BALANCING PAIRWISE COMPARISON MATRICES BY TRANSITIVE MATRICES

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Abstract

We discuss the development and use of a recursive rank-one residue iteration (triple R-I) to balancing pairwise comparison matrices (PCMs). This class of positive matrices is in the centre of interest of a widely used multi-criteria decision making method called analytic hierarchy process (AHP). To find a series of the 'best' transitive matrix approximations to the original PCM the Newton-Kantorovich (N-K) method is employed for the solution to the formulated nonlinear problem. Applying a useful choice for the update in the iteration, we show that the matrix balancing problem can be transformed to minimizing the Frobenius norm, and, equivalently, for certain matrices the l_1- and the l_∞ -norms. Convergence proofs for this scaling algorithm are given. A comprehensive numerical example is included.

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1 Introduction

We call an element-wise positive $n \times n$ matrix $\mathbf{A} = (a_{ij})$, $i, j = 1, 2, \ldots, n$ *symmetrically reciprocal* (SR) if $a_{ij}a_{ji} = 1$, $i \neq j$ for all $i, j = 1, 2, \ldots, n$ and $a_{ii} = 1$ for all $i = 1, 2, \ldots, n$. We call an element-wise positive $n \times n$ matrix $\mathbf{B} = (b_{ij})$ *transitive* if $b_{ij}b_{jk} = b_{ik}$, for all $i, j, k = 1, 2, \ldots, n$. As we proved in [4, p.425], a transitive matrix is necessarily in SR and has rank one, hence it may be expressed as $\mathbf{B} = \mathbf{u}\mathbf{v}^\top$, where $\mathbf{u}$ (resp. $\mathbf{v}^\top$) is its first column (resp. row).

Non-transitive SR matrices are used in Saaty's multi-criteria decision making method called the analytic hierarchy process (AHP) [18]. Such matrices also occur in the field of macroeconomics, in both the static and the dynamic input-output analyses (see [5] and [21]). In the AHP, an entry a_{ij} of the so called *pairwise comparison matrix* $\mathbf{A}$ (PCM) represents the strength with which an alternative *i* dominates an other alternative *j*. After a PCM has been constructed it is required to derive implicit *weights*, $w_1, \ldots, w_n$, associated with the n decision alternatives $A_1, \ldots, A_n$, respectively. Saaty proposed the u_i elements of the principal (Perron) eigenvector of $\mathbf{A}$ to give the priority weights of the alternatives with respect to a given criterion. This solution for the weights is unique up to a multiplicative constant. The entries of the weight vector $\mathbf{w} = (w_i)$, $w_i > 0$, $i = 1, 2, \ldots, n$, are then usually normalized so that the sum of its elements is unity.

The nonnegative matrix balancing problem has attracted immense interest over the past decades. Osborne [16] was the first who presented remarkable results in pre-conditioning matrices and showed that a matrix balanced in the l_2 -norm has minimal Frobenius norm. For balancing in the l_1 -norm, a relevant result related to similarity scalings was established in [9]. There, it was proven that if $\mathbf{A}$ is irreducible, then a diagonal balancing matrix exists and is unique up to scalar multiples. Explicit characterizations of nonnegative matrices for which such scalings do exist were obtained by [1]. Necessary and sufficient conditions that a matrix is a similarity-scaling of another matrix were presented in [19]. Then, Eaves et al. [3] provided characterization theorems on nonnegative balanceable matrices.

Iterative scaling algorithms as well as optimization algorithms for different matrix balancing problems are well-known in the related literature (see the Osborne, the Parlett-Reinsch, the Krylov-based, cycle based and weighted balancing algorithms). For a detailed discussion and comparisons of these procedures, we refer to the paper [20] and the work of [2]. Recently, Genma et al. [8] have proposed an algorithm for a fractional minimization problem equivalent to minimizing the sum of linear ratios over the positive orthant as a matrix balancing problem. This approach was then applied to the so-called binary AHP method (see in [10] and

[22]) which is an oversimplified version of the traditional AHP having any comparison value $a_{ij} \in [\frac{1}{\alpha}, \alpha]$.

In this paper we present a new scaling algorithm, termed a *recursive rank-one residue iteration* (triple R-I) to balancing such SR matrices. For this purpose, we employ a system of inhomogeneous nonlinear equations and discuss the related least-squares optimization problem. We show that a sequence of transitive approximations to the original PCM will monotonically improve the objective function value, and, simultaneously, minimize the Frobenius-norm of the balanceable matrix. In practice, this means that repeated iteration of the proposed update rule guarantees convergence of a locally optimal matrix balancing. The balanced matrices have some useful properties for the application of the AHP. The algorithm seems to be efficient in computational time and easy usage.

This article is organized as follows. In Section 2, we introduce some notation and necessary definitions. In Section 3, we outline our earlier results related to the Newton-Kantorovich (N-K) procedure. The iterative scaling algorithm is presented in Section 4. We give proofs for the main results in Section 5, concerning convergence of the algorithm, existence of the similarity scalings and obtaining a limit point which comprises a stationary vector. Finally, in Section 6, a comprehensive numerical example provides a means of clear understanding for the readers underlying the use of our findings in the AHP.

2 Notations and definitions

Two mutually connected notations will be used for the weights: $\mathbf{w} = (w_i)$, $w_i > 0$, $i = 1, 2, \ldots, n$ is the weight (column) vector from $\mathbb{R}^n$, whereas $\mathbf{W} = \mathrm{diag}[w_i]$, denotes a diagonal matrix with the diagonal entries $w_1, w_2, \ldots, w_n$. Thus, $\mathbf{W}$ is a positive definite diagonal matrix if and only if $\mathbf{w}$ is an element-wise positive column vector.

The n vector $\mathbf{e}^\top = [1, 1, \ldots, 1]$ is defined to be the row vector of $\mathbb{R}^n$, and the $n \times n$ matrix $\mathbf{E} = (e_{ij})$=$\mathbf{e}\mathbf{e}^\top$ to be the square matrix of $\mathbb{R}^n$ with all entries equal to one. The $n \times n$ matrix $\mathbf{I}_n$ of $\mathbb{R}^n$ denotes the identity matrix with ones on the main diagonal and zeros elsewhere.

An $n \times n$ matrix $\mathbf{A}$ with nonnegative entries is said to be *balanced* if for each $i = 1, 2, \ldots, n$, the sum of the elements in the ith row of $\mathbf{A}$ equals the sum of the elements in the ith column of $\mathbf{A}$, i.e., if $\mathbf{A}$ is *line-sum-symmetric* so that

$$\mathbf{A}\mathbf{e} = \mathbf{A}^\top\mathbf{e}. \tag{2.1}$$

A matrix $\mathbf{A}$ is said to be *balanceable via diagonal similarity-scaling* if there exists a nonsingular diagonal matrix $\mathbf{W}$ with positive diagonal elements such that $\mathbf{WAW}^{-1}$ is balanced, i.e., if

$$\mathbf{WAW}^{-1}\mathbf{e} = \mathbf{W}^{-1}\mathbf{A}^{\top}\mathbf{We}. \tag{2.2}$$

For a real number $p \geq 1$ the l_p -norm of a vector $\mathbf{w}$ is defined by $\|\mathbf{w}\|_p = (|w_1|^p + |w_2|^p + \cdots + |w_n|^p)^{\frac{1}{p}}$. If $\|\mathbf{A}\|$ denotes the norm of matrix $\mathbf{A}$, then the l_1 -norm is: $\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$, the l_∞ -norm is: $\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$. Using the p norms, in the special case $p = 2$, the Frobenius norm is: $\|\mathbf{A}\|_\mathrm{F} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}$.

Kalantari et al. [12] have defined the matrix balancing problem in more generality. According to their definition an $n \times n$ matrix $\mathbf{Q}$ with arbitrary real entries is said to be *balanced in the l_p norm* ($p > 0$) if for each $i = 1, \ldots, n$, the ith row and column of $\mathbf{Q}$ have the same l_p -norm . An invertible (nonsingular) diagonal matrix $\mathbf{X} = \mathrm{diag}\,[x_1, x_2, \ldots, x_n]$ *balances* $\mathbf{Q}$ *in the* l_p *-norm* if for each $i = 1, \ldots, n$, the l_p -norm of the ith row and column of $\mathbf{XQX}^{-1}$ are identical, i.e.

$$\sum_{j=1}^{n} \left| q_{ij}\frac{x_i}{x_j} \right|^p = \sum_{j=1}^{n} \left| q_{ji}\frac{x_j}{x_i} \right|^p, \quad i = 1, \ldots, n. \tag{2.3}$$

Clearly, an invertible diagonal matrix $\mathbf{X} = \mathrm{diag}\,[x_1, x_2, \ldots, x_n]$ balances $\mathbf{Q}$ in the l_p -norm if and only if the positive diagonal matrix $\mathbf{W} = \mathrm{diag}\,[w_1, w_2, \ldots, w_n]$ balances the nonnegative matrix $\mathbf{A} = (|q_{ij}|^p)$ in l_1 -norm. The general matrix balancing problem in l_p -norm can thus be reduced to the case of nonnegative matrix balancing via a positive diagonal matrix.

An $n \times n$ matrix $\mathbf{A}$ is said to be *reducible* if and only if for some permutation matrix $\mathbf{A}$, the matrix $\mathbf{P}^{\top}\mathbf{AP}=\left[\begin{smallmatrix}\mathbf{B} & \mathbf{C}\\ \mathbf{0} & \mathbf{D}\end{smallmatrix}\right]$ is block upper triangular, where $\mathbf{B}$ and $\mathbf{D}$ are both square. Otherwise $\mathbf{A}$ is said to be an *irreducible* matrix. The graph $\mathcal{G}(\mathbf{A})$ of $\mathbf{A}$ is defined to be the *directed graph* of N nodes in which there is a directed arc leading from node i to node j if and only if $a_{ij} > 0$. $\mathcal{G}(\mathbf{A})$ is called *strongly connected* if for each pair of nodes (i, k) there is a sequence of directed arcs leading from node i to node k. $\mathbf{A}$ is an irreducible matrix if and only if $\mathcal{G}(\mathbf{A})$ is strongly connected, i.e., if it contains exactly one strongly connected component which includes all of the nodes.

An $n \times n$ nonnegative matrix $\mathbf{A}$ is called *completely reducible* if $i \in N$ has access to $j \in N$ if and only if j has access to i. In particular, every irreducible matrix is completely reducible.

3 Preliminaries

An approach to producing a 'best' transitive rank-one matrix approximation $\mathbf{B}$ to a SR matrix $\mathbf{A}$ (to a general PCM) was developed and presented in [6]. There, the 'best' is assessed in a least-squares (LS) sense. Thereby, in order to extract a vector of the weights $\mathbf{w}$ from $\mathbf{A}$ the following expression, i.e. the Euclidean-distance between matrices $\mathbf{A}$ and $\mathbf{B}$ should be minimized:

$$S^2(\mathbf{w}) := \|\mathbf{A} - \mathbf{B}\|_{\mathrm{F}}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_j}{w_i}\right)^2, \tag{3.1}$$

where the subscript F denotes the Frobenius norm; the square root of the sum of the squares of the elements. As we have shown in [6, p.694] a stationary value $\mathbf{w}$ of the error functional $S^2(\mathbf{w})$, denoted by $\mathbf{w}^*$, satisfies the *homogeneous* nonlinear equation

$$\mathbf{R}(\mathbf{w})\mathbf{w} = 0, \tag{3.2}$$

where the variable dependent coefficient matrix, $\mathbf{R}(\mathbf{w})$=(r_{ij}), $i, j = 1, 2, \ldots, n$, from $\mathbb{R}^n$, has the form:

$$\mathbf{R}(\mathbf{w}) = \mathbf{W}^{-2}(\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^\top\mathbf{W}) - (\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^\top\mathbf{W})^\top\mathbf{W}^{-2}.$$

with the off-diagonal elements

$$r_{ij} = \begin{cases} -(w_j - a_{ij}w_i)\left(\dfrac{1}{w_i^3} + \dfrac{1}{a_{ij}w_j^3}\right) & \text{for } i < j, \\ (w_i - a_{ji}w_j)\left(\dfrac{1}{w_j^3} + \dfrac{1}{a_{ji}w_i^3}\right) & \text{for } i > j, \end{cases}$$

and the diagonal elements

$$r_{ii} = 0, \qquad i = 1, 2, \ldots, n.$$

Since $\mathbf{A}$ is an SR matrix, obviously, $\mathbf{R}(\mathbf{w})$ is a *skew-symmetric* matrix.

However, the solution of equation (3.2) cannot be unique, since any constant multiple of this solution would produce an other solution. To overcome these shortcomings a nonzero vector $\mathbf{c} \in \mathbb{R}^n$ is introduced. Moreover, let (3.2) have a positive solution $\mathbf{w}$ normalized so that $\mathbf{c}^\top\mathbf{w} = 1$. (A convenient choice is $j = 1$ and thus $\mathbf{c}^\top = [1, 0, \ldots, 0]$, i.e. this condition is then $w_1 = 1$.) This way, for any

$j,\ 1 \le j \le n$, apparently, a stationary vector $\mathbf{w}^*$ is a solution to the following *inhomogeneous* system of n nonlinear equations

$$\mathbf{c}^\top \mathbf{w} = 1, \quad \mathbf{f}(\mathbf{w}) = \mathbf{R}_{k*}(\mathbf{w})\mathbf{w} = 0, \qquad k \neq j, \quad 1 \le k \le n, \tag{3.3}$$

where a conventional notation for the *j*th row of matrix $\mathbf{M}$ by $\mathbf{M}_{j*}$ is used. Observe in (3.3) that $\mathbf{f}(\mathbf{w})$ is at least twice differentiable.

To search for a vector root $\mathbf{w}^{*\top} = [w_1^*, w_2^*, \ldots, w_n^*]$ of equation (3.3) we used the Newton-Kantorovich (N-K) method, which employs the recurrent procedure:

$$\mathbf{w}^{(p+1)} = \mathbf{w}^{(p)} - \left(\nabla^2 \mathbf{f}(\mathbf{w})^{(p)}\right)^{-1} \nabla \mathbf{f}(\mathbf{w})^{(p)}, \qquad p = 0, 1, 2, \ldots, \tag{3.4}$$

where $\nabla^2 \mathbf{f}$ denotes the second Frèchet derivative (i.e. the Hessian matrix in the finite-dimensional case). The main convergence result for iteration (3.4) originated with Kantorovich [14]. As it is well-known, an appropriately chosen initial approximation, say $\mathbf{w}^{(0)}$, is critical for the convergence of the procedure. This means that the norm of the vector function, $\|\mathbf{f}(\mathbf{w}^{(0)})\|_{\mathrm{F}}$ should be small enough, i.e., $\mathbf{w}^{(0)}$ must be close to a solution. We always have chosen the solution of the 'best' linear approximation to problem (3.1), denoted by ϕ_0 in [6, p.693], as an initial value for running the procedure. Applying this strategy to the minimization problem (3.3) subject to an equality constraint on the weights, these N-K sequences were always *convergent* and produced *local* minima. Furthermore, computations with different choices for $\mathbf{c}$ and *j* have always led to scalar multiples of the same solution $\mathbf{w}^*$, giving some confidence the conjecture, that in these cases, this stationary vector $\mathbf{w}^*$ is associated with a *global* minimum of S^2. We mention here that the interested reader may find a relatively new approach to the N-K method, where global convergence can be achieved for functions that are not necessarily convex and the iteration converges globally for an arbitrary initial point [15]. For further problem background we refer to the excellent paper of Polyak [17].

It is apparent that by the weight vector $\mathbf{w}$ (and thus by the matrix $\mathbf{W}$) the 'best' approximating transitive matrix $\mathbf{B}$ to a matrix $\mathbf{A}$ in a LS sense can be obtained as

$$\mathbf{B} = \mathbf{W}^{-1}\mathbf{E}\mathbf{W} = \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^\top\mathbf{W} = \left[\frac{w_j}{w_i}\right], \qquad i, j = 1, 2, \ldots, n. \tag{3.5}$$

From (3.5), it is easy to see that $\mathbf{B}\mathbf{W}^{-1}\mathbf{e} = n\mathbf{W}^{-1}\mathbf{e}$, i.e. the only nonzero (principal) eigenvalue of $\mathbf{B}$ is n and its associated Perron-eigenvector is $\mathbf{W}^{-1}\mathbf{e}$, i.e. a vector whose elements are the reciprocals of the weights. For the nontransitive cases, however, $\lambda_{\max} > n$ (see a proof for this relationship e.g. in [5, p.407]).

4 A recursive rank-one residue iteration

In this section we discuss the development of a particular scaling method to balancing SR matrices. For this purpose, we formulate a least-squares (LS) optimization algorithm called a *recursive rank-one residue iteration* (triple R-I).

Let the set Ω denote the feasible region for problem (3.1):

$$\Omega = \left\{ \mathbf{w} \in \mathbb{R}^n \,\middle|\, \mathbf{c}^\top \mathbf{w} = 1, \mathbf{w} > 0 \right\}.$$

The triple R-I starts by using the N-K method for solving equation (3.3) to find a stationary vector $\mathbf{w}^{*(0)}$ (and thus the diagonal matrix $\mathbf{W}_0^*$) at the initial step, $k = 0$. The normalization condition $\mathbf{c}^\top \mathbf{w} = 1$ is imposed in order to hold $\{\mathbf{w}^{*(k)}\}$, $k = 0, 1, 2, \ldots$, in a bounded set throughout the entire process. By $\mathbf{W}_0^*$ and with the expression (3.5), the 'best' transitive matrix approximation $\mathbf{B}_0$ to the original SR matrix $\mathbf{A}$ in a LS sense can thus be determined.

A strategy to design an iterative procedure by establishing a successively adjusted sequence of rank-one matrices is the following. It is clear that the 'best' approximation of an entry a_{ij} of matrix $\mathbf{A}$ is $w_j^{*(0)}/w_i^{*(0)}$ $i, j = 1, 2, \ldots, n$. Since we may reasonably expect that $(w_i^{*(0)}/w_j^{*(0)})a_{ij}$ produces a 'good' approximation of 1, it is readily apparent that

$$\left[\frac{w_i^{*(0)}}{w_j^{*(0)}} a_{ij} \right] = \mathbf{W}_0^* \mathbf{A} \mathbf{W}_0^{*(-1)} \approx \mathbf{E}, \qquad i, j = 1, 2, \ldots, n. \tag{4.1}$$

The main idea is to achieve continuous improvement in further approximating $\mathbf{E}$. For this purpose, let a positive $n \times n$ matrix $\mathbf{H}_k = (h_{ij}^{(k)})$, $k = 0, 1, 2, \ldots$, called a *residue* be defined. It is convenient to set $\mathbf{H}_0 = \mathbf{A}$, at $k = 0$. Hence, necessarily, $\mathbf{H}_k$ is also in SR. Next, at the consecutive steps of the iteration process each entries of $\mathbf{H}_k$ will simultaneously be updated by performing a similarity transformation (diagonal similarity scaling) of the previous update $\mathbf{H}_{k-1}$ with the generating diagonal matrices $\mathbf{W}_{k-1}$ and $\mathbf{W}_{k-1}^{-1}$. This yields the *updating rule*:

$$\mathbf{H}_k = \mathbf{W}_{k-1} \mathbf{H}_{k-1} \mathbf{W}_{k-1}^{-1} = \left[\frac{w_i^{(k-1)}}{w_j^{(k-1)}} \right] \circ \left[h_{ij}^{(k-1)} \right], \qquad k = 1, 2, \ldots. \tag{4.2}$$

Note here that (4.2) can also be written in the form of a Hadamard product. For updating matrix $\mathbf{H}_{k-1}$, formula (4.2) is referred to as the *step-operator*, $\mathbf{S}_k(\mathbf{H}_k)$, in order to the rank-one matrix $\widetilde{\mathbf{B}}_k$ be recursively adjusted to the original matrix $\mathbf{A}$ at the consecutive iteration steps, so that

$$\widetilde{\mathbf{B}}_k = \widetilde{\mathbf{W}}_k^{-1} \mathbf{E} \widetilde{\mathbf{W}}_k, \quad k = 0, 1, \dots, \quad \text{where } \widetilde{\mathbf{W}}_k = \prod_{i=0}^{k-1} \mathbf{W}_i \text{ and } \widetilde{\mathbf{W}}_k^{-1} = \prod_{i=0}^{k-1} \mathbf{W}_i^{-1}. \tag{4.3}$$

It can be readily seen that each of the adjustment errors, $S(\widetilde{\mathbf{w}}^{(k)}) = \|\mathbf{A} - \widetilde{\mathbf{B}}_k\|_{\mathrm{F}}$, will be greater for $k = 1, 2, \dots,$ than that of for $k = 0$. An other transitive matrix,

$$\mathbf{B}_k^P = \mathbf{W}_k^{-1} \mathbf{E} \mathbf{W}_k, = \left[\frac{w_j^{(k)}}{w_i^{(k)}} \right] \circ \left[e_{ij}^{(k)} \right], \qquad k = 1, 2, \dots, \tag{4.4}$$

called a *pattern* will represent the 'best' transitive matrix approximation to its corresponding residue $\mathbf{H}_k$. Its approximation error is: $S(\mathbf{w}^{(k)}) = \|\mathbf{H}_k - \mathbf{B}_k^P\|_{\mathrm{F}}$. Obviously, $\mathbf{B}_0 = \mathbf{B}_0^P = \widetilde{\mathbf{B}}_0$. It is evident that the updating rule (4.2) will force all entries of $\widetilde{\mathbf{B}}_k$ to be set to 1, while the elements of the Perron-eigenvectors of the pattern $\mathbf{B}_k^P$ will successively approach to those of matrix $\mathbf{A}$.

The process is repeated until some convergence criterion is met. The stopping rule is to halt the algorithm at iteration step $k = q$ once the numerical error falls below a predefined tolerance (a reasonably small positive number, $\varepsilon > 0$) yielding the "stabilized" matrices $\widetilde{\mathbf{W}}_q, \mathbf{H}_q, \widetilde{\mathbf{B}}_q$ and $\mathbf{B}_q^P$.

The formal description of the algorithm is presented below:

Triple R-I Algorithm

Input module. Enter the SR matrix $\mathbf{A}$. Calculate its Perron-eigenvalue, $\lambda_{\max}(\mathbf{A})$, and its normalized right and left Perron-eigenvectors, $\mathbf{u}_{\max}(\mathbf{A})$ and $\mathbf{v}_{\max}(\mathbf{A})$.

Initial module. For $k = 0$. Given a positive initial value ϕ_0 and a reasonably small $\varepsilon > 0$. Using the N-K method find the stationary vector $\mathbf{w}^{*(0)}$ (and thus the diagonal matrix $\mathbf{W}^{*(0)}$) by solving the following system of nonlinear equations:

$$\{\mathbf{W}_0^{-2}(\mathbf{A} - \mathbf{W}_0^{-1}\mathbf{E}\mathbf{W}_0) - (\mathbf{A} - \mathbf{W}_0^{-1}\mathbf{E}\mathbf{W}_0)^\top \mathbf{W}_0^{-2}\}\mathbf{W}_0\mathbf{e} = 0, \tag{4.5}$$

where in (4.5), $\mathbf{W}_0\mathbf{e} = \mathbf{w}^{(0)} = [w_i^{(0)}]$, $i = 1, 2, \dots, n$, is normalized so that $\mathbf{c}^\top \mathbf{w}^{(0)} = 1$.

a) If $\mathbf{w}^{(0)}$ is stationary, compute

$$\widetilde{\mathbf{B}}_0 = \mathbf{B}_0^P = \mathbf{W}_0^{*(-1)}\mathbf{E}\mathbf{W}_0^* = \left[\frac{w_j^{*(0)}}{w_i^{*(0)}}\right], \qquad i, j = 1, 2, \ldots, n. \tag{4.6}$$

b) Else choose an other promising positive initial value and repeat the N-K procedure until $\mathbf{w}^{(0)}$ is stationary, then compute $\widetilde{\mathbf{B}}_0 = \mathbf{B}_0^P$ according to (4.6).

c) Calculate the error of the 'best' transitive matrix approximation $\widetilde{\mathbf{B}}_0$ to the original matrix $\mathbf{A}$ as: $S(\mathbf{w}^{(0)}) = S(\widetilde{\mathbf{w}}^{(0)})$=$\|\mathbf{A} - \widetilde{\mathbf{B}}_0\|_{\mathrm{F}}$.

Set $\mathbf{H}_0 = \mathbf{A}$.

Recursion module. For $k = 1, 2, \ldots$. Using the N-K method find the stationary vector $\mathbf{w}^{*(k)}$ (and $\mathbf{W}^{*(k)}$) by solving the following system of nonlinear equations:

$$\{\mathbf{W}_k^{-2}(\mathbf{H}_k - \mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k) - (\mathbf{H}_k - \mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k)^\top \mathbf{W}_k^{-2}\}\mathbf{W}_k\mathbf{e} = 0, \tag{4.7}$$

where in (4.7), the vector $\mathbf{W}_k\mathbf{e} = \mathbf{w}^{(k)} = [w_i^{(k)}]$, $i = 1, 2, \ldots, n$, is normalized so that $\mathbf{c}^\top\mathbf{w}^{(k)} = 1$ and the residue (updating rule) is given by the formula (4.2).

a) If $\|\mathbf{W}_k - \mathbf{I}_n\| < \varepsilon$, for $k > N(\varepsilon)$, set $k = q$, then compute $\widetilde{\mathbf{W}}_q$, $\mathbf{H}_q$, $\widetilde{\mathbf{B}}_q$ and $\mathbf{B}_q^P$, then calculate the Perron-eigenvalue, $\lambda_{\max}(\mathbf{H}_q)$, and its normalized right and left Perron-eigenvectors, $\mathbf{u}_{\max}(\mathbf{H}_q)$ and $\mathbf{v}_{\max}(\mathbf{H}_q)$ and stop.

b) Else compute $\widetilde{\mathbf{W}}_k$, $\mathbf{H}_k$, $\widetilde{\mathbf{B}}_k$ and $\mathbf{B}_k^P$.

c) Calculate the adjustment error of the rank-one matrix $\widetilde{\mathbf{B}}_k$ to the original matrix $\mathbf{A}$ as: $S(\widetilde{\mathbf{w}}^{(k)})$=$\|\mathbf{A} - \widetilde{\mathbf{B}}_k\|_{\mathrm{F}}$.

d) Continue the iteration process for $k + 1$.

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5 Diagonal similarity scaling of pairwise comparison matrices

This section discusses the matrix balancing problem done through successive adjustments of the residue and the pattern matrices. We will show that matrices $\mathbf{A}$ and $\mathbf{B}_0$ are balanceable in the sense of (2.2) and can be balanced by virtue of (2.1). The balanced matrices have useful properties which provide some novel contributions to the theory of the AHP as well. In particular, we will give proofs that our triple R-I algorithm with a user specified termination criterion results in the similarity scalings, $\mathbf{B}_q^P$ and $\mathbf{H}_q$.

We start by focusing on the sequence of the pattern $\{\mathbf{B}_k^P\}$, $k = 1, 2, \ldots$, generated via line-sum-symmetric diagonal similarity scaling (DSS). We will assume that matrix $\mathbf{B}_k^P$ is irreducible by permutation matrices. Since $\mathbf{B}_k^P > 0$, according to the Perron-theorem this condition holds. The reason for this restriction is that in the triple R-I it is desirable to have the entries of the sequence of diagonal matrices bounded. The iterative algorithm looks for a sequence of $\{\mathbf{W}_k\}$, so that at each step k, the pattern $\mathbf{B}_k^P = \mathbf{W}_k^{-1}\mathbf{E}\mathbf{W}_k$ is yielded. It is easy to recognize here that this is equivalent with the DSS: $\mathbf{B}_k^{P\top} = \mathbf{W}_k\mathbf{E}\mathbf{W}_k^{-1}$. Accordingly, the following results will be related to the transpose of the pattern. Thus, the first row (and *not* its resp. column as for $\mathbf{B}_k^P$) of this matrix will contain the reciprocals of the weights, w_i^{-1}, $i = 1, 2, \ldots, n$, i.e. the priorities of the decision alternatives in the AHP.

We first present an important result to characterize existence of line-sum-symmetric similarity scalings of balanceable matrices [3]:

Corollary 1. (Eaves et al. [3, p.133]). *Let $\mathbf{A}$ be an $n \times n$ nonnegative matrix and let f be the real valued function defined on Ω $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \gg 0\}$ by*

$$f(\mathbf{w}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i a_{ij} w_j^{-1}. \tag{5.1}$$

Then the following are equivalent:

(a) $\mathbf{A}$ *has a line-sum-symmetric similarity-scaling,*

(b) $\mathbf{A}$ *is completely reducible, and*

(c) f *attains a minimum over* Ω.

An other famous result originated with the same authors relates to the characterization of the set of diagonal matrices yielding line-sum-symmetric similarity scalings of balanceable matrices [3]:

Theorem 1. (Eaves et al. [3, p.134]). *Let* $\mathbf{A}$ *be an* $n \times n$ *nonnegative matrix and let* f *be the real valued function defined on* Ω $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \gg 0\}$ *by* (5.1). *Consider the following properties of a vector* $\mathbf{w}^* \in \Omega$*:*

(a) $\mathbf{w}^*$ *minimizes the function f over* Ω*, and*

(b) *the matrix* $\mathbf{W}(\mathbf{w}^*)\mathbf{A}\mathbf{W}(\mathbf{w}^*)^{-1}$ *is line-sum-symmetric.*

The $\mathbf{w}^*$ *satisfies* (a) *if and only if* $\mathbf{w}^*$ *satisfies* (b).

Proofs of the aforesaid results are given in [3, pp.133-134].

Theorem 1 shows that the problem of searching for a line-sum-symmetric similarity scaling of a given balanceable matrix is equivalent to the problem of minimizing a (nonlinear) function over the positive orthant. Corollary 1 shows that these problems have a solution if and only if the underlying matrix is completely reducible. Theorem 1 states also that a stationary vector, $\mathbf{w}^*$ minimizes the function f over Ω which is a unique optimal solution up to a positive scalar multiple, if and only if the matrix $\mathbf{W}_k\mathbf{A}\mathbf{W}_k^{-1}$ is in line-sum-symmetry.

We now define the following vector functions of the row and column sums for a positive vector $\mathbf{w}$ of the diagonal similarity scaling $\mathbf{B}_k^{P\top}{=}\mathbf{W}_k\mathbf{E}\mathbf{W}_k^{-1}$. (To simplify notation, hereafter we will omit the iteration step index k):

$$\mathbf{r}(\mathbf{w}) \equiv \left(\sum_{j=1}^{n}\left(e_{1j}\frac{w_j}{w_1}\right)^2, \ldots, \sum_{j=1}^{n}\left(e_{nj}\frac{w_j}{w_n}\right)^2\right) = \left(\sum_{j=1}^{n}\left(\frac{w_j}{w_1}\right)^2, \ldots, \sum_{j=1}^{n}\left(\frac{w_j}{w_n}\right)^2\right), \tag{5.2}$$

$$\mathbf{c}(\mathbf{w}) \equiv \left(\sum_{i=1}^{n}\left(e_{i1}\frac{w_1}{w_i}\right)^2, \ldots, \sum_{i=1}^{n}\left(e_{in}\frac{w_n}{w_i}\right)^2\right) = \left(\sum_{i=1}^{n}\left(\frac{w_1}{w_i}\right)^2, \ldots, \sum_{i=1}^{n}\left(\frac{w_n}{w_i}\right)^2\right). \tag{5.3}$$

Observe now that any positive $\mathbf{w}$ satisfies $f(\mathbf{w}){=}\mathbf{r}(\mathbf{w})\mathbf{e}{=}\mathbf{c}(\mathbf{w})\mathbf{e}$ and for a positive vector $\mathbf{w}^*$, $\mathbf{r}(\mathbf{w}^*){=}\mathbf{c}(\mathbf{w}^*)$ holds, if and only if $\mathbf{w}^*$ is a stationary vector. The triple R-I iterates over all rows and columns of the balanceable matrix at a particular step k to find an appropriate scale vector, i.e. each entry of $\mathbf{W}_k$ and thus $\mathbf{B}_k^P$

is updated and so the rows/columns are scaled simultaneously, making this algorithm very efficient. The following two lemmas and their proofs are extensions of similar results that were presented by Genma et al. [8].

Lemma 1. *For a positive vector* $\mathbf{w}$*, at each iteration step* k*,* $k=1,2,\ldots,$ *of the triple R-I algorithm, let* $\mathbf{r}{=}\mathbf{r}(\mathbf{w})$ *and* $\mathbf{c}{=}\mathbf{c}(\mathbf{w})$*, defined as* (5.2) *and* (5.3)*. For any particular row and column, assume that* $r_l{\neq}c_l$*,* $l=1,2,\ldots,n$*. Let an index* $\kappa(l){=}\sqrt{r_l/c_l}$ *be introduced and let* $w_i(\kappa(l)){=}\kappa(l)w_i$ *for* $i=l$*. Then,*

$$\min_{\kappa(l)>0} f\left(\mathbf{w}(\kappa(l)\right) - f(\mathbf{w}) \tag{5.4}$$

has a unique optimal solution $\mathbf{w}^{*(-1)\top}{=}\left(\sqrt{r_1/c_1},\ldots,\sqrt{r_n/c_n}\right)$ *and its optimal objective function value is* $Z^*{=}\sum\limits_{l=1}^{n} -\left(r_l{-}c_l\right)^2$*.*

Proof. We first note that in $\kappa(l){=}\sqrt{r_l/c_l}$, the restriction imposed on $\mathbf{B}_k^P$ prevents c_l from equaling 0. After some algebraic manipulations we may obtain that

$$\begin{aligned} & f\left(\mathbf{w}(\kappa(l))\right) - f(\mathbf{w}) = \\ & = \sum_{j=1}^{n}\left(\frac{w_j}{\kappa(l)w_l}\right)^2 + \sum_{i=1}^{n}\left(\frac{\kappa(l)w_l}{w_i}\right)^2 - \left(\sum_{j=1}^{n}\left(\frac{w_j}{w_l}\right)^2 + \sum_{i=1}^{n}\left(\frac{w_l}{w_i}\right)^2\right) = \\ & = \frac{1}{\kappa^2(l)}r_l^2 + \kappa^2(l)c_l^2 {-} r_l^2 {-} c_l^2, \qquad l=1,\ldots,n. \end{aligned} \tag{5.5}$$

To the following steps, it is convenient to introduce the functions resulted in (5.5) as: $g(\kappa(l)){=}\frac{1}{\kappa^2(l)}r_l^2{+}\kappa^2(l)c_l^2{-}r_l^2{-}c_l^2$. Since, obviously, both $r_l^2{>}0$ and $c_l^2{>}0$, therefore, the $g(\kappa(l))$'s are each strictly convex. Minimizing them, we get that

$$\frac{dg}{d\kappa(l)}{=}{-}\frac{2}{\kappa^3(l)}r_l^2{+}2\kappa(l)c_l^2{=}0, \qquad l=1,\ldots,n. \tag{5.6}$$

Now, it is easy to see that a stationary value, $\kappa^*(l){=}\sqrt{r_l/c_l}$, satisfies (5.6). Hence,

$$Z^*{=}\sum_{l=1}^{n} g(\kappa^*(l)){=}\sum_{l=1}^{n}\frac{1}{\frac{r_l}{c_l}}c_l^2{+}\frac{r_l}{c_l}r_l^2{-}r_l^2{-}c_l^2{=}\sum_{l=1}^{n}{-}r_l^2{-}c_l^2{+}2r_lc_l{=}\sum_{l=1}^{n}{-}(r_l{-}c_l)^2. \tag{5.7}$$

□

With $\kappa^*(l)$, the minimum that can be achieved in (5.5) yields $2r_lc_l$. The updating rule (4.2) ensures the reduction of the value of the objective function Z^*. Thus, the sequence $\{\mathbf{W}_k\}$ $k = 1, 2, \ldots,$ generated by the recursive process induces a decrease in the Frobenius norm, $\left\|\mathbf{B}_k^{P\top}\right\|_{\mathrm{F}}$, at each step k, as is shown by the following lemma.

Lemma 2. *At a particular iteration step k, $k = 1, 2, \ldots,$ of the triple R-I iteration and utilizing the index $\kappa(l)$, $l = 1, 2, \ldots, n$, it yields*

$$\sum_{l=1}^{n} -\left(r_l^{(k)} - c_l^{(k)}\right)^2 = f\left(\mathbf{w}^{(k+1)}\right) - f\left(\mathbf{w}^{(k)}\right). \tag{5.8}$$

Therefore, $f\left(\mathbf{w}^{(k)}\right) > f\left(\mathbf{w}^{(k+1)}\right)$, $\qquad k = 1, 2, \ldots$.

Proof. Since $\mathbf{w}^{(k+1)}$ is equal to a positive scalar multiple of $\mathbf{w}\left(\sqrt{\frac{r_l}{c_l}}\right)$ at step $k+1$, it follows from $f\left(\mathbf{w}^{(k+1)}\right) = f\left(\mathbf{w}\left(\sqrt{\frac{r_l}{c_l}}\right)\right)$ and Lemma 1 that

$$\begin{aligned} 0 > \sum_{l=1}^{n} -\left(r_l^{(k)} - c_l^{(k)}\right)^2 &= \min_{\kappa(l)>0} f\left(\mathbf{w}^{(k)}(\kappa(l))\right) - f\left(\mathbf{w}^{(k)}\right) = \\ = f\left(\mathbf{w}\left(\sqrt{\frac{r_l}{c_l}}\right)\right) - f(\mathbf{w}^{(k)}) &= f\left(\mathbf{w}^{(k+1)}\right) - f\left(\mathbf{w}^{(k)}\right). \end{aligned} \tag{5.9}$$

As it can be seen from (5.9) the norm is never increased. For the cases $r_l^{(k)} \neq c_l^{(k)}$ there is a reduction in $\left\|\mathbf{B}_k^P\right\|_{\mathrm{F}}^2$. The restriction of irreducibility placed upon $\mathbf{B}_k^P$ prevents $r_l^{(k)}$ and $c_l^{(k)}$ from identically vanishing. Thus,

$$\left\|\mathbf{B}_k^{P\top}\right\|_{\mathrm{F}}^2 - \left\|\mathbf{B}_{k+1}^{P\top}\right\|_{\mathrm{F}}^2 = \sum_{l=1}^{n}\left(r_l^{(k)} - c_l^{(k)}\right)^2 \geq 0, \qquad k = 1, 2, \ldots. \tag{5.10}$$

□

We remark that using similar arguments and applying the technique of proofs used for Lemmas 1 and 2 accordingly, one could easily verify that there is a strict decrease in the l_1 -norm, $\left\|\mathbf{B}_k^P\right\|_1$, and in the l_∞ -norm, $\left\|\mathbf{B}_k^P\right\|_\infty$, as well (see also Corollary 1 and Theorem 1), if one defines:

$$r_l^{(k)} = \sum_{j=1}^{n}\left|b_{lj}^{P(k)}\right| \text{ and } c_l^{(k)} = \sum_{j=1}^{n}\left|b_{jl}^{P(k)}\right|, \qquad l = 1, 2, \ldots, n. \tag{5.11}$$

The convergence theorem can now be stated.

Theorem 2. *Let $\mathbf{B}_k^{P\top} = (b_{ji}^P)$, $i, j = 1, 2, \ldots, n$, be a transitive matrix with positive entries and called a pattern. Assume that $\mathbf{B}_k^{P\top}$ is irreducible by permutations. Through the triple R-I iteration, the sequence $\{\mathbf{B}_k^{P\top}\}$, $k = 1, 2, \ldots$, generated by the diagonal matrices $\mathbf{W}_k$ and $\mathbf{W}_k^{-1}$ defined in* (4.3) *converges to some $\mathbf{B}_q^{P*}$ over the feasible set Ω and q indicates the step of the termination of the algorithm for a prescribed reasonably small tolerance $\varepsilon > 0$. Then,*

(i) $\lim\limits_{k\to\infty} \mathbf{B}_k^{P\top} = \mathbf{B}_q^{P\top}$ *exists;*

(ii) $\mathbf{B}_q^{P\top}{=}\mathbf{W}_q\mathbf{E}\mathbf{W}_q^{-1}$*;*

(iii) $\mathbf{W}_q{=}\lim\limits_{k\to\infty} \mathbf{W}_k{=}\lim\limits_{k\to\infty} \mathbf{W}_k^{-1}{=}\mathbf{I}_n$*;*

(iv) $\left\|\mathbf{B}_q^{P\top}\right\|_{\mathrm{F}} {=}\inf\limits_{\mathbf{W}_\psi} \left\|\mathbf{W}_\psi\mathbf{E}\mathbf{W}_\psi^{-1}\right\|_{\mathrm{F}}$*;*

(v) $\mathbf{B}_q^{P\top}{=}\mathbf{B}_q^P{=}\mathbf{E}{=}\mathbf{B}^{P*}$*;*

(vi) $\mathbf{B}^{P*}$ *is in sum-symmetry;*

where in (iv)*, $\mathbf{W}_\psi$ ranges over the class of all nonsingular diagonal matrices.*

Proof. As a consequence of (5.10) in Lemma 2, the sequence $\{\left\|\mathbf{W}_k\mathbf{E}\mathbf{W}_k^{-1}\right\|_{\mathrm{F}}\}$ of similarity scalings is bounded. Therefore, since $w_1^{(k)}{=}1$ is fixed for all k, there exists $\alpha > 0$ and $\beta > 0$ such that $\alpha \leq w_i^{(k)}(\kappa(l)) \leq \beta$ for all k and $i, l = 1, 2, \ldots, n$. From this, it is clear that there exists $\gamma > 0$ such that $r_l > \gamma$ and $c_l > \gamma$ for all k. Following the line of the technique that was used in [16], from $\left\|\mathbf{B}_1^{P\top}\right\|_{\mathrm{F}} \geq \left\|\mathbf{B}_2^{P\top}\right\|_{\mathrm{F}} \geq \ldots \geq 0$ follows the existence of L such that

$$\lim_{k\to\infty} \left\|\mathbf{B}_k^{P\top}\right\|_{\mathrm{F}} {=}L \geq 0, \qquad \left\|\mathbf{B}_k^{P\top}\right\|_{\mathrm{F}} \geq L \quad \text{for all } k. \tag{5.12}$$

According to (5.10), this implies that

$$\lim_{k\to\infty} \sum_{l=1}^{n} (r_l - c_l)^2 = \lim_{k\to\infty} \left(\left\|\mathbf{B}_{k+1}^P\right\|_{\mathrm{F}}^2 - \left\|\mathbf{B}_k^P\right\|_{\mathrm{F}}^2\right) {=}0, \tag{5.13}$$

or

$$r_l{=}c_l + \eta^{(k)}, \qquad \lim_{k\to\infty} \left|\eta^{(k)}\right| {=}0, \qquad \text{for all } l \text{ and } k. \tag{5.14}$$

But

$$w_i^{(-1)2}(\kappa(l)){=}\frac{r_l}{c_l}{=}1 + \frac{\eta^{(k)}}{c_l} \qquad \text{and } \ r_l > \gamma. \tag{5.15}$$

Therefore,

$$\lim_{k\to\infty} w_i^{(-1)}(\kappa(l)){=}1, \qquad i,l=1,2,\ldots,n. \tag{5.16}$$

The values of $|r_l - c_l|$ pass through zero at least once for every k consecutive steps of the iterative process. The sums r_l and c_l are bounded for all k so that (5.16) implies that the changes in r_l and c_l over any k successive steps approach zero as k increases without limit. This implies that

$$\lim_{k\to\infty} |r_l - c_l|{=}0, \qquad l=1,2,\ldots,n. \tag{5.17}$$

Another consequence of the fact that the diagonal elements of $\{\mathbf{W}_k\}$, $k=1,2,\ldots$, are bounded away from zero and from above is that a subsequence $\{\mathbf{W}_{\bar{k}}\}$ of $\{\mathbf{W}_k\}$ can be chosen such that

$$\lim_{\bar{k}\to\infty} \mathbf{W}_{\bar{k}}{=}\mathbf{W}_q, \qquad \mathbf{W}_q \text{ is nonsingular.} \tag{5.18}$$

Also, since the entries of $\mathbf{B}_k^{P\top}$ are continuous functions of the diagonal elements of $\mathbf{W}_k$, it follows that

$$\lim_{k\to\infty} \mathbf{B}_{\bar{k}}^{P\top}{=}\mathbf{W}_q\mathbf{E}\mathbf{W}_q^{-1}{=}\mathbf{B}_q^{P\top} \tag{5.19}$$

exists, representing a unique limit point of the triple R-I iteration, denoted as $\mathbf{B}_q^{P*}$. The iterate $\mathbf{W}_k$ converges to $\mathbf{I}_n$ in a limiting sense. The algorithm stops for a stipulated arbitrarily small $\varepsilon > 0$ if a certain N can be found such that

$$\|\mathbf{W}_q - \mathbf{I}_n\| < \varepsilon, \qquad \text{for } q > N(\varepsilon).$$

Thus, $S^2(\mathbf{w}^{(q+p)}) = S^2(\mathbf{w}^{(q)})$, for $p > 0$. A direct implication of (5.16) is that $\mathbf{B}_q^{P\top}{=}\mathbf{B}_q^{P}{=}\mathbf{E}{=}\mathbf{B}_q^{P*}$. Also, it is straightforward that the matrices $\mathbf{B}_q^{P\top}$ and $\mathbf{B}_q^{P}$ are in sum-symmetry, since their entries are all ones. Indeed, it is easy to see that $\mathbf{B}_q^{P}$ embodies the sole line-sum-symmetric transitive matrix over the class of all transitive matrices. This completes the proof. □

In the sequel, we consider the residue matrix $\mathbf{H}_k$. We will show that the updating rule of the triple R-I algorithm is essentially analogous to the *fixed point iteration*

$$\mathbf{H}_{k+1} = \mathbf{S}_k(\mathbf{H}_k) = \mathbf{W}_k\mathbf{H}_k\mathbf{W}_k^{-1}, \quad k=0,1,\ldots \tag{5.20}$$

where $\mathbf{S}_k(\mathbf{H}_k)$ is the step operator of the triple R-I. The objective is to minimize the Frobenius norm:

$$\|\mathbf{H}_{k+1} - \mathbf{H}_k\|_{\mathrm{F}} \Rightarrow \text{minimum}, \qquad k=0,1,\ldots. \tag{5.21}$$

The convergence theorem for the sequence $\{\mathbf{H}_k\}$ is stated below.

Theorem 3. *Let* $\mathbf{H} = (h_{ij})$, $i, j = 1, 2, \ldots, n$ *be an SR matrix with positive entries and called a residue. The sequence* $\{\mathbf{H}_k\}$, $k = 1, 2, \ldots$*, generated by the fixed point iteration* (5.20) *using the step operator* $\mathbf{S}_k$ *converges to some* $\mathbf{H}_q^* \in \mathcal{H}^*$*, where* $\mathcal{H}^*$ *is the set of stationary points of problem* (5.21) *over the feasible set* Ω*, and* q *indicates the step of the termination of the triple R-I for a prescribed reasonably small tolerance* $\varepsilon > 0$.

Proof. As follows from its construct, $\mathbf{S}_k(\mathbf{H}_k)$ is non-expansive, therefore $\{\mathbf{H}_k\}$ lies in a compact set and must have a limit point, say $\hat{\mathbf{H}} = \lim_{j\to\infty} \mathbf{H}_{k_j}$. Additionally, for any $\mathbf{H}_q^* \in \mathcal{H}^*$,

$$\left\|\mathbf{H}_{k+1} - \mathbf{H}_q^*\right\|_{\mathrm{F}} = \left\|(\mathbf{S}_k(\mathbf{H}_k) - \mathbf{S}_{k_j}(\mathbf{H}_q^*)\right\|_{\mathrm{F}} \le \left\|\mathbf{H}_k - \mathbf{H}_q^*\right\|_{\mathrm{F}},$$

which implies that the sequence $\{\|\mathbf{H}_k - \mathbf{H}_q^*\|_{\mathrm{F}}\}$ is monotonically non-increasing under the updating rule (4.2). Hence,

$$\lim_{k\to\infty} \left\|\mathbf{H}_k - \mathbf{H}_q^*\right\|_{\mathrm{F}} = \left\|\hat{\mathbf{H}} - \mathbf{H}_q^*\right\|_{\mathrm{F}}, \tag{5.22}$$

where $\hat{\mathbf{H}}$ can be any limit point of $\{\mathbf{H}_k\}$. Considering that $\mathbf{S}_k(\mathbf{H}_k)$ is continuous, the step operator for $\hat{\mathbf{H}}$,

$$\mathbf{S}_{k_j}(\hat{\mathbf{H}}) = \lim_{j\to\infty} \mathbf{S}_{k_j}(\mathbf{H}_{k_j}) = \lim_{j\to\infty} \mathbf{H}_{k_j+1},$$

produces also a limit point of $\{\mathbf{H}_k\}$. Therefore, we have

$$\left\|\mathbf{S}_{k_j}(\hat{\mathbf{H}}) - \mathbf{S}_q(\mathbf{H}_q^*)\right\|_{\mathrm{F}} = \left\|\mathbf{S}_{k_j}(\hat{\mathbf{H}}) - \mathbf{H}_q^*\right\|_{\mathrm{F}} = \left\|\hat{\mathbf{H}} - \mathbf{H}_q^*\right\|_{\mathrm{F}}$$

which shows that $\hat{\mathbf{H}}$ is a stationary point of problem (5.21). Finally, by setting $\mathbf{H}_q^* = \hat{\mathbf{H}} \in \mathcal{H}^*$ in (5.22), we obtain

$$\lim_{k\to\infty} \left\|\mathbf{H}_k - \hat{\mathbf{H}}\right\|_{\mathrm{F}} = \lim_{j\to\infty} \left\|\mathbf{H}_{k_j} - \hat{\mathbf{H}}\right\|_{\mathrm{F}} = 0,$$

i.e. $\{\mathbf{H}_k\}$ converges to its limit point $\hat{\mathbf{H}}$. In each step k of the recursive algorithm the N-K method is used to solve the system of nonlinear equations (4.7). Therefore, at step $k = q$, when the iteration has converged to any limit point $\mathbf{H}_q^*$ in the interior of the feasible region Ω, this point is necessarily a stationary point (see Farkas et al. [6, p.695]). This completes the proof. □

The following lemma refers to the limit of the sequence of the entries of matrices $\{\mathbf{H}_k\}$, $k = 1, 2, \ldots$, and verifies that this is also a convergent sequence.

Lemma 3. *For the convergence of a sequence of matrices* $\{\mathbf{H}_k\}$*,* $k = 0, 1, 2, \ldots$ *it is necessary and sufficient that the generalized Cauchy test hold, namely for any* $\varepsilon > 0$ *there must be a number* $N = N(\varepsilon)$ *such that for* $k > N$*,* $p > 0$

$$\|\mathbf{H}_{k+p} - \mathbf{H}_k\| < \varepsilon, \tag{5.23}$$

where the matrix norm can be any canonical norm.

Proof. Indeed, since according to Theorem 3 the sequence $\{\|\mathbf{H}_k\|_{\mathrm{F}}\}$ is decreasing and therefore, inequality (5.23) is valid. Thus, for every element $h_{ij}^{(k)}$ of the matrices of the sequence $\{\mathbf{H}_k\}$ the Cauchy test (see e.g. in [13]) will hold, and hence, there exists

$$\lim_{k\to\infty} \mathbf{H}_k = [\lim_{k\to\infty} h_{ij}^{(k)}] = \mathbf{H}_q. \tag{5.24}$$

The matrix $\mathbf{H}_q$ is stabilized at step q, and repeats itself in the succeeding steps, if we would continue the iteration. Therefore, $\mathbf{H}_q = \mathbf{H}_{q+p}$, for $p > 0$. $\square$

Next, we show that matrix $\mathbf{H}_q^*$ is in line-sum-symmetry.

Corollary 2. *For the limit matrix* $\mathbf{H}_q^*$*, of the triple R-I the right and the left eigenvectors associated with the zero eigenvalue of the skew-symmetric matrix* $(\mathbf{H}_q^* - \mathbf{H}_q^{*\top})$*, are the vectors* $\mathbf{e}$ *and* $\mathbf{e}^\top$*, respectively.*

Proof. Using the diagonal matrix $\overset{\sim}{\mathbf{W}}$ defined as (4.3) we can write the product of the diagonal matrices $\mathbf{W}_k$ in a limiting sense as

$$\lim_{k\to\infty} (\mathbf{W}_{k-1}\mathbf{W}_{k-2} \ldots \mathbf{W}_2\mathbf{W}_1\mathbf{W}_0) = \overset{\sim}{\mathbf{W}}. \tag{5.25}$$

By taking the limit of (4.2) we have

$$\lim_{k\to\infty} \mathbf{H}_k = \overset{\sim}{\mathbf{W}}_q \mathbf{A} \overset{\sim}{\mathbf{W}}_q^{-1}. \tag{5.26}$$

Applying (5.26) for $k > N$, the system of nonlinear equations (4.3) leads to the following equation:

$$\Big(\overset{\sim}{\mathbf{W}}_q \mathbf{A} \overset{\sim}{\mathbf{W}}_q^{-1} - \overset{\sim}{\mathbf{W}}_q^{-1} \mathbf{A}^\top \overset{\sim}{\mathbf{W}}_q\Big)\mathbf{e} = (\mathbf{H}_q^* - \mathbf{H}_q^{*\top})\mathbf{e} = 0, \tag{5.27}$$

where it is apparent that the right eigenvector associated with the zero eigenvalue of the skew-symmetric matrix $(\mathbf{H}_q^* - \mathbf{H}_q^{*\top})$ is $\mathbf{e}$, while the left eigenvector is $\mathbf{e}^\top$.

As it is readily seen from equation (5.27) the matrix $\mathbf{H}_q^*$ is balanced, since it is in line-sum-symmetry in a sense of (2.2). $\square$

Writing the recursion formula (4.2) in an element-wise form we have

$$\begin{cases} h_{ij}^{(k)} = h_{ij}^{(k-1)} \left(\dfrac{w_j^{(k-1)}}{w_i^{(k-1)}} \right) & \text{for } \ i < j, \\ h_{ji}^{(k)} = h_{ji}^{(k-1)} \left(\dfrac{w_i^{(k-1)}}{w_j^{(k-1)}} \right) & \text{for } \ i > j, \end{cases}$$

and for the diagonal elements

$$h_{ii} = 1, \qquad i = 0, 1, \ldots, n.$$

Making use of Lemma 1 and from Theorems 1 and 2 it can be seen that the sequences $\{h_{ij}^{(k)}\}$ and $\{h_{ji}^{(k)}\}$, $k = 0, 1, 2, \ldots$, for all $i, j = 1, 2, \ldots, n$, as well as $\{w_i^{(k)}\}$, $k = 0, 1, 2, \ldots$, for all $i, j = 1, 2, \ldots, n$, converge to their limits. We now show that the quotients $\left(\frac{w_j^{(k)}}{w_i^{(k)}} \right)$ and their products with $\{h_{ij}^{(k)}\}$ also converge to a limit point. For this purpose, we apply the following well-known theorem.

Theorem 4. *Suppose that $\{w_i^{(k)}\}$ and $\{w_j^{(k)}\}$ are two convergent sequences generated by the triple R-I over the feasible set Ω. Let these sequences simply be denoted by $(\{a_n\})$ and $(\{b_n\})$, respectively, with limits A and B. Then, the following rules apply:*

(i) Product Rule. *The product $(\{a_n\}\ \{b_n\})$ is convergent and* $\lim\limits_{n\to\infty} (\{a_n\}\{b_n\}) = AB$;

(ii) Quotient Rule. *If $B \neq 0$, then $\left(\frac{\{a_n\}}{\{b_n\}}\right)$ is also convergent, and* $\lim\limits_{n\to\infty} \left(\frac{\{a_n\}}{\{b_n\}}\right) = \frac{A}{B}$.

Proof. To see that (i) is valid, observe that the following relation holds:

$$\lim_{n\to\infty} (\{a_n\}) = A \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} (\{a_n\} - A) = 0. \tag{5.28}$$

Applying (5.28) to our problem, we have to admit that

$$\lim_{n\to\infty} (\{a_n\}\{b_n\} - AB) = 0. \tag{5.29}$$

Using the triangular inequality:

$$\begin{aligned}|\{a_n\}\{b_n\}-AB| &= |\{a_n\}\{b_n\}-A\{b_n\}+A\{b_n\}-AB| \\ &= |\{b_n\}(\{a_n\}-A)+A(\{b_n\}-B)| \\ &\leq |\{b_n\}|\,|\{a_n\}-A|+|A|\,|\{b_n\}-B|\,.\end{aligned} \tag{5.30}$$

Since $|\{b_n\}|$ is a bounded sequence (as it is convergent), whereas $|\{a_n\}-A|$ is a null sequence, their product yields a null sequence. The constant sequence $|A|$ is, obviously bounded, while the sequence $|\{b_n\}-B|$ is a null sequence, thus their product is also a null sequence. Whence, the sum is a null sequence as well. Therefore, we can write

$$(|\{a_n\}\{b_n\}-AB|)\;\;\text{nullsequence}\quad\Longrightarrow\quad\lim_{n\to\infty}(\{\mathrm{a}_n\}\{\mathrm{b}_n\}-\mathrm{AB})=0, \tag{5.31}$$

since a null sequence is invariant to the absolute value. To see that (ii) holds, we make the following rearrangements:

$$\begin{aligned}&\left|\frac{\{a_n\}}{\{b_n\}}-\frac{A}{B}\right| = \frac{|\{a_n\}B-A\{b_n\}|}{|\{b_n\}B|} = \frac{|\{a_n\}B-AB+BA-A\{b_n\}|}{|\{b_n\}B|} \\ &= \\ &\frac{|B(\{a_n\}-A)+A(B-\{b_n\})|}{|\{b_n\}B|} \leq \frac{|B|\,|\{a_n\}-A|}{|\{b_n\}|\,|B|}+\frac{|A|\,|B-\{b_n\}|}{|\{b_n\}|\,|B|} \\ &= \underbrace{\underbrace{\frac{1}{|\{b_n\}|}}_{\text{bounded}}\cdot\underbrace{|\{a_n\}-A|}_{\text{nullsequence}}+\underbrace{\frac{|A|}{|B|}}_{\text{bounded}}\cdot\underbrace{\frac{1}{|\{b_n\}|}}_{\text{bounded}}\cdot\underbrace{|\{b_n\}-B|}_{\text{nullsequence}}}_{\text{nullsequence}}\end{aligned} \tag{5.32}$$

From the last expression in (5.32) it is seen that $\left(\left|\frac{\{a_n\}}{\{b_n\}}-\frac{A}{B}\right|\right)$ is a null sequence. By the Product rule (i), then, $\lim_{n\to\infty}\frac{\{a_n\}}{\{b_n\}}=\frac{A}{B}$. This completes the proof. □

Finally, we show that using the triple R-I the sequence of matrices $\{\mathbf{H}_k\}$, $k=1,2,\ldots$, achieves a minimum, represented by the stabilized matrix $\mathbf{H}_q^*$ for a prescribed reasonably small tolerance $\varepsilon>0$. The following theorem utilizes the element-wise form of the recursion rule: $h_{ij}^{(k+1)}{=}w_i^{(k)}h_{ij}^{(k)}w_j^{-1(k)}$.

Theorem 5. *Let the elements* $h_{ij}^{(k)} > 0$ *and* $w_i^{(k)} > 0$, $i, j = 1, 2, \ldots, n, k = 0, 1, 2, \ldots,$ *be generated by the triple R-I algorithm. The sum of products*

$$\sum_{l=1}^{n^2} \left(h_{ij}^{(k)}\right)_l \left(\frac{w_i^{(k)}}{w_j^{(k)}}\right)_l \quad i, j = 1, 2, \ldots, n, \quad k = 0, 1, 2, \ldots, \quad l = 1, 2, \ldots, n^2, \tag{5.33}$$

attains a minimum if

$$\begin{gathered} \left(h_{ij}^{(k)}\right)_1 \leq \left(h_{ij}^{(k)}\right)_2 \leq \ldots \leq \left(h_{ij}^{(k)}\right)_{n^2} \quad \text{and} \\ \left(\frac{w_i^{(k)}}{w_j^{(k)}}\right)_1 \geq \left(\frac{w_i^{(k)}}{w_j^{(k)}}\right)_2 \geq \ldots \geq \left(\frac{w_i^{(k)}}{w_j^{(k)}}\right)_{n^2} \end{gathered} \tag{5.34}$$

hold.

Proof. Suppose that the statement of Theorem 5 is not true. This would mean that the minimum of (5.33) is generated by such a sum of products for which the (5.34) orders are not held, i.e. (thereafter we use a simplified notation for the members of the products):

$$h_1 \acute{w}_1 + h_2 \acute{w}_2 + \ldots + h_j \acute{w}_{j+1} + h_{j+1} \acute{w}_j + \ldots + h_l \acute{w}_l \tag{5.35}$$

If the statement of the theorem is false, then the sum (5.35) is less than that of (5.33), i.e.

$$h_1 \acute{w}_1 + \ldots + h_j \acute{w}_{j+1} + h_{j+1} \acute{w}_j + \ldots + h_l \acute{w}_l < \sum_{l=1}^{n^2} h_l \acute{w}_l \tag{5.36}$$

Since the right hand and the left hand sides of the inequality (5.36) differ in two members only, therefore

$$h_j \acute{w}_{j+1} + h_{j+1} \acute{w}_j < h_j \acute{w}_j + h_{j+1} \acute{w}_{j+1} \tag{5.37}$$

must hold, i.e.

$$0 < h_j(\acute{w}_j - \acute{w}_{j+1}) + h_{j+1}(\acute{w}_{j+1} - \acute{w}_j) \tag{5.38}$$

and

$$0 < (h_j - h_{j+1})(\acute{w}_j - \acute{w}_{j+1}). \tag{5.39}$$

Since by (5.34) $h_j - h_{j+1} \leq 0$ and $\acute{w}_j - \acute{w}_{j+1} \geq 0$, inequality (5.39) is certain not to happen. Thus, an opposite statement than the statement of the theorem leads to a contradiction. Therefore, by Lemma 3 and considering the fact that at iteration step q, $w_i^{(q)}$=1 for all i, the sum of products (5.33) attains a minimum. This completes the proof. □

Finally, it should be noted that depending upon a certain degree of perturbation of a PCM (termed a level of inconsistency of matrix $\mathbf{A}$ in the field of decision sciences) the algorithm may produce more than one limit point. We refer to [7] for some details of this phenomenon, however, this issue is subject to further research.

6 Numerical illustration

The following example demonstrates the results discussed in the previous sections. The prescribed accuracy is: $\varepsilon = 10^{-6}$. Numerical results are presented to four digits. For comparison, weights are normalized so that the sum of their elements is one.

Example. We consider a 5×5 SR matrix $\mathbf{A}$. The objective is to prioritize five given alternatives. Saaty's nine-point scale $[1/9, \ldots, 1/2, 1, 2, \ldots, 9]$ is used for the entries of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 4 & 2 \\ 1/3 & 1 & 4 & 4 & 6 \\ 1/5 & 1/4 & 1 & 2 & 2 \\ 1/4 & 1/4 & 1/2 & 1 & 2 \\ 1/2 & 1/6 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

In the input module of the algorithm the spectral properties of $\mathbf{A}$ are calculated. The principal eigenvalue of $\mathbf{A}$ is: $\lambda_{\max} = 5.5737$. The right and left Perron-eigenvectors of $\mathbf{A}$, where the right one represents the weights (priority scores) of the alternatives, are, respectively (observe here that the eigenvectors are not element-wise reciprocal, see [11] for more detail):

$$\mathbf{u}^\top(\mathbf{A}) = [0.4254, 0.3030, 0.1071, 0.0859, 0.0786],$$
$$\mathbf{v}^\top(\mathbf{A}) = [0.0695, 0.0859, 0.2190, 0.2701, 0.3561].$$

The output of the initial module of the algorithm yields the stationary vector $\mathbf{w}^{*(0)}$ which appears in the first row of the 'best' transitive matrix approximation

$\mathbf{B}_0$ to matrix $\mathbf{A}$ in an LS sense given in a non-normalized form:

$$\mathbf{w}^{*\top(0)} = [1.0000, 0.8574, 3.7688, 3.6920, 4.0897].$$

Its inverse, displayed in a normalized form, represents the 'best' estimate of the weights in a LS sense:

$$\mathbf{w}^{(-1)*\top(0)} = [0.3393, 0.3958, 0.0903, 0.0919, 0.0830],$$

which is apparently not a good adjustment to the right Perron-eigenvector, $\mathbf{u}(\mathbf{A})$, of the original matrix $\mathbf{A}$.

The triple R-I terminates at step $q = 66$, producing the stationary vector:

$$\mathbf{w}^{*\top(q)} = [1.0000, 1.0000, 1.0000, 1.0000, 1.0000],$$

which would repeat itself if one continues the iteration. Therefore, $\mathbf{B}_q^P{=}\mathbf{E}{=}\mathbf{B}^{P*}$, are clearly in line-sum-symmetry, thus the pattern matrix $\mathbf{B}^P$ has been balanced.

The stabilized SR matrix $\mathbf{H}_q^*$ yields:

$$\mathbf{H}_q^* = \begin{bmatrix} 1 & 2.2972 & 1.4109 & 0.8946 & 0.3690 \\ 0.4353 & 1 & 1.4740 & 1.1683 & 1.4457 \\ 0.7088 & 0.6784 & 1 & 1.5853 & 1.3078 \\ 1.1178 & 0.8559 & 0.6308 & 1 & 1.6499 \\ 2.7099 & 0.6917 & 0.7647 & 0.6061 & 1 \end{bmatrix}.$$

The principal eigenvalue of $\mathbf{H}_q^*$ is: $\lambda_{\max} = 5.5737$. It is easy to check that $\mathbf{H}_q^*$ is in line-sum symmetry, since it has been balanced. The right and the left Perron-eigenvectors of $\mathbf{H}_q^*$ as function of the selected ε are very close to $\mathbf{e}$. They are, respectively,

$$\mathbf{u}_q^\top(\mathbf{H}_q^*) = [0.2045, 0.2014, 0.1983, 0.1997, 0.1961],$$
$$\mathbf{v}_q^\top(\mathbf{H}_q^*) = [0.2034, 0.2027, 0.1988, 0.1995, 0.1956].$$

Up to the termination step of the iteration, $q = 66$,the inverse matrix $\widetilde{\mathbf{W}}_q^{-1}$ of the product of the diagonal matrices $\mathbf{W}_k^{-1}$, $k = 0, 1, 2, \ldots, q$, is obtained as

$$\widetilde{\mathbf{W}}_q^{-1} = \begin{bmatrix} 1 & & & & 0 \\ & 0.7658 & & & \\ & & 0.2822 & & \\ & & & 0.2237 & \\ 0 & & & & 0.1845 \end{bmatrix}.$$

The rank-one matrix $\widetilde{\mathbf{B}}_q{=}\widetilde{\mathbf{W}}_q^{-1}\mathbf{E}\widetilde{\mathbf{W}}_q$, representing the 'best' transitive matrix approximation to the original matrix $\mathbf{A}$, at the termination step q, yields

$$\widetilde{\mathbf{B}}_q = \begin{bmatrix} 1 & 1.3064 & 3.5449 & 4.4721 & 5.4223 \\ 0.7655 & 1 & 2.7135 & 3.4233 & 4.1507 \\ 0.2821 & 0.3685 & 1 & 1.2616 & 1.5296 \\ 0.2236 & 0.2921 & 0.7927 & 1 & 1.2125 \\ 0.1844 & 0.2409 & 0.6538 & 0.8248 & 1 \end{bmatrix}.$$

From $\widetilde{\mathbf{B}}_q$, the 'best' adjustment of the Perron-eigenvectors to those of the original SR matrix $\mathbf{A}$ are:

$$\widetilde{\mathbf{w}}_q^{(-1)\top} = [0.4072, 0.3117, 0.1149, 0.0911, 0.0751].$$
$$\widetilde{\mathbf{w}}_q^{\top} = [0.0635, 0.0830, 0.2251, 0.2840, 0.3444],$$

It appears that they provide very good adjustments to $\mathbf{u}(\mathbf{A})$ and $\mathbf{v}^\top(\mathbf{A})$.

For the steps $k = 0$, $k = 1$ and $k = q$, the consistency adjustment errors to matrix $\mathbf{A}$, $S\left(\widetilde{\mathbf{w}}^{(k)}\right) {=} \|\mathbf{A} - \widetilde{\mathbf{B}}_k\|_{\mathrm{F}}$, are:

$$S\left(\mathbf{w}^{(0)}\right) = S\left(\widetilde{\mathbf{w}}^{(0)}\right) = 4.0477, \quad S\left(\widetilde{\mathbf{w}}^{(1)}\right) = 5.9592 \quad \text{and} \quad S\left(\widetilde{\mathbf{w}}^{(q)}\right) = 4.9234.$$

In accordance with our strategy followed in the triple R-I, $S(\mathbf{w}^{(0)})$ produces the smallest, whereas $S(\widetilde{\mathbf{w}}^{(1)})$ produces the largest consistency adjustment error.

7 Conclusions

We have shown that our recursive iteration when applied to a specific class of positive matrices called pairwise comparison matrices is equivalent with a line-sum-symmetric diagonal similarity scaling. We have proven that a series of transitive matrix approximations to the original SR matrix produces a convergent process, which, with a prescribed accuracy, yields stabilized matrices. Since both of these matrices were in line-sum-symmetry, they have been balanced. By the conformity to a simple fixed-point iteration the existence of a limit point for the sequence of the successively updated matrices has also been shown. It was proven that the limit point of the residue corresponds to the minimum Frobenius norm of the original pairwise comparison matrix. The N-K method for solving a system of inhomogeneous nonlinear equations appears to be robust as concerns the strategy employed for the choices of the starting vectors. Furthermore, our triple R-I algorithm seems to be more generally useful as it can be applied to merely positive matrices also. To facilitate readers' understanding, a numerical illustration demonstrated the theoretical results.

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