MATHEMATICS OF OPERATIONS RESEARCH
Vol. 32, No. 1, February 2007, pp. 233–256
ISSN 0364-765X | EISSN 1526-5471 | 07 | 3201 | 0233

DOI 10.1287/moor.1060.0244


# The Symmetric Traveling Salesman Polytope: New Facets from the Graphical Relaxation

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The *path*, the *wheelbarrow*, and the *bicycle* inequalities have been shown by Cornuéjols, Fonlupt, and Naddef to be facet-defining for the *graphical relaxation* of STSP($n$), the polytope of the symmetric traveling salesman problem on an $n$-node complete graph. We show that these inequalities, and some generalizations of them, define facets also for STSP($n$). In conclusion, we characterize a large family of facet-defining inequalities for STSP($n$) that include, as special cases, most of the inequalities currently known to have this property as the *comb*, the *clique tree*, and the *chain* inequalities. Most of the results given here come from a strong relationship of STSP($n$) with its graphical relaxation that we have pointed out in another paper, where the basic proof techniques are also described.

*Key words*: traveling salesman problem; Hamiltonian cycle; polyhedron; facet; linear inequality
*MSC2000 subject classification*: Primary: 90C57; secondary: 52B12
*OR/MS subject classification*: Primary: networks/graphs; secondary: traveling salesman
*History*: Received October 29, 2004; revised February 17, 2006, and July 25, 2006.

**1. Introduction.** "The most spectacular success of the cutting plane technique has certainly been achieved for the traveling salesman problem" Grötschel and Lovász observe in their survey on combinatorial optimization (Grötschel and Lovász [8]). This success is due mostly to the exploitation of the current (partial) knowledge of the structure of the traveling salesman polytope and is the main motivation for pushing this knowledge a bit further.

Cornuéjols et al. [5] define a class of valid inequalities for the symmetric traveling salesman polytope on a graph with $n$ nodes (STSP($n$)) known as the *path*, the *wheelbarrow*, and the *bicycle* inequalities (the PWB inequalities for short). They show that these inequalities are facet-defining for a relaxation of STSP($n$), namely, the *graphical relaxation polyhedron* (GTSP($n$)).

The main result of this paper is that these inequalities, and many of their generalizations, are facet-defining for STSP($n$). This result was contained in the research report of Naddef and Rinaldi [16], which had the same title as this article and was widely referenced in the literature, both in research and in survey articles. However, it was very technical and difficult to read and, thus, was never submitted for publication. This paper is a substantial revision of that report.

The motivation for adding the PWB inequalities to the list of those defining facets of STSP($n$), in addition to the fact that they provide a huge and natural generalization of the *comb* inequalities, comes from the fact that they are useful in a polyhedral cutting plane algorithm for the solution of the traveling salesman problem, as shown by Naddef and Thienel [19, 20] (see also Clochard and Naddef [4]). This experimental evidence confirms the theoretical expectation expressed by Goemans [6], where he addresses the problem of measuring the quality of a class of inequalities. As a measure, he proposes the ratio of the lower bound for the graphical traveling salesman problem obtained by using all the inequalities of a given class along with the subtour elimination inequalities and the lower bound produced with only the subtour elimination inequalities. The best value, among all the inequalities of GTSP($n$) known to date, is obtained for the PWB inequalities.

In Naddef and Rinaldi [18] we show how the two polyhedra STSP($n$) and GTSP($n$) are very tightly related. On this basis, we develop a technique to prove that some given facet-defining inequalities for GTSP($n$) are also facet-defining for STSP($n$). Moreover, we state sufficient conditions under which the composition of facet-defining inequalities that we introduced in Naddef and Rinaldi [17] for GTSP($n$) also applies to the case of STSP($n$). Finally, we show how to apply several kinds of liftings to the inequalities that define facets of STSP($n$).

In this paper we first use the technique of Naddef and Rinaldi [18] to prove that the path, the wheelbarrow, and the bicycle inequalities are facet-defining for STSP($n$) (§2). Since these inequalities contain the comb inequalities as a special case, as a byproduct we get another proof that the comb inequalities define facets of STSP($n$). Another such proof for comb inequalities can be found in Naddef and Wild [21], and also, of course, in the original proof of Grötschel and Padberg [9, 10].

In Naddef and Rinaldi [18], we define an edge cloning operation to extend an inequality to a higher dimensional space, and we state sufficient conditions under which the application of such operation preserves the facet-defining property of an inequality. Here we apply this operation to the PWB inequalities. The so-called *chain inequalities* of Padberg and Hong [24] are a particular case of the inequalities obtained in this way.

We finally prove that the repeated 2-sum composition of PWB inequalities yields facet-defining inequalities. *Clique trees*, with at least one node outside all handles and teeth, are a special case of these inequalities. Therefore, we get an alternative proof that they are facet-defining; the original one was given by Grötschel and Pulleyblank [11].

Let $G = (V, E)$ be a graph on $n$ nodes. By $e = (u, v)$ we denote the edge of $G$ having $u$ and $v$ as end nodes, and we let $\mathbb{R}^E$ be the set of all real vectors whose components are indexed by the edge set $E$. For every real vector $x$ in $\mathbb{R}^E$, by $x_e$, or by $x(u, v)$, we denote the component of $x$ indexed by $e = (u, v)$. For a subset $F \subseteq E$, we let $x(F)$ be the sum $\sum_{e\in F} x_e$. When $G$ is complete, we denote it by $K_n = (V, E_n)$.

With every Hamiltonian cycle $H$ of $K_n$ we associate a unique incidence vector $\chi^H$ in $\mathbb{R}^{E_n}$. The components of $\chi^H$ indexed by the edges of $H$ have value 1 while all the other components have value 0. The *symmetric traveling salesman polytope* (also called the *Hamiltonian cycle polytope*) associated with $K_n$ is denoted by STSP$(n)$ and is the convex hull of the set of the incidence vectors of all the Hamiltonian cycles of $K_n$.

The description of STSP$(n)$ with linear inequalities is a classical topic of polyhedral combinatorics and has attracted a lot of interest. For the fundamentals of polyhedral combinatorics we refer the reader to the book of Nemhauser and Wolsey [22]. While for small values of $n$ ($n \leq 9$) a complete description of the system of inequalities defining facets of STSP$(n)$ has been generated by means of a computer program (see, e.g., Christof and Reinelt [2]), for arbitrary values of $n$ the knowledge of such a system is far from being complete, and it is very unlikely that it will ever be. In the last 40 years many papers appeared in which new valid or facet-defining inequalities for STSP$(n)$ were introduced. We refer to Jünger et al. [12], Lawler et al. [13], Naddef [14], and Naddef and Pochet [15] for a list of them and for further details on the traveling salesman polytope. As new inequalities are discovered, it becomes more and more difficult to keep track of all of them in a unifying framework; moreover, the proof techniques are usually specific for each class of inequalities.

This paper aims at giving a compact combinatorial description for a large family of facet-defining inequalities that includes most of the known ones, as well as at providing a standard proof technique to show that an inequality defines a facet of STSP$(n)$. As STSP$(n)$ is not full-dimensional, to study its polyhedral structure it is customary to embed it into a full-dimensional polyhedron called a *relaxation*, which is obtained by dropping some conditions on the solution set, and then to find sufficient conditions for an inequality facet-defining for the relaxation to maintain such a property for STSP$(n)$.

The two major relaxations that have been considered in the study of the polyhedral structure of STSP$(n)$ are the *monotone traveling salesman polytope*, introduced by Grötschel [7], and the *graphical traveling salesman polyhedron*. Sufficient conditions for a facet-defining inequality for the monotone relaxation to be facet-defining for STSP$(n)$ are given by Balas and Fischetti [1].

A desirable property of a relaxation $R$ is that every facet of STSP$(n)$ be contained in exactly one of the facets of $R$ that do not contain the entire polytope STSP$(n)$. If this property holds, then there is a one-to-one correspondence between a subset of the facets of $R$ and all facets of STSP$(n)$. Unfortunately, the monotone traveling salesman polytope does not have such a property. On the contrary, the graphical traveling salesman polyhedron does. For this reason and for some nice connections with STSP$(n)$, which will be mentioned later, this polyhedron appears to be the most natural and useful relaxation for studying the polyhedral structure of STSP$(n)$.

We exploit here such connections between STSP$(n)$ and the graphical traveling salesman polyhedron. The latter polyhedron has been studied by Cornuéjols et al. [5] and by Naddef and Rinaldi [17]. In Naddef and Rinaldi [18] we studied its connections with STSP$(n)$. Most of our proofs are based on the results of these three papers.

To formally define the graphical traveling salesman polyhedron, we need the following definitions.

A *multiset of edges* of $G = (V, E)$ is a collection $F$ of elements of $E$ that may contain several copies of the same element. For every element $e$ of $E$, we call *multiplicity* of $e$ in $F$ the number of times $e$ appears in $F$. Clearly, a *set of edges* of $G$ is a multiset where every element has multiplicity 1. Let $F_1$ and $F_2$ be two multisets of edges of $G$ and let $F_1 + F_2$ denote the multiset for which the multiplicity of every element is given by the sum of its multiplicities in $F_1$ and $F_2$. By $F + e$ and, if $e \in F$, by $F - e$ we denote the multisets for which the element $e$ has multiplicity one more and one less than in $F$, respectively. Finally, $t\{e\}$ denotes the multiset containing only the element $e$ with multiplicity $t$.

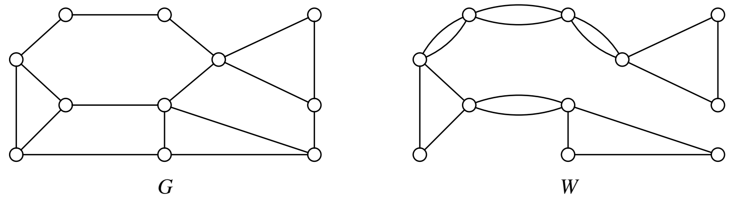


FIGURE 1. Example of a closed walk $W$ in a graph $G$.

Let $F$ be a multiset of edges of $G=(V,E)$. By $G[F]$ we denote the multigraph having node set $V$ and having, for every pair of distinct nodes $u$ and $v$ in $V$, as many edges with end nodes $u$ and $v$ as the multiplicity of $(u,v)$ in $F$. For every node $v$ in $V$, the *degree of* $v$ *in* $F$ is the degree of $v$ in the multigraph $G[F]$, and the *neighbors of* $v$ *in* $F$ are the neighbors of $v$ in the multigraph $G[F]$. With every multiset $F$ of edges of $G$ we associate a unique representative vector $\chi^F \in \mathbb{R}^E$ by setting $\chi^F_e$ equal to the multiplicity of $e$ in $F$ for every $e\in E$. If $c$ is a vector in $\mathbb{R}^E$, the *c-length* of $F$, also denoted by $c(F)$, is defined as $c(F)=c\chi^F$. For any two multisets $F_1$ and $F_2$ of edges of $G$, if $\chi^{F_1}\le \chi^{F_2}$ we say that $F_1$ is contained in $F_2$.

A *spanning closed walk* of a graph $G=(V,E)$ is a multiset $W$ of edges of $G$ such that

(i) the degree in $W$ of every $v\in V$ is positive and even;

(ii) $G[W]$ is connected.

We simply use the term *walk* for a spanning closed walk since all our walks are of this kind.

Thus, a Hamiltonian cycle in $G$ is a walk where every node has degree 2, while a walk is not, in general, a Hamiltonian cycle.

Figure 1 shows a graph and a closed walk which is not a Hamiltonian cycle.

The *graphical traveling salesman polyhedron* associated with the graph $G$, denoted by GTSP($G$) or GTSP($n$) when $G=K_n$, is the convex hull of the set of the representative vectors of all the walks of $G$ and is the polyhedron associated with the *graphical traveling salesman problem*.

If $G$ is connected, then $2T=\{2\{e\}\mid e\in T\}$ is a walk for any spanning tree $T$ of $G$. Moreover, $W+2t\{e\}$ is a walk for any walk $W$, for any edge $e$ of $G$, and for any nonnegative integer $t$. Therefore, GTSP($G$) is a full-dimensional unbounded polyhedron. Clearly, GTSP($n$) is a relaxation of STSP($n$) (the degree of each node is no longer required to be 2 but only positive and even) and STSP($n$) $=\{x\in \text{GTSP}(n)\mid x(E)=n\}$ (see Cornuéjols et al. [5]). Therefore, STSP($n$) is a face of GTSP($n$).

For any inequality $fx\ge f_0$ and for each node $u\in V$, we define the edge set $\Delta_f(u)=\{(v,w)\in E_n\mid u\neq v,$ $u\neq w,\ f(v,w)=f(u,v)+f(u,w)\}$.

DEFINITION 1.1. An inequality $fx\ge f_0$ defined on $\mathbb{R}^{E_n}$ is said to be *tight triangular* (abbreviated $TT$) or *in tight triangular form* (abbreviated $TT$ form) if:

(a) The coefficients $f_e$ satisfy the triangular inequality, i.e., $f(u,v)\le f(u,w)+f(w,v)$ for every triple $u$, $v$, $w$ of distinct nodes in $V$;

(b) $\Delta_f(u)\neq\varnothing$ for all $u$ in $V$.

In Naddef and Rinaldi [18] we prove that, except for the trivial inequalities $x_e\ge 0$ and for the degree inequalities $x(\delta(\{u\}))\ge 2$, all facet-defining inequalities of GTSP($n$) are in $TT$ form (here and in the following, by $\delta(U)=\{(u,v)\in E\mid u\in U,\ v\in V\setminus U\}$ we denote the cocycle of a subset $U$ of $V$ in the graph $G=(V,E)$). Moreover, we show that every nontrivial inequality $cx\ge c_0$ facet-defining for STSP($n$) has a unique equivalent inequality $fx\ge f_0$ in $TT$ form, up to scaling by a nonnegative constant $\pi$, where $f$ and $f_0$ are defined as follows:

$$\begin{aligned} f(u,v)&=\pi(\lambda_u+\lambda_v+c(u,v))\quad \text{for all } (u,v)\in E\\ f_0&=\pi\Big(2\sum_{u\in V}\lambda_u+c_0\Big), \end{aligned}\tag{1}$$

where $\lambda\in\mathbb{R}^V$ satisfies

$$\lambda_u=\frac{1}{2}\max\{c(v,w)-c(u,v)-c(u,w)\mid u,v,w\in V,\, u\neq v\neq w\}\quad \text{for all } u\in V.\tag{2}$$

We now show the $TT$ form of the comb inequality, probably the best-known inequality of the linear description of STSP($n$). We consider the simplest comb inequality, the one with three teeth. Such an inequality is defined on a subset of vertices $H$ called the *handle* and on three mutually disjoint subsets of vertices $T_1$, $T_2$, and $T_3$, called the *teeth*, which intersect $H$ (see Figure 2).

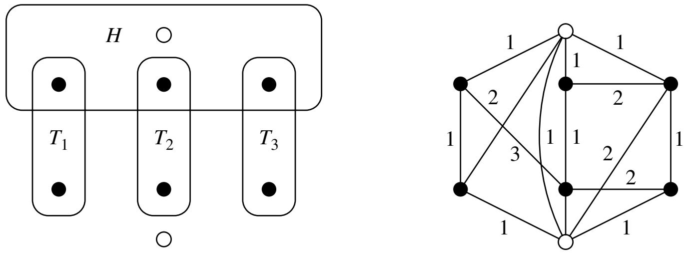


FIGURE 2. A 3-tooth comb and the coefficients of the inequality in $TT$ form.

When first defined by Chvátal [3] and then by Grötschel and Padberg [9], the inequality, where $k$ is odd and stands for the number of teeth (in our example $k = 3$) and where $\gamma(U)$ denotes the edge set $\{(u, w) \mid u, w \in U\}$, was given as

$$x(\gamma(H)) + \sum_{i=1}^{k} x(\gamma(T_i)) \le |H| + \sum_{i=1}^{k}(|T_i| - 1) - (k+1)/2.$$

After multiplying both sides of the inequality by $-1$ and applying (2), one obtains for $\lambda_u$ a value given by half the number of sets (handle and teeth) to which $u$ belongs. Then, by applying (1) (with $\pi = 2$ to produce integral coefficients), one obtains the following version of the inequality in $TT$ form:

$$x(\delta(H)) + \sum_{j=1}^{k} x(\delta(T_j)) \ge 3k + 1.$$

In the right-hand side part of Figure 2 some edges of $G$ are drawn with their coefficients in the $TT$-form of the comb inequality defined by the sets in the left-hand side. The coefficient of any edge $e$ is given by the number of sets whose border is crossed by $e$. To unclutter the figure, only a few examples of such coefficients are shown.

In Naddef and Rinaldi [18] we show that the $TT$ form of any facet-defining inequality for STSP($n$) is also facet-defining for GTSP($n$).[1] Finally, we give sufficient conditions for an inequality facet-defining for GTSP($n$) to define a facet of STSP($n$).

Based on these results, the process of finding new valid inequalities for STSP($n$) and of proving that they are facet-defining can go along the following four steps.

1. One first restricts the attention to a (possibly sparse) spanning graph $\bar{G} = (V, \bar{E})$ and proves that an inequality, say $cx \ge c_0$, defines a facet of GTSP($\bar{G}$). This task may take advantage from the sparsity of $\bar{G}$ and from the fact that using walks, rather than Hamiltonian cycles, simplifies the proofs considerably. Note that $\bar{G}$ only has to be connected to have a full-dimensional polyhedron GTSP($\bar{G}$) and actually may not be Hamiltonian at all. The graph $\bar{G}$ is called the *skeleton* of the inequality (see Naddef and Rinaldi [17, pp. 373–374]).

2. Using a standard sequential lifting procedure, one then extends the inequality $cx \ge c_0$ to become a facet-defining inequality for GTSP($n$). This is done by choosing an ordering $\langle e_1, e_2, \ldots, e_r \rangle$ of the edges of the set $E_n \backslash \bar{E}$ and then, for each $l = 1, \ldots, r$, by assigning the smallest coefficient to $e_l$ such that the length of the shortest walk in $G_l = (V, \bar{E} \cup \bigcup_{t=1}^{l} e_t)$ is $c_0$. The resulting inequality is facet-defining for GTSP($n$) by construction. Different orderings of the edges in $E_n \backslash \bar{E}$ yield, in general, different inequalities. When this is not the case, we say that the skeleton is *stable* (see Naddef and Rinaldi [17]). Examples of a skeleton that is stable and of one that is not are given in §2.

3. Once a facet-defining inequality for GTSP($n$) has been obtained, one tries to show that it defines also a facet of STSP($n$) by proving that one of the mentioned sufficient conditions is satisfied.

4. To describe in a compact way a large family of facet-defining inequalities, one can apply some operations that we describe in Naddef and Rinaldi [17, 18] that generate new inequalities from known ones, using the inequalities produced at Step 3 as building blocks.

It is easy to see that a $TT$ inequality cannot have negative coefficients. However, it can have coefficients with value zero. A $TT$ inequality $cx \ge c_0$ defined on $\mathbb{R}^{E_n}$ with $c_e > 0$ for all $e \in E_n$ is called *simple*. Simple inequalities have a peculiar geometric property. They define all bounded facets of GTSP($n$).

Let $(U : W)$ denote the edge set $\{(u, w) \mid u \in U, w \in W\}$. Suppose we are given a $TT$ inequality $cx \ge c_0$ which is not simple. It is easy to see that $V$ can be partitioned into $p$ sets $V^1, \ldots, V^p$ such that:

[1] Until very recently, no examples were known of facet-defining inequalities for GTSP($n$) that are provably not facet-defining for STSP($n$). Some of such inequalities are exhibited by Oswald et al. [23].

(a) $c_e = 0$ for all $e \in \gamma(V^i)$, $i = 1, \ldots, p$;
(b) for all $i \neq j \in \{1, \ldots, p\}$, $c_e = c_f > 0$ for $e, f \in (V^i : V^j)$.
Since the edges linking two vertices contained in a same subset $V^i$ have a zero coefficient, these edges do not play any structural role in the definition of the inequality, only the edges linking the various subsets do. In fact, as far as GTSP($n$) is concerned, given a facet-defining inequality, one can add as many vertices as desired to any subsets $V^i$ and obtain a facet-defining inequality for GTSP($n'$), $n' > n$. This suggests to first study the case in which $|V^i| = 1$ for all $i$, i.e., to study first simple facet-defining inequalities.

The subgraph of $G$ induced by the set obtained by taking one representative node for each of the sets $V_i$ for $i = 1, \ldots, p$ is a complete graph $K_p$ on $p$ nodes. The *simple inequality associated with* $cx \geq c_0$ is the inequality $\bar{c}\bar{x} \geq c_0$ defined by it on the complete graph $K_p$ with

$$\bar{c}(u_i, u_j) = c_e, \quad e \in (V^i : V^j) \qquad \text{for all } 1 \leq i < j \leq p.$$

From now on, every time we refer to an inequality of type $\mathscr{A}$ without using the attribute "simple" we mean that the inequality may have zero coefficients and that it is associated with a simple inequality of type $\mathscr{A}$.

The paper is organized as follows. In §2 we show that the simple *path*, *wheelbarrow*, and *bicycle* inequalities (simple PWB for short), proved by Cornuéjols et al. [5] to be facet-defining for GTSP($n$), define also facets of STSP($n$).

In §3 we extend the result to all PWB inequalities. In §4 we prove that the same holds for a generalization of these inequalities, which we call *extended* PWB inequalities. PWB and extended PWB are obtained from the simple PWB inequalities by applying the operations mentioned at step 4 of our "facet-hunting" process described above. In §5 we study some compositions of PWB inequalities that yield a large superclass of the clique-tree inequalities.

**2. The simple PWB inequalities.** The PWB inequalities were first defined by Cornuéjols et al. [5]. We give here the alternate definition proposed by Naddef and Pochet [15] that has been more widely used in the recent literature on the traveling salesman problem and, being based on handles and teeth, is more similar to the classical definition of the comb inequalities, of which the PWB are a generalization.

Definition 2.1. A *k-PWB configuration* is a quadruple $\langle \mathscr{H}, \mathscr{T}, \alpha, \beta \rangle$, where $\mathscr{H} = \{H_r \mid r = 1, \ldots, h\}$ and $\mathscr{T} = \{T_i \mid i = 1, \ldots, k\}$ are two collections of subsets of $V$ called the *handles* and the *teeth* of the configuration, respectively, and $\alpha$ and $\beta$ are integer vectors of $h$ and $k$ components, respectively. Each handle $H_r$ has associated the component $\alpha_r$ and each tooth $T_i$ has associated the component $\beta_i$. The vectors $\alpha$ and $\beta$ satisfy the *equality interval property* (see Definition 2.2 below); moreover, the handles and the teeth satisfy the following conditions:

$$H_r \subset H_{r+1} \quad \text{for } 1 \leq r \leq h-1 \tag{3}$$

$$H_1 \cap T_i \neq \varnothing \quad \text{for } i = 1, \ldots, k \tag{4}$$

$$T_i \backslash H_h \neq \varnothing \quad \text{for } i = 1, \ldots, k \tag{5}$$

$$T_i \cap T_j = \varnothing \quad \text{for } 1 \leq i < j \leq k \tag{6}$$

$$(H_{r+1} \backslash H_r) \Big\backslash \bigcup_{i=1}^{k} T_i = \varnothing \quad \text{for } 1 \leq r \leq h-1 \tag{7}$$

The above conditions state that the handles are linearly nested (3), that the innermost handle intersects all teeth (4), that each tooth has at least one node not contained in any handle (5), that the teeth are pairwise disjoint (6), and, finally, that the node of each handle is contained either in $H_1$ or in a tooth (7).

In Figure 3, an example of a PWB configuration with three teeth and six handles is shown. The black-filled points in the figure represent nonempty sets of nodes while the white filled points represent possibly empty sets of nodes. The union of all these node sets gives $V$. It is easy to verify that the handles and the teeth satisfy the conditions (3)–(7). In particular, by Condition (7), the nodes that are not contained in any tooth can either belong to $H_1$ (in the figure, the set of these nodes is marked as $Y$) or belong to the complement of $H_h$ (the set of these nodes is marked as $Z$).

To complete Definition 2.1, we have to define the *equality interval property*. To this purpose, we need the notion of *handle interval* relatively to a given tooth $T_i$. A *handle interval* relatively to a given tooth $T_i$ of a $k$-PWB configuration is the index set $R \subseteq \{1, \ldots, h\}$ of a maximal set of handles that have the same intersection with $T_i$. By definition, a handle interval is made of consecutive indices. Moreover, the index set $\{1, \ldots, h\}$ is partitioned into $s_i$ handle intervals relatively to a tooth $T_i$. For example, in the PWB configuration of Figure 3, the handle intervals relative to $T_1$ are $\{1, 2, 3\}$ and $\{4, 5, 6\}$; those relative to $T_3$ are $\{1\}$, $\{2, 3\}$, $\{4, 5\}$, and $\{6\}$.

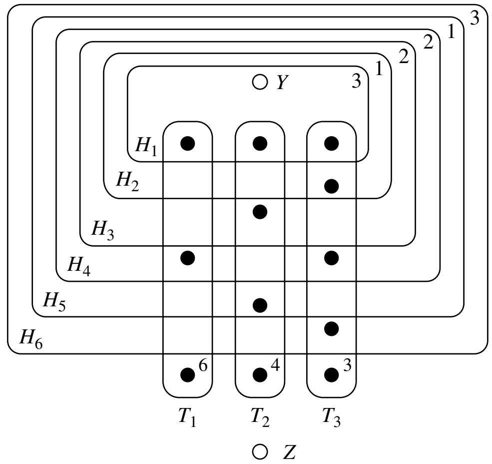


FIGURE 3. Example of a PWB configuration.

DEFINITION 2.2. The vectors $\alpha$ and $\beta$ of a $k$-PWB configuration $\langle \mathcal{H}, \mathcal{T}, \alpha, \beta\rangle$ satisfy the *equality interval property* if $\sum_{r\in R} \alpha_r = \beta_i$ holds for each tooth $T_i$ and for each handle interval $R$ relative to $T_i$.

It is easy to check that the integers associated with each handle and tooth in Figure 3 define vectors $\alpha$ and $\beta$ that do satisfy the equality interval property; thus, the one shown is indeed a PWB configuration.

DEFINITION 2.3. A $k$-PWB inequality associated with the $k$-PWB configuration $\langle \mathcal{H}, \mathcal{T}, \alpha, \beta\rangle$ is the inequality

$$\sum_{r=1}^{h} \alpha_r x(\delta(H_r)) + \sum_{i=1}^{k} \beta_i x(\delta(T_i)) \geq (k+1)\sum_{r=1}^{h} \alpha_r + 2\sum_{i=1}^{k} \beta_i \tag{8}$$

Note that the coefficient of any edge $e$ is easily computed by adding up the coefficient of all the sets whose border is crossed by $e$.

The coefficient vector of inequality (8) is the positive weighted sum of incident vectors of cocycles in $G$. Thus, each coefficient satisfies the triangular inequality. It is not difficult to verify that also condition (b) of Definition 1.1 is satisfied, hence (8) is in $TT$ form.

A comb inequality is a special case of a PWB-inequality. Its configuration has $|\mathcal{H}| = 1$, and the components of $\alpha$ and $\beta$ have all value 1.

REMARK 2.1. If the number of teeth $k$ is even, then the inequality (8) is not valid for the STSP($n$).

DEFINITION 2.4. A $k$-PWB configuration (inequality) is called a *$k$-path*, a *$k$-wheelbarrow*, or a *$k$-bicycle configuration* (*inequality*), depending on whether both the subsets $H_1 \setminus \bigcup_{i=1}^{k} T_i$ and $(V\setminus H_1)\setminus \bigcup_{i=1}^{k} T_i$ are nonempty, or only one of two is empty, or they are both empty.

The equality handle interval property is very binding, and for an arbitrarily chosen quadruple $\langle \mathcal{H}, \mathcal{T}, \alpha, \beta\rangle$ that satisfies the conditions (3)–(7) there are very few chances that it is satisfied. For generating all possible $k$-PWB configuration, a different approach might be more useful where the elements of the quadruple $\langle \mathcal{H}, \alpha, \mathcal{T}, \beta\rangle$ are generated by the following simple procedure.

PROCEDURE 2.1. Input: *any odd $k \geq 3$, any $k$-tuple of positive integers $(n_1, \ldots, n_k)$, with $n_i \geq 2$ for $i \in \{1, \ldots, k\}$ and any partition of $V$ into the sets $S_Y$, $S_Z$, $S_1^1, \ldots, S_{n_1}^1$, $S_1^2, \ldots, S_{n_2}^2, \ldots, S_1^k, \ldots, S_{n_k}^k$, where only $S_Y$ and $S_Z$ are allowed to be empty.* Output: *a quadruple $\langle \mathcal{H}, \mathcal{T}, \alpha, \beta\rangle$ satisfying the equality handle interval property and* (3)–(7).

(i) *Let $\zeta$ be the least common multiple of the integers $n_1 - 1, \ldots, n_k - 1$;*
(ii) *for $i = 1, \ldots, k$, let $T_i = \bigcup\{S_j^i \mid j = 1, \ldots, n_i\}$ and $\beta_j = \zeta/(n_i - 1)$;*
(iii) *let $H_1 = \bigcup\{S_1^i \mid i = 1, \ldots, k\} \cup \{Y\}$ and $\alpha_1 = \min\{\beta_i \mid i = 1, \ldots, k\}$;*
(iv) *for $i = 1, \ldots, k$, let $j_i = 1$ and $\beta_i' = \beta_i - \alpha_1$;*
(v) *let $l = 1$;*

(vi) *while* $j_i < n_i - 1$ *for some* $i \in \{1, \ldots, k\}$ *do the following*:
(a) *increment* $l$ *by* 1;
(b) *for any* $i$ *such that* $\beta'_i = 0$, *let* $\beta'_i = \beta_i$ *and increment* $j_i$ *by* 1;
(c) *let* $H_l = H_{l-1} \cup \bigcup\{S^i_{j_i} \mid i = 1, \ldots, k\}$ *and* $\alpha_l = \min\{\beta'_i \mid i = 1, \ldots, k\}$;
(d) *for* $i = 1, \ldots, k$, *decrement* $\beta'_i$ *by* $\alpha_l$;

In the case of Figure 3, let us set $k = 3$ and consider the partition of $V$ given by the sets represented by the black- and white-filled points. Let $S_Y$ and $S_Z$ be the sets marked as $Y$ and $Z$, respectively. Finally, for $j = 1, \ldots, k$, let $S^j_1, \ldots, S^j_{n_j}$ be the sets contained in the tooth $T_j$ and ordered from the top to the bottom. It is easy to verify that the above procedure generates the teeth, the handles, and the coefficients $\alpha$ and $\beta$ as those shown in the drawing.

If the end nodes of an edge of $G$ belong to the same handles and teeth of a $k$-PWB configuration, then the coefficient of this edge is zero in (8). On the contrary, if there is a handle or a tooth that contains only one of the end nodes, then the edge belongs to one of the cocycles of the handles and of the teeth, and its coefficient is positive in (8). Consequently, if there are no two nodes of $G$ that are contained in the same handles and teeth of a $k$-PWB configuration, the corresponding $k$-PWB inequality is simple. We call *simple* the $k$-PWB configuration in this case. For example, the PWB configuration of Figure 3 is simple if each black-filled point represents a single node of $V$ and each white-filled point represents a single node of $V$ or the empty set. In this section we only consider simple $k$-PWB configurations and inequalities.

A simple $k$-PWB configuration induces a labeling of the nodes of $G$ that will be widely used throughout the paper. If $H_1 \setminus \bigcup_{i=1}^k T_i$ is nonempty, its only node is labeled $Y$; similarly, if $(V \setminus H_1) \setminus \bigcup_{i=1}^k T_i$ is nonempty, its only node is labeled $Z$. The $n_i$ nodes of a tooth $T_i$ can be linearly ordered in such a way that $w \in T_i$ precedes $v \in T_i$ if it is contained in more handles than $v$. Each node of $T_i$ is labeled $u^i_j$, where $j$ is the position of the node in such an ordering. Therefore, $u^i_1$ is the node contained in $H_1$ and $u^i_{n_i}$ is contained in no handles. In what follows, paths (teeth) indices are taken modulo $k$. Therefore, the value of the superscript of a label $u^i_j$ is defined as $((i-1) \bmod k) + 1$; thus, for example, $u^{k+i}_j = u^i_j$. Note that this labeling is in agreement with the names given to the sets $S^i_j$ in Procedure 2.1. Consequently, given any odd $k \geq 3$ and any $k$-tuple of positive integers $(n_1, \ldots, n_k)$, with $n_i \geq 2$ for $i \in \{1, \ldots, k\}$ and the labeling

$$\{Y, Z\} \cup \{u^i_j \mid j \in \{1, \ldots, n_i\}, \quad i \in \{1, \ldots, k\}\}. \tag{9}$$

Procedure 2.1 uniquely identifies a $k$-PWB configuration $\langle \mathcal{H}, \mathcal{T}, \alpha, \beta \rangle$. Consequently, we refer to the labeling (9) as to a $k$-PWB configuration.

Depending on whether a $k$-PWB is a $k$-path (labels $Y$ and $Z$ are present), a $k$-wheelbarrow (only label $Z$ is present), or a $k$-bicycle ($Y$ and $Z$ are missing) configuration, the labeling (9) is denoted by $P(n_1, \ldots, n_k)$, $W(n_1, \ldots, n_k)$, or $B(n_1, \ldots, n_k)$, respectively.

For convenience we label $u^i_0 = Y$, and $u^i_{n_{i+1}} = Z$, for all $i = 1, \ldots, k$. Following the terminology used by Cornuéjols et al. [5], we call the nodes $Y$ and $Z$ the *odd nodes* of the configuration, and we call all the other nodes *even* (see Figure 4).

In the following, to simplify the notation, by $\pi^i$ we denote the edge set

$$\pi^i = \{(u^i_j, u^i_{j+1}) \mid j \in \{1, \ldots, n_i - 1\}\} \quad \text{for } i \in \{1, \ldots, k\}.$$

An edge of any set $\pi^i$ or any edge $(u^i_0, u^i_1)$ or $(u^i_{n_i}, u^i_{n_{i+1}})$ for $i \in \{i, \ldots, k\}$ is called a *path edge*. By $C_Y$ and $C_Z$ we denote the cycles whose edge sets are $\{(u^i_1, u^{i+1}_1) \mid i \in \{1, \ldots, k\}\}$ and $\{(u^i_{n_i}, u^{i+1}_{n_{i+1}}) \mid i \in \{1, \ldots, k\}\}$,

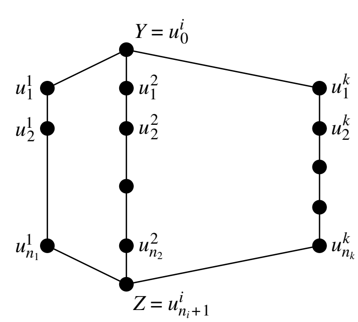


FIGURE 4. The labeling induced by the 3-path configuration of Figure 3 and the skeleton of the corresponding 3-path inequality.

respectively. An edge of any of these two cycles is call a *cycle edge*. An edge of the type $(u_1^i, u_1^r)$ or $(u_{n_i}^i, u_{n_r}^r)$ with $2 \le ((r-i) \bmod k) \le k-2$ is called a *chord* (a *short chord* if one of the two inequalities holds with equality) of the cycle $C_Y$ or $C_Z$, respectively.

Now it is easy to relate the definition above to the following original definition of the path inequality given by Cornuéjols et al. [5]:

Definition 2.5. The *simple k-path inequality* associated with $P(n_1, \ldots, n_k)$ is the following inequality on $\mathbb{R}^{E_n}$, with $n = 2 + \sum_{i=1}^k n_i$, in which $\eta_i = 1/(n_i - 1)$ for $i \in \{1, \ldots, k\}$:

$$cx \ge c_0 = k + 1 + 2\sum_{i=1}^{k} \eta_i, \tag{10}$$

where

$$c_e = \begin{cases} |j-q|\eta_i & \text{for } e = (u_j^i, u_q^i),\ \ j \ne q,\ \ i \in \{1, \ldots, k\},\\ & \quad j, q \in \{0, \ldots, n_i + 1\},\\ \eta_i + \eta_r + |(j-1)\eta_i - (q-1)\eta_r| & \text{for } e = (u_j^i, u_q^r),\ \ i \ne r,\ \ i, r \in \{1, \ldots, k\},\\ & \quad j \in \{1, \ldots, n_i\},\ \ q \in \{1, \ldots, n_r\},\\ 1 & \text{for } e = (Y, Z) \end{cases} \tag{11}$$

The coefficient vectors of the simple $k$-wheelbarrow and of the simple $k$-bicycle inequalities are restrictions of the coefficient vector of a simple $k$-path inequality, obtained by removing all the edges incident with $Y$ and all the edges incident with $Y$ and $Z$, respectively.

The coefficients of the Equation (8) are exactly those of Definition 2.5 multiplied by the least common multiple of the numbers $n_i - 1$, $i = 1, \ldots, k$. Therefore, the two inequalities of the Definitions 2.4 and 2.5 define the same face of STSP($n$).

If $n_i = p$ for $i \in \{1, \ldots, k\}$, the simple path, wheelbarrow, and bicycle configurations and their corresponding inequalities are called *p-regular*. The 2-regular PWB inequalities are the comb inequalities.

For the special case of regular configurations, the coefficients of the associated inequalities can be written as follows:

$$c_0 = (k+1)(p+1) - 2, \tag{12}$$

$$c_e = \begin{cases} |j-q| & \text{for } e = (u_j^i, u_q^i),\ \ i \in \{1, \ldots, k\},\ \ j \ne q,\ \ j, q \in \{0, \ldots, p+1\},\\ |j-q| + 2 & \text{for } e = (u_j^i, u_q^r),\ \ i \ne r,\ \ i, r \in \{1, \ldots, k\},\ \ j, q \in \{1, \ldots, p\},\\ p-1 & \text{for } e = (Y, Z). \end{cases} \tag{13}$$

**2.1. The GTSP case.** Cornuéjols et al. [5] show that the $k$-path, the $k$-wheelbarrow, and the $k$-bicycle inequalities define facets of GTSP($G$), provided that $G$ has a stable skeleton of the inequality as a subgraph. The stable skeleton for a $k$-path is given by the union of all the path edges (see an example in Figure 4), for the $k$-wheelbarrow it is given by the union of all the path edges and $C_Y$, and for the $k$-bicycle it is given by the union of all the path edges, $C_Y$, $C_Z$, and the short chords of these two cycles.

We give here a proof that the $k$-bicycle inequalities define facets of GTSP($n$), which is slightly different from that of Cornuéjols et al. [5] as it also specifies a special set of tight walks that is used in the following proofs.

We make use the following result proven by Cornuéjols et al. [5, Theorem 3.4]:

Theorem 2.2. *The simple bicycle inequalities are valid for* GTSP($n$).

To prove the next theorem we need a few more definitions and a simple lemma.

A walk $W$ is said to be *minimal* if it does not contain another walk. For every inequality $fx \ge f_0$ in $\mathbb{R}^{E_n}$, valid for STSP($n$) or for GTSP($G$), we call *tight* the Hamiltonian cycles and the minimal walks whose representative vectors satisfy the inequality with equality. The set of tight Hamiltonian cycles and tight walks for $f$ are denoted by $\mathcal{H}_f^=$ and $\mathcal{W}_f^=$, respectively. The dimension of GTSP($G$) is $|E|$ (i.e., GTSP($G$) is full-dimensional) as long as $G$ is connected. A *walk basis* of an inequality $cx \ge c_0$ defining a facet of GTSP($G$) is any set $\mathcal{B}_c$ of $|E|$ walks in $\mathcal{W}_c^=$ whose representative vectors are linearly independent. Note that linear and affine independence of the representative vectors are equivalent since the zero vector does not belong to GTSP($G$). A *Hamiltonian cycle* *basis* of an inequality $cx \geq c_0$ defining a facet of STSP($n$) is a set $\mathscr{C}_c$ of $|E_n| - |V|$ Hamiltonian cycles in $\mathscr{H}_c^=$ whose representative vectors are linearly independent.

If $z$ is a vector in $\mathbb{R}^{E_n}$, by $z_{\bar{E}}$ we denote its restriction (projection) to the subspace $\mathbb{R}^{\bar{E}}$ for $\bar{E} \subset E_n$.

Lemma 2.1. *Let $G = (V, \bar{E})$ be a graph with $\bar{E} \subset E_n$, let $e^*$ be an edge in $E_n \backslash \bar{E}$ and $fx \geq f_0$ be a valid inequality for* GTSP($n$). *If $f_{\bar{E}} x_{\bar{E}} \geq f_0$ is facet-defining for* GTSP($G$) *and there exists a walk of $\mathscr{W}_f^=$ containing $e^*$ and only edges of $\bar{E}$, then the inequality $f_{E'} x_{E'} \geq f_0$ is facet-defining for* GTSP($G'$), *where $E' = \bar{E} \cup \{e^*\}$ and $G' = (V, E')$.*

Proof. Let $\mathscr{B}$ be a walk basis of $f_{\bar{E}} x_{\bar{E}} \geq f_0$ and let $W$ be a walk of $\mathscr{W}_f^=$ containing $e^*$ and only edges of $\bar{E}$. Then $\mathscr{B} \cup \{W\}$ is a set of walks whose representative vectors are linearly independent, i.e., it is a walk basis for $f_{E'} x_{E'} \geq f_0$. □

The use of Lemma 2.1 is quite evident in our context. Given the stable skeleton $\bar{G}$, once we prove that the inequality is facet-defining for GTSP($\bar{G}$), it is straightforward to prove that it is facet-defining for GTSP($n$).

In the following theorem and quite often throughout the paper we make use of some special walks given by the following

Definition 2.6. For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i - 1\}$, we define (see Figure 5)

$$\begin{aligned} W_A^i = 2\pi^i &\cup \{(u_{n_s}^s, u_{n_{s+1}}^{s+1}) \mid s \in \{i, i+2, \ldots, i+k-1\}\} \\ &\cup \{(u_1^s, u_1^{s+1}) \mid s \in \{i+1, i+3, \ldots, i+k-2\}\} \\ &\cup \{\pi^l \mid l \in \{1, \ldots, k\},\ l \neq i\}, \end{aligned}$$

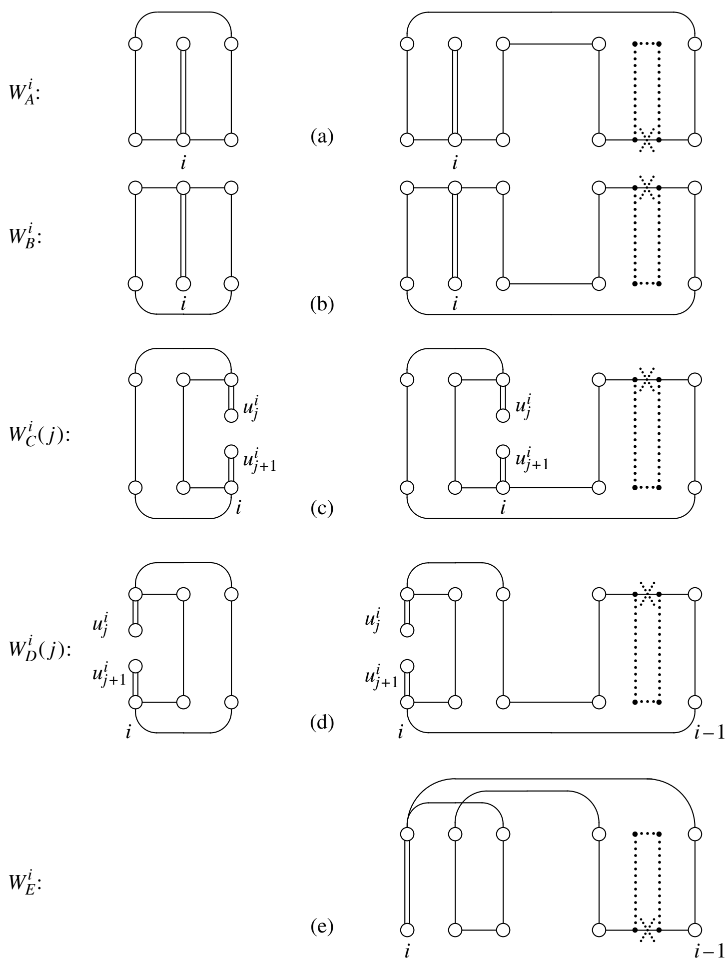


Figure 5. The tight walks of Definition 2.6. In (c)–(d) and in (e) the paths of (a)–(b) are circularly permuted to the left and to the right, respectively.

$$\begin{aligned}
W_B^i = 2\pi^i &\cup \{(u_1^s, u_1^{s+1}) \mid s \in \{i, i+2, \dots, i+k-1\}\} \\
&\cup \{(u_{n_s}^s, u_{n_{s+1}}^{s+1}) \mid s \in \{i+1, i+3, \dots, i+k-2\}\} \\
&\cup \{\pi^l \mid l \in \{1, \dots, k\},\ l \neq i\},
\end{aligned}$$

$$W_C^i(j) = W_A^i - (u_1^{i-2}, u_1^{i-1}) - 2(u_j^i, u_{j+1}^i) + (u_1^{i-2}, u_1^i) + (u_1^{i-1}, u_1^i),$$

$$W_D^i(j) = W_A^i - (u_1^{i+1}, u_1^{i+2}) - 2(u_j^i, u_{j+1}^i) + (u_1^i, u_1^{i+2}) + (u_1^i, u_1^{i+1}),$$

$$W_E^i = W_B^i - (u_1^i, u_1^{i+1}) - (u_1^{i+2}, u_1^{i+3}) + (u_1^i, u_1^{i+2}) + (u_1^{i+1}, u_1^{i+3}).$$

The walks $W_A^i$, $W_B^i$, $W_C^i(2)$, $W_D^i(2)$, and $W_E^i$ are shown in Figures 5(a), (b), (c), (d), and (e), respectively, for both the case $k = 3$ and $k \geq 5$. Observe that all the edges of these walks are either path edges, or cycle edges, or short chords of the cycle $C_Y$. To unclutter the figures, all the edge sets $\pi^i$ have cardinality 1 in most of them. However, any path edge may be replaced by a path with an arbitrary number of edges. The nodes $u_1^i$ for $i \in \{1, \dots, k\}$ are those at the top of the drawings. Moreover, we have operated in the figures a circular permutation of the paths (teeth). For example, in the Figure 5(a and b) the teeth are ordered from left to right with the indices $i-1, i, \dots, k, 1, \dots, i-2$. In Figure 5(d and e) the order is $i, i+1, \dots, k, 1, \dots, i-1$. To extend the walks shown in the figures to the case when $k$ is greater than 5, split a cycle edge into three edges, then remove the central edge (marked with a cross in the figures), and finally connect its end nodes by the path marked by dotted lines in the figures. In the following we will use the same convention for all the figures involving PWB configurations.

Remark 2.2. The walks of Definition 2.6 satisfy a bicycle inequality at equality. For the sake of simplicity, in the following we refer to these walks also dealing with wheelbarrow and path inequalities. In these cases, though, we will always assume that each of these walks be modified by inserting each required odd node, by removing any cycle edge (of the cycle associated to the odd node) from the walk, and by adding an edge from each of its end nodes to the odd node.

Theorem 2.3. *A simple bicycle inequality $cx \geq c_0$ associated with the simple $k$-bicycle configuration $B(n_1, \dots, n_k)$ is facet-defining for* GTSP$(n)$ *and has a walk basis $\mathcal{B}_c$ with the following properties*:
(a) *Every walk of $\mathcal{B}_c$ intersects the edge set of the cycle $C_Y$.*
(b) *Every walk of $\mathcal{B}_c$ intersects the edge set of the cycle $C_Z$.*
(c) *For each edge $e \in C_Y$ there exists $e' \in C_Y$ and a walk $W \in \mathcal{B}_c$ containing both $e$ and $e'$.*
(d) *For each edge $e \in C_Z$ there exists $e' \in C_Z$ and a walk $W \in \mathcal{B}_c$ containing both $e$ and $e'$.*

Proof. We prove first that the inequality $c_S x_S \geq c_0$ is facet-defining for GTSP$(G)$, where $G = (V, S)$ is the subgraph of $K_n$ whose edges are all the path edges, all the cycle edges, and all the short chords of the cycle $C_Y$.

The inequality $c_S x_S \geq c_0$ is valid by Theorem 2.2 and is supporting since all the walks of Definition 2.6 belong to $\mathcal{W}_c^=$. We show that the set of all these walks constitutes a basis $\mathcal{B}_1$ for $c_S x_S \geq c_0$.

Let $fx \geq c_0$ be an inequality defining a facet of GTSP$(G)$ that contains the face defined by $c_S x_S \geq c_0$, i.e., such that $\mathcal{W}_c^= \subseteq \mathcal{W}_f^=$.

From $f(W_C^i(j')) = f(W_C^i(j''))$ for $j' \neq j'' \in \{1, \dots, n_i - 1\}$ it follows that

$$f(u_{j'}^i, u_{j'+1}^i) = f(u_{j''}^i, u_{j''+1}^i) = g_i \quad \text{for } i \in \{1, \dots, k\},\ \ j' \neq j'' \in \{1, \dots, n_i - 1\}. \tag{14}$$

For notational convenience we set $b_i = f(u_{n_i}^i, u_{n_{i+1}}^{i+1})$, $d_i = f(u_1^i, u_1^{i+1})$, $e_i = f(u_1^i, u_1^{i+2})$, for $i \in \{1, \dots, k\}$ (see Figure 6). From $f(W_A^i) = f(W_B^{i+1})$ for $i \in \{1, \dots, k\}$, it follows that

$$(n_i - 1)g_i + b_i = (n_{i+1} - 1)g_{i+1} + d_i \quad \text{for } i \in \{1, \dots, k\}. \tag{15}$$

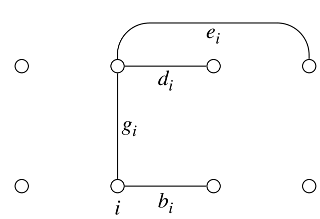


Figure 6. The coefficients of some relevant edges.

From $f(W_B^i) = f(W_A^{i+1})$ for $i \in \{1, \ldots, k-1\}$, it follows that

$$(n_i - 1)g_i + d_i = (n_{i+1} - 1)g_{i+1} + b_i \quad \text{for } i \in \{1, \ldots, k-1\}. \tag{16}$$

The Equations (15) and (16) imply

$$b_i = d_i \quad \text{for } i \in \{1, \ldots, k-1\} \tag{17}$$

and

$$g_i = \frac{h}{n_i - 1} \quad \text{for } i \in \{1, \ldots, k-1\}. \tag{18}$$

From $f(W_A^{i+1}) = f(W_C^{i+1})$ for $i \in \{1, \ldots, k\}$ it follows that

$$e_{i-1} = d_{i-1} - d_i + \frac{2h}{n_{i+1} - 1} \quad \text{for } i \in \{1, \ldots, k\}, \tag{19}$$

and from $f(W_C^{i+1}) = f(W_D^{i-1})$ for $i \in \{2, \ldots, k\}$ we have

$$d_i = d_{i-1} + \frac{h}{n_i + 1} - \frac{h}{n_i - 1} \quad \text{for } i \in \{2, \ldots, k\}. \tag{20}$$

If $k = 3$ then $d_1 = e_2$, otherwise from $f(W_E^1) = f(W_B^1)$ it follows that $d_1 + d_3 = e_1 + e_2$ and from (19) we have

$$d_i = \frac{h}{n_i - 1} + \frac{h}{n_{i+1} - 1} \quad \text{for } i \in \{1, \ldots, k\}. \tag{21}$$

If for the bicycle inequality $cx \geq c_0$ we use the Definitions 2.1 and 2.3, with the vectors $\alpha$ and $\beta$ computed with Procedure 2.1, then setting $h = r\zeta$, where $\zeta$ is as defined in Procedure 2.1, one can see that $f_e = rc_e$, for $e$ in the skeleton. Since we chose the same right-hand sides for both the inequalities, we have $r = 1$. Similarly, if we use Definition 2.5, then setting $r = 1/\zeta$, i.e., setting $h = 1$, we obtain the same result. Therefore, $f \equiv c_S$, the inequality $c_S x_S \geq c_0$ is facet-defining for GTSP($G$), and the $|S|$ walks of the set $\mathscr{B}_1$ are linearly independent.

To complete the proof, we take a sequence $\langle e_1, e_2, \ldots, e_s \rangle$ of the edges in $E_n \backslash S$ and we construct the sequences of nested edge sets $F_0, F_1, \ldots, F_s$, with $F_0 = S$ and $F_l = F_{l-1} \cup \{e_l\}$, for $l = 1, 2, \ldots, s$. Then for every $e_l$ we exhibit a walk of $\mathscr{W}_c^=$ containing $e_l$ and only edges in $F_{l-1}$. Consequently, by Lemma 2.1, $cx \geq c_0$ defines a facet of GTSP($n$). We describe now these walks.

Let $e = (u_j^i, u_q^r)$ be any edge of $E_n \backslash S$ and let $d(e, k) = \min\{(i - r) \bmod k, (r - i) \bmod k\}$. Depending on the value of $d(e, k)$ we have different kinds of walks. Clearly, $0 \leq d(e, k) \leq (k-1)/2$.

For an edge $e = (u_j^i, u_{j+q}^i)$ the value of $d(e, k)$ is zero, and we consider the walk

$$W_0(e) = W_A^i \backslash \{(u_{j+t}^i, u_{j+t+1}^i) \mid t = 1, \ldots, q-1\} + e.$$

All the edges of $W_0(e)$, except $e$, belong to $S$.

For $d(e, k) = 1, \ldots, 4$ the corresponding walks $W_1(e), \ldots, W_4(e)$ are shown in Figure 7(a), ..., (d), respectively. The reader should not be confused by the fact that in the figures the end nodes of $e = (u_j^i, u_q^r)$ always have $1 < j < n_i$ and $1 < q < n_r$. Actually, the walks shown can be easily modified to accommodate the cases for all possible values of $j$ and $q$ in the sets $\{1, \ldots, n_i\}$ and $\{1, \ldots, n_r\}$, respectively, as long as the inequality $(q-1)\eta_r \geq (j-1)\eta_i$ is satisfied. If this is not the case, one can consider the mirror images of the walks of Figure 7, which would correspond to take the opposite orientation of the cycles $C_Y$ and $C_Z$ to draw the walks.

The walks corresponding to a chord of the cycle $C_Y$ (when $j = q = 1$) and to a chord of the cycle $C_Z$ (when $j = n_i$ and $q = n_r$) are of the kind of those shown in Figure 7. All walks $W_l(e)$, for $l = 0, \ldots, 4$ have the edge $e$ and only edges in $S$.

If $k \leq 9$, then $d(e, k)$ is never greater than 4. Therefore, the above walks are sufficient to complete the proof that the inequality is facet-defining, no matter how the sequence of the edges in the set $E_n \backslash S$ is chosen.

It is easy to verify that if $d(e, k) \geq 5$, it is not possible to construct a walk that uses only $e$ and edges in $S$. However, it is always possible to construct a walk $W_5(e)$ that contains $e$, some edges in $S$, and short chords of the cycle $C_Z$. Such a walk is shown in Figure 8, where $k$ has any value greater than 11 and $d(e, k)$ has any value greater than 5. To apply Lemma 2.1, the edges of $E_n \backslash S$ have to be ordered in such a way that any edge $e$ with $d(e, k) \geq 5$ follows all of the short chords of cycle $C_Z$ in the sequence $\langle e_1, e_2, \ldots, e_s \rangle$.

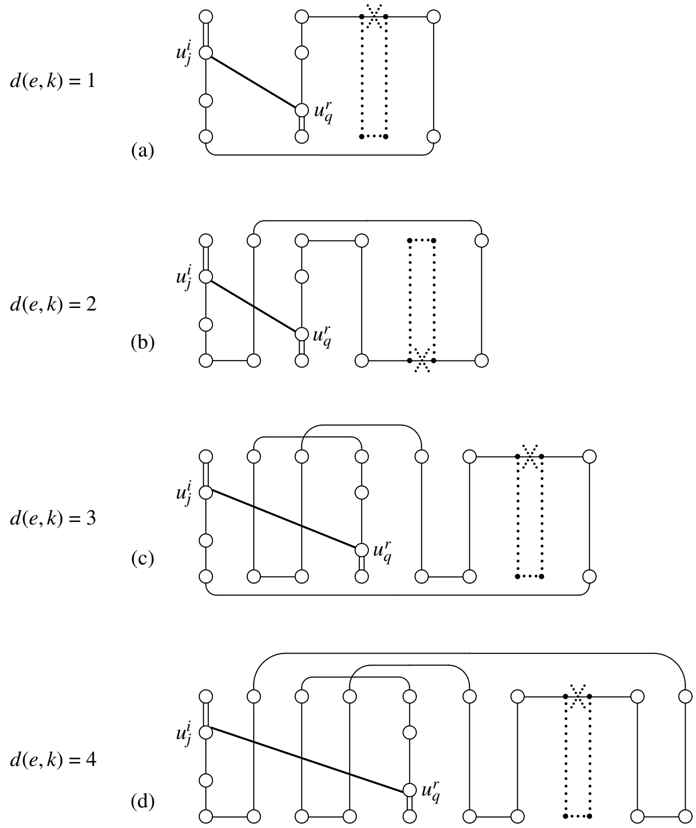


FIGURE 7. Tight walks containing an edge across two paths and only edges of a stable skeleton.

The union of $\mathcal{B}_1$ and all of the walks associated with the edges $e_1, e_2, \ldots, e_s$ is a walk basis $\mathcal{B}_c$ of the inequality $cx \geq c_0$. It is easy to verify, by inspection, that all its walks intersect both the cycles $C_Y$ and $C_Z$.

Finally, for any edge $e = (u_1^i, u_1^{i+1})$ of the cycle $C_Y$, the walk $W_B^i$ contains $e$ and the edge $(u_1^{i-1}, u_1^i)$, which also belongs to $C_Y$. Analogously, for any edge $e = (u_{n_i}^i, u_{n_{i+1}}^{i+1})$ of the cycle $C_Z$, the walk $W_A^i$ contains $e$ and the edge $(u_{n_{i-1}}^{i-1}, u_{n_i}^i)$, which also belongs to $C_Z$. □

REMARK 2.3. As a by-product of Theorem 2.3, we have that a simple bicycle inequality is facet-defining for GTSP$(G)$, where $G = (V, S)$ is the subgraph of $K_n$, whose edges are all of the path edges, all of the cycle edges, and all of the short chords of the cycle $C_Y$. This result is a bit stronger than that given by Cornuéjols et al. [5], where $G$ is required to have also all of the short chords of the cycle $C_Z$.

As it is claimed in the proof of Theorem 2.3, the skeleton $(V, S)$ (where $S$ is the union of the path edges, the edge cycles, and the short chords of $C_Y$) is stable only for $k \leq 9$. One may wonder what happens for $k \geq 11$, or, more precisely, what happens if the lifting sequence described in the proof of the theorem starts with an edge $e$ with $d(e,k) \geq 5$. If the edge $e$ is a chord of $C_Y$, then it is still possible to find a tight walk of the bicycle inequality made of $e$ and of edges in $S$ (see Figure 9a). This implies that the lifting coefficient resulting for $e$ is the same as that given in Definition 2.5. The same is true if $e$ is a chord of $C_Z$ but only for $k = 11$ (see Figure 9b). However, this is no longer true in general; e.g., take the case where $n_i = 2$ for all $i$ and $k = 11$. If one starts the lifting sequence with $e = (u_1^1, u_2^6)$, then the result is $c_e = 1$; if $(u_1^{11}, u_2^6)$ is the second edge in the sequence, the coefficient is again 1. At this point, no matter how the remaining edges are ordered, the

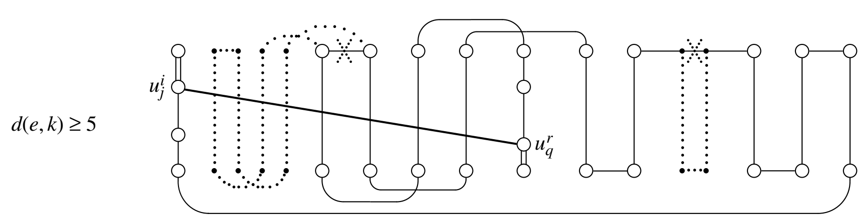


FIGURE 8. Tight walk made of an edge across two paths, edges of a stable skeleton and short chords of $C_Z$.

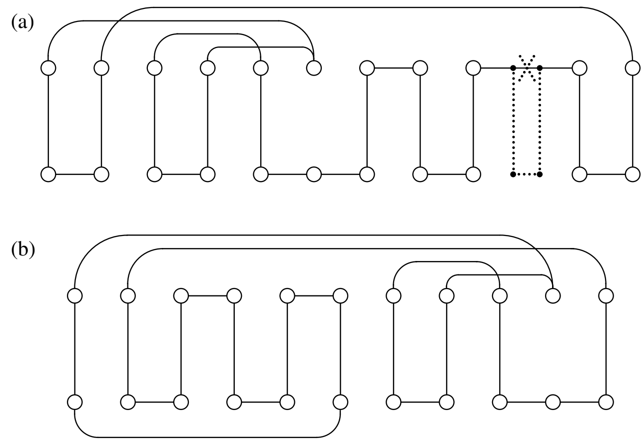


FIGURE 9. Tight walks containing a long chord and only edges of a stable skeleton.

coefficients they get either are the same as in Definition 2.5, or have value 4. The edges having a coefficient that differs from the one of Definition 2.5 are shown in Figure 10, where the edges with coefficient equal to 1 and 4 are drawn with dashed and solid lines, respectively (note that, to unclutter Figure 10, the arrangement of the nodes has been changed with respect to Figure 9). Thus, the inequality in this way described by Figure 10 and Definition 2.5 is a new facet-defining inequality for STSP(22) that shares the same skeleton $(V, S)$ with the bicycle inequality. On the other hand, $(V, S \cup \{(u_1^1, u_2^6), (u_1^{11}, u_2^6)\})$ is a stable skeleton for the new inequality.

**2.2. The STSP case.** For $u \in V$, we say that a walk $W$ is *almost Hamiltonian in $u$* if $u$ has degree 4 in $W$ and every other node of $V$ has degree 2 in $W$.

A basis $\mathscr{C}_c$ of an inequality $cx \geq c_0$, defining a facet of GTSP($n$), is called *canonical* if it contains $|E_n| - n$ Hamiltonian cycles and $n$ almost Hamiltonian walks (i.e., one for each $u \in V$).

The following notion of a set of edges being $c$-connected in a given node is the key for the sufficient conditions, which we give in Naddef and Rinaldi [18], for an inequality facet-defining for GTSP($n$) to preserve such a property for STSP($n$). The notion of $c$-connectedness is explained in more details in that paper.

Let $e = (u, v)$ and $f = (w, y)$ be two distinct edges in $E_n$. We say that $e$ and $f$ are *$c$-adjacent* if they belong to a tight Hamiltonian cycle $H \in \mathscr{H}_c^=$. Let $z$ be a node in $V$; we say that $e$ and $f$ are *$c$-adjacent in $z$* if:

(i) $e$ and $f$ belong to $\Delta_c(z)$;

(ii) there exists a walk $W_z \in \mathscr{W}_c^=$ almost Hamiltonian in $z$ that contains the edges $(z, u)$, $(z, v)$, $(z, w)$, and $(z, y)$;

(iii) $W_z - (z, u) - (z, v) + e$ is a Hamiltonian cycle (and such is also $W_z - (z, w) - (z, y) + f$).

A set of edges $J \subseteq E_n$ is said to be *$c$-connected* if for every pair of distinct edges $f_1$ and $f_2 \in J$ there exists a sequence of $t$ edges $e_1, \ldots, e_t$ in $J$, with $e_1 \equiv f_1$ and $e_t \equiv f_2$, where $e_i$ is $c$-adjacent to $e_{i+1}$, for $i = 1, \ldots, t-1$. A set of edges $J \subseteq E_n$ is said to be *$c$-connected in $z$* if for every pair of distinct edges $f_1$ and $f_2 \in J$ there exists a sequence of $t$ edges $e_1, \ldots, e_t$ in $E_n$ (not necessarily belonging to $J$), with $e_1 \equiv f_1$ and $e_t \equiv f_2$, where $e_i$ and $e_{i+1}$ are $c$-adjacent in $z$, for $i = 1, \ldots, t-1$. Observe that the notion of $c$-connectedness in $z$ is "weaker" than the one of $c$-connectedness, in the sense that, contrary to what happens for the usual concept of connectivity, in this case every subset of a set $c$-connected in $z$ is also $c$-connected in $z$.

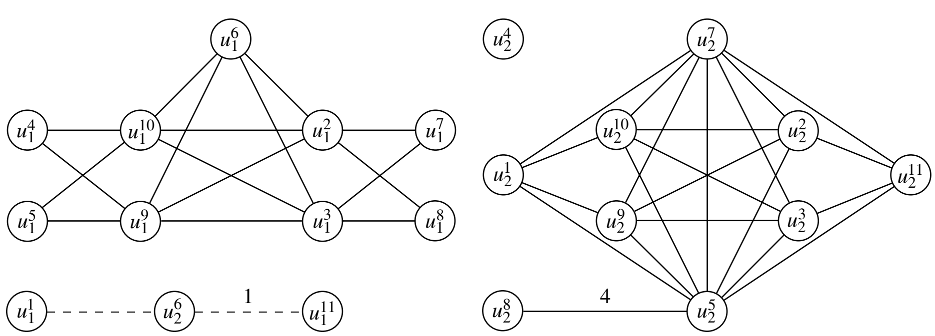


FIGURE 10. Edges in which a new facet-defining inequality differs from that of an 11-bicycle.

Lemma 2.2 (Naddef and Rinaldi [18, Lemma 2.14]). *Let $cx \geq c_0$ be a TT inequality defining a facet of* GTSP($n$). *If $\Delta_c(u)$ is c-connected in u for every $u \in V$, then $cx \geq c_0$ has a canonical basis; thus it is facet-defining for* STSP($n$).

For every ordered triple $\langle u, v, w\rangle$ of distinct nodes in $V$, we call *shortcut* on $\langle u, v, w\rangle$ the vector $s_{uvw} \in \mathbb{R}^{E_n}$ defined by

$$s_{uvw}(e) = \begin{cases} 1 & \text{if } e = (v, w) \\ -1 & \text{if } e \in \{(u, v), (u, w)\} \\ 0 & \text{otherwise.} \end{cases}$$

Adding a shortcut to a walk that is tight for an inequality can produce a new tight walk with fewer edges. More precisely, we have the following

Lemma 2.3 (Naddef and Rinaldi [18, Lemma 2.5]). *Let $cx \geq c_0$ be a tight triangular inequality that is supporting for* GTSP($n$), *and let $W \in \mathcal{W}_c^=$ be a walk having $t > n$ edges and containing the edge e. For every node $u \in V$ with degree $k \geq 4$ in W, there exists a shortcut $s_{uvw}$ such that the edge multiset having representative vector $\chi^W + s_{uvw}$ is a walk with $t-1$ edges belonging to $\mathcal{W}_c^=$ and containing the edge e. Necessarily, the edge $(v, w)$ belongs to $\Delta_c(u)$.*

The sets $\Delta_c(u)$ may contain several edges; therefore, the application of Lemma 2.3 may require a lengthy procedure. The following lemma requires a weaker condition that is also easier to verify.

Lemma 2.4 (Naddef and Rinaldi [18, Lemma 2.15]). *Let $cx \geq c_0$ be a TT inequality defining a facet of* GTSP($n$). *If there exists a basis $\mathcal{B}_c$ of $cx \geq c_0$ such that for every $u \in V$ there exists a nonempty set of edges $J_u \subseteq \Delta_c(u)$ c-connected in u and every walk $W \in \mathcal{B}_c$ can be reduced to an element of $\mathcal{H}_c^=$ by using only shortcuts in the set $\{s_{uvw} \mid (v, w) \in J_u,\ u \in V\}$, then $cx \geq c_0$ has a canonical basis; hence, it is facet-defining for* STSP($n$).

We use now Lemma 2.4 to prove that the simple bicycle inequality defines a facet of STSP($n$) and to derive some properties of one of its canonical bases that will be used in the following to prove a similar theorem for the simple wheelbarrow inequality.

Theorem 2.4. *A simple bicycle inequality $cx \geq c_0$ associated with the simple k-bicycle configuration $B(n_1, n_2, \ldots, n_k)$ is facet-defining for* STSP($n$) *and has a Hamiltonian cycle basis $\mathcal{C}_c$ with the following properties*:

(a) *Every Hamiltonian cycle of $\mathcal{C}_c$ intersects the edge set of the cycle $C_Y$.*
(b) *Every Hamiltonian cycle of $\mathcal{C}_c$ intersects the edge set of the cycle $C_Z$.*
(c) *For each edge $e \in C_Y$ there exists $e' \in C_Y$ and a Hamiltonian cycle $H \in \mathcal{C}_c$ containing both e and $e'$.*
(d) *For each edge $e \in C_Z$ there exists $e' \in C_Z$ and a Hamiltonian cycle $H \in \mathcal{C}_c$ containing both e and $e'$.*

Proof. Let $\mathcal{B}_c$ the walk basis constructed in the proof of Theorem 2.3. For every node $w \in V$ we define a set of edges $J_w \subseteq \Delta_c(w)$ such that, by using shortcuts in the set $\{s_{wyz} \mid (y, z) \in J_w\}$, every walk in $\mathcal{B}_c$ where $w$ has degree 4 can be reduced to a walk where $w$ has degree 2. Then we show that $J_w$ is $c$-connected in $w$ for all $w \in V$. By Lemma 2.4, this implies that the inequality defines a facet of STSP($n$).

For $w = u_j^i$, $i = 1, 2, \ldots, k$, $j = 2, \ldots, n_i - 1$ we set $J_w = \{(u_{j-1}^i, u_{j+1}^i)\}$. In these cases $J_w \subseteq \Delta_c(w)$ and has cardinality 1, and so it is $c$-connected in $w$.

For $w = u_1^i$, $i = 1, 2, \ldots, k$, we set $J_w = \{(u_1^{i-1}, u_2^i), (u_1^{i+1}, u_2^i), (u_1^{i-2}, u_2^i)\} \subseteq \Delta_c(w)$. If $n_i = 2$ let $W_F^i$ be the walk $W_B^i$ of Definition 2.6. If $n_i \geq 3$ we define $W_F^i = W_B^i - 2(u_2^i, u_3^i) - (u_{n_{i-1}}^{i-1}, u_{n_{i-2}}^{i-2}) - \{(u_j^i, u_{j+1}^i) \mid j = 3, \ldots, n_i - 1\} + (u_{n_{i-1}}^{i-1}, u_3^i) + (u_{n_i}^i, u_{n_{i-2}}^{i-2})$, where the set $\{(u_j^i, u_{j+1}^i) \mid j = 3, \ldots, n_i - 1\}$ is empty if $n_i = 3$ (see Figure 11,

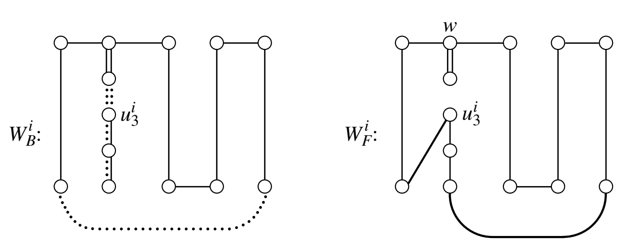


Figure 11. The tight walk $W_F^i$ built from $W_B^i$.

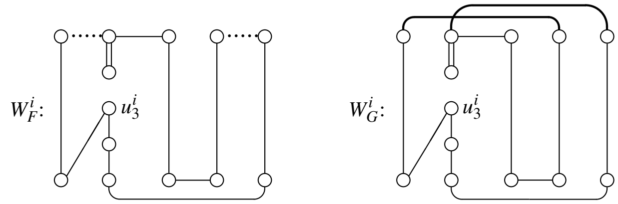

FIGURE 12. The tight walk $W_G^i$ built from $W_F^i$.

where the edges to be removed in $W_B^i$ and those to be added to obtain $W_F^i$ are represented by dotted and thick lines, respectively). The walk $W_F^i$ belongs to $\mathcal{W}_c^=$, as it can be checked using (11), and contains the edges $(w, u_1^{i-1})$ and $(w, u_1^{i+1})$ and two copies of edge $(w, u_2^i)$. Moreover, the edge sets $W_F^i - (w, u_1^{i-1}) - (w, u_2^i) + (u_1^{i-1}, u_2^i)$ and $W_F^i - (w, u_1^{i+1}) - (w, u_2^i) + (u_1^{i+1}, u_2^i)$ are both Hamiltonian cycles. This implies that $(u_1^{i-1}, u_2^i)$ and $(u_1^{i+1}, u_2^i)$ are $c$-adjacent in $w$. Similarly, to show that the edges $(u_1^{i-2}, u_2^i)$ and $(u^{i+1}, u_2^i)$ are $c$-adjacent in $w$, we have to exhibit a suitable walk that is almost Hamiltonian in $w$. We can assume that $k \geq 5$, since for $k = 3$ the edges $(u_1^{i-2}, u_2^i)$ and $(u^{i+1}, u_2^i)$ coincide. Consider the walk $W_G^i$ obtained from $W_F^i$ by replacing two edges of the cycle $C^Y$ with two of its short chords, i.e., $W_G^i = W_F^i - (u_1^{i-1}, u_1^i) - (u_1^{i-2}, u_1^{i-3}) + (u_1^i, u_1^{i-2}) + (u_1^{i-1}, u_1^{i-3})$ (see Figure 12, where we used dotted and thick lines with the same convention as in Figure 11). From $W_G^i$ we construct, as before, two Hamiltonian cycles to show that $(u_1^{i-2}, u_2^i)$ and $(u^{i+1}, u_2^i)$ are $c$-adjacent in $w$. It follows that the set $J_w$ is $c$-connected in $w$.

For $w = u_{n_i}^i$, $i = 1, 2, \ldots, k$, we set $J_w = \{(u_{n_{i-1}}^{i-1}, u_{n_i-1}^i), (u_{n_{i+1}}^{i+1}, u_{n_i-1}^i)\} \subseteq \Delta_c(w)$. The proof that in this cases the two edges of $J_w$ are $c$-adjacent in $w$ goes like for the cases $w = u_1^i$, $i = 1, 2, \ldots, k$; for each $i$ only one walk is used, actually the walk $W_G^i$ taken "upside down."

It is easy to verify that each of the walks of the basis $\mathcal{B}_c$ constructed in the proof of Theorem 2.3 can be reduced to an Hamiltonian cycle using the shortcuts defined by the sets $J_w$, $w \in V$. An example of such a reduction is shown in Figure 13 for the walk $W_3((u_1^1, u_1^4))$ of Figure 7(c). Again, for the dotted and the thick lines we follow the convention used for Figure 11. Observe how the sets $J_w$ have three edges for $w$ belonging to the cycle $C_Y$, while have only two edges for the nodes of the cycle $C_Z$. This is because, for all the walks of $\mathcal{B}_c$, if a node $w$ of $C_Z$ has degree four, then the walk has a path edge and an edge of $C_Z$ that are incident to $w$, which is not always the case if $w$ belongs to $C_Y$.

Finally, by Theorem 2.3, all the walks of $\mathcal{B}_c$ intersect the cycles $C_Y$ and $C_Z$. It is easy to check that also the Hamiltonian cycles of $\mathcal{C}_c$ obtained by shortcuts from those walks share this property. This proves that the properties (a) and (b) of the statement hold for $\mathcal{C}_c$. To prove that $\mathcal{C}_c$ has also the properties (c) and (d), consider the walk $W_C^i(1)$ of Figure 5 for $i = 1, 2, \ldots, k$. This walk contains the two cycle edges $(u_1^i, u_1^{i-1})$ and $(u_1^i, u_1^{i+1})$ of $C_Y$ whose end nodes have both degree 2. Therefore, the Hamiltonian cycle of $\mathcal{C}_c$ obtained by shortcuts from this walk also contains these two edges. Analogously, the Hamiltonian cycle of $\mathcal{C}_c$ obtained from the walk $W_D^i(n_i - 1)$ contains both of the cycle edges $(u_{n_i}^i, u_{n_{i-1}}^{i-1})$ and $(u_{n_i}^i, u_{n_{i+1}}^{i+1})$ of $C_Z$. □

**2.3. The 1-node lifting.** We have seen in §2 that a simple bicycle inequality is the restriction of a simple wheelbarrow inequality obtained by removing the coefficients of the edges incident with the node $Z$. On the other hand, one can view a simple wheelbarrow as the extension of a simple bicycle inequality obtained by adding a node to its corresponding configuration. The operation of adding one node (and the edges incident with it) to the configuration of a generic facet-defining inequality is called 1-*node lifting* in Naddef and Rinaldi [18], where we investigate the conditions on the coefficients of the new edges for the new inequality to preserve the facet-defining property. In particular, the theorem that follows states sufficient conditions for this to happen. When one of the edges incident with the new node has a zero coefficient, we have a special case of 1-node lifting, called *zero-lifting*, that is exploited in §3.

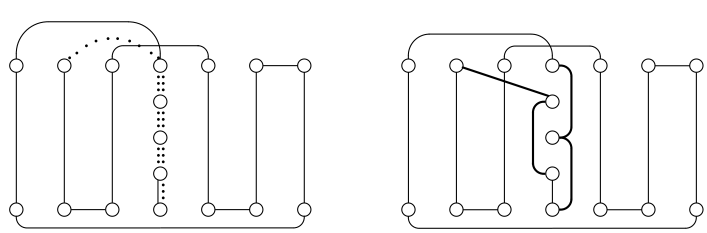

FIGURE 13. Example of reduction by shortcuts.

THEOREM 2.5 (NADDEF AND RINALDI [18, THEOREM 4.4]). *Let $cx \geq c_0$ be a TT inequality that is facet-defining for* STSP($n$); *an inequality $c^*x^* \geq c_0$, which is obtained by* 1*-node lifting of $cx \geq c_0$ is facet-defining for* STSP($n+1$) *if it is tight triangular and there exist an edge set $F \subseteq \Delta_{c^*}(u_{n+1})$ and a Hamiltonian cycle basis $\mathscr{C}_c$ of $cx \geq c_0$ such that*:

(i) *$F \cap H \neq \varnothing$ for all $H \in \mathscr{C}_c$*;
(ii) *for all $e \in F$ there exist $e' \neq e$, $e' \in \Delta_{c^*}(u_{n+1})$ and $H \in \mathscr{H}_c^=$ such that $e$ and $e'$ belong to $H$*;
(iii) *every connected component of the graph $(V, F)$ contains at least one odd cycle*;
(iv) *$F$ is $c^*$-connected in $u_{n+1}$.*

REMARK 2.4. In Naddef and Rinaldi [18] we prove Theorem 2.5 by showing that a set of walks, obtained by first removing one or two edges of the set $F$ from some walk of the canonical basis $\mathscr{C}_c$ of $cx \geq c_0$, and then connecting the end nodes of the removed edges to the node $u_{n+1}$ contains a canonical basis $\mathscr{C}_{c^*}$ of the inequality $c^*x^* \geq c_0$. Therefore, each walk of $\mathscr{C}_{c^*}$ shares all the edges of some walk of $\mathscr{C}_c$, except some of those that intersect the set $F$.

Using Theorem 2.5 and Remark 2.4 we can now show that the simple wheelbarrow inequalities define facets, of STSP($n$) and we can derive some properties of one of their canonical bases that will be used in the following to prove a similar theorem for the simple path inequalities.

THEOREM 2.6. *A simple wheelbarrow inequality $cx \geq c_0$ associated with the simple $k$-wheelbarrow configuration $W(n_1, n_2, \ldots, n_k)$ is facet-defining for* STSP($n$) *and has a Hamiltonian cycle basis $\mathscr{C}_c$ with the following properties*:

(a) *Every Hamiltonian cycle of $\mathscr{C}_c$ intersects the edge set of the cycle $C_Z$.*
(b) *For each edge $e \in C_Z$ there exists $e' \in C_Z$ and a Hamiltonian cycle $H \in \mathscr{C}_c$ containing both $e$ and $e'$.*

PROOF. A simple wheelbarrow inequality with the coefficients given in (11) is in $TT$ form, since it is facet-defining for GTSP($n$) (see Naddef and Rinaldi [18, Proposition 2.2]). As it has been already observed, it can be obtained by 1-node lifting of a bicycle inequality $\hat{c}\hat{x} \geq c_0$. To prove that it is facet-defining for STSP($n$), we use Theorem 2.5. Let the set $F$, required by that theorem, be defined as follows:

$$F = C_Y \cup \{(u_1^i, u_j^{i+1}) \mid i \in \{1, 2, \ldots, k\},\ j \in \{2, \ldots, n_{i+1}\}\}.$$

The set $F$ belongs to $\Delta_c(Y)$ and satisfies the conditions (i), (ii), and (iii) due to the properties (a) and (c) of Theorem 2.4 and due to the fact that the cycle $C_Y$ has odd length. We only need prove that $F$ is $c$-connected in $Y$. For $i = 1, 2, \ldots, k$, consider the walk $W_C^i(1)$ of Figure 5. Such a walk is tight for the bicycle inequality $\hat{c}\hat{x} \geq c_0$. This walk contains the two cycle edges $e = (u_1^i, u_1^{i-1})$ and $e' = (u_1^i, u_1^{i-2})$ of $C_Y$, whose end nodes have both degree 2. Therefore, the Hamiltonian cycle $H \in \mathscr{H}_{\hat{c}}^=$, obtained by shortcuts from this walk, also contains these two edges. The walk obtained by removing $e$ and $e'$ from $H$ and adding the four edges that connect the end nodes of the removed edges to $Y$ is almost Hamiltonian in $Y$, belongs to $\mathscr{W}_c^=$, and implies that the two edges $e$ and $e'$ are $c$-adjacent in $Y$. Repeating this argument for the walk $W_D^{i-2}(1)$, we see that $(u_1^{i-2}, u_1^{i-1})$ and $(u_1^i, u_1^{i-2})$ are $c$-adjacent in $Y$ as well. Consequently, the edge set $C_Y$ is $c$-connected in $Y$. To show that any edge $e \in F \backslash C_Y$ is $c$-adjacent in $Y$ to some edge of $C_Y$, it is sufficient to start with a Hamiltonian cycle $H \in \mathscr{H}_{\hat{c}}^=$ that contains $e$ and an edge $e' \in C_Y$. Such a Hamiltonian cycle exists by Property (a) of Theorem 2.4. By removing $e$ and $e'$ from $H$ and adding the edges that connect their end nodes to $Y$ we produce a walk almost Hamiltonian in $Y$, implying that $e$ and $e'$ are $c$-adjacent in $Y$. Since the cycle $C_Z$ has empty intersection with the edge set $F$, the property (a) and (b) of the Hamiltonian cycle basis $\mathscr{C}_c$ follow from Remark 2.4 and the properties (b) and (d) of Theorem 2.4. □

THEOREM 2.7. *A simple path inequality $cx \geq c_0$ associated with the simple $k$-path configuration $P(n_1, n_2, \ldots, n_k)$ is facet-defining for* STSP($n$).

PROOF. A simple path inequality with the coefficients given in (11) is in $TT$ form, since it is facet-defining for GTSP($n$) (see Naddef and Rinaldi [18, Proposition 2.2]). In addition it can be obtained by 1-node lifting of a simple wheelbarrow inequality $\hat{c}\hat{x} \geq c_0$. To prove that it is facet-defining for STSP($n$), we use again Theorem 2.5. Let the set $F$ required by that theorem be defined as follows:

$$F = C_Z \cup \{(u_{n_i}^i, u_j^{i+1}) \mid i \in \{1, 2, \ldots, k\},\ j \in \{1, \ldots, n_{i+1} - 1\}\} \cup \{(Y, u_{n_1}^1)\}.$$

The set $F$ belongs to $\Delta_c(Z)$ and satisfies the conditions (i), (ii), and (iii) due to the properties (a) and (b) of Theorem 2.6 and due to the fact that the cycle $C_Z$ has odd length. We only need prove that $F$ is $c$-connected in $Z$. For $i = 1, 2, \ldots, k$, consider the walk obtained from $W_D^i(n_i - 1)$ of Figure 5 by deleting an edge belonging to $C_Y$ and connecting its end nodes to $Y$. Such a walk is tight for the simple wheelbarrow inequality $\hat{c}\hat{x} \geq c_0$. This walk contains the two cycle edges $e = (u^i_{n_i}, u^{i-1}_{n_{i-1}})$ and $e' = (u^i_{n_i}, u^{i+1}_{n_{i+1}})$ of $C_Z$, whose end nodes have both degree 2. Therefore, the Hamiltonian cycle $H \in \mathscr{H}^=_{\hat{c}}$, obtained by shortcuts from this walk, also contains these two edges. The walk obtained by removing $e$ and $e'$ from $H$ and adding the four edges that connect the end nodes of the removed edges to $Z$ is almost Hamiltonian in $Z$, belongs to $\mathscr{W}^=_c$, and implies that the two edges $e$ and $e'$ are $c$-adjacent in $Z$. Consequently, $C_Z$ is $c$-connected in $Z$. To show that any edge $e \in F \setminus C_Z$ and some edge of $C_Y$ are $c$-adjacent in $Z$, it is sufficient to start with a Hamiltonian cycle $H \in \mathscr{H}^=_{\hat{c}}$ that contains $e$ and an edge $e' \in C_Z$. Such a Hamiltonian cycle exists by Property (a) of Theorem 2.6. By removing $e$ and $e'$ from $H$ and adding the edges that connect their end nodes to $Z$, we produce a walk almost Hamiltonian in $Z$ that implies that $e$ and $e'$ are $c$-adjacent in $Z$. □

The Theorems 2.4, 2.6, and 2.7 are summarized in the following.

THEOREM 2.8. *The simple PWB inequalities are facet-defining for* STSP($n$).

**3. The PWB inequalities.** As we have seen in §1, the configuration of a PWB inequality $c^*x \geq c_0$ is obtained from the configuration of a simple one $cx \geq c_0$ (same right-hand side) by replacing each nodes $i$ with a node set $V^i$ of arbitrary size. As the edges in $\gamma(V^i)$ have a zero coefficient, the operation of replacing a node $i$ by a set $V^i$ in a configuration is called *zero-lifting* of $i$.

The following theorem, which we prove in Naddef and Rinaldi [18], gives a sufficient condition to derive the facet-defining property of an inequality from the facet-defining property of its simple archetype.

THEOREM 3.1 (NADDEF AND RINALDI [18, THEOREM 4.9]). *A TT inequality $cx \geq c_0$ is facet-defining for* STSP($n$) *if its associated simple inequality $\bar{c}\bar{x} \geq c_0$ is nontrivial and facet-defining for* STSP($p$) *and for every $v \in V_p$ the set $\delta(v)$ in $K_p$ is $\bar{c}$-connected.*

We first apply this theorem to the bicycle inequalities.

THEOREM 3.2. *The bicycle inequalities are facet-defining for* STSP($n$).

PROOF. Let $c^*x^* \geq c_0$ be a bicycle inequality and $cx \geq c_0$ be its associated simple bicycle inequality. By converting the walks of the Figures 5, 7, and 8 to Hamiltonian cycles using shortcuts, it is easy to verify the following:

(a) For $i = 1, 2, \ldots, k$ and for each edge $e$ incident with $u^i_1$ with $e \neq (u^i_1, u^i_2)$, there exists a Hamiltonian cycle of $\mathscr{H}^=_c$ containing both $e$ and $(u^i_1, u^i_2)$.

(b) For $i = 1, 2, \ldots, k$ and for each edge $e$ incident with $u^i_{n_i}$ with $e \neq (u^i_{n_i-1}, u^i_{n_i})$, there exists a Hamiltonian cycle of $\mathscr{H}^=_c$ containing both $e$ and $(u^i_{n_i-1}, u^i_{n_i})$.

(c) For $i = 1, 2, \ldots, k$ with $n_i \geq 3$ and for each edge $e$ incident with $u^i_j$ with $1 < j < n_i$, $e \neq (u^i_j, u^i_{j-1})$, and $e \neq (u^i_j, u^i_{j+1})$, there exists a Hamiltonian cycle of $\mathscr{H}^=_c$ containing either the pair of edges $e$ and $(u^i_j, u^i_{j-1})$ or the pair $e$ and $(u^i_j, u^i_{j+1})$.

(d) For $i = 1, 2, \ldots, k$ with $n_i \geq 3$, there exists a Hamiltonian cycle of $\mathscr{H}^=_c$ containing both $(u^i_j, u^i_{j-1})$ and $(u^i_j, u^i_{j+1})$.

From (a), (b), (c), and (d) it follows that $\delta(u^i_j)$ is $c$-connected for $i = 1, 2, \ldots, k$ and for $j = 1, \ldots, n_i$. Therefore, by Theorem 3.1, the theorem follows. □

Now we make use of the following lemma to extend the facet-defining property to all the PWB inequalities.

LEMMA 3.1 (NADDEF AND RINALDI [18, LEMMA 4.8]). *Let $cx \geq c_0$ be a TT inequality facet-defining for* STSP($n$), *$c^*x^* \geq c_0$ be an inequality obtained by 1-node lifting of $cx \geq c_0$ and $F$ be a subset of $\Delta_{c^*}(u_{n+1})$ that satisfies the conditions of Theorem 2.5. Then the following hold*:

(a) *The edge set $\delta(u_{n+1}) \subseteq E_{n+1}$ is $c^*$-connected if the graph $(V, F)$ is connected.*

(b) *For every $v \in V$ the edge set $\delta(v) \subseteq E_{n+1}$ is $c^*$-connected if the edge set $\delta(v) \subseteq E_n$ is $c$-connected.*

THEOREM 3.3. *The PWB inequalities are facet-defining for* STSP($n$).

PROOF. It follows from Theorem 3.2, from Lemma 3.1 and from the connectivity of the graph induced by the edge set $F$ defined in the Theorems 2.6 and 2.7. □

Many of the inequalities known to define facets of STSP($n$) are special cases of PWB inequalities, as it can be easily verified by putting them in $TT$ form. In particular,

(i) the 2-*matching* inequalities, introduced by Edmonds, are 2-regular PWB inequalities obtained from a simple inequality possibly by zero-lifting of the nodes $Y$ and $Z$;

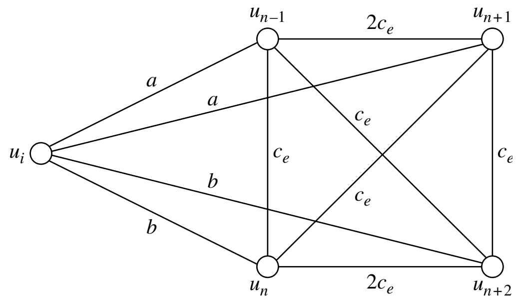


FIGURE 14. Cloning edge $(u_{n-1}, u_n)$ one time.

(ii) the *Chvátal comb* inequalities, introduced by Chvátal, are 2-regular PWB inequalities obtained from a simple inequality possibly by zero-lifting of $Y$, $Z$, and of the nodes of the cycle $C_Y$;
(iii) the *comb* inequalities, introduced by Grötschel and Padberg, are 2-regular PWB inequalities.

**4. The extended PWB inequalities.** Some simple inequalities can be obtained by extending other simple inequalities by simultaneously adding two nodes to their configuration. In Naddef and Rinaldi [18] we call *edge cloning* an example of such an extension, and we define it as follows:

DEFINITION 4.1. Let $cx \geq c_0$ be a $TT$ inequality defined on $\mathbb{R}^{E_n}$ and $e$ be an edge in $E_n$. We say that the inequality $c^* x^* \geq c_0^*$ defined on $\mathbb{R}^{E_{n+2t}}$, with $t \geq 1$, is obtained from $cx \geq c_0$ by *cloning* $t$ times edge $e = (u_{n-1}, u_n)$ if (see Figure 14, in which $t = 1$):

$$c_0^* = c_0 + 2tc_e$$
$$c^*(u_i, u_{n+j}) = c(u_i, u_{n-1}) \quad \text{for } 1 \leq i \leq n-2, \ \ 1 \leq j \leq 2t-1 \ \text{ and } \ j \text{ odd},$$
$$c^*(u_i, u_{n+j}) = c(u_i, u_n) \quad \text{for } 1 \leq i \leq n-2, \ \ 2 \leq j \leq 2t \ \text{ and } \ j \text{ even},$$
$$c^*(u_{n+i}, u_{n+j}) = 2c_e \quad \text{for } -1 \leq i < j \leq 2t \ \text{ and } \ j-i \text{ even},$$
$$c^*(u_{n+i}, u_{n+j}) = c_e \quad \text{for } -1 \leq i < j \leq 2t \ \text{ and } \ j-i \text{ odd}.$$

The nodes $u_{n+j}$ for $1 \leq j < 2t$ and $j$ odd and the nodes $u_{n+j}$ for $1 < j \leq 2t$ and $j$ even are called the *clones* of the nodes $u_{n-1}$ and $u_n$, respectively. The edge $e = (u_{n-1}, u_n)$ is said to be *cloned*.

Let $cx \geq c_0$ be a $TT$ inequality defining a facet of STSP$(n)$; an edge $e = (u, v)$ is called $c$-*clonable* if the $c$-length of every walk $W$ of $K_n$ is at least $c_0 + (d_e(W) - 2)c_e$, where $d_e(W)$ is the minimum of the degrees of $u$ and $v$ in $W$; we say that a node $v \in V$ is $\alpha$-*critical* for the inequality if the $c$-length of a minimum $c$-length walk of $K_n - \{v\}$ is $c_0 - \alpha$.

THEOREM 4.1 (NADDEF AND RINALDI [18, THEOREMS 4.12 AND 4.13]). *Let $cx \geq c_0$ be a nontrivial TT inequality facet-defining for* STSP$(n)$ *and let $e = (u_{n-1}, u_n)$ be a c-clonable edge such that $u_{n-1}$ and $u_n$ are $2c_e$-critical for $cx \geq c_0$. Then the following hold*:
(a) *the inequality $c^* x^* \geq c_0^*$ obtained by cloning $e$ ($t$ times) is facet-defining for* STSP$(n + 2t)$;
(b) *the edge subsets $\delta(u_{n-1}), \ldots, \delta(u_{n+2t})$ of $E_{n+2t}$ are $c^*$-connected*;
(c) *for $v \in V - \{u_{n-1}, u_n\}$, if $\delta(v)$ in $K_n$ is c-connected, then $\delta(v)$ in $K_{n+2t}$ is $c^*$-connected*;
(d) *if $f = (z_1, z_2) \neq e$ is an edge in $E_n$ such that $z_1$ and $z_2$ are $2c_f$-critical for $cx \geq c_0$, then $z_1$ and $z_2$ are $2c_f^*$-critical for $c^* x \geq c_0^*$.*

DEFINITION 4.2. A *simple extended PWB inequality* is the inequality obtained by cloning $t_i$ times the edge $(u_1^i, u_2^i)$ for each $i \in I$ where $I$ is any subset of $\{1, \ldots, k\}$ such that $n_i = 2$ for all $i \in I$. An inequality is an *extended PWB* if its associated simple inequality is a simple extended PWB.

THEOREM 4.2. *The extended PWB inequalities are facet-defining for* STSP$(n)$.

PROOF. Let $cx \geq c_0$ be a simple PWB inequality of STSP$(n)$ with $n_i = 2$ for some $i$. We first prove the following two claims.

CLAIM 4.1. *Let $e = (u_1^i, u_2^i)$, with $n_i = 2$, be an edge of an extended PWB configuration that is not cloned and let $cx \geq c_0$ be the inequality associated with the configuration. Then the validity of $cx \geq c_0$ implies that e is c-clonable.*

PROOF. We have to show that if $e$ is not $c$-clonable, i.e., if there exists a walk $W$ of $K_n$ such that $c(W) < c_0 - (d_e(W)-2)c_e$, then the inequality is not valid, i.e., there exists a walk $W^*$ with $c(W^*) < c_0$. If $d_e(W) = 2$, then obviously $W^* = W$. If $d_e(W) \geq 4$ we show how to construct a walk $W'$ from $W$ where the minimum degree of $u_1^i$ and $u_2^i$ is $d_e(W) - 2$ and $c(W') \leq c(W) - 2c_e$. Thus, we can construct the walk $W^*$ by recursively applying this process. We consider three distinct cases.

*Case* (a). $W$ contains at least three copies of $e$. The walk $W'$ is obtained from $W$ by removing two copies of $e$. The multigraph $K_n[W']$ is connected and $c(W') = c(W) - 2c_e$.

*Case* (b). $W$ contains two copies of $e$. Let $u_r^q \notin \{u_1^i, u_2^i\}$ and $u_t^s \notin \{u_1^i, u_2^i\}$ be neighbors in $W$ of $u_1^i$ and $u_2^i$, respectively, such that the multigraph $K_n[W']$ is connected, where $W' = W - e - (u_1^i, u_r^q) - (u_2^i, u_t^s) + (u_r^q, u_t^s)$. Since $n_i = 2$, it follows that either $u_r^q \in \{Y, Z\}$ or $q \neq i$ and either $u_t^s \in \{Y, Z\}$ or $s \neq i$. Thus, it is easy to verify from (11) that $c(W') \leq c(W) - 2c_e$ and that equality holds only if $u_r^q = Y$ and $u_t^s = Z$.

*Case* (c). $W$ contains only one copy of $e$. Among the edges of $W$ different from $e$ that are incident with $u_1^i$, there are always two, say $(u_1^i, w)$ and $(u_1^i, z)$, or, possibly, two copies of the same edge (in which case $w = z$), such that $K_n[W'']$ is connected, where $W'' = W - (u_1^i, w) - (u_1^i, z) + (w, z)$. Specularly, among the edges of $W$ different from $e$ that are incident with $u_2^i$, there are always two, say $(u_2^i, w')$ and $(u_3^i, z')$, or, possibly, two copies of the same edge (in which case $w' = z'$), such that $K_n[W']$ is connected, where $W' = W'' - (u_2^i, w') - (u_2^i, z') + (w', z')$. It is easy to verify from (11) that $c(W') \leq c(W) - 2c_e$. □

Observe that if $n_i > 2$, then none of the edges of the path $\pi^i$ is $c$-clonable, because there are tight walks where both the end nodes of a path edge have degree four (see, e.g., Figure 5). This explains the exclusion of these edges from the cloning process in Definition 4.2.

CLAIM 4.2. *Both the end nodes of the edge $e = (u_1^i, u_2^i)$ are $2c_e$-critical.*

PROOF. The walk $W_A^i$ of Definition 2.6 contains two copies of $e$. The Hamiltonian cycle $H_1$ of $K_n - \{u_1^i\}$, obtained from $W_A^i$ by removing the two copies of $e$, has length $c_0 - 2c_e$. It follows that $u_1^i$ is $2c_e$-critical. Analogously, using the walk $W_B^i$, one proves that also $u_2^i$ is $2c_e$-critical. □

END OF PROOF OF THEOREM 4.2. Let $l$ be the number of edges that have been cloned to obtain the inequality $cx \geq c_0$, i.e., $l = |\{i \mid t_i > 0\}|$. If $l = 0$, the inequality is valid since it is a simple PWB. Therefore, by Claim 4.1, $e = (u_1^i, u_2^i)$ with $n_i = 2$ is $c$-clonable and, by Claim 4.2, both its end nodes are $2c_e$-critical. Thus, applying Theorem 4.1, the inequality $c^*x^* \geq c_0^*$, obtained from $cx \geq c_0$ by cloning $e$ for $t_i$ times, is facet-defining for STSP$(n+2t_i)$ and, for any node $v$, the edge set $\delta(v)$ in $K_{n+2t_i}$ is $c^*$-connected. In addition, if the end nodes of an edge $f$ are $2c_f$-critical, then they are also $2c_f^*$-critical. Finally, by induction on $l$ one shows that any simple extended PWB inequality $\hat{c}\hat{x} \geq \hat{c}_0$ is facet-defining and that all its nodes are $\hat{c}$-connected. Thus, any extended PWB inequality is facet-defining. □

Padberg and Hong [24] define the *chain inequality* as follows:

Let $S_i \subseteq V$ for $i = 0, 1, \ldots, q$ be any proper subsets of $V$ satisfying $S_i \cap S_0 = \varnothing$ for $i = 1, \ldots, p$, $|S_i \cap S_0| \geq 1$ and $|S_i \backslash S_0| \geq 1$ for $i = p+1, \ldots, q$, and $S_i \cap S_j = \varnothing$ for $1 \leq i \leq j \leq q$. Let $R \subseteq S_0$ be a subset of $S_0$ satisfying $|R| = p$ and $R \cap S_i = \varnothing$ for $i = 1, \ldots, q$. Then the *chain inequality* is

$$\sum_{i=0}^{q} x(\gamma(S_i)) + \sum_{i=1}^{p} x((R:S_i)) \leq |S_0| + |R| + \sum_{i=1}^{q} (|S_i| - 1) - \left\lceil \frac{q-p+1}{2} \right\rceil. \tag{22}$$

Padberg and Hong [24] show that the chain inequality is a valid inequality for STSP$(n)$ with $n \geq 8$, when $2 \leq p < q$ (for $p \leq 1$ the chain inequality coincides with a comb inequality). If $q - p$ is odd, then the inequality is dominated by a nonnegative linear combination of subtour elimination and degree constraints (see, e.g., Naddef and Pochet [15], where a generalization of these inequalities is described).

It is not difficult to verify, by putting (22) in $TT$ form, that the chain inequality is an extended 2-regular PWB inequality obtained by cloning only one path edge of a simple 2-regular PWB configuration, say $(u_1^1, u_2^1)$, and then zero-lifting all nodes except $u_1^1$ and its clones (these are the members of the set $R$ in the definition of the chain inequalities).

Extended PWB inequalities generalize chain inequalities not only because the cloning process can involve all path edges of a 2-regular PWB inequality and because all nodes can be zero-lifted, but also because these inequalities can be derived from any nonregular PWB inequality having at least one path of length 2. This generalization of the chain inequalities differs from the one described by Naddef and Pochet [15].

**5. Composition of PWB inequalities.** In Naddef and Rinaldi [17] we describe an operation, called 2-*sum*, which yields facet-defining inequalities for GTSP($n$) and involves two facet-defining inequalities defined on smaller configurations. In Naddef and Rinaldi [18] we give sufficient conditions under which an inequality, obtained as the 2-sum of two other inequalities, defines a facet of STSP($n$). We apply the 2-sum operation to PWB inequalities. For the sake of completeness, we give the formal definition of the 2-sum operation.

Two weighted graphs $G^1 = (V^1, E^1, c^1)$ and $G^2 = (V^2, E^2, c^2)$ are *isomorphic* if there exists a one-to-one correspondence $\rho$ between their node sets that preserves the weight function, i.e., for every edge $(u, v) \in E^1$, the edge $(\rho(u), \rho(v))$ belongs to $E^2$ and $c^1(u, v) = c^2(\rho(u), \rho(v))$.

Definition 5.1. Let $c^1x^1 \geq c_0^1$ and $c^2x^2 \geq c_0^2$ be two $TT$ inequalities facet-defining for STSP($n_1$) and STSP($n_2$), respectively, and let $e_1 = (u_1, v_1) \in E_{n_1}$ and $e_2 = (u_2, v_2) \in E_{n_2}$ be two edges such that $c^1(u_1, v_1) = c^2(u_2, v_2) = \theta > 0$. Denote by $V_{n_1}$ and $V_{n_2}$ the node sets of the two graphs $K_{n_1}$ and $K_{n_2}$, respectively, by $V^1$ the set $V_{n_1} - \{u_1, v_1\}$ and by $V^2$ the set $V_{n_2} - \{u_2, v_2\}$. Then the 2-*sum* of the two inequalities, obtained by *identifying* $u_1$ with $u_2$ and $v_1$ with $v_2$, is the inequality $cx \geq c_0^1 + c_0^2 - 2\theta$ defined on $\mathbb{R}^{E_n}$, with $n = n_1 + n_2 - 2$, whose support graph $G_c = (V, E_n, c)$ is defined as follows:

(i) $V = V^1 + V^2 + \{u, v\}$;

(ii) the subgraph of $G_c$ induced by $V^1 + \{u, v\}$ is isomorphic to $G_{c^1}$ and $u$ and $v$ correspond to $u_1$ and $v_1$, respectively, in the isomorphism;

(iii) the subgraph of $G_c$ induced by $V^2 + \{u, v\}$ is isomorphic to $G_{c^2}$ and $u$ and $v$ correspond to $u_2$ and $v_2$, respectively, in the isomorphism;

(iv) the coefficients of the edges with one end node in $V_1$ and the other in $V_2$, that we call the *crossing edges* of the 2-sum, are computed by a sequential lifting procedure.

The inequalities $c^1x^1 \geq c_0^1$ and $c^2x^2 \geq c_0^2$ are called the *component inequalities* of the 2-sum.

For the sake of simplicity, from now on every time that the correspondence between nodes, edges, and walks of each of the graph $G_{c_1}$ and $G_{c_2}$ and their corresponding isomorphic subgraphs of $G_c$ is evident, we omit to mention it explicitly.

In Naddef and Rinaldi [17] we use the 2-sum operation in a recursive way to generate huge families of inequalities defining facets of GTSP($n$). The most interesting of such families is perhaps the one of the *regular parity path tree* (or *regular parity PWB-tree*) inequalities. These inequalities are peculiar because the lifting procedure does not have to be carried out explicitly, since the coefficient of each crossing edge is given by the $c$-length of the shortest path between its end nodes in the $c$-weighted graph produced by each 2-sum. This implies that the coefficients of the crossing edges do not depend on the order of the sequential lifting procedure.

Definition 5.2. A *regular parity PWB-tree inequality* is one of the following:

(i) A regular PWB inequality $\mathscr{P}_0$. In this case the PWB-tree inequality has length 0. The path edges as well as the even and the odd nodes of $\mathscr{P}_0$ coincide with the corresponding ones of the PWB inequality.

(ii) The 2-sum $\mathscr{P}_{l+1}$ of a regular parity PWB-tree inequality $\mathscr{P}_l$, having length $l$ and of a regular PWB inequality obtained by identifying the end nodes of a path edge of one inequality with the end nodes of a path edge of the other, with the constraints that the identified nodes have the same parity. In this case the PWB-tree inequality has length $l + 1$. The nodes that result from the two identifications inherit the parity of the nodes from which they derive; the edge between them is a path edge of $\mathscr{P}_{l+1}$. The path edges as well as the even and the odd nodes of $\mathscr{P}_l$ and of the PWB inequality are path edges, even and odd nodes of $\mathscr{P}_{l+1}$.

As it was mentioned before, for the regular parity PWB-tree inequalities the following theorem holds:

Theorem 5.1 (Naddef and Rinaldi [17, Theorem 5.3]). *Let $c^1x^1 \geq c_0^1$ and $c^2x^2 \geq c_0^2$ be two regular parity PWB-tree inequalities, let $cx \geq c_0$ be their 2-sum, and let $(u, v)$ be the edge that results from the identification. Then, using the notation of Definition 5.1, for any $x \in V^1$ and $y \in V^2$ the following holds*:

$$c(x, y) = \min\{c^1(x, u) + c^2(u, y), c^1(x, v) + c^2(v, y)\}. \tag{23}$$

The main result of this section is a proof that regular parity PWB-tree inequalities are facet-defining for STSP($n$). To do so, we use the following theorem that gives conditions for a 2-sum inequality to be facet-defining for STSP($n$).

We call a 2-sum inequality *h-liftable* if the coefficients of its crossing edges do not change if Hamiltonian cycles are used in the lifting procedure, instead of walks.

Theorem 5.2 (Naddef and Rinaldi [18, Theorem 3.5]). *Under the assumptions of Definition 5.1, let $c^1x^1 \geq c_0^1$ and $c^2x^2 \geq c_0^2$ be nontrivial inequalities defining facets of* STSP($n_1$) *and* STSP($n_2$), *respectively.*

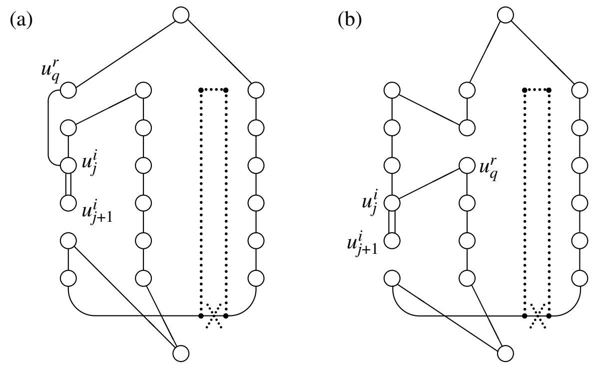


FIGURE 15. Tight walks containing edge $(u_j^i, u_q^r)$ and twice edge $(u_j^i, u_{j+1}^i)$.

*The* 2*-sum inequality* $cx \geq c_0$ *is facet-defining for* STSP$(n)$ *if it is h-liftable and*:

(a) $v_1$ *is* $2\theta$*-critical for* $c^1x^1 \geq c_0^1$,

(b) $\delta(u_2)$ *is* $c^2$*-connected,*

*and either*

*Case* (A):

(c′) $u_2$ *is* $2\theta$*-critical for* $c^2x^2 \geq c_0^2$,

(d′) $\delta(v_1)$ *is* $c^1$*-connected,*

*or*

*Case* (B):

(c″) $v_2$ *is* $2\theta$*-critical for* $c^2x^2 \geq c_0^2$,

(d″) $\delta(u_1)$ *is* $c^1$*-connected,*

(e″) *there exists a Hamiltonian cycle* $H_1 \in \mathcal{H}_{c_1}^{=}$ *containing the edge* $(u_1, v_1)$ *and any edge* $e_1 \in \Delta_{c_1}(v_1)$,

(f″) *there exists a Hamiltonian cycle* $H_2 \in \mathcal{H}_{c_2}^{=}$ *containing the edge* $(u_2, v_2)$ *and any edge* $e_2 \in \Delta_{c_2}(v_2)$.

Before stating the main theorem, we prove a lemma concerning the $h$-liftability of simple regular parity PWB-tree inequalities and recall three lemmata that we state in Naddef and Rinaldi [18] and that will be used in the main proof as well.

LEMMA 5.1. *The* 2*-sum of a simple regular parity PWB-tree and of a simple PWB inequality is h-liftable.*

PROOF. We consider the 2-sum of a simple regular parity PWB-tree inequality $\mathcal{P}_l$ with a simple regular PWB inequality $cx \geq c_0$, obtained by identifying the end nodes of a path edge of $\mathcal{P}_l$ with the end nodes of the edge $(u_j^i, u_{j+1}^i)$ of the PWB inequality. Without loss of generality, we can assume that $j < n_i$, and thus that $u_{j+1}^i$ is always even. Let $e = (a, u_q^r)$ be a crossing edge of the 2-sum inequality. We consider two cases, depending on whether node $u_j^i$ (as well as the corresponding node of $\mathcal{P}_l$) is even or odd. We use the walks of the Figures 15–17. These walks belong to $\mathcal{W}_c^{=}$ when $cx \geq c_0$ is a path inequality and can be easily adjusted to the cases when the inequality is either a wheelbarrow or a bicycle.

*Case* A. Node $u_j^i$ is even. Without loss of generality, we assume that $q \leq j$ (or else, just exchange the roles of $j$ and $j+1$ and those of $Z$ and $Y$). Then it is easy to see, by (23) and (13), that $c_e = c^1(a, u_j^i) + c^2(u_j^i, u_q^r)$.

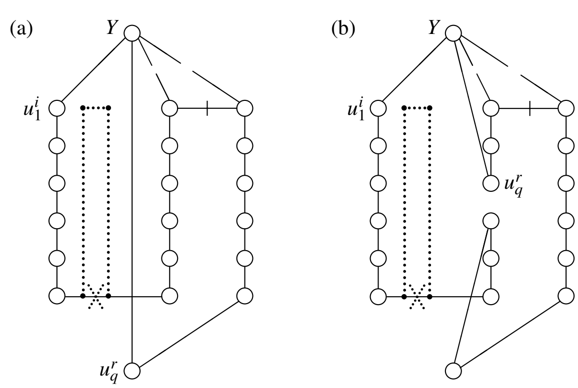


FIGURE 16. Tight walks containing the edges $(u_1^i, Y)$ and $(Y, u_q^r)$.

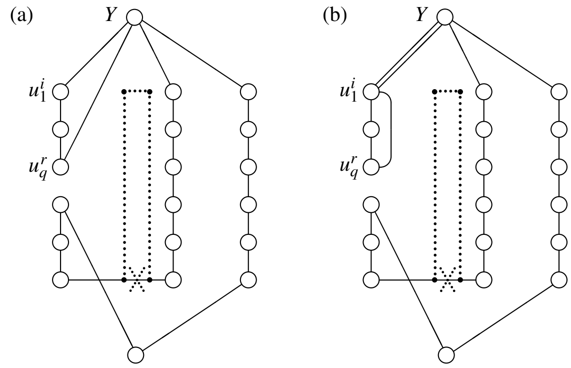


FIGURE 17. A tight walk containing the edges $(u_1^i, Y)$ and $(Y, u_q^r)$ and one containing twice edge $(u_1^i, Y)$ and edge $(u_1^i, u_q^r)$.

The inequality $\mathcal{P}_l$ is facet-defining for GTSP($n$) (see Naddef and Rinaldi [17]). Therefore, there exists a tight walk containing $(a, u_j^i)$ that, by Lemma 2.3, can be turned into a Hamiltonian cycle $H$ containing $(a, u_j^i)$. If $r = i$, then let $W_1$ be the walk of Figure 15(a); otherwise, let $W_1$ be the walk of Figure 15(b). This walk contains edge $(u_j^i, u_{j+1}^i)$ twice. The Hamiltonian cycle $H + W_1 - 2(u_j^i, u_{j+1}^i) - (a, u_j^i) - (u_j^i, u_q^r) + (a, u_q^r)$ is tight for the 2-sum inequality and contains the crossing edge $e$.

*Case* B. Node $u_j^i$ is odd, i.e., $u_j^i = Y$ and $j = 0$. In Naddef and Rinaldi [17, Lemma 5.6] we show that there exists a tight walk $W$ for the inequality $\mathcal{P}_l$ that contains both edges $(a, Y)$ and $(Y, u_1^i)$. We first assume that either $r \neq i$ or $q = n_r + 1$, in the latter case we have $u_q^r = Z$. Then it is easy to see that, by (23) and (11), $c_e = c^1(a, Y) + c^2(Y, u_q^r)$. If $q = n_r + 1$, then let $W_2$ be the walk of Figure 16(a); otherwise, let it be the walk of Figure 16(b) (the edges represented by broken lines have to replace the marked edge if node $Y$ has degree 2 in $W$). The walk $W + W_2 - 2(u_1^i, Y) - (a, Y) - (Y, u_q^r) + (a, u_q^r)$ is tight for the 2-sum inequality, contains the crossing edge $e$, and, by Lemma 2.3, can be transformed into a tight Hamiltonian cycle by shortcuts involving only edges of $E^1$ since only one edge on $E^2$ is incident with $u_1^i$ and $Y$, respectively. Now let us assume that $r = i$ and $q > 0$. If $c_e = c^1(a, Z) + c^2(Z, u_q^r)$, then let $W_3$ be the walk of Figure 17(a). The walk $W' = W + W_3 - 2(u_1^i, Y) - (a, Y) - (Y, u_q^r) + (a, u_q^r)$ is tight for the 2-sum inequality and contains the crossing edge $e$. Finally, if $c_e = c^1(a, u_1^i) + c^2(u_1^i, u_q^r)$, let $H$ be the Hamiltonian cycle defined for Case (A) and $W_4$ the walk of Figure 17(b). Then also the walk $W'' = H + W_4 - 2(u_1^i, Y) - (a, u_1^i) - (u_1^i, u_q^r) + (a, u_q^r)$ is tight for the 2-sum and contains $e$. Both walks $W'$ and $W''$ can be turned into tight walks where every node but $Y$ has degree 2, by applying shortcuts involving only edges of $E^1$, since only one edge on $E^2$ is incident with $u_1^i$. To reduce each of these walks to a Hamiltonian cycle, a shortcut involving a crossing edge $(w, u_t^s)$, where $s \neq i$, is necessary. However, since the coefficients of the inequality do not depend on the lifting sequence, we can assume, without loss of generality, that the lifting coefficients for all the crossing edges $(w, u_t^s)$, with $s \neq i$, are computed first, and the lemma is proved. □

The following three lemmata give conditions under which some properties of the components of a 2-sum are carried over to the resulting inequality. The notation used in their statements is that of Definition 5.1.

LEMMA 5.2 (NADDEF AND RINALDI [18, LEMMA 3.5]). *Under the assumptions of Definition 5.1, let $w \in V_{n_1}$ be $\alpha$-critical for $c^1x^1 \geq c_0^1$, and let $c_e^1 = \alpha/2$ for some edge $e \in \delta(w)$ in $K_{n_1}$. The corresponding node $w \in V$ is $\alpha$-critical for $cx \geq c_0$ if $cx \geq c_0$ is supporting for* GTSP($n$) *and any of the following conditions holds*:

(a) $w \notin \{u_1, v_1\}$,
(b) $w = u_1$ *and* $u_2$ *is* $2\theta$*-critical for* $c^2x^2 \geq c_0^2$,
(c) $w = v_1$ *and* $v_2$ *is* $2\theta$*-critical for* $c^2x^2 \geq c_0^2$.

LEMMA 5.3 (NADDEF AND RINALDI [18, LEMMA 3.6]). *Under the assumptions of Definition 5.1, if there exists a Hamiltonian cycle $H \in \mathcal{H}_{c_1}^{=}$ containing two nonadjacent edges $e$ and $f \in E_{n_1}$ and at least one of the two nodes $u_2$ and $v_2$ is $2\theta$-critical for $c^2x^2 \geq c_0^2$, then there exists a Hamiltonian cycle $H^* \in \mathcal{H}_c^{=}$ containing the edges in $E_n$ corresponding to $e$ and $f$, respectively.*

LEMMA 5.4 (NADDEF AND RINALDI [18, LEMMA 3.7]). *Under the assumptions of Definition 5.1, let $\delta(w)$ in $K_{n_1}$ be $c^1$-connected for every node $w \in V_{n_1}$ and let $\delta(w)$ in $K_{n_2}$ be $c^2$-connected for every node $w \in V_{n_2}$. Then $\delta(w)$ in $K_n$ is $c$-connected for every node $w \in V$ if the following conditions hold*:

(i) $cx \geq c_0$ *is* $h$*-liftable*;
(ii) *at least one of the two nodes* $u_1$ *and* $v_1$ *is* $2\theta$*-critical for* $c^1x^1 \geq c_0^1$;
(iii) *at least one of the two nodes* $u_2$ *and* $v_2$ *is* $2\theta$*-critical for* $c^2x^2 \geq c_0^2$.

We have now all the basic tools to prove the following

THEOREM 5.3. *The regular parity PWB-tree inequalities are facet-defining for* STSP$(n)$.

PROOF. Let $\mathcal{P}_l$ be a simple regular parity PWB-tree inequality of length $l$. We proceed by induction on $l$. The theorem is true for $l=0$ since $\mathcal{P}_0$ is a PWB inequality. We consider now the case $l=1$. Thus, by Definition 5.2, $\mathcal{P}_1$ is the 2-sum of an inequality $\mathcal{P}_0$, which we denote by $c^1x^1 \ge c_0^1$, and of a simple PWB inequality $c^2x^2 \ge c_0^2$, obtained by identifying the end nodes of the path edge $(u_1, v_1)$ of the first inequality to the end nodes of the path edge $(u_2, v_2)$ of the second. Using the notation of Definition 5.1, we show now that the conditions of Theorem 5.2 are satisfied. By Lemma 5.1, $\mathcal{P}_l$ is $h$-liftable for all values of $l$.

CLAIM 5.1. *An even end node of a path edge $e$ of a simple PWB inequality $cx \ge c_0$ is $2c_e$-critical.*

PROOF. Without loss of generality, we can assume that $e = (u_j^i, u_{j+1}^i)$, with $j \in \{0, \ldots, n_i - 1\}$. We consider two cases.

The first case arises when $j \in \{1, \ldots, n_i - 1\}$; in this case both the end nodes of $e$ are even. Consider the walk $W_A^i$ of Definition 2.6 if $j=1$, the walk $W_B^i$ if $j = n_i - 1$, and $W_C^i(j+1)$ if $j \in \{2, \ldots, n_i - 2\}$. Such a walk contains two copies of edge $e$; in addition, by removing these two copies, we obtain a walk of $V \backslash \{u_{j+1}^i\}$ of $c$-length $c_0 - 2c_e$. Therefore, $u_{j+1}^i$ is $2c_e$-connected. Similarly, one shows that also $u_j^i$ is $2c_e$-connected.

In the second case, arising when $j=0$, we consider the walk $W_C^i(1)$, the only even node is $u_1^i$. We add the (odd) node $Y = u_0^i$; then we remove the two cycle edges $(u_1^i, u_1^{i-1})$ and $(u_1^i, u_1^{i+1})$; and finally we add the edges $(Y, u_1^{i-1})$, $(Y, u_1^{i+1})$; and two copies of the edge $e = (Y, u_1^i)$. The resulting walk has $c$-length $c_0$ and contains two copies of $e$; moreover, the removal of these two copies of $e$ yields a walk of $V \backslash \{u_j^i\}$ of $c$-length $c_0 - 2c_e$. ☐

CLAIM 5.2. *An even end node of a path edge $e$ of a simple regular parity PWB-tree inequality $cx \ge c_0$ is $2c_e$-critical.*

PROOF. It follows immediately by applying the recursion of Definition 5.2, Claim 5.2, and Lemma 5.2. ☐

Since a path edge has at most one odd end node, we can assume, without loss of generality, that the nodes $u_1$ and $u_2$ are both even. Thus, by Claims 5.1 and 5.2, the conditions (a) and (c″) are satisfied for all values of $l$. Depending on the parity of the nodes $v^1$ and $v^2$ we have two distinct cases.

*Case* A. $v^1$ and $v^2$ are even. In this case, by Claims 5.1 and 5.2 Condition (c′) is satisfied.

*Case* B. $v^1$ and $v^2$ are odd. First, we prove the following

CLAIM 5.3. *Let $cx \ge c_0$ be a simple PWB inequality and let $(u_1^i, Y)$ be one of its path edges incident with an odd node. Then there exists a Hamiltonian cycle $H \in \mathcal{H}_c^=$ containing $(u_1^i, Y)$ and an edge of $\Delta_c(Y)$.*

PROOF. Consider the walk of Figure 7(a), where $j$ and $q$ are set to 1 and $n_r$, respectively. With these settings of $j$ and $q$, the walk is actually a Hamiltonian cycle. Replace $(u_1^i, u_{n_r}^r)$ by the two edges $(u_1^i, Y)$ and $(Y, u_{n_r}^r)$. For the case of a path inequality, insert node $Z$ into the cycle in a similar manner, by removing any cycle edge of $C_Z$ (by Theorem 2.3 at least one edge of $C_Z$ belongs to the Hamiltonian cycle). The resulting Hamiltonian cycle belongs to $\mathcal{H}_c^=$, contains the edge $(u_1^i, Y)$, and, again by Theorem 2.3, intersects $C_Y \subseteq \Delta_c(Y)$. ☐

CLAIM 5.4. *Let $cx \ge c_0$ be a simple regular parity PWB-tree inequality and $(u, v)$ be one of its path edges incident with an odd node $v$. Then there exists a Hamiltonian cycle $H \in \mathcal{H}_c^=$ containing $(u, v)$ and an edge of $\Delta_c(v)$.*

PROOF. It follows immediately by applying the recursion of Definition 5.2, Claim 5.1, and Lemma 5.3. ☐

END OF PROOF OF THEOREM 5.3. Claims 5.3 and 5.4 imply that both conditions (e″) and (f″) are satisfied for all values of $l$.

For $l=1$ the two inequalities are of PWB type, hence, by Theorems 3.2 and 3.3, the conditions (b), (d′), and (d″) are satisfied. Thus, the inequality is facet-defining, and each node of the graph is $c$-connected with respect to the coefficient vector $c$ of the inequality. This implies that the same holds for $l > 1$, by applying the induction and the Lemmata 5.1 and 5.4. ☐

A clique-tree inequality (see Grötschel and Pulleyblank [11]), as long as there is a node not contained in any handle or tooth, is a special case of a regular parity path tree obtained as follows (see Naddef and Rinaldi [17]):

—Only 2-regular PWB inequalities with at least one odd node are used as components.

—Each 2-sum involves an odd node.

—All odd nodes involved in the 2-sums correspond to a single odd node in the resulting inequality.

It is evident how the regular parity PWB-tree inequalities generalize the clique tree inequalities in many possible ways.

**6. Conclusions.** The operations described in this paper can be extended in several directions to prove that further classes of inequalities derived from PWB inequalities are facet-defining for the STSP polytope. For example, the components of a 2-sum can be extended PWB inequalities, or edge cloning can be applied to regular parity PWB inequalities.

**Acknowledgment.** This work was supported in part by the Marie Curie Research Training Network "Algorithmic Discrete Optimization" (ADONET), Grant MRTN-CT-2003-504438, of the European Union Research Commission, which we gratefully acknowledge.

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