Limited Failure-Censored Life Test for the Weibull Distribution

Jong-Wuu Wu, Tzong-Ru Tsai, and Liang-Yuh Ouyang

***Abstract*—When the distribution of lifetimes is 2-parameter exponential, Balasooriya [6] provided a failure-censored reliability sampling-plan to save test time. This paper extends the Balasooriya sampling plan to the Weibull distribution and provides a limited failure-censored reliability sampling plan (LFCR) to do life testing when test facilities are scarce. The s-expected test time of the LFCR is computed, and the optimal stopping rule of LFCR corresponding to the shortest test time is established. The s-confidence intervals for the parameters are generated.**

***Index Terms*—Best linear unbiased estimator, life-test sampling plan, Monte Carlo simulation, order statistic, Weibull distribution.**

ACRONYMS AND ABBREVIATIONS[1]

BLUE	best linear unbiased estimator
CI	s-confidence interval
LFCR	limited failure-censored reliability (sampling plan)
ETT	s-expected test time
MSE	mean square error
pdf	probability density function
r.v.	random variable
WD	Weibull distribution

Notation

$X_{1:n}^{(i)}$	first order statistic in random sample i, each of size n, $i = 1, \ldots, r$
$[x]$	greatest integer lower bound for x
r_*	$[r/k]$
T_1	$\sum_{i=1}^{r} X_{1:n}^{(i)}$
T_2	total test time when r random samples, each of size n, are used simultaneously
T_3	$\sum_{i=1}^{k} X_{r_i:n}^{(i)}$; r_i is the number of failures in the random sample i, each of size n, $i = 1, 2, \ldots, k$
α	scale parameter of Weibull distribution
β	shape parameter of Weibull distribution
χ_ν^2	chi-square r.v, with ν degrees of freedom

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[1]The singular and plural of an acronym are always spelled the same.

I. INTRODUCTION

LIFETIME data, with censoring, create special problems in the data analysis. Generally speaking, censoring results in exact lifetimes being known for only a portion of the products; the remainder of the lifetimes are known only to exceed the censoring time. There are several types of censoring. In life testing, experiments involving type II censoring (failure-censored) are often used: n units are placed on test, but instead of continuing until all n units have failed, the test is stopped at the time of failure r. Such a test can save time, because it could take a long time for all units to fail [1], [2].

Sometimes the life of a product is quite long. Thus, a type II censoring life-test plan for such a product can be too long. Johnson [4] proposed a sampling plan in which the experimenter can decide to group the test units into several sets (each set is an assembly of test units), and then run all the test units simultaneously until the first failure in each group. Such plans are usually feasible when test facilities are scarce but test material is relatively cheap. Balasooriya [6] examined the failure-censored sampling plan for the 2-parameter exponential distribution based on testing r random samples, each of size n, one after the other. That procedure is based on exact results, and only the first failure time of each sample is needed. The Balasooriya sampling plan is compared with traditional sampling plans using a sample of size $r \cdot n$.

This paper extends the results of [6] to the WD; then gives a new, more efficient procedure than [6] when r test sets can be run simultaneously. In some test situations, the test facilities are scarce. We must get r failures from $k(\leq r)$ test sets, each of size n. An LFCR is proposed to solve this problem.

Section II extends the limited failure-censored life test to WD. It proves that the LFCR is the best one for saving time when $\beta = 1$. If $\beta \neq 1$, the formula of ETT is very complicated. When $\beta \leq 1$, the LFCR still works. Monte Carlo simulation is used to evaluate these cases in Section IV. When $\beta > 1$, the stopping rule of LFCR depends on both the failure numbers in each test line and β. We can not get the unique optimal solution and it is not discussed in this paper.

Section III shows how to get 3 feasible point estimations and CI estimation procedures when an LFCR is performed. If the distribution of lifetimes is exponential, then the proposed estimators in Section III are still the BLUE. If the distribution of lifetimes is Weibull with $\beta \neq 1$, then the pooled transformed linear estimators are used to estimate the unknown parameters. Two alternative minimal MSE estimators are proposed. The corresponding CI estimations of parameters in WD are derived in Section III.

II. LIMITED FAILURE-CENSORED RELIABILITY SAMPLING PLAN

A. The Failure-Censored Sampling Plan

Let X be the lifetime of a product, it is WD with pdf:

$$f(x;\beta,\alpha)=\frac{\beta}{\alpha}\cdot\left(\frac{x}{\alpha}\right)^{\beta-1}\cdot\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right],\qquad x>0. \quad (1)$$

Then $X_{1:n}^{(i)}$, $i=1,2,\ldots,r$, is a random sample of size r with pdf:

$$f(x_{1:n};\beta,\alpha\cdot n^{-1/\beta}) = \frac{n\cdot\beta}{\alpha}\cdot\left(\frac{x_{1:n}}{\alpha}\right)^{\beta-1}\cdot\exp\left[-n\cdot\left(\frac{x_{1:n}}{\alpha}\right)^{\beta}\right],\quad x_{1:n}>0. \quad (2)$$

Hence, the distribution of $X_{1:n}^{(i)}$ is Weibull with scale parameter $\alpha\cdot n^{-1/\beta}$, $i=1,\ldots,r$. If the lifetime of a product is quite long, and test facilities are scarce, but testing materials are relatively cheap, then one can test $r\cdot n$ experimental units by testing r sets, each containing n experimental units, one after the other. Let the test stop when the first failure in each test set is achieved (the total number of failures is r). Let the distribution of lifetime of specimens be (1). Then the ETT is:

$$E[T_1]=r\cdot\alpha\cdot n^{-1/\beta}\cdot\Gamma\left(1+\frac{1}{\beta}\right). \quad (3)$$

When the test facilities are available to run r test sets simultaneously, according to the test procedure in [6], then T_2 equals the order statistic r of $X_{1:n}^{(i)}$, $i=1,\ldots,r$; i.e., the largest order statistic of $X_{1:n}^{(1)},\ldots,X_{1:n}^{(r)}$. Use the s-expected value of the order statistic r in (1) [5, p. 638],

$$E[X_{r:n}]=\alpha\cdot n\cdot\binom{n-1}{r-1}\cdot\Gamma\left(1+\frac{1}{\beta}\right)\cdot\sum_{i=0}^{r-1}(-1)^i\cdot\binom{r-1}{i}\cdot\frac{1}{(n-r+i+1)^{1+1/\beta}}, \quad (4)$$

$$E[T_2]=r\cdot\alpha\cdot n^{-1/\beta}\cdot\Gamma\left(1+\frac{1}{\beta}\right)\cdot\sum_{i=0}^{r-1}(-1)^i\cdot\binom{r-1}{i}\cdot\frac{1}{(i+1)^{1+1/\beta}}. \quad (5)$$

These $r\cdot n$ specimens are tested in r test sets s-independently and simultaneously. The duration of the experiment using order statistic r in all $r\cdot n$ specimens is shorter than for taking the order statistic r in $X_{1:n}^{(1)},\ldots,X_{1:n}^{(r)}$. Hence, take order statistic r on all $r\cdot n$ test specimens, and revise $E[T_2]$ as:

$$E[T_2]=\alpha\cdot r\cdot n\binom{rn-1}{r-1}\cdot\Gamma\left(1+\frac{1}{\beta}\right)\cdot\sum_{i=0}^{r-1}(-1)^i\cdot\binom{r-1}{i}\cdot\frac{1}{(rn-r+i+1)^{1+1/\beta}}. \quad (6)$$

The ETT based on (6) is shorter than the ETT based on the Balasooriya design when the test facilities are available.

B. Limited Failure-Censored Reliability Sampling Plan

When the test facilities are scarce and we need to take r failures from $k(\le r)$ test sets, it is inappropriate to use the Balasooriya sampling plan. Since only the first failure is observed in each test set, the total number of failures is less than r. Thus, the test can not be stopped. More failures must be accumulated in some test sets; this sampling plan is LFCR. In LFCR, consider 2 situations:

1) r/k is an integer,
2) r/k is not an integer.

Use theorem 1 and corollary 1 to discuss the 2 situations, respectively. When $\beta=1$, the WD is exponential. Then

$$E[X_{r:n}]=\alpha\cdot\sum_{i=1}^{r}\frac{1}{n-i+1}. \quad (7)$$

Use (7) to get

$$E[T_3]=\alpha\cdot\sum_{i=1}^{k}\left[\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{n-r_i+1}\right]. \quad (8)$$

Theorem 1: A batch of n items are tested consecutively, and the replicate times are limited to $k(\le r)$. $r_i(\ge 1)$ is the stopping number in the test set $i=1,2,\ldots,k$. r/k is an integer. Then the minimal ETT of the LFCR is for $r_i=r/k$, $i=1,2,\ldots,k$.

Proof: See the Appendix, Section 1.

Corollary 1: In theorem 1, if the remainder of r divided by k is c_r, then the minimal ETT of the LFCR is achieved when c_r test sets are stopped at r_*+1 failures obtained in each test set, and the other $k-c_r$ test sets are stopped at r_* failures obtained in each test set.

Proof: See the Appendix, Section 2.

C. Discussion

If $\beta\ne 1$, then the formula of ETT is very complicated. Use Monte Carlo simulation to solve the equation. According to our simulation results, the optimal stopping rule of LFCR such that the minimal ETT is achieved depends on both β and the failure numbers in test sets. Theorem 1 and corollary 1 are valid when $\beta\le 1$. When $\beta>1$, the "optimal stopping rule of LFCR such that the minimal ETT is achieved" does not have a unique solution. The solution depends on β and the failure numbers in test sets. Therefore, Section IV presents only the result of Monte Carlo simulations when $\beta\le 1$.

III. PARAMETER AND s-CONFIDENCE INTERVALS ESTIMATION IN LFCR

In an LFCR, without loss of generality, let the first c_r test sets have r_*+1 failures and the last $k-c_r$ test sets have r_* failures. When $\beta=1$, use the estimators of α in each test set as follows:

$$\hat\alpha_i=\begin{cases}\dfrac{1}{r_*+1}\cdot\displaystyle\sum_{h=1}^{r_*+1}X_{h:n}^{(i)}+(n-r_*-1)\cdot X_{r_*+1:n}^{(i)}, & i=1,\ldots,c_r,\\[2ex] \dfrac{1}{r_*}\cdot\displaystyle\sum_{l=1}^{r_*}X_{l:n}^{(i)}+(n-r_*)\cdot X_{r_*:n}^{(i)}, & i=c_r+1,\ldots,k.\end{cases} \quad (9)$$

Since each test set with n test specimens is tested s-independently, then $\hat\alpha_i$ is the BLUE of α, $i=1,\ldots,k$ [7]. Under

LFCR, a total of $k \cdot n$ specimens are tested. Hence, the pooled estimators are used to estimate α.

$$\hat{\hat{\alpha}} = \frac{(r_* + 1) \cdot \sum_{i=1}^{c_r} \hat{\alpha}_i + r_* \cdot \sum_{j=c_r+1}^{k} \hat{\alpha}_j}{k \cdot r_* + c_r}. \quad (10)$$

By the Gauss–Markov theorem, $\hat{\hat{\alpha}}$ is the BLUE of α for the total $k \cdot n$ specimens tested. The $[2(r_* + 1) \cdot \hat{\alpha}_i]/\alpha$ has a χ^2 distribution with $2(r_* + 1)$ degrees of freedom, $i = 1, \ldots, c_r$, and $[2r_* \hat{\alpha}_j]/\alpha$ has a χ^2 distribution with $2r_*$, $j = c_r + 1, \ldots, k$ degrees of freedom. Therefore, $[2(c_r + k \cdot r_*) \cdot \hat{\hat{\alpha}}]/\alpha$ has a χ^2 distribution with $2(c_r + k \cdot r_*)$ degrees of freedom. The $(1 - \gamma)$ CI for α is:

$$\left(\frac{2(c_r + k \cdot r_*)}{\chi^2_{\gamma/2, (c_r + k \cdot r_*)}}, \quad \frac{2(c_r + k \cdot r_*)}{\chi^2_{(1-\gamma)/2, (c_r + k \cdot r_*)}} \right). \quad (11)$$

$\chi^2_{\gamma, \nu}$ satisfies $\Pr\{\chi^2_\nu > \chi^2_{\gamma, \nu}\} = \gamma$.

When the distribution of lifetimes is (1) and $\beta < 1$, then the pooled transformed linear estimators are used to estimate the unknown parameters. The corresponding CI are derived. The pooled transformed linear estimators are shown to be the minimal MSE estimators.

Transform WD (take the "$-\log$" transformation) to an extreme value distribution with location parameter: $\lambda = \log(\alpha)$, scale parameter: $\delta = 1/\beta$. The BLUE estimators of λ and δ in each test-set are:

$$\hat{\lambda}_i = \begin{cases} \sum_{h=1}^{r_*+1} a(h; n, r_* + 1) \cdot Y_{(h)}, & i = 1, \ldots, c_r, \\ \sum_{l=1}^{r_*} a(l; n, r_*) \cdot Y_{(l)}, & i = c_r + 1, \ldots, k. \end{cases} \quad (12)$$

$$\hat{\delta}_i = \begin{cases} \sum_{h=1}^{r_*+1} b(h; n, r_* + 1) \cdot Y_{(h)}, & i = 1, \ldots, c_r, \\ \sum_{l=1}^{r_*+1} b(l; n, r_*) \cdot Y_{(l)}, & i = c_r + 1, \ldots, k. \end{cases} \quad (13)$$

The $a(i; n, r)$, $b(i; n, r)$ are in [7, A12a]. The transformed linear estimators of α and β in each test set are:

$$\hat{\alpha}_i = \exp(\hat{\lambda}_i), \quad \hat{\beta}_i = \frac{1}{\hat{\delta}_i}, \qquad i = 1, \ldots, k.$$

When r_* is large, then $\hat{\alpha}_i$ is approximately s-normal distributed with mean α and

$$\mathrm{Var}[\hat{\alpha}_i] = A_{n,p} \left(\frac{\alpha}{\beta} \right)^2; \qquad p = r_* + 1 \text{ if } i \le c_r;$$

otherwise $p = r_*$, and $A_{n,p}$ is in [7, A12].

When r_* is large, then $\hat{\beta}_i$ is approximately s-normal distributed with mean β and

$$\mathrm{Var}[\hat{\beta}_i] = \beta^2 \cdot B_{n,p}; \qquad p = r_* + 1 \text{ if } i \le c_r;$$

otherwise $p = r_*$, and $B_{n,p}$ is in [7, A12].

If $k \cdot n$ specimens are tested, then use the pooled estimators to estimate α and β. The $\hat{\alpha}_i$, $i = 1, \ldots, k$ are s-independently and asymptotically s-normal. The $\hat{\beta}_i$, $i = 1, 2, \ldots, k$ are also s-independently and asymptotically s-normal. The weighted estimator

$$\hat{\hat{\alpha}}^* = \sum_{i=1}^{k} a_i \cdot \hat{\alpha}_i; \qquad \left[\sum_{i=1}^{k} a_i = 1 \right]$$

can be used to estimate α. By the Cauchy–Schwarz inequality, the variance of $\hat{\hat{\alpha}}^*$ is minimal when

$$a_i = \frac{t}{A_{n,p}}; \qquad t = \left(\frac{c_r}{A_{n, r_*+1}} + \frac{k - c_r}{A_{n, r_*}} \right)^{-1}.$$

The weighted estimator

$$\hat{\hat{\beta}}^* = \sum_{i=1}^{k} b_i \cdot \hat{\beta}_i$$

can be used to estimate β;

$$b_i \equiv \frac{t_*}{B_{n,p}}, \qquad t_* \equiv \left(\frac{c_r}{B_{n, r_*+1}} + \frac{(k - c_r)}{B_{n, r_*}} \right)^{-1}; \qquad \sum_{i=1}^{k} b_i = 1.$$

When r_* is large, then $\hat{\hat{\alpha}}^*$ closes to a s-normal distribution with mean α and variance

$$\sigma^2_\alpha = \sum_{i=1}^{k} a_i^2 \cdot \mathrm{Var}[\hat{\alpha}_i].$$

Replace α and β by $\hat{\hat{\alpha}}^*$ and $\hat{\hat{\beta}}^*$, respectively; then the variance estimator $\hat{\sigma}^2_\alpha$ can be calculated. Thus, the asymptotic $(1 - \gamma)$ CI for α is

$$\left(\hat{\hat{\alpha}}^* - z_{\gamma/2} \cdot \hat{\sigma}_\alpha, \, \hat{\hat{\alpha}}^* + z_{\gamma/2} \hat{\sigma}_\alpha \right); \quad z_\gamma \text{ satisfies } \Pr\{Z > z_\gamma\} = \gamma,$$

and Z is the standard s-normal r.v. Similarly, $\hat{\hat{\beta}}^*$ is asymptotically s-normally distributed with mean β and variance

$$\sigma^2_\beta = \sum_{i=1}^{k} b_i^2 \cdot \mathrm{Var}[\hat{\beta}_i].$$

The variance estimator $\hat{\sigma}^2_\beta$ can be computed by replacing β by $\hat{\hat{\beta}}^*$. Thus, the asymptotic $(1 - \gamma)$ CI for β is

$$\left(\hat{\hat{\beta}}^* - z_{\gamma/2} \cdot \hat{\sigma}_\beta, \quad \hat{\hat{\beta}}^* + z_{\gamma/2} \cdot \hat{\sigma}_\beta \right).$$

This paragraph presents another method to get the minimal MSE estimators of α and β. The estimators of α, β are, respectively:

$$\hat{\hat{\alpha}} = a \cdot \sum_{i=1}^{k} \hat{\alpha}_i, \qquad \hat{\hat{\beta}} = b \cdot \sum_{j=1}^{k} \hat{\beta}_j.$$

Let a be real and satisfy $E[(\hat{\hat{\alpha}} - \alpha)^2]$ is minimum. Let b satisfy $E[(\hat{\hat{\beta}} - \beta)^2]$ is minimum. Let $g(a) \equiv E[(\hat{\hat{\alpha}} - \alpha)^2]$. When r_* is large, $\hat{\hat{\alpha}}$ is approximately s-normal with mean $a \cdot k \cdot \alpha$ and variance

$$a^2 \cdot \sum_{i=1}^{k} \mathrm{Var}[\hat{\alpha}_i].$$

TABLE I
SIMULATION RESULTS FOR LFCR USING THE WD WITH $n = 30, k = 2$ FOR $r = 12, 21$

	$\alpha = 1;\ \beta = \Downarrow$				$\alpha = 5;\ \beta = \Downarrow$				$\alpha = 10;\ \beta = \Downarrow$			
(r_1, r_2)	1/4	1/2	3/4	1	1/4	1/2	3/4	1	1/4	1/2	3/4	1
	$r = 12$											
(1,11)	6.1462	1.9708	1.3220	1.0970	6.1462	1.9708	1.3220	1.0970	6.1462	1.9708	1.3220	1.0970
(2,10)	4.0086	1.6133	1.2023	1.0616	4.0086	1.6133	1.2023	1.0616	4.0086	1.6133	1.2023	1.0616
(3, 9)	2.5734	1.3415	1.1124	1.0344	2.5734	1.3415	1.1124	1.0344	2.5734	1.3415	1.1124	1.0344
(4, 8)	1.6629	1.1502	1.0495	1.0152	1.6629	1.1507	1.0495	1.0152	1.6629	1.1507	1.0495	1.0152
(5, 7)	1.1603	1.0375	1.0123	1.0038	1.1603	1.0375	1.0123	1.0038	1.1603	1.0375	1.0123	1.0038
(6, 6)	1	1	1	1	1	1	1	1	1	1	1	1
	$r = 21$											
(1,20)	16.1485	3.0609	1.7208	1.2996	16.1485	3.0609	1.7208	1.2996	16.1485	3.0609	1.7208	1.2996
(2,19)	11.4266	2.5789	1.5584	1.2334	11.4266	2.5789	1.5584	1.2334	11.4226	2.5789	1.5584	1.2334
(3,18)	8.0979	2.1839	1.4238	1.1792	8.0979	2.1839	1.4238	1.1792	8.0979	2.1839	1.4238	1.1792
(4,17)	5.7370	1.8606	1.3104	1.1312	5.7370	1.8606	1.3104	1.1312	5.7370	1.8606	1.3104	1.1312
(5,16)	4.0602	1.5990	1.2178	1.0926	4.0602	1.5990	1.2178	1.0926	4.0602	1.5990	1.2178	1.0926
(6,15)	2.8754	1.3910	1.1430	1.0607	2.8754	1.3910	1.1430	1.0607	2.8754	1.3910	1.1430	1.0607
(7,14)	2.0515	1.2307	1.0848	1.0361	2.0515	1.2307	1.0848	1.0361	2.0515	1.2307	1.0848	1.0361
(8,13)	1.4995	1.1139	1.0420	1.0179	1.4995	1.1139	1.0420	1.0179	1.4995	1.1139	1.0420	1.0179
(9,12)	1.1609	1.0376	1.0139	1.0059	1.1609	1.0376	1.0139	1.0059	1.1609	1.0376	1.0139	1.0059
(10,11)	1	1	1	1	1	1	1	1	1	1	1	1

TABLE II
SIMULATION RESULTS FOR LFCR USING THE WD WITH $n = 30, k = 3$ FOR $r = 10, 15$

	$\alpha = 1;\ \beta = \Downarrow$				$\alpha = 5;\ \beta = \Downarrow$				$\alpha = 10;\ \beta = \Downarrow$			
(r_1, r_2, r_3)	1/4	1/2	3/4	1	1/4	1/2	3/4	1	1/4	1/2	3/4	1
	$r = 10$											
(1,1,8)	7.2403	2.0380	1.3019	1.0667	7.2403	2.0380	1.3019	1.0667	7.2403	2.0380	1.3019	1.0667
(1,2,7)	4.3039	1.6295	1.1875	1.0408	4.3039	1.6295	1.1875	1.0408	4.3039	1.6295	1.1875	1.0408
(1,3,6)	2.5711	1.3606	1.1135	1.0237	2.5711	1.3606	1.1135	1.0237	2.5711	1.3606	1.1135	1.0237
(1,4,5)	1.7700	1.2272	1.0771	1.0152	1.7700	1.2272	1.0771	1.0152	1.7700	1.2272	1.0771	1.0152
(2,2,6)	2.4743	1.3096	1.0924	1.0202	2.4742	1.3096	1.0924	1.0202	2.4743	1.3096	1.0924	1.0202
(2,3,5)	1.4911	1.1196	1.0371	1.0079	1.4911	1.1196	1.0371	1.0079	1.4911	1.1196	1.0371	1.0079
(2,4,4)	1.1820	1.0566	1.0189	1.0038	1.1820	1.0566	1.0189	1.0038	1.1820	1.0566	1.0189	1.0038
(3,3,4)	1	1	1	1	1	1	1	1	1	1	1	1
	$r = 15$											
(1,1,13)	17.7281	2.9219	1.5519	1.1583	17.7281	2.921993	1.551939	1.158346	63.8529	6.3288	2.7067	1.7896
(1,2,12)	12.0149	2.4278	1.4168	1.1190	12.0148	2.4278	1.4168	1.1190	43.2750	5.2585	2.4711	1.7289
(1,3,11)	8.0363	2.0360	1.3103	1.0875	8.0362	2.0360	1.3103	1.0875	28.9449	4.4100	2.2853	1.6802
(1,4,10)	5.3401	1.7388	1.2295	1.0634	5.3401	1.7388	1.2295	1.0634	19.2341	3.7662	2.1445	1.6429
(1,5, 9)	3.6144	1.5303	1.1729	1.0463	3.6144	1.5303	1.1729	1.0463	13.0184	3.3147	2.0456	1.6166
(1,6, 8)	2.6554	1.4068	1.1393	1.0362	2.6553	1.4068	1.1393	1.0362	9.5641	3.0471	1.9870	1.6009
(1,7, 7)	2.3481	1.3659	1.1281	1.0328	2.3480	1.3659	1.1281	1.0328	8.4573	2.9584	1.9676	1.5957
(2,2,11)	8.0094	2.0125	1.2982	1.0852	8.0093	2.0125	1.2982	1.0852	28.8480	4.3590	2.2642	1.6767
(2,3,10)	5.2627	1.6891	1.2066	1.0586	5.2627	1.6891	1.2066	1.0586	18.9552	3.6586	2.1044	1.6356
(2,4, 9)	3.4511	1.4516	1.1395	1.0389	3.4511	1.4516	1.1395	1.0389	12.4304	3.1441	1.9874	1.6051
(2,5, 8)	2.3555	1.2955	1.0955	1.0259	2.3555	1.2955	1.0955	1.0259	8.4841	2.8061	1.9106	1.5850
(2,6,7)	1.8401	1.2182	1.0736	1.0194	1.8401	1.2182	1.0736	1.0194	6.6277	2.6386	1.8726	1.5750
(3,3,9)	3.4006	1.4254	1.1286	1.0364	3.4006	1.4254	1.1289	1.0364	12.2483	3.0874	1.9685	1.6013
(3,4,8)	2.2191	1.2403	1.0742	1.0208	2.2191	1.2403	1.0742	1.0208	7.9929	2.6864	1.8735	1.5771
(3,5,7)	1.5671	1.1304	1.0419	1.0114	1.5671	1.1304	1.0419	1.0114	5.6445	2.4485	1.8172	1.5627
(3,6,6)	1.3590	1.0940	1.0312	1.0083	1.3590	1.0940	1.0312	1.0083	4.8949	2.3697	1.7986	1.5579
(4,4,7)	1.4813	1.1013	1.0315	1.0088	1.4810	1.1013	1.0315	1.0088	5.3554	2.3855	1.7990	1.5586
(4,5,6)	1.1366	1.0324	1.0104	1.0028	1.1366	1.0324	1.0104	1.0028	4.0938	2.2363	1.7622	1.5494
(5,5,5)	1	1	1	1	1	1	1	1	1	1	1	1

Let

$$g(a) \equiv a^2 \cdot \sum_{i=1}^{k} \mathrm{Var}[\hat{\alpha}_i] + (a \cdot k - 1)^2 \cdot \alpha^2. \tag{14}$$

$g(a)$ is minimal when

$$a = \frac{k \cdot \beta^2}{(c_r \cdot A_{n,r_*+1} + (k - c_r) \cdot A_{n,r_*}) + k^2 \cdot \beta^2}.$$

Therefore, when β is known,

$$\hat{\hat{\alpha}} = a \cdot \sum_{i=1}^{k} \hat{\alpha}_i$$

can be used to estimate α, and this estimator has the smallest MSE. A similar procedure is used to obtain the MSE estimator of β. Let

$$h(b) = b^2 \cdot \sum_{i=1}^{k} \mathrm{Var}[\hat{\beta}_i] + (b \cdot k - 1)^2 \cdot \beta^2. \tag{15}$$

$h(b)$ is minimum when

$$b = \frac{k}{(c_r \cdot B_{n,r_*+1} + (k - c_r) \cdot B_{n,r_*}) + k^2}.$$

Hence

$$\hat{\hat{\beta}} = b \cdot \sum_{j=1}^{k} \hat{\beta}_j$$

is the MSE estimator of β.

IV. Monte Carlo Simulation

This section conducts some Monte Carlo simulations when LFCR is performed with $\beta \le 1$. Let $\beta = 1/4, 1/2, 3/4, 1$, $\alpha = 1, 5, 10$. For r different failures, all combinations are simulated with $n = 30$, $k = 2, 3$. Table I gives the simulation results. More detailed simulation results are in [3]. Table I also gives the ratios of ETT to the minimal ETT. When r/k is an integer, then the minimal ratio $(= 1)$ occurs when each random sample takes the same r/k failures. For example, $r = 12$, $k = 2$, $\beta = 1/4, 1/2, 3/4, 1$, then the minimal ETT occurs when $r_1 = r_2 = 6$. When the remainder of r/k is c_r, then the minimal ratio $(= 1)$ occurs when we take $r_* + 1$ failures in each of the c_r random samples, and r_* failures in each one of the other $k - c_r$ random samples. For example, let $r = 21$, $k = 2$, $\beta = 1/4, 1/2, 3/4, 1$, then $c_r = 1$. Hence, we can take a random sample that stops with 10 failures and another random sample that stops with 11 failures. Without loss of generality, let $r_1 = 10$ and $r_2 = 11$. All simulation results meet the optimal stopping rules in theorem 1 and corollary 1 when $\beta \le 1$.

Appendix

1) Proof of Theorem 1: Let

$$g(r_1, r_2, \ldots, r_k) = \sum_{i=1}^{k} \left[\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n - r_i + 1} \right], \quad \frac{r}{k} \text{ be an integer.}$$

We show that $g(r_1, \ldots, r_k)$ is minimum when $r_i = r/k$, $i = 1, \ldots, k$. Without loss of generality, let there be:

- p test sets with number of failures $< r/k$,
- d test sets with number of failures $= r/k$,
- q test sets with number of failures $> r/k$;

$p + d + q = k$ and $p, d, q \ge 0$. Hence,

$$g(r_1, r_2, \ldots, r_k) - g\left(\frac{r}{k}, \frac{r}{k}, \ldots, \frac{r}{k}\right) = -\sum_{i \in A} \frac{1}{t_i} + \sum_{j \in B} \frac{1}{e_j}, \tag{A.1}$$

$A \equiv$ set of differences between $g(r_1, \ldots, r_k)$ and $g(r/k, \ldots, r/k)$ in p test sets.
$B \equiv$ set of differences between $g(r_1, \ldots, r_k)$ and $g(r/k, \ldots, r/k)$ in q test sets.

Because d test sets have the same number of failures r/k between $g(r_1, \ldots, r_k)$ and $g(r/k, \ldots, r/k)$, the number of nonzero terms in (A-1) is less than k. For example, let $n = 12, k = 3, r = 9, r_1 = 2, r_2 = 3, r_3 = 4$, then $p = 1, d = 1, q = 1$ and

$$\begin{aligned} & g(2, 3, 4) - g(3, 3, 3) \\ &= \left[\left(\tfrac{1}{12} + \tfrac{1}{11}\right) + \left(\tfrac{1}{12} + \tfrac{1}{11} + \tfrac{1}{10}\right) \right. \\ &\quad \left. + \left(\tfrac{1}{12} + \tfrac{1}{11} + \tfrac{1}{10} + \tfrac{1}{9}\right) \right] - 3\left[\tfrac{1}{12} + \tfrac{1}{11} + \tfrac{1}{10}\right] \\ &= -\tfrac{1}{10} + \tfrac{1}{9}. \end{aligned}$$

Hence, $A = \{1\}$, $t_1 = 10$, $B = \{1\}$, $e_1 = 9$; and $1/e_j > 1/t_i$ for all i, j.

Therefore $g(r_1, \ldots, r_k) \ge g(r/k, \ldots, r/k)$, and equality holds when $r_i = r/k$, $i = 1, \ldots, k$. QED

2) Proof of Corollary 1: If r/k is not an integer, assign r_* failures to each test set. Next, assign the $c_r (< k)$ failures to k test sets such that the differences of number-of-failures between test sets are as small as possible. Therefore, c_r test sets are stopped until $r_* + 1$ failures are obtained in each test set, and the other $k - c_r$ test sets are stopped until r_* failures are obtained in each test set. QED

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