG},\bar{s}}$ for $\bar{s} \in \mathrm{S}$):

$$\mathrm{R}' := \{\ \mathsf{eq}_{\bar{s}}(X_{\bar{s}},X_{\bar{s}}) = -\ \mid\ \bar{s} \in \mathrm{S}\ \}$$

Since the sort restrictions do not allow $\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X},\beta}$ to make any use of terms of the sort s_{new} when rewriting terms of an "old" sort, we get

$$\forall\beta \preceq \omega{+}\omega : \quad \longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X},\beta} \cap\, (\mathcal{T}(\mathrm{sig},\mathrm{X}) \times \mathcal{T}(\mathrm{sig},\mathrm{X})) \;=\; \longrightarrow_{\mathrm{R},\mathrm{X},\beta/\mathrm{sig}/\mathrm{cons}}$$

(the latter being defined over the non-enriched signatures). Therefore (as no new critical peaks occur) the critical peaks keep being quasi overlay joinable. We are going to show confluence of $\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X}}$, which implies confluence of $\longrightarrow_{\mathrm{R},\mathrm{X}/\mathrm{sig}/\mathrm{cons}}$. We know that $\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X}}$ is noetherian on each "old" sort; and by this we also know $\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X}}$ to be noetherian on $s_{\rm new}$ because $-$ is irreducible and terms of the form "$\mathsf{eq}_{\bar{s}}(u,v)$" allow at most one reduction step via $\longrightarrow_{\rm R\cup R',X} \setminus \longrightarrow_{\rm R,X/sig'/cons'}$.

We define $\longrightarrow_{0} := \longrightarrow_{\rm R',X}$; $\longrightarrow_{\beta} := \longrightarrow_{\rm R\cup R',X,\beta}$ for any ordinal β with $0 \prec \beta \prec \omega{+}\omega$ (where $\prec$ is the ordering of ordinal numbers); and $\longrightarrow := \longrightarrow_{\omega+\omega} := \longrightarrow_{\rm R\cup R',X}$.

For $v,u,s,t \in \mathcal{T}$ with $v \overset{\oplus}{\longleftarrow} u$ and $s \overset{\oplus}{\longrightarrow} t$; $\Pi \subseteq \mathcal{POS}(u)$ with $\forall p,q \in \Pi : (p \neq q \Rightarrow p|q)$ and $\forall o \in \Pi : u/o = s$; we say that $P(v,u,s,t,\Pi)$ holds :iff $v \downarrow u[\,o \leftarrow t \mid o \in \Pi\,]$. Now (by $\Pi := \{\emptyset\}$) it suffices to show that $P(v,u,s,t,\Pi)$ holds for all appropriate v,u,s,t,Π. We will show this by noetherian induction over the lexicographic combination of the following orderings:

1. $(\longrightarrow \cup \rhd_{\rm ST})^{\oplus}$ (Cf. Lemma 7.6(1))
2. $\succ$
3. $\succ$

using the following measure on (v,u,s,t,Π):

1. s
2. the smallest ordinal $\beta \preceq \omega{+}\omega$ for which $v \overset{\oplus}{\longleftarrow}_{\beta} u$
3. the smallest $n \in \mathbb{N}$ for which $v \overset{n}{\longleftarrow}_{\beta} u$ for the β of (2)

For the limit ordinals 0, ω, $\omega{+}\omega$ in the second position of the measure, the induction step is trivial ($\overset{\oplus}{\longleftarrow}_{0} \subseteq \longleftarrow_{0} \cup \mathrm{id}$; $\overset{\oplus}{\longleftarrow}_{\omega} \subseteq \bigcup_{i\in\mathbb{N}\setminus\{0\}} \overset{\oplus}{\longleftarrow}_{i}$; $\overset{\oplus}{\longleftarrow}_{\omega+\omega} \subseteq \bigcup_{i\in\mathbb{N}} \overset{\oplus}{\longleftarrow}_{\omega+i}$). Thus, as we now suppose a smallest (v,u,s,t,Π) with $P(v,u,s,t,\Pi)$ not holding for, the second position of the measure must be a non-limit ordinal $\beta{+}1$.

As $P(v,u,s,t,\Pi)$ holds trivially for $u = v$ or $s = t$ we have some u', s' with $v \overset{n}{\longleftarrow}_{\beta+1} u' \longleftarrow_{\beta+1} u$ $(n \in \mathbb{N})$ (with $\forall m \in \mathbb{N} : (v \overset{m}{\longleftarrow}_{\beta+1} u \;\Rightarrow\; m > n)$) and $s \longrightarrow s' \overset{\oplus}{\longrightarrow} t$. Now for a contradiction it is sufficient to show

<u>**Claim**</u>: There is some z with $v \overset{\oplus}{\longrightarrow} z \overset{\oplus}{\longleftarrow} u[\,o \leftarrow s' \mid o \in \Pi\,]$.

because then we have $z \downarrow u[\,o \leftarrow t \mid o \in \Pi\,]$ by $P(z, u[\,o \leftarrow s' \mid o \in \Pi\,], s', t, \Pi)$, which is smaller than (v,u,s,t,Π) in the first position of the measure by $s \longrightarrow s'$.

If we had $s \longrightarrow s'$ by some redex below the top of s, then **Claim** would hold by induction hypothesis with $P(\ldots)$ being smaller in the first position using $\rhd_{\rm ST}$. Thus, there are some $((l_0,r_0),C_0) \in \mathrm{R}\cup\mathrm{R}'$; $\mu_0 \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $s = l_0\mu_0$; $s' = r_0\mu_0$; and (by Lemma 5.10) $C_0\mu_0$ is fulfilled w. r. t. $\longrightarrow$. Furthermore, we have some $q \in \mathcal{POS}(u)$; $((l_1,r_1),C_1) \in \mathrm{R}\cup\mathrm{R}'$; $\mu_1 \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $u/q = l_1\mu_1$; $u' = u[\,q \leftarrow r_1\mu_1\,]$; if C_1 contains some inequality $(u{\neq}v)$ then $\omega \preceq \beta$; and (by Lemma 5.10) $C_1\mu_1$ fulfilled w. r. t. $\longrightarrow_{\beta}$. We now distinguish two cases by the relative position of q and Π:

<u>"q not strictly below any $p \in \Pi$"</u>: There is no p' with $pp' = q$, $p' \neq \emptyset$, and $p \in \Pi$

Define $\Pi' := \{\, p \mid qp \in \Pi \,\}$.

<u>The critical peak case:</u> There is some $p \in \Pi' \cap \mathcal{POS}(l_1)$ with $l_1/p \notin \mathrm{V}$.

<u>Sub-claim 1:</u> There is some w with $s\ (\longrightarrow \cup \rhd_{\rm ST})^{\oplus}\ r_1\mu_1/p \overset{\oplus}{\longrightarrow} w \overset{\oplus}{\longleftarrow} s'$ and $r_1\mu_1 = l_1\mu_1[\,p \leftarrow r_1\mu_1/p\,]$.

<u>Proof of Sub-claim 1</u>: Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)] \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$. Let ϱ be given by $(x \in \mathrm{V})$: $x\varrho := \left\{\begin{matrix} x\mu_1 & \text{if } x \in \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) \\ x\xi^{-1}\mu_0 & \text{otherwise}\end{matrix}\right\}$.

By $l_0\xi\varrho = l_0\xi\xi^{-1}\mu_0 = s = u/qp = l_1\mu_1/p = (l_1/p)\varrho$ let $\mathrm{Y} := \mathcal{V}((l_0{=}r_0{\longleftarrow}C_0)\xi, l_1{=}r_1{\longleftarrow}C_1)$, $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/p)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $(\sigma\varphi)|_{\mathrm{Y}} = \varrho|_{\mathrm{Y}}$. Let $(((t_0,t_1),D),\hat{t}) := (((l_1[\,p \leftarrow r_0\xi\,], r_1), C_0\xi\, C_1), l_1)\sigma$. Now we have $(((t_0,t_1),D),\hat{t})\varphi = (((l_1\mu_1[\,p \leftarrow r_0\mu_0\,], r_1\mu_1), C_0\mu_0\, C_1\mu_1), l_1\mu_1)$. Therefore $D\varphi$ is fulfilled by Lemma 5.10.

If $t_0 = t_1$, then $s \longrightarrow s' = r_0\mu_0 = t_0\varphi/p = t_1\varphi/p = r_1\mu_1/p$; and $r_1\mu_1 = r_1\mu_1[p \leftarrow r_1\mu_1/p] = t_1\varphi[p \leftarrow r_1\mu_1/p] = t_0\varphi[p \leftarrow r_1\mu_1/p] = l_1\mu_1[p \leftarrow r_1\mu_1/p]$. Otherwise, if $t_0 \neq t_1$, $(((t_0,t_1),D),\hat{t},p)$ is a critical peak and we get $s = u/qp = l_1\mu_1/p = \hat{t}\varphi/p \ (\longrightarrow \cup \rhd_{\mathrm{ST}})^{\oplus} \ t_1\varphi/p \downarrow (t_0/p)\varphi = r_0\mu_0 = s'$ where $t_1\varphi/p = r_1\mu_1/p$; and $r_1\mu_1 = t_1\varphi = t_0\varphi[p \leftarrow t_1\varphi/p] = l_1\mu_1[p \leftarrow r_1\mu_1/p]$. Q. e. d. (Sub-claim 1)

Sub-claim 2: For $o \in \Pi \setminus \{qp\}$ we have $u'/o = s$.

Proof of Sub-claim 2: If there is some p' with $o = qp'$, then by Sub-claim 1: $u'/o = u'/qp' = r_1\mu_1/p' = l_1\mu_1[p \leftarrow \ldots]/p' = l_1\mu_1/p' = u/qp' = u/o = s$. Otherwise, by $qp \in \Pi$ we know $o|q$, and therefore $u'/o = u/o = s$. Q. e. d. (Sub-claim 2)

By Sub-claim 1 we get $u'[qp \leftarrow w] = u[q \leftarrow r_1\mu_1[p \leftarrow w]] = u[q \leftarrow l_1\mu_1[p \leftarrow w]] = u[qp \leftarrow w]$. Hence, for $\hat{u} := u'[qp \leftarrow w][o \leftarrow s' \mid o \in \Pi \setminus \{qp\}]$ we get $\hat{u} = u[qp \leftarrow w][o \leftarrow s' \mid o \in \Pi \setminus \{qp\}] \overset{\oplus}{\longleftarrow} u[o \leftarrow s' \mid o \in \Pi]$ by $w \overset{\oplus}{\longleftarrow} s'$ (by Sub-claim 1). Thus, for **Claim** we only have to show $v \downarrow \hat{u}$. For $u'' := u'[o \leftarrow s' \mid o \in \Pi \setminus \{qp\}]$ (cf. Sub-claim 2) there is some w' with $v \overset{\oplus}{\longrightarrow} w' \overset{\oplus}{\longleftarrow} u''$ by Sub-claim 2 and $P(v, u', s, s', \Pi \setminus \{qp\})$, which is smaller in the second or third position. Finally, we get $w' \downarrow \hat{u}$ by $u''/qp = u'/qp = r_1\mu_1/p$, Sub-claim 1, and $P(w', u'', r_1\mu_1/p, w, \{qp\})$, which is smaller in the first position by Sub-claim 1. Q. e. d. (The critical peak case)

The variable overlap (if any) case: $\forall p \in \Pi' \cap \mathcal{POS}(l_1) : l_1/p \in \mathrm{V}$

Define a function ? on V by $(x \in \mathrm{V})$: $?(x) := \{ p' \mid \exists p : (l_1/p = x \ \wedge \ pp' \in \Pi') \}$. Define μ_1' by $(x \in \mathrm{V})$: $x\mu_1' := x\mu_1[p' \leftarrow s' \mid p' \in ?(x)]$. Define $\hat{u} := u[q \leftarrow r_1\mu_1'][o \leftarrow s' \mid o \in \Pi \setminus (q\Pi')]$ and $\check{u} := u[q \leftarrow l_1\mu_1'][o \leftarrow s' \mid o \in \Pi \setminus (q\Pi')]$.
We are going to show $v \downarrow \hat{u} \longleftarrow \check{u} \overset{\oplus}{\longleftarrow} u[o \leftarrow s' \mid o \in \Pi]$ for **Claim**. Since for $p' \in ?(x)$ we always have some p with $l_1/p = x$; $x\mu_1/p' = l_1\mu_1/pp' = u/qpp' = s \longrightarrow s'$; we get $v \downarrow \hat{u}$ by $P(v, u', s, s', (\Pi \setminus (q\Pi')) \cup \{ qpp' \mid \exists x : (r_1/p = x \ \wedge \ p' \in ?(x)) \})$, which is smaller in the second or third position. We get $\hat{u} \longleftarrow \check{u}$ by Lemma 5.4 and

Sub-claim 3: $C_1\mu_1'$ is fulfilled.

Finally, $\check{u} \overset{\oplus}{\longleftarrow} u[q \leftarrow l_1\mu_1[o \leftarrow s' \mid o \in \Pi']][o \leftarrow s' \mid o \in \Pi \setminus (q\Pi')] = u[o \leftarrow s' \mid o \in \Pi]$.

Proof of Sub-claim 3: For $(\bar{u}{=}\bar{v})$ in C_1 we have $\bar{u}\mu_1 {\downarrow_\beta} \bar{v}\mu_1$ and hence for the sort $\bar{s}$ of $\bar{u}$: $- \overset{\oplus}{\longleftarrow}_{\beta} (\mathsf{eq}_{\bar{s}}(\bar{u},\bar{v}))\mu_1$. We get $- \downarrow (\mathsf{eq}_{\bar{s}}(\bar{u},\bar{v}))\mu_1'$ by $P(-,\ (\mathsf{eq}_{\bar{s}}(\bar{u},\bar{v}))\mu_1,\ s,\ s'\ , \{ pp' \mid \exists x : ((\mathsf{eq}_{\bar{s}}(\bar{u},\bar{v}))/p = x \ \wedge \ p' \in ?(x)) \})$, which is smaller in the second position. Since there are no rules for $-$ and only one for $\mathsf{eq}_{\bar{s}}$, this means $\bar{u}\mu_1' \downarrow \bar{v}\mu_1'$. For (Def $\bar{u}$) in C_1 we know the existence of some $\vec{\bar{u}} \in \mathcal{GT}(\mathrm{cons})$ with $\vec{\bar{u}} \overset{\oplus}{\longleftarrow}_{\beta} \bar{u}\mu_1$. We get some $\hat{\bar{u}}$ with $\vec{\bar{u}} \overset{\oplus}{\longrightarrow} \hat{\bar{u}} \overset{\oplus}{\longleftarrow} \bar{u}\mu_1'$ by $P(\vec{\bar{u}}, \bar{u}\mu_1, s, s', \{ pp' \mid \exists x : (\bar{u}/p = x \ \wedge \ p' \in ?(x)) \})$, which is smaller in the second position. By Lemma 5.8 we get $\hat{\bar{u}} \in \mathcal{GT}(\mathrm{cons})$. Finally, for $(\bar{u}{\neq}\bar{v})$ in C_1 we have some $\vec{\bar{u}}, \vec{\bar{v}} \in \mathcal{GT}(\mathrm{cons})$ with $\bar{u}\mu_1 \overset{\oplus}{\longrightarrow}_{\beta} \vec{\bar{u}} \not\downarrow \vec{\bar{v}} \overset{\oplus}{\longleftarrow}_{\beta} \bar{v}\mu_1$ (by Lemma 5.9 and $\omega \preceq \beta$). By applying the same procedure as before twice we get $\hat{\bar{u}}, \hat{\bar{v}} \in \mathcal{GT}(\mathrm{cons})$ with $\bar{u}\mu_1' \overset{\oplus}{\longrightarrow} \hat{\bar{u}} \overset{\oplus}{\longleftarrow} \vec{\bar{u}} \not\downarrow \vec{\bar{v}} \overset{\oplus}{\longrightarrow} \hat{\bar{v}} \overset{\oplus}{\longleftarrow} \bar{v}\mu_1'$, i. e. $\bar{u}\mu_1' \overset{\oplus}{\longrightarrow} \hat{\bar{u}} \not\downarrow \hat{\bar{v}} \overset{\oplus}{\longleftarrow} \bar{v}\mu_1'$. Q. e. d. (Sub-claim 3)

Q. e. d. (The variable overlap (if any) case)

Q. e. d. ("q not strictly below any $p \in \Pi$")

"q strictly below $p \in \Pi$": There is some p' with $pp' = q$, $p' \neq \emptyset$, and $p \in \Pi$

Sub-claim 4: For $o \in \Pi \setminus \{p\}$ we have $u'/o = s = l_0\mu_0$.

Proof of Sub-claim 4: Since $o|p$, we have $u'/o = u[pp' \leftarrow \ldots]/o = u/o = s = l_0\mu_0$.

Q. e. d. (Sub-claim 4)

The (second) critical peak case: $p' \in \mathcal{POS}(l_0) \ \wedge\ l_0/p' \notin \mathrm{V}$

Sub-claim 5: There is some w with $s \ (\longrightarrow \cup \rhd_{\mathrm{ST}})^{\oplus} \ s[\,p' \leftarrow r_1\mu_1\,] \overset{\oplus}{\longrightarrow} w \overset{\oplus}{\longleftarrow} s'$.
Proof of Sub-claim 5: Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)] \cap \mathcal{V}(l_0{=}r_0{\longleftarrow}C_0) = \emptyset$. Let ϱ be given by $(x \in \mathrm{V})$: $x\varrho := \begin{Bmatrix} x\mu_0 & \text{if } x \in \mathcal{V}(l_0{=}r_0{\longleftarrow}C_0) \\ x\xi^{-1}\mu_1 & \text{otherwise} \end{Bmatrix}$. By $l_1\xi\varrho = l_1\xi\xi^{-1}\mu_1 = u/q = u/pp' = l_0\mu_0/p' = (l_0/p')\varrho$ let $\mathrm{Y} := \mathcal{V}((l_1{=}r_1{\longleftarrow}C_1)\xi, l_0{=}r_0{\longleftarrow}C_0)$, $\sigma := \mathrm{mgu}(\{(l_1\xi, l_0/p')\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $(\sigma\varphi)|_{\mathrm{Y}} = \varrho|_{\mathrm{Y}}$. Let $(((t_0,t_1),D),\hat{t}) := (((l_0[\,p' \leftarrow r_1\xi\,], r_0), C_1\xi\, C_0), l_0)\sigma$. Now we have $(((t_0,t_1),D),\hat{t})\varphi = (((l_0\mu_0[\,p' \leftarrow r_1\mu_1\,], r_0\mu_0), C_1\mu_1\, C_0\mu_0), l_0\mu_0)$. Therefore $D\varphi$ is fulfilled by Lemma 5.10. Since $s/p' = u/pp' = u/q = l_1\mu_1$ we have $s \longrightarrow s[\,p' \leftarrow r_1\mu_1\,]$. Because $s[\,p' \leftarrow r_1\mu_1\,] = l_0\mu_0[\,p' \leftarrow r_1\mu_1\,] = t_0\varphi$ and $t_1\varphi = r_0\mu_0 = s'$ we now only have to show $t_0\varphi \downarrow t_1\varphi$. If $t_0 = t_1$, this is trivial. Otherwise, if $t_0 \neq t_1$, $(((t_0,t_1),D),\hat{t},p')$ is a critical peak and we get $t_0\varphi \downarrow t_1\varphi$ by Lemma 6.2. Q. e. d. (Sub-claim 5)

For $\hat{u} := u[\,p \leftarrow w\,][\,o \leftarrow s' \mid o \in \Pi \setminus \{p\}\,]$ we get $\hat{u} \overset{\oplus}{\longleftarrow} u[\,o \leftarrow s' \mid o \in \Pi\,]$ by $w \overset{\oplus}{\longleftarrow} s'$ (by Sub-claim 5). Thus, for **Claim** we only have to show $v \downarrow \hat{u}$. For $u'' := u'[\,o \leftarrow s' \mid o \in \Pi \setminus \{p\}\,]$ (Cf. Sub-claim 4) there is some w' with $v \overset{\oplus}{\longrightarrow} w' \overset{\oplus}{\longleftarrow} u''$ by Sub-claim 4 and $P(v, u', s, s', \Pi \setminus \{p\})$, which is smaller in the second or third position. Finally, we get $w' \downarrow \hat{u}$ by $u''/p = u'/p = u[\,pp' \leftarrow r_1\mu_1\,]/p = u/p[\,p' \leftarrow r_1\mu_1\,] = s[\,p' \leftarrow r_1\mu_1\,]$, Sub-claim 5, and $P(w', u'', s[\,p' \leftarrow r_1\mu_1\,], w, \{p\})$, which is smaller in the first position by Sub-claim 5. Q. e. d. (The (second) critical peak case)

The (second) variable overlap case: There are $\check{p}, \hat{p}, x$ with $p' = \check{p}\hat{p}$ and $l_0/\check{p} = x \in \mathrm{V}$

Define μ_0' by $(y \in \mathrm{V})$: $y\mu_0' := \begin{Bmatrix} x\mu_0[\,\hat{p} \leftarrow r_1\mu_1\,] & \text{if } y = x \\ y\mu_0 & \text{else} \end{Bmatrix}$. Define $\hat{u} := u[\,o \leftarrow r_0\mu_0' \mid o \in \Pi\,]$ and $\check{u} := u[\,o \leftarrow l_0\mu_0' \mid o \in \Pi\,]$. Since $x\mu_0/\hat{p} = l_0\mu_0/\check{p}\hat{p} = s/p' = u/pp' = u/q = l_1\mu_1 \longrightarrow r_1\mu_1$ and $u' = u[\,p\check{p}\hat{p} \leftarrow r_1\mu_1\,] = u[\,p \leftarrow l_0\mu_0[\,\check{p} \leftarrow x\mu_0'\,]\,][\,o \leftarrow l_0\mu_0 \mid o \in \Pi \setminus \{p\}\,]$ (by Sub-claim 4), we get some w with $v \overset{\oplus}{\longrightarrow} w \overset{\oplus}{\longleftarrow} \check{u}$ by $P(v, u', l_1\mu_1, r_1\mu_1, \{\, o\check{o}\hat{p} \mid o \in \Pi \ \wedge\ l_0/\check{o} = x \ \wedge\ o\check{o} \neq p\check{p} \,\})$, which is smaller in the first position by $s \rhd_{\mathrm{ST}} s/p' = u/pp' = u/q = l_1\mu_1$. Now by $\hat{u} \overset{\oplus}{\longleftarrow} u[\,o \leftarrow r_0\mu_0 \mid o \in \Pi\,] = u[\,o \leftarrow s' \mid o \in \Pi\,]$ for **Claim** we only have to show $w \downarrow \hat{u}$. But this is given by Sub-claim 6 below (which implies $l_0\mu_0' \longrightarrow r_0\mu_0'$ by Lemma 5.4) and $P(w, \check{u}, l_0\mu_0', r_0\mu_0', \Pi)$, which is smaller in the first position of the measure by $s = l_0\mu_0 \overset{\oplus}{\longrightarrow} l_0\mu_0'$.
Sub-claim 6: $C_0\mu_0'$ is fulfilled.

Proof of Sub-claim 6: For $(\bar{u}{=}\bar{v})$ in C_0 we have some w with $\bar{u}\mu_0 \overset{\oplus}{\longrightarrow} w \overset{\oplus}{\longleftarrow} \bar{v}\mu_0$. We get some w' with $w \overset{\oplus}{\longrightarrow} w' \overset{\oplus}{\longleftarrow} \bar{u}\mu_0'$ by $P(w, \bar{u}\mu_0, l_1\mu_1, r_1\mu_1, \{\, o\hat{p} \mid \bar{u}/o = x \,\})$, and then $w' \downarrow \bar{v}\mu_0'$ by $P(w', \bar{v}\mu_0, l_1\mu_1, r_1\mu_1, \{\, o\hat{p} \mid \bar{v}/o = x \,\})$, which are smaller in the first position of the measure by $s \rhd_{\mathrm{ST}} l_1\mu_1$ (shown above). For (Def $\bar{u}$) in C_0 we know the existence of some $\vec{u} \in \mathcal{GT}(\mathrm{cons})$ with $\vec{u} \overset{\oplus}{\longleftarrow} \bar{u}\mu_0$. We get some $\hat{\vec{u}}$ with $\vec{u} \overset{\oplus}{\longrightarrow} \hat{\vec{u}} \overset{\oplus}{\longleftarrow} \bar{u}\mu_0'$ by $P(\vec{u}, \bar{u}\mu_0, l_1\mu_1, r_1\mu_1, \{\, o\hat{p} \mid \bar{u}/o = x \,\})$, which is smaller in the first position (as before). By Lemma 5.8 we get $\hat{\vec{u}} \in \mathcal{GT}(\mathrm{cons})$. Finally, for $(\bar{u}{\neq}\bar{v})$ in C_0 we have some $\vec{u}, \vec{v} \in \mathcal{GT}(\mathrm{cons})$ with $\bar{u}\mu_0 \overset{\oplus}{\longrightarrow} \vec{u} \not\downarrow \vec{v} \overset{\oplus}{\longleftarrow} \bar{v}\mu_0$. By applying the same procedure as before twice we get $\hat{\vec{u}}, \hat{\vec{v}} \in \mathcal{GT}(\mathrm{cons})$ with $\bar{u}\mu_0' \overset{\oplus}{\longrightarrow} \hat{\vec{u}} \overset{\oplus}{\longleftarrow} \vec{u} \not\downarrow \vec{v} \overset{\oplus}{\longrightarrow} \hat{\vec{v}} \overset{\oplus}{\longleftarrow} \bar{v}\mu_0'$, i. e. $\bar{u}\mu_0' \overset{\oplus}{\longrightarrow} \hat{\vec{u}} \not\downarrow \hat{\vec{v}} \overset{\oplus}{\longleftarrow} \bar{v}\mu_0'$. Q. e. d. (Sub-claim 6)
Q. e. d. (The (second) variable overlap case)

Q. e. d. ("q strictly below $p \in \Pi$") Q. e. d. (**Proof of Theorem 6.3**)

Proof of Theorem 6.5

Claim: For $\beta \preceq \omega+\omega$ and $s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\beta} t$ we have $\mathcal{A}_\kappa(s) = \mathcal{A}_\kappa(t)$.

Proof of Claim: By induction on β. By induction on the number of derivation steps, it suffices to do the proof for $\longrightarrow_{\mathrm{R,X},\beta}$ instead of $\overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\beta}$. If β is one of the limit ordinals $0, \omega$, $\omega+\omega$, the induction step is trivial. If β is a non-limit ordinal $\gamma+1$, the induction step is as follows: For $s \longrightarrow_{\mathrm{R,X},\gamma+1} t$, either $s \longrightarrow_{\mathrm{R,X},\omega} t$ and $\omega \preceq \gamma$, and then (by induction hypothesis) $\mathcal{A}_\kappa(s) = \mathcal{A}_\kappa(t)$; or there is a substitution $\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$, a rule $((l,r),C) \in \mathrm{R}$, and a $p \in \mathcal{POS}(t)$ with $s/p = l\sigma$, $t = s[p \leftarrow r\sigma]$, and $C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X},\gamma}$. Since $\mathcal{A}$ is a sig/cons-model of R, we only have to show that $C\sigma$ is true w. r. t. $\mathcal{A}_\kappa$. For $(u{=}v)$ in $C\sigma$ we have $u{\downarrow_{\mathrm{R,X},\gamma}}v$ and hence by induction hypothesis we have $\mathcal{A}_\kappa(u) = \mathcal{A}_\kappa(v)$. For $\bar{u} \in \mathrm{S}$, $u \in \mathcal{T}(\mathrm{sig},\mathrm{X})_{\bar{u}}$, $(\mathrm{Def}\, u)$ in $C\sigma$ there is some $\hat{u} \in \mathcal{GT}_{\mathrm{CONS},\bar{u}}$ with $u \overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\gamma} \hat{u}$ and hence by induction hypothesis we have $\mathcal{A}_\kappa(u) = \mathcal{A}_\kappa(\hat{u}) \in \mathcal{A}(\mathrm{CONS}, \bar{u})$. For $(u{\neq}v)$ in $C\sigma$ we know $\omega \preceq \gamma$ and there are $\hat{u}, \hat{v} \in \mathcal{GT}(\mathrm{cons})$ with $u \overset{\oplus}{\longrightarrow}_{\mathrm{R,X},\gamma} \hat{u} \not\downarrow_{\mathrm{R,X},\gamma} \hat{v} \overset{\oplus}{\longleftarrow}_{\mathrm{R,X},\gamma} v$ and w. l. o. g. (since $\longrightarrow_{\mathrm{R,X}}[\cap(D_{\mathrm{X}} \times D_{\mathrm{X}})]$ is noetherian and by Lemma 5.8) $\hat{u}, \hat{v} \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$; thus, (since $\hat{u}$ and $\hat{v}$ are of the same sort and unequal) we have by induction hypothesis: $\mathcal{A}_\kappa(u) = \mathcal{A}_\kappa(\hat{u}) \neq \mathcal{A}_\kappa(\hat{v}) = \mathcal{A}_\kappa(v)$. Q. e. d. (Claim)

Now we show confluence for part (1) of the theorem. Suppose $u \overset{\oplus}{\longleftarrow}_{\mathrm{R,X}} s \overset{\oplus}{\longrightarrow}_{\mathrm{R,X}} v$. Since $\longrightarrow_{\mathrm{R,X}}$ is noetherian, there are $\hat{u}, \hat{v} \in \mathcal{T}(\mathrm{sig},\mathrm{X}) \backslash \mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$ with $u \overset{\circledast}{\longrightarrow}_{\mathrm{R,X}} \hat{u}$ and $v \overset{\circledast}{\longrightarrow}_{\mathrm{R,X}} \hat{v}$. By Claim we have $\mathcal{A}_\kappa(\hat{u}) = \mathcal{A}_\kappa(u) = \mathcal{A}_\kappa(s) = \mathcal{A}_\kappa(v) = \mathcal{A}_\kappa(\hat{v})$ and therefore $\hat{u} = \hat{v}$. Finally for the proof of part (2) of the theorem, in the above we can additionally assume $s \in D_{\mathrm{X}}$ and thus the existence of some $t \in \mathcal{GT}(\mathrm{cons}) \setminus \mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$ (since $\longrightarrow_{\mathrm{R,X}} \cap (D_{\mathrm{X}} \times D_{\mathrm{X}})$ is noetherian and by Lemma 5.8) with $s \overset{\circledast}{\longleftrightarrow}_{\mathrm{R,X}} t$. Since $\longrightarrow_{\mathrm{R,X}} \cap (D_{\mathrm{X}} \times D_{\mathrm{X}})$ is noetherian we get $\hat{u}, \hat{v}$ like above. Then by Claim we have $\mathcal{A}_\kappa(\hat{u}) = \mathcal{A}_\kappa(u) = \mathcal{A}_\kappa(s) = \mathcal{A}_\kappa(t)$ and $\mathcal{A}_\kappa(\hat{v}) = \mathcal{A}_\kappa(v) = \mathcal{A}_\kappa(s) = \mathcal{A}_\kappa(t)$, i. e. $\hat{u} = t = \hat{v}$.

Proof of Lemma 7.1 and Lemma 7.2

It is standard to encode a universal deterministic Turing machine with a finite set of left-linear, non-overlapping rules. This can be done in the following way, where stop, left, right, nil, 0, − are constant symbols, s is a unary function symbol, c and nth are binary, cmd, state are ternary, T is sexary; and T(l,a,r,c,s,p) encodes the Turing machine with

	meaning	intended range
l	being the tape to the left of the head	nil or $\mathsf{c}(\mathsf{s}^n(0)$, list-of-integers)
a	being the symbol under the head	$\mathsf{s}^n(0)$
r	being the tape to the right of the head	nil or $\mathsf{c}(\mathsf{s}^n(0)$, list-of-integers)
c	being the next command to be executed	stop or left or right or $\mathsf{s}^n(0)$
s	being the next state to be in	$\mathsf{s}^n(0)$
p	being the program	c(table-of-commands, table-of-states)

The rules are:

$$
\begin{array}{llllll}
\mathsf{T}(l, & a, r, & \mathsf{stop}, & s, p) & = & - \\
\mathsf{T}(\mathsf{nil}, & a, r, & \mathsf{left}, & s, p) & = & \mathsf{T}(\mathsf{nil}, 0, \mathsf{c}(a,r), \mathsf{cmd}(0,s,p), \mathsf{state}(0,s,p), p) \\
\mathsf{T}(\mathsf{c}(b,l), & a, r, & \mathsf{left}, & s, p) & = & \mathsf{T}(l, b, \mathsf{c}(a,r), \mathsf{cmd}(b,s,p), \mathsf{state}(b,s,p), p) \\
\mathsf{T}(l, & a, \mathsf{nil}, & \mathsf{right}, & s, p) & = & \mathsf{T}(\mathsf{c}(a,l), 0, \mathsf{nil}, \mathsf{cmd}(0,s,p), \mathsf{state}(0,s,p), p) \\
\mathsf{T}(l, & a, \mathsf{c}(b,r), & \mathsf{right}, & s, p) & = & \mathsf{T}(\mathsf{c}(a,l), b, r, \mathsf{cmd}(b,s,p), \mathsf{state}(b,s,p), p) \\
\mathsf{T}(l, & a, r, & 0, & s, p) & = & \mathsf{T}(l, 0, r, \mathsf{cmd}(0,s,p), \mathsf{state}(0,s,p), p) \\
\mathsf{T}(l, & a, r, & \mathsf{s}(x), & s, p) & = & \mathsf{T}(l, \mathsf{s}(x), r, \mathsf{cmd}(\mathsf{s}(x),s,p), \mathsf{state}(\mathsf{s}(x),s,p), p)
\end{array}
$$

with the following auxiliary functions:

$\mathsf{nth}(\mathsf{0},\mathsf{c}(a,l)) = a$
$\mathsf{nth}(\mathsf{s}(x),\mathsf{c}(a,l)) = \mathsf{nth}(x,l)$
$\mathsf{cmd}(a,s,\mathsf{c}(\textit{table-of-commands},\textit{table-of-states})) = \mathsf{nth}(a,\mathsf{nth}(s,\textit{table-of-commands}))$
$\mathsf{state}(a,s,\mathsf{c}(\textit{table-of-commands},\textit{table-of-states})) = \mathsf{nth}(a,\mathsf{nth}(s,\textit{table-of-states}))$

We use $-$ instead of a reasonable output because we are interested in the halting problem only. We assume all function symbols so far to be constructor symbols.

From this system we are now going to construct our positive-conditional rule system by exchanging the recursive T-rules of the form " $\mathsf{T}(\mathrm{l,a,r,c,s,p}) = \mathsf{T}(\mathrm{l',a',r',c',s',p'})$ " for rules of the form " $\mathsf{T}(\mathrm{l,a,r,c,s,p}) = - \longleftarrow \mathsf{T}(\mathrm{l',a',r',c',s',p'}){=}-$ ".

Now the above Turing machine halts in the described configuration iff the ground term " $\mathsf{T}(\mathrm{l,a,r,c,s,p})$ " is reducible. Therefore (the halting problem being not co-semi-decidable) the reducibility of ground terms cannot be co-semi-decidable.

For Lemma 7.2 we now add the following new rule:

$$\mathsf{foreverp}(l,a,r,c,s,p) = \mathsf{true} \longleftarrow \mathsf{T}(l,a,r,c,s,p) \neq -,\ \mathrm{Def}\,\mathsf{T}(l,a,r,c,s,p),\ \mathrm{Def}\,-$$

Now " $\mathsf{foreverp}(\mathrm{l,a,r,c,s,p})$ " is reducible iff the above Turing machine does not halt. Therefore the reducibility of ground terms cannot be semi-decidable.

Proof of Theorem 7.3

Proof of *(1)*: The function g that firstly tests whether its single argument s is in the enumerable set $\mathcal{T}(\mathrm{sig},\mathrm{X})$, secondly tries to compute $f(s)$, and thirdly (if $s \in \mathcal{T}(\mathrm{sig},\mathrm{X})$ and $f(s)$ is defined) is defined iff the test $f(s) = s$ succeeds, is defined exactly on the irreducible terms from $\mathcal{T}(\mathrm{sig},\mathrm{X})$.

Proof of *(2)*:

Claim 1: The following sets are universally (cf. footnote † of Lemma 7.19) enumerable for all $\beta \preceq \omega{+}\omega$ and $s \in \mathcal{T}(\mathrm{sig},\mathrm{X})$:

$$Y_\beta(s) := \{\ t \mid s \longrightarrow_{\mathrm{R,X},\beta} t\ \}$$
$$Z_\beta(s) := \{\ t \mid s \stackrel{\oplus}{\longrightarrow}_{\mathrm{R,X},\beta} t\ \}$$

If Claim 1 holds, we can enumerate $Z_{\omega+\omega}(s)$ and simultaneously the irreducible terms from $\mathcal{T}(\mathrm{sig},\mathrm{X})$ until one term t occurs in both enumerations and we can return $f(s) := t$.

Proof of Claim 1: By induction on β. It suffices to show that $Y_\beta(s)$ is enumerable for all $s \in \mathcal{T}(\mathrm{sig},\mathrm{X})$. The induction step for the limit ordinals 0, ω, $\omega{+}\omega$ is trivial ($Y_0(s) = \emptyset$; $Y_\omega(s) = \bigcup_{i \prec \omega} Y_i(s)$; $Y_{\omega+\omega}(s) = \bigcup_{i \prec \omega} Y_{\omega+i}(s)$). The induction step for the non-limit ordinals $\beta{+}1$ can be done the following way: For all rules $((l,r),C) \in \mathrm{R}$ and all $p \in \mathcal{POS}(s)$ and all σ in the enumerable set $\mathcal{SUB}(\mathcal{V}(((l,r),C)),\mathcal{T}(\mathrm{X}))$ with $s/p = l\sigma$ we test in an enumerative fashion whether $C\sigma$ is fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X},\beta}$ and enumerate $s[\,p \leftarrow r\sigma\,]$ if the test succeeds. For $\omega \preceq \beta$ we merge this enumeration with that of $Y_\omega(s)$. The test of "$C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X},\beta}$" can be semi-decided the following way: For $(u{=}v)$ in $C\sigma$ we test the enumerable set $Z_\beta(u) \cap Z_\beta(v)$ for non-emptiness. For $(\mathrm{Def}\,u)$ in $C\sigma$ we test the enumerable set $Z_\beta(u) \cap \mathcal{GT}(\mathrm{cons})$ for non-emptiness. For $(u{\neq}v)$ in $C\sigma$ we test for the existence of $\hat{u} \in A(u)$ and $\hat{v} \in A(v)$ with $\hat{u} \neq \hat{v}$ (syntactically) for the enumerable sets $A(w) := Z_\beta(w) \cap (\mathcal{GT}(\mathrm{cons}) \setminus \mathrm{dom}(\longrightarrow_{\mathrm{R,X}}))$. This last test succeeds only if $\exists \hat{u},\hat{v} \in \mathcal{GT}(\mathrm{cons}) : u \stackrel{\oplus}{\longrightarrow}_{\mathrm{R,X},\beta} \hat{u} \not\downarrow_{\mathrm{R,X},\beta} \hat{v} \stackrel{\oplus}{\longleftarrow}_{\mathrm{R,X},\beta} v$ holds. It also succeeds if this property holds because of $\forall s \in \mathcal{GT}(\mathrm{cons}) : \exists t : s \stackrel{\oplus}{\longrightarrow}_{\mathrm{R,X}} t \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$, Lemma 5.8, and $\omega \preceq \beta$.

Proof of Lemma 7.5

There exists some $\tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$. It suffices to show for all $t, t' \in \mathcal{T}$:

Claim 1: For $t \longrightarrow_{\mathrm{R,V}} t'$ we have $t\tau \longrightarrow_{\mathrm{R,X}} t'\tau$.

Claim 2: For $q \in \mathcal{POS}(t)$ and $t/q\imath_{\mathrm{R,V}} \ \ t'$ we have $t\tau\imath_{\mathrm{R,X}} \ \ t'\tau$.

Proof of Claim 1: By Corollary 5.7 we get $t\tau \longrightarrow_{\mathrm{R,V}} t'\tau$, then by Lemma 5.12 $t\tau \longrightarrow_{\mathrm{R,X}} t'\tau$.

Proof of Claim 2: There are $((l,r),C) \in \mathrm{R}$; $\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T})$; $p \in \mathcal{POS}(t/q)$; $u \in \mathcal{TERMS}(C)$ with $t/qp = l\sigma$; $t' = u\sigma$; $C\sigma$ fulfilled w. r. t. $\longrightarrow_{\mathrm{R,V}}$. By Corollary 5.7, $C\sigma\tau$ is fulfilled w. r. t. $\longrightarrow_{\mathrm{R,V}}$. By Lemma 5.12, $C\sigma\tau$ is fulfilled w. r. t. $\longrightarrow_{\mathrm{R,X}}$. Thus; since $t\tau \in \mathcal{T}(\mathrm{sig},\mathrm{X})$; $\sigma\tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; $qp \in \mathcal{POS}(t\tau)$; $t\tau/qp = l\sigma\tau$; $t'\tau = u\sigma\tau$; we get $t\tau\imath_{\mathrm{R,X}} \ \ t'\tau$.

Proof of Lemma 7.6

1.: Suppose that $\succ$ is not noetherian. As $\rhd_{\mathrm{ST}}$ and ´ are noetherian, there must be $r, s : \mathbb{N} \to \mathcal{T}(\mathrm{sig},\mathrm{X})$ with $\forall i \in \mathbb{N} : r_i \rhd_{\mathrm{ST}} s_i$ ´$^{\oplus}$ r_{i+1} . Then there is a $p : \mathbb{N} \to \mathbb{N}^+$ with $\forall i \in \mathbb{N} : r_i/p_i = s_i$. Define $t_n := r_0\,[\,p_0 \leftarrow r_1\,[\,p_1 \leftarrow r_2 \ldots [\,p_{n-1} \leftarrow r_n\,] \ldots]]$. Because of $r_i/p_i = s_i$ ´$^{\oplus}$ r_{i+1} we get $t_n \in \mathcal{T}(\mathrm{sig},\mathrm{X})$ (as ´$^{\oplus}$ sort-invariant) and t_n ´$^{\oplus}$ t_{n+1} (as ´$^{\oplus}$ X-monotonic). This contradicts ´ being noetherian.

If ´ is Y-stable, additionally, then $\succ$ is Y-stable too, because $\rhd_{\mathrm{ST}}$ is.

Here is an example for $\succ$ not sort-invariant and not $\emptyset$-monotonic: Let A, B be two different sorts. Let $\alpha(\mathsf{a}) = A$, $\alpha(\mathsf{f}) = A \longrightarrow B$, $\alpha(\mathsf{g}) = A \longrightarrow A$. Define ´$:= \emptyset$. Then we have $\succ = \rhd_{\mathrm{ST}}$ and therefrom: $\mathsf{f}(\mathsf{a}) \succ \mathsf{a}$ (hence not sort-invariant); and $\mathsf{g}(\mathsf{a}) \succ \mathsf{a}$ but $\mathsf{f}(\mathsf{g}(\mathsf{a})) \not\succ \mathsf{f}(\mathsf{a})$ (hence not $\emptyset$-monotonic).

2.: Take the signature from the example in the proof of (1). ´$:= \{(\mathsf{a}, \mathsf{f}(\mathsf{a}))\}$ is a V-monotonic (indeed!), well-founded ordering on $\mathcal{T}$ that is not sort-invariant. Now $\succ$ is not irreflexive: a ´ $\mathsf{f}(\mathsf{a}) \rhd_{\mathrm{ST}} \mathsf{a}$. If one changes $\alpha(\mathsf{f})$ to be $\alpha(\mathsf{f}) = A \longrightarrow A$, then ´ is a sort-invariant, well-founded ordering on $\mathcal{T}$ that is not V-monotonic.

3.: For $\mathcal{T}(\mathrm{sig},\mathrm{X}) \ni t \rhd_{\mathrm{ST}} t'$ ´ t'' there is a $p \in \mathcal{POS}(t)$; $p \neq \emptyset$ with $t' = t/p$. By sort-invariance and X-monotonicity of ´ we get $t = t[\,p \leftarrow t'\,]$ ´ $t[\,p \leftarrow t''\,] \rhd_{\mathrm{ST}} t''$.

4.: W. r. t. the noetherian word rewriting system $\{(\rhd_{\mathrm{ST}} \rhd_{\mathrm{ST}},\ \rhd_{\mathrm{ST}}),\ (´\ \rhd_{\mathrm{ST}}\ ´,\ ´\ ´\ \rhd_{\mathrm{ST}})\}$ (or else for X=V: $\{(\rhd_{\mathrm{ST}} \rhd_{\mathrm{ST}},\ \rhd_{\mathrm{ST}}),\ (\rhd_{\mathrm{ST}}\ ´,\ ´\ \rhd_{\mathrm{ST}})\}$), generated from *(3)*, words from $\{´, \rhd_{\mathrm{ST}}\}^+$ have normal forms only in $\{\rhd_{\mathrm{ST}}\} \cup (\{\rhd_{\mathrm{ST}}\}^* \circ \{´\}^+ \circ \{\rhd_{\mathrm{ST}}\}^*)$ (or else $\{\rhd_{\mathrm{ST}}\} \cup (\{´\}^+ \circ \{\rhd_{\mathrm{ST}}\}^*)$).

Proof of Lemma 7.7

Suppose $\rhd$ is not noetherian. Then there is a $t : \mathbb{N} \to \mathcal{T}$ with $\forall i \in \mathbb{N} :\ t_i(´ \cup \hookrightarrow \cup \rhd_{\mathrm{ST}})\, t_{i+1}$. Define $k : \mathbb{N} \to \mathbb{N}$ by: $k_0 = 0$, $k_{i+1} = 1 + \min\{\, j \mid j \geq k_i \ \wedge\ t_j \hookrightarrow t_{j+1}\,\}$. The above minimum always exists because $(´ \cup \rhd_{\mathrm{ST}})$ is noetherian. We have $t_{k_i}((´ \cup \rhd_{\mathrm{ST}})^{\oplus} \circ \hookrightarrow)\, t_{k_{i+1}}$. By $\rhd_{\mathrm{ST}} \circ$ ´ $\subseteq$ ´ $\circ \rhd_{\mathrm{ST}}$ we get $t_{k_i}(´^{\oplus} \circ \rhd_{\mathrm{ST}}{}^{\oplus} \circ \hookrightarrow)\, t_{k_{i+1}}$, which means $t_{k_i}(´ \ \cup (\,\trianglerighteq_{\mathrm{ST}} \circ \hookrightarrow))^{\oplus} t_{k_{i+1}}$, which contradicts the assertion that $(´ \ \cup (\,\trianglerighteq_{\mathrm{ST}} \circ \hookrightarrow))$ is noetherian.

Proof of Lemma 7.15

$\longrightarrow_{\mathrm{R,Y}} \ \subseteq\ >$ is trivial by induction on the construction of $\longrightarrow_{\mathrm{R,Y}}$ using Lemma 5.10. $\longrightarrow_{\mathrm{R,Y}} \cup\, \imath_{\mathrm{R,Y}} \ \ \cup \rhd_{\mathrm{ST}} \ \subseteq\ \rhd$ is trivial by definition of $\imath_{\mathrm{R,Y}}$. R is X-compatible with T by Lemma 5.12. By Lemma 7.5 we know that $(\longrightarrow_{\mathrm{R,V}} \ \cup\ (\,\trianglerighteq_{\mathrm{ST}} \circ \imath_{\mathrm{R,V}} \ \ ))$ is noetherian. By Lemma 7.6 and Lemma 7.7 we know that $(\ \overset{\oplus}{\longrightarrow}_{\mathrm{R,V}},\ (\longrightarrow_{\mathrm{R,V}} \cup\, \imath_{\mathrm{R,V}} \ \ \cup \rhd_{\mathrm{ST}})^{\oplus}\ )$ is a termination-pair over sig/V, with which R is V-compatible by Lemma 5.4.

Proof of Theorem 7.17

(1) ⇒ *(2)*: By Lemma 6.1. Q. e. d. (*(1)* ⇒ *(2)*)

(2) ⇒ *(1)*: First notice that the usual modularisation of the proof for the unconditional analogue of the theorem (by showing first that local confluence is guaranteed except for the cases that are matched by critical peaks (the so-called "critical pair lemma")) is not possible here because we need the confluence-property to hold for the condition terms even for the cases that are not matched by critical peaks. Now to the proof: Let s be minimal in $\rhd$ such that $\longrightarrow$ is not confluent below s. Because of $\longrightarrow\ \subseteq\ \rhd$ (by Lemma 7.15) and minimality of s, $\longrightarrow$ is not even locally confluent below s. Let $p,q \in \mathcal{POS}(s)$; $t_0 {}_p\!\!\longleftarrow s \longrightarrow_q t_1$; $t_0 \not\downarrow t_1$. Now as one of p,q must be a prefix of the other, w. l. o. g. say that q is a prefix of p. As $s \trianglerighteq s/q$, by the minimality of s we have $q = \emptyset$. Now for $k < 2$ there must be $((l_k, r_k), C_k) \in \mathrm{R}$; $\mu_k \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; with $C_k\mu_k$ fulfilled; $s = l_1\mu_1$; $s/p = l_0\mu_0$; $t_0 = l_1\mu_1[\,p \leftarrow r_0\mu_0\,]$; $t_1 = r_1\mu_1$.

The inductive case: $p = q_0q_1$; $l_1/q_0 = x \in \mathrm{V}$: By $x\mu_1/q_1 = l_1\mu_1/q_0q_1 = s/p = l_0\mu_0$ and Lemma 5.8 (in case of $x \in \mathrm{V_{CONS}}$), we can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by ($y \in \mathrm{V}$): $y\nu := \left\{ \begin{array}{ll} x\mu_1[\,q_1 \leftarrow r_0\mu_0\,] & \text{if } y = x \\ y\mu_1 & \text{otherwise} \end{array} \right\}$ and get $y\mu_1 \stackrel{\leq 1}{\longrightarrow} y\nu$ for $y \in \mathrm{V}$. By Corollary 5.6:
$t_0 = l_1\mu_1[\,q_0q_1 \leftarrow r_0\mu_0\,] = l_1[\,q_0 \leftarrow x\nu\,][\,q' \leftarrow y\mu_1 \mid l_1/q' = y \in \mathrm{V}\ \wedge\ q' \neq q_0\,]$
$\stackrel{\oplus}{\longrightarrow} l_1[\,q' \leftarrow y\nu \mid l_1/q' = y \in \mathrm{V}\,] = l_1\nu$;
$t_1 = r_1\mu_1 \stackrel{\oplus}{\longrightarrow} r_1\nu$. It suffices to show $l_1\nu \longrightarrow r_1\nu$, which follows from:
Claim: $C_1\nu$ is fulfilled.
Proof of Claim: For each L in C_1 we have to show that $L\nu$ is fulfilled. By our compatibility-property we get $\forall u \in \mathcal{TERMS}(L) : l_1\mu_1 \rhd u\mu_1$.
$L = (u{=}v)$: We know $u\nu \stackrel{\oplus}{\longleftarrow} u\mu_1 \downarrow v\mu_1 \stackrel{\oplus}{\longrightarrow} v\nu$. As $s = l_1\mu_1 \rhd u\mu_1$ we have $u\nu \downarrow v\mu_1 \stackrel{\oplus}{\longrightarrow} v\nu$. As $s = l_1\mu_1 \rhd v\mu_1$ we have $u\nu \downarrow v\nu$.
$L = (\mathrm{Def}\,u)$: We know the existence of $\hat{u} \in \mathcal{GT}(\mathrm{cons})$ with $u\nu \stackrel{\oplus}{\longleftarrow} u\mu_1 \stackrel{\oplus}{\longrightarrow} \hat{u}$. By Lemma 5.8 and Lemma 7.15 we can additionally assume $\hat{u}$ to be irreducible. By $s = l_1\mu_1 \rhd u\mu_1$ we get $u\nu \stackrel{\oplus}{\longrightarrow} \hat{u}$.
$L = (u{\neq}v)$: We know the existence of $\hat{u}, \hat{v} \in \mathcal{GT}(\mathrm{cons})$ with
$u\nu \stackrel{\oplus}{\longleftarrow} u\mu_1 \stackrel{\oplus}{\longrightarrow} \hat{u} \not\downarrow \hat{v} \stackrel{\oplus}{\longleftarrow} v\mu_1 \stackrel{\oplus}{\longrightarrow} v\nu$. Just like above we can additionally assume $\hat{u}, \hat{v}$ to be irreducible. By $s = l_1\mu_1 \rhd u\mu_1, v\mu_1$ we get $u\nu \downarrow \hat{u}$; $\hat{v} \downarrow v\nu$; and hence $u\nu \stackrel{\oplus}{\longrightarrow} \hat{u} \not\downarrow \hat{v} \stackrel{\oplus}{\longleftarrow} v\nu$.
Q. e. d. (Claim; The inductive case)

The critical peak case: $p \in \mathcal{POS}(l_1)$; $l_1/p \notin \mathrm{V}$: Let $\xi \in \mathcal{SUB}(\mathrm{V}, \mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)] \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$. Define $\mathrm{Y} := \mathcal{V}((l_0{=}r_0{\longleftarrow}C_0)\xi, l_1{=}r_1{\longleftarrow}C_1)$. Let ϱ be given by $x\varrho = \left\{ \begin{array}{ll} x\mu_1 & \text{if } x \in \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) \\ x\xi^{-1}\mu_0 & \text{else} \end{array} \right\}$ ($x \in \mathrm{V}$). By $l_0\xi\varrho = l_0\xi\xi^{-1}\mu_0 = s/p = l_1\mu_1/p = l_1\varrho/p = (l_1/p)\varrho$ let $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/p)\}, \mathrm{Y})$ and $\tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $(\sigma\tau)|_{\mathrm{Y}} = \varrho|_{\mathrm{Y}}$. Let $((l', r'), D) := ((l_1[\,p \leftarrow r_0\xi\,], r_1), C_0\xi\ C_1)\sigma$ and $\hat{t} := l_1\sigma$. Now we have: $((l', r'), D)\tau = ((l_1[\,p \leftarrow r_0\xi\,], r_1), C_0\xi\ C_1)\varrho = ((l_1\mu_1[\,p \leftarrow r_0\mu_0\,], r_1\mu_1), C_0\mu_0\ C_1\mu_1)$ $= ((t_0, t_1), C_0\mu_0\ C_1\mu_1)$. Hence $D\tau$ is fulfilled. Furthermore, by $\hat{t}\tau = l_1\varrho = l_1\mu_1 = s$ and induction hypothesis we get $\forall u \lhd \hat{t}\tau :\ \longrightarrow$ is confluent below u . If $t_0 = t_1$, then $t_0 \downarrow t_1$ trivially. Otherwise $(((l', r'), D), \hat{t}, p) \in \mathrm{CP(R)}$ and then by *(2)* we get $t_0 \downarrow t_1$, too.
Q. e. d. (The critical peak case; *(2)* ⇒ *(1)*)

Proof of Theorem 7.18

Since the analoguos conclusions of Lemma 7.15 still hold, the proof reads just like the proof of Theorem 7.17 with the exception of the "Proof of Claim" in "The inductive case" of " (2) $\Rightarrow$ (1) ", where after the first sentence we have to add the following:
Since we have free constructors and $x\mu_1 \longrightarrow x\nu$, we know $x \in \mathrm{V_{SIG}}$. Now, if $\mathcal{V}(L) \subseteq \mathrm{V_{CONS}}$, we get $x \notin \mathcal{V}(L)$, and thus $L\nu = L\mu_1$ is fulfilled. Therefore we can assume $\mathcal{V}(L) \not\subseteq \mathrm{V_{CONS}}$ in the sequel.

Proof of Lemma 7.19

Statement *(1)* follows from statement *(2)*. By Lemma 7.15 we can now show statements *(2)* and *(3)* by noetherian induction on s w. r. t. $\triangleright$.
Proof of *(2)*: There is only a finite number of of positions $p \in \mathcal{POS}(s)$ and of rules $l{=}r{\longleftarrow}C$ of R matching s/p, and the set of matching substitutions $\sigma \in \mathcal{SUB}(\mathcal{V}(l{=}r{\longleftarrow}C), \mathcal{T}(\mathrm{X}))$ with $s/p = l\sigma$ is enumerable. Since $s \;\triangleright\; s[\,p \leftarrow r\sigma\,]$ for $C\sigma$ being fulfilled, by induction hypothesis we only have to be able to semi-decide whether $C\sigma$ is fulfilled. In case of left-right-compatibility we semi-decide the fulfilledness of the literals from left to right; in case of compatibility only, we wait with our parallel enumeration until we have been able to establish the condition terms to be $\triangleleft$-smaller than $l\sigma$. Thus, for semi-deciding the fulfilledness of a literal L in $C\sigma$ we can assume its terms to be $\triangleleft$-smaller than s. By induction hypothesis it is obvious now how to semi-decide fulfilledness of '='- and 'Def'-literals in $C\sigma$, and how to semi-decide '$\neq$'-literals using statement *(3)*.
Proof of *(3)*: By Lemma 5.8, the constructor rules being extra-variable free, and R and $\mathcal{POS}(s)$ being finite, there can be only a finite number of terms t with $s \longrightarrow t$, and by the induction hypothesis this also holds for those t with $s \stackrel{\oplus}{\longrightarrow} t$. In case of left-right-compatibility we decide the fulfilledness of the literals from left to right; in case of compatibility only, we know that $C\sigma$ cannot be fulfilled if the test fails whether all its terms are $\triangleleft$-smaller than $l\sigma$. Thus, for deciding the fulfilledness of a literal L in $C\sigma$ we can assume its terms to be $\triangleleft$-smaller than s. By induction hypothesis it is obvious now how to decide fulfilledness of '='- and 'Def'-literals in $C\sigma$.

Proof of Lemma 7.20

Take the unconditional rule system of the proof of Lemma 7.1. From this system we are now going to construct our positive-conditional rule system by adding as new first argument a step-counter to our Turing machine by exchanging the recursive T-rules of the form " $\mathsf{T}(\mathrm{l,a,r,c,s,p}) = \mathsf{T}(\mathrm{l',a',r',c',s',p'})$ " for rules of the form " $\mathsf{T}(\mathsf{s}(x),\mathrm{l,a,r,c,s,p}) = \mathsf{T}(x,\mathrm{l',a',r',c',s',p'})$ " and the non-recursive T-rule by $\mathsf{T}(x,l,a,r,\mathsf{stop},s,p) = -$. Finally we add the rule $\mathsf{terminatesp}(l,a,r,c,s,p) = \mathsf{true} \longleftarrow \mathsf{T}(x,l,a,r,c,s,p){=}-$, where '$\mathsf{terminatesp}$' and 'true' are of a new sort. For the decidable ordering $\triangleright$ of the termination-pair take the lexicographic path ordering where '$\mathsf{terminatesp}$' is bigger than all other function symbols and the variables of the old sorts; 'T' is bigger than all function symbols except '$\mathsf{terminatesp}$'; and 'cmd' and 'state' are bigger than 'nth'.
Now, since " $\mathsf{terminatesp}(\mathrm{l,a,r,c,s,p})$ " is reducible iff the Turing machine of the proof of Lemma 7.1 halts, reducibility of ground terms cannot be co-semi-decidable.

Proof of Lemma 7.21

The proof is very similar to that of Lemma 7.19. Finally, *(3)* follows from *(2)*, Theorem 7.17, and $\mathcal{T}(\mathrm{sig}, \mathrm{X})$ being enumerable.

Proof of Lemma 7.22

Take the unconditional part of the rule system of the proof of Lemma 7.20. For getting our positive conditional CRS R, add the rules $\mathsf{terminatesp}(x) = \mathsf{false}$ and $\mathsf{terminatesp}(x) = \mathsf{true} \longleftarrow \mathsf{T}(x,\mathrm{l},\mathrm{a},\mathrm{r},\mathrm{c},\mathrm{s},\mathrm{p}) = -$, where 'terminatesp', 'false', and 'true' are of a new sort; and l, a, r, c, s, p are ground terms. For the decidable ordering $\rhd$ of the termination-pair take the lexicographic path ordering where 'terminatesp' is bigger than all other function symbols and the variables of the old sorts; 'T' is bigger than all function symbols except 'terminatesp'; and 'cmd' and 'state' are bigger than 'nth'. Now, using Theorem 7.17, the following sentences are logically equivalent: $\longrightarrow_{\mathrm{R,V}}$ is confluent. The critical peak $(((\mathsf{true}, \mathsf{false}), \mathsf{T}(x,\mathrm{l},\mathrm{a},\mathrm{r},\mathrm{c},\mathrm{s},\mathrm{p}) = -), \mathsf{terminatesp}(x), \emptyset)$ is joinable w. r. t. R, V. There is no term $t \in \mathcal{T}(\mathrm{sig}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathrm{CONS}})$ with $\mathsf{T}(t,\mathrm{l},\mathrm{a},\mathrm{r},\mathrm{c},\mathrm{s},\mathrm{p}) \overset{\oplus}{\longrightarrow}_{\mathrm{R,V}} -$. The Turing machine of the proof of Lemma 7.1 does not halt.

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