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Some **P**-properties for linear transformations on Euclidean Jordan algebras

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The first author dedicates this paper to Professor Walter Rudin, University of Wisconsin-Madison

Abstract

A real square matrix is said to be a **P**-matrix if all its principal minors are positive. It is well known that this property is equivalent to: the nonsign-reversal property based on the componentwise product of vectors, the order **P**-property based on the minimum and maximum of vectors, uniqueness property in the standard linear complementarity problem, (Lipschitzian) homeomorphism property of the normal map corresponding to the nonnegative orthant. In this article, we extend these notions to a linear transformation defined on a Euclidean Jordan algebra. We study some interconnections between these extended concepts and specialize them to the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices with the semidefinite cone $\mathcal{S}^n_+$ and to the space R^n with the Lorentz cone.

1. Introduction

A real $n \times n$ matrix M is said to be a **P**-matrix if all its principal minors are positive. Since their introduction by Fiedler and Pták [8] in 1962 (see also [10]), **P**-matrices have found many applications in various fields. There are numerous ways

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to describe a **P**-matrix, see e.g., [1]. For our discussion, we consider the following equivalent conditions on $M \in R^{n\times n}$:

(1) All principal minors of M are positive.
(2) The implication $x \in R^n$, $x * Mx \leqslant 0 \Rightarrow x = 0$ holds, where '$*$' denotes the componentwise product.
(3) The implication $x \wedge Mx \leqslant 0 \leqslant x \vee Mx \Rightarrow x = 0$ holds, where '$\wedge$' and '$\vee$' denote the componentwise minimum and maximum respectively.
(4) For all $q \in R^n$, there exists a unique $x \in R^n$ such that
$x \geqslant 0$, $Mx + q \geqslant 0$, and $\langle x, Mx + q\rangle = 0$.
(5) The function $F(x) := Mx^+ + x - x^+$ is invertible in a neighborhood of zero, where x^+ is the orthogonal projection of x onto the nonnegative orthant in R^n.
(6) The function $F(x) := Mx^+ + x - x^+$ is invertible in a neighborhood of zero with Lipschitzian inverse.

The equivalence of (1) and (2) was established by Fiedler and Pták [8], see also [10]. The item (3) is a simple reformulation of (2). The equivalence of (1) and (4) was established in [25], see also [17,31]. The equivalence of (1) and (5) is due to Samelson et al. [31]. That (5) and (6) are equivalent follows from the fact that the inverse of a piecewise affine function $F(x)$ is piecewise affine and hence Lipschitzian.

As can be seen, the above conditions deal with the cone R^n_+ (of nonnegative vectors in R^n), the order induced by R^n_+, the componentwise product $x * y$, and the (usual) inner product in R^n. With appropriate modifications, conditions (4)–(6) have been generalized to closed convex sets. In this general setting, one deals with the projection onto a given closed convex set (in place of x^+) and a variational inequality problem instead of a linear complementarity problem that appears in (4). Even in the general setting one has the implications (6) $\Rightarrow$ (5) $\Leftrightarrow$ (4), see [5]. In this article, we introduce and study analogs of properties (1)–(6) for a linear transformation defined on a Euclidean Jordan algebra, which is a finite dimensional inner product space equipped with a (Jordan) product, the corresponding (symmetric) cone of squares, and order. The space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices is an example of a Euclidean Jordan algebra where $\mathcal{S}^n_+$ (the set of all positive semidefinite matrices in $\mathcal{S}^n$) is the cone of squares. In this setting, property (2) was extended, see [13], to a linear transformation L on $\mathcal{S}^n$ by means of the condition

$$X \in \mathcal{S}^n, \quad XL(X) = L(X)X \preceq 0 \ \Rightarrow \ X = 0,$$

where $Z \preceq 0$ means that Z is negative semidefinite. It was shown in [13] that the analog of (4) $\Rightarrow$ (2) holds in this setting. The above property in $\mathcal{S}^n$ and its non-symmetric version were studied extensively in [12,14–16,27]. A question arose as to whether the above property could be introduced in other Euclidean Jordan algebras, in particular in R^n where the cone of squares is the Lorentz cone (also called the second-order cone or ice-cream cone). In this paper, we introduce two generalizations of property (2), to be called the **P**-property and the Jordan **P**-property, that are valid in any Euclidean Jordan algebra. Since the cone of squares in a Euclidean Jordan algebra is, in general, nonpolyhedral, the order induced by the cone will not be a lattice order, that is, the 'minimum' and 'maximum' are not defined uniquely. Still, noting the property $x \wedge y = x - (x-y)^+$ in R^n, we suitably extend property (3) to the setting of a Euclidean Jordan algebra and show that this property implies the Jordan **P**-property. The positive principal minor property (1) is much more delicate to generalize. Given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V and a linear transformation L on V, we restrict L to the eigenspace $V^{(l)} := \{x \in V : x \circ (e_1 + e_2 + \cdots + e_l) = x\}$ to get a principal subtransformation of L. By calling the determinant of this restriction a principal minor of L, we introduce the positive principal minor property of L. By using ideas from nonsmooth analysis, we will show that an analog of (6) $\Rightarrow$ (1) holds in this general setting. The above analysis is partly motivated by the desire to understand the analog of (6) for symmetric cones, which figures prominently, at least in the semidefinite and Lorentz cones, in the stability/regularity of a solution of a (nonlinear) complementarity problem, see [26].

The organization of the paper is as follows. In the following section we introduce the operations '$\sqcap$' and '$\sqcup$' as analogs of the componentwise minimum and maximum of vectors. We also give a brief introduction to Euclidean Jordan algebras. In Section 4, we introduce the order **P**-property, Jordan **P**-property, and the **P**-property as generalizations of properties (3) and (2), and study some interconnections between them. Section 5 deals with the linear complementarity problems over symmetric cones. In Section 6, we study the complementarity properties of transformations with the **P**-property and introduce the **GUS**-property generalizing property (4). In Section 7, we introduce the Lipschitzian **GUS**-property as a generalization of property (6). In this section, we also introduce the principal subtransformation and the positive principal minor property of a linear transformation. We show here that Lipschitzian **GUS**-property implies the positive principal minor property. In Section 8, we specialize our results to symmetric linear transformations, to monotone transformations, and also to polyhedral cones.

2. The projection map and Euclidean Jordan algebras

2.1. The projection map Π_K

Consider a finite dimensional inner product space $(H, \langle\cdot,\cdot\rangle)$ and a closed convex cone K in H. This K induces a (partial) order on H:

$$x \leqslant y \text{ (or } y \geqslant x) \quad \Leftrightarrow \quad y - x \in K.$$

We use the notation $x < y$ (or $y > x$) when $y - x \in \mathrm{int}(K)$ (if exists).

Corresponding to K, let Π_K denote the metric projection onto K: For an $x \in H$, $x^* = \Pi_K(x)$ if and only if $x^* \in K$ and $\|x - x^*\| \leqslant \|x - y\|$ for all $y \in K$. It is well known that x^* (which belongs to K) is unique and is characterized by the so-called obtuse angle property:

$$\langle y - x^*, x - x^* \rangle \leqslant 0 \quad \forall y \in K. \tag{1}$$

Now, let

$$K^* := \{x : \langle x, y \rangle \geqslant 0 \text{ for all } y \in K\}$$

denote the *dual cone* of K.

We then have the Moreau decomposition [24]: Any $x \in H$ can be written as

$$x = \Pi_K(x) - \Pi_{K^*}(-x) \quad \text{with } \langle \Pi_K(x), \Pi_{K^*}(-x) \rangle = 0.$$

Moreover, $x = x_1 - x_2$ with $x_1 \in K, x_2 \in K^*$ and $\langle x_1, x_2 \rangle = 0$ if and only if $x_1 = \Pi_K(x)$ and $x_2 = \Pi_{K^*}(-x)$.

Definition 1. Suppose that K (which is a closed convex cone in H) is self-dual, i.e., $K^* = K$. For any $x \in H$, we define the *nonnegative part of* x, *nonpositive part of* x, *and the absolute value of* x by

$$x^+ := \Pi_K(x), \quad x^- := x^+ - x, \quad \text{and} \quad |x| := x^+ + x^-. \tag{2}$$

For $x, y \in H$, let

$$x \sqcap y := x - (x - y)^+ \quad \text{and} \quad x \sqcup y := y + (x - y)^+. \tag{3}$$

In the case of $H = R^n$ with the usual inner product, and $K = R^n_+$, the above operations '$\sqcap$' and '$\sqcup$' become the usual componentwise minimum and maximum operations on vectors. We note that H becomes a vector lattice in the order induced by K if and only if the cone K is polyhedral [28]. In particular, if K is nonpolyhedral, the implication $x \leqslant y, x \leqslant z \Rightarrow x \leqslant y \sqcap z$ may be false.

The following proposition describes some basic properties of the above two operations.

Proposition 2. *Let K be a closed convex self-dual cone in H. Then*

(a) *For any element $x \in H$, we have $x = x^+ - x^-$, $x^+, x^- \geqslant 0$, and $\langle x^+, x^- \rangle = 0$. This decomposition is unique in the sense that if $x = a - b$, $a, b \geqslant 0$, and $\langle a, b \rangle = 0$, then $a = x^+$ and $b = x^-$.*
(b) $x \sqcap y = y \sqcap x$.
(c) $-(x \sqcup y) = (-x) \sqcap (-y)$.

2.2. Euclidean Jordan algebras

In this subsection, we briefly describe some concepts, properties, and results from Euclidean Jordan algebras that are needed in this paper. All these can be found in the book [4] by Faraut and Korányi. Excellent summaries can be found in the articles [7,32].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$ where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in V$.

In addition, we assume that there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$.

Henceforth, we assume that V is a Euclidean Jordan algebra and call $x \circ y$ the Jordan product of x and y. In V, the set of squares

$$K := \{x^2 : x \in V\}$$

is a *symmetric cone* [4, p. 46]. This means that K is a self-dual closed convex cone and for any two elements $x, y \in \operatorname{int} K$, there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$.
For $x \in V$, we define

$$m(x) := \min\left\{k > 0 : \{e, x, \ldots, x^k\} \text{ is linearly dependent}\right\}$$

and *rank* of V by $r = \max\{m(x) : x \in V\}$. An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j \quad \text{and} \quad \sum_1^m e_i = e.$$

Note that $\langle e_i, e_j\rangle = \langle e_i \circ e_j, e\rangle = 0$ whenever $i \neq j$.

Theorem 3 (The spectral decomposition theorem). *Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{4}$$

The numbers λ_i are called the eigenvalues of x.

The expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the *spectral decomposition* (*or the spectral expansion*) of x. Given (4), we easily verify the following:

$$x^+ = \sum_{i=1}^r \lambda_i^+ e_i, \quad x^- = \sum_{i=1}^r \lambda_i^- e_i, \quad \text{and} \quad |x| = \sum_{i=1}^r |\lambda_i| e_i.$$

In particular, if $x \geqslant 0$, then every $\lambda_i \geqslant 0$ (which can be seen by noting $0 \leqslant \langle x, e_i \rangle = \lambda_i \|e_i\|^2$). When $\lambda_i \geqslant 0$ for all i, we define the (unique) square root of x by $\sqrt{x} = \sum_{i=1}^{r} \sqrt{\lambda_i} e_i$. Note that $|x| = \sqrt{x^2}$. We say that an element x is invertible if there is a polynomial in x, say y, such that $x \circ y = e$, or equivalently, every eigenvalue of x is nonzero [4].

Example 0.0. Consider R^n with the (usual) inner product and Jordan product defined respectively by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad x \circ y = x * y,$$

where x_i denotes the ith component of x etc., and $x * y$ denotes the componentwise product of vectors x and y. Then R^n is a Euclidean Jordan algebra with R^n_+ as its cone of squares.

Example 1.0. Let $\mathscr{S}^n$ be the set of all $n \times n$ real symmetric matrices with the inner and Jordan product given by

$$\langle X, Y \rangle := \operatorname{trace}(XY) \quad \text{and} \quad X \circ Y := \tfrac{1}{2}(XY + YX).$$

In this setting, the cone of squares $\mathscr{S}^n_+$ is the set of all positive semidefinite matrices in $\mathscr{S}^n$. The identity matrix is the unit element. The set $\{E_1, E_2, \ldots, E_n\}$ is a Jordan frame in $\mathscr{S}^n$ where E_i is the diagonal matrix with 1 in the (i, i)-slot and zeros elsewhere. Note that the rank of $\mathscr{S}^n$ is n. Given any $X \in \mathscr{S}^n$, there exists an orthogonal matrix U with columns $u_1, u_2, \ldots, u_n$ and a real diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $X = UDU^{\mathrm{T}}$. Clearly,

$$X = \lambda_1 u_1 u_1^{\mathrm{T}} + \cdots + \lambda_n u_n u_n^{\mathrm{T}}$$

is the spectral decomposition of X. Note that we may think of R^n (of Example 0.0) as the product of n copies of $\mathscr{S}^1$.

Example 2.0. Consider R^n $(n > 1)$ where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}$$

with $x_0 \in R$ and $\bar{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by $\mathscr{L}^n$. In this algebra, the cone of squares, denoted by $\mathscr{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathscr{L}^n_+ = \{x : \|\bar{x}\| \leqslant x_0\}.$$

The unit element in $\mathscr{L}^n$ is $e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We note the spectral decomposition of any x with $\bar{x} \neq 0$:

$$x = \lambda_1 e_1 + \lambda_2 e_2,$$

where

$$\lambda_1 := x_0 + \|\bar{x}\|, \quad \lambda_2 := x_0 - \|\bar{x}\|$$

and

$$e_1 := \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix} \quad \text{and} \quad e_2 := \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}.$$

In a Euclidean Jordan algebra V, for an $x \in V$, we define the corresponding *Lyapunov transformation* $L_x : V \to V$ by

$$L_x(z) = x \circ z.$$

(Traditionally, the notation $L(x)$ has been used to denote the Lyapunov transformation, see [4]. In this paper, we reserve the notation L_x for the Lyapunov transformation and write $L(x)$ to denote the image of an element $x \in V$ under a linear transformation $L : V \to V$.) We note that L_x is a self-adjoint linear transformation on V.

We say that elements x and y *operator commute* if L_x and L_y commute, i.e.,

$$L_x L_y = L_y L_x.$$

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame ([4, Lemma X.2.2] or [32, Theorem 27]). In the case of $\mathscr{S}^n$, matrices X and Y operator commute if and only if $XY = YX$. In the case of $\mathscr{L}^n$, vectors x and y (see Example 2.0) operator commute if and only if either $\bar{y}$ is a multiple of $\bar{x}$ or $\bar{x}$ is a multiple of $\bar{y}$.

The Peirce decomposition. Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R e_i$$

and when $i \neq j$,

$$V_{ij} := \left\{x \in V : x \circ e_i = \tfrac{1}{2} x = x \circ e_j\right\}.$$

Then we have the following

Theorem 4 [4, Theorem IV.2.1]. *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leqslant j)$. Furthermore,*

$$V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj},$$
$$V_{ij} \circ V_{jk} \subset V_{ik} \quad \text{if } i \neq k,$$
$$V_{ij} \circ V_{kl} = \{0\} \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset.$$

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i + \sum_{i<j} x_{ij},$$

where $x_i \in R\, e_i$ and $x_{ij} \in V_{ij}$. One can think of x as a symmetric $r \times r$ matrix whose diagonal elements are the x_i and whose off-diagonal elements are the x_{ij} (appearing in the (i, j) and (j, i) positions). The above theorem shows that the product in V has many of the properties of the Jordan product of matrices.

Simple Jordan algebras and the structure theorem. A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. The classification theorem [4, Chapter V] says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(1) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices (Example 1.0).
(2) The algebra $\mathcal{L}^n$ (Example 2.0).
(3) The algebra $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(4) The algebra $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(5) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

Theorem 5 [4, Propositions III.4.4 and III.4.5, Theorem V.3.7]. *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

We note that the 'direct sum' in the theorem refers to the orthogonal as well as the Jordan product direct sum. Thus given a Euclidean Jordan algebra V and the corresponding symmetric cone K, we may write

$$V = V_1 \times V_2 \times \cdots \times V_n \quad \text{and} \quad K = K_1 \times K_2 \times \cdots \times K_n,$$

where each V_i is a simple Jordan Algebra with the corresponding symmetric cone K_i. Moreover, for $x = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ and $y = (y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ in V with $x^{(i)}, y^{(i)} \in V_i$, we have

$$x \circ y = (x^{(1)} \circ y^{(1)}, \ldots, x^{(n)} \circ y^{(n)}) \quad \text{and} \quad \|x\|^2 = \sum \|x^{(i)}\|^2.$$

This leads to

$$\Pi_K = (\Pi_{K_1}, \Pi_{K_2}, \ldots, \Pi_{K_n}).$$

Automorphisms. A linear transformation $\Lambda : V \to V$ is said to be an *automorphism* of V if Λ is invertible and $\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y)$ for all $x,\ y \in V$. The set of all automorphisms is denoted by $\mathrm{Aut}(V)$.

A linear transformation $\Gamma : V \to V$ is said to be an *automorphism* of K if $\Gamma(K) = K$. Note that such a transformation is necessarily invertible. We denote the set of all automorphisms of K by $\mathrm{Aut}(K)$.

Since V carries an inner product, we can talk about the orthogonal group $\mathrm{Orth}(V)$ consisting of all linear transformations on V that preserve the inner product of V. While

$$\mathrm{Aut}(V) \subset \mathrm{Aut}(K)$$

always, $\mathrm{Aut}(V)$ need not be contained in $\mathrm{Orth}(V)$, see [4, p. 56] for an example. However, if $\langle x, y\rangle = \mathrm{tr}(x \circ y)$ or $\mathrm{tr}(L_{x\circ y})$ for all $x, y \in V$, then

$$\mathrm{Aut}(V) = \mathrm{Aut}(K) \cap \mathrm{Orth}(V)$$

see [4, p. 57]. In this setting [4], any $\Gamma \in \mathrm{Aut}(K)$ can be written as

$$\Gamma = P_x \Lambda,$$

where $x \in \mathrm{int}\, K,\ P_x := 2L_x^2 - L_{x^2}$ is the quadratic representation of x, and $\Lambda \in \mathrm{Aut}(V)$.

To illustrate these concepts, we consider the following examples.

Example 0.1. Consider R^n with the usual inner product and Jordan product (see Example 0.0). Then it is easily seen that the permutation matrices are the automorphisms of R^n and any automorphism of R^n_+ is a product of positive definite diagonal matrix and a permutation matrix.

Example 1.1. Consider $V = \mathcal{S}^n$. In this case, it is known [22,29] that corresponding to any $\Gamma \in \mathrm{Aut}(\mathcal{S}^n_+)$, there exists an invertible matrix $Q \in R^{n\times n}$ such that

$$\Gamma(Z) = QZQ^{\mathrm{T}} \quad (\forall Z \in \mathcal{S}^n).$$

In particular, for $\Lambda \in \mathrm{Aut}(\mathcal{S}^n)$, there exists a real orthogonal matrix U such that

$$\Lambda(Z) = UZU^{\mathrm{T}} \quad (\forall Z \in \mathcal{S}^n).$$

Example 2.1. Consider $V = \mathcal{L}^n$. In this case, it is known [23] that an $n \times n$ matrix A (or $-A$) belongs to $\mathrm{Aut}(\mathcal{L}^n_+)$ if and only if there exists a $\mu > 0$ such that

$$A^{\mathrm{T}} J_n A = \mu J_n,$$

where $J_n = \mathrm{diag}(1, -1, -1, \ldots, -1)$. In particular, if $A \in \mathrm{Aut}(\mathcal{L}^n)$, then (because of $Ae = e$), it can be easily seen that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}$$

where $D : R^{n-1} \to R^{n-1}$ is an orthogonal matrix.

3. Some preliminary resuts

In this section, we present some preliminary results that are needed in the paper. *As before, we assume that V is a Euclidean Jordan algebra and K is the corresponding cone of squares.*

The next two results are well known when V is: R^n with cone R^n_+, $\mathscr{S}^n$ with semidefinite cone $\mathscr{S}^n_+$, or $\mathscr{L}^n$ with the Lorentz cone $\mathscr{L}^n_+$, see [3,9,30]. In each case, the results are proved in an ad-hoc fashion. Below, we present a unified argument which also shows the operator commutativity of the variables involved.

Proposition 6. *For $x,\ y \in V$, the following conditions are equivalent*:

(1) $x \sqcap y = 0$.
(2) $x \geqslant 0,\ y \geqslant 0$, *and* $\langle x, y\rangle = 0$.
(3) $x \geqslant 0,\ y \geqslant 0$, *and* $x \circ y = 0$.
(4) $x + y - \sqrt{x^2 + y^2} = 0$.
(5) $x + y \geqslant 0$, *and* $x \circ y = 0$.

In each case, elements x and y operator commute.

Proof. (1) $\Rightarrow$ (2): $x \sqcap y = 0$ implies $x = (x - y)^+ \geqslant 0$. From $x \sqcap y = y \sqcap x$, it follows that $y \geqslant 0$. Since $(x - y)^+ = x$,

$$-\langle y, k - x\rangle = \langle x - y - x, k - x\rangle = \langle x - y - (x - y)^+, k - (x - y)^+\rangle \leqslant 0$$

for any $k \in K$. Putting $k = 0$ and $k = 2x$, we get $\langle x, y\rangle = 0$.

(2) $\Rightarrow$ (3): This is Exercise 3 in [4, p. 59], see also Lemma 2.2 in [7]. For the sake of completeness, we provide a proof. Assume $x \geqslant 0$, $y \geqslant 0$, and $\langle x, y\rangle = 0$. By the Spectral Decomposition Theorem, $x = \sum \lambda_i e_i$ and $y = \sum \mu_j f_j$, where $\{e_i\}$ and $\{f_i\}$ are Jordan frames. Clearly, λ_i and μ_j are nonnegative. Now

$$0 = \langle x, y\rangle = \sum \lambda_i \mu_j \langle e_i, f_j\rangle.$$

Since $\langle e_i, f_j\rangle \geqslant 0$, we have $\lambda_i\mu_j = 0$ or $\langle e_i, f_j\rangle = 0$. Suppose, for some i and j, $\langle e_i, f_j\rangle = 0$. Then

$$0 = \langle e_i, f_j\rangle = \langle e_i \circ e_i, f_j\rangle = \langle e_i, L_{f_j}(e_i)\rangle.$$

Since L_{f_j} is a (self-adjoint) positive semidefinite operator on V (cf. [4, Proposition III.2.2]), $L_{f_j}(e_i) = 0$. Hence, $e_i \circ f_j = 0$. So $\lambda_i\mu_j\langle e_i, f_j\rangle = 0$ implies that $\lambda_i\mu_j e_i \circ f_j = 0$. It follows that $x \circ y = \sum \lambda_i \mu_j (e_i \circ f_j) = 0$.

(3) $\Rightarrow$ (4): Since $x \circ y = 0$, we have $(x + y)^2 = x^2 + y^2$. Since $x + y \geqslant 0$, by the uniqueness of square root, $x + y = \sqrt{x^2 + y^2}$.

(4) $\Rightarrow$ (5): Follows easily from squaring both sides of the equality $x + y = \sqrt{x^2 + y^2}$ and using the commutativity of the Jordan product.

(5) $\Rightarrow$ (3): Assume $x + y \geqslant 0$ and $x \circ y = 0$. Let $x = \sum \lambda_i e_i$. We will show that every λ_i is nonnegative. Suppose, without loss of generality that $i = 1$ and $\lambda_1 \neq 0$. Since $x \circ e_1 = \lambda_1 e_1$, we have

$$0 \leqslant \langle x + y, e_1\rangle = \langle x, e_1\rangle + \langle y, e_1\rangle = \lambda_1 \|e_1\|^2 + \frac{1}{\lambda_1}\langle y, x \circ e_1\rangle$$
$$= \lambda_1 \|e_1\|^2 + \frac{1}{\lambda_1}\langle x \circ y, e_1\rangle.$$

Since $x \circ y = 0$, we see that $\lambda_1 > 0$. Hence $x \geqslant 0$ and similarly, $y \geqslant 0$.

(3) $\Rightarrow$ (2): The implication follows from $\langle x, y\rangle = \langle x \circ y, e\rangle$.

(2) $\Rightarrow$ (1): Follows from the Moreau decomposition of $x - y$.

Now assuming (2), we show that L_x and L_y commute. We first claim that $x \circ \sqrt{y} = 0$.

Following the notation in the proof of (2) $\Rightarrow$ (3), we have $x = \sum \lambda_i e_i$ and $y = \sum \mu_j f_j$ with $\lambda_i \geqslant 0$, $\mu_j \geqslant 0$, and $\lambda_i \mu_j (e_i \circ f_j) = 0$. Thus, $\lambda_i \sqrt{\mu_j}(e_i \circ f_j) = 0$, so $x \circ \sqrt{y} = 0$, where $\sqrt{y} = \sum \sqrt{\mu_j} f_j$.

We will make use of the following important identity in Euclidean Jordan algebras [4]:

$$[L_u, L_{v^2}] + 2[L_v, L_{u \circ v}] = 0, \tag{5}$$

where $[A, B] := AB - BA$. Putting $u = x$ and $v = \sqrt{y}$, we get

$$[L_x, L_y] + 2[L_{\sqrt{y}}, L_{x \circ \sqrt{y}}] = 0.$$

Since $x \circ \sqrt{y} = 0$, we see that $[L_x, L_y] = 0$, that is, $L_x L_y = L_y L_x$. □

We now present a perturbed version of implication (3) $\Leftrightarrow$ (4) in the previous proposition. While it is not needed in the rest of the paper, we feel that it might be useful in the study of interior-point trajectories in Euclidean Jordan algebras.

Proposition 7. *For $\varepsilon > 0$, the following are equivalent in any Euclidean Jordan algebra*:

(1) $x + y - \sqrt{x^2 + y^2 + 2\varepsilon e} = 0$,
(2) $x, y > 0$ *and* $x \circ y = \varepsilon e$.

In each case, x and y operator commute and $x^{-1} = y/\varepsilon$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate.

Now assume (1). Upon squaring both sides of $x + y = \sqrt{x^2 + y^2 + 2\varepsilon e}$, we get $x \circ y = \varepsilon e$. Since $e > 0$, we have $x + y > 0$. First we show that $x, y \geqslant 0$. As in the proof of Proposition 6, let $x = \sum \lambda_i e_i$. Fix an eigenvalue, say, λ_1. If λ_1 is zero, then $\langle x, e_1\rangle = 0$ and

$$0 = \langle y, x \circ e_1\rangle = \langle x \circ y, e_1\rangle = \langle \varepsilon e, e_1\rangle = \varepsilon \|e_1\|^2$$

which is clearly a contradiction. Now suppose that $\lambda_1 < 0$. Then, as in the proof of (5) $\Rightarrow$ (3) in Proposition 6, we get

$$0 \leqslant \langle x + y, e_1\rangle = \lambda_1 \|e_1\|^2 + \frac{1}{\lambda_1}\langle x \circ y, e_1\rangle = \lambda_1 \|e_1\|^2 + \frac{\varepsilon}{\lambda_1}\langle e, e_1\rangle < 0,$$

which is a contradiction. Thus, $\lambda_1 > 0$. Similarly every λ_i is positive so that $x > 0$. Likewise, $y > 0$. To show that x and y operator commute, we will use the identity (5)

$$[L_y, L_{x^2}] + 2[L_x, L_{y\circ x}] = 0.$$

From $x \circ y = \varepsilon e$, we get $L_{y\circ x} = \varepsilon I$ and $[L_x, L_{y\circ x}] = 0$. Thus, L_y and L_{x^2} commute, so there is a common Jordan frame $\{e_j\}$ such that $y = \sum \nu_i e_i$ and $x^2 = \sum \mu_i e_i$. Since $x > 0$, $x = \sum_{i=1}^r \sqrt{\mu_i} e_i$. Thus, x and y have their spectral expansions with respect to a common Jordan frame, so x and y operator commute.

Now $x > 0$ implies that $\det(x)$, being the product of eigenvalues of x, is positive; hence x^{-1} exists (cf. [4, Proposition II.2.4]). Let z be the inverse of x so that $x \circ z = e$. Since x and y operator commute (as well as x and z), we have

$$\begin{aligned} y &= y \circ e = y \circ (x \circ z) = x \circ (y \circ z) = x \circ (z \circ y) \\ &= z \circ (x \circ y) = z \circ \varepsilon e = \varepsilon z. \end{aligned}$$

Hence $x^{-1} = y/\varepsilon$. $\square$

The following result is known. It is a special case of (Löwner–Heinz inequality) Corollary 9 in [21] which itself is a special case of a result in [20]. For the Euclidean Jordan algebra $\mathscr{S}^n$ with cone $\mathscr{S}^n_+$, it appears in Lemma 6.1 of [30] and for $\mathscr{L}^n$ it appears in Proposition 3.4 of [9]. We provide below a self-contained elementary proof based on the proof of a similar result for symmetric matrices, see [33].

Proposition 8. *In V, if $x \geqslant 0$, $y \geqslant 0$, and $x \geqslant y$, then $\sqrt{x} \geqslant \sqrt{y}$.*

Proof. Let $p = \sqrt{x}$, $q = \sqrt{y}$, and $z = p - q$. Then we have

$$0 \leqslant x - y = p^2 - (p - z)^2 = p^2 - (p^2 - 2p \circ z + z^2) = 2p \circ z - z^2.$$

Let $z = \sum \lambda_i e_i$, where $\{e_i\}$ is a Jordan frame. We claim that $\lambda_i \geqslant 0$ for all i. If this is not the case, let, without loss of generality, $\lambda_1 < 0$. We have

$$\begin{aligned} 0 \leqslant \langle 2p \circ z - z^2, e_1\rangle &= \langle 2p \circ z, e_1\rangle - \langle z^2, e_1\rangle \\ &= \langle 2p, z \circ e_1\rangle - \langle z^2, e_1\rangle \\ &= \lambda_1\langle 2p, e_1\rangle - {\lambda_1}^2\langle e_1, e_1\rangle. \end{aligned}$$

Since $\langle e_1, e_1\rangle = \|e_1\|^2 > 0$, and $p \geqslant 0$, $e_1 \geqslant 0 \Rightarrow \langle 2p, e_1\rangle \geqslant 0$, we have $\lambda_1\langle 2p, e_1\rangle - {\lambda_1}^2\langle e_1, e_1\rangle < 0$. This is a contradiction. Hence $\lambda_i \geqslant 0$ for all i. This proves that $z \geqslant 0$, that is, $\sqrt{x} \geqslant \sqrt{y}$. $\square$

The following lemma is crucially needed in the following section.

Lemma 9. *For $x, y \in V$, consider the following statements*:

(1) *x and y operator commute, and $x \circ y \leqslant 0$.*
(2) $x \circ y \leqslant 0$.
(3) $x \sqcap y \leqslant 0 \leqslant x \sqcup y$.
(4) $\langle x, y\rangle \leqslant 0$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Let $x \circ y \leqslant 0$. Then $(x+y)^2 \leqslant (x-y)^2 \Rightarrow |x+y| \leqslant |x-y|$, by Proposition 8. Also,

$$x \sqcap y = x - (x-y)^+ = x - \left[\frac{(x-y)+|x-y|}{2}\right] = \frac{x+y-|x-y|}{2} \leqslant 0.$$

Now,

$$\begin{aligned} x \circ y \leqslant 0 &\Rightarrow (-x)\circ(-y) \leqslant 0 \Rightarrow (-x)\sqcap(-y) \leqslant 0 \\ &\Rightarrow -(x \sqcup y) \leqslant 0 \Rightarrow x \sqcup y \geqslant 0. \end{aligned}$$

(3) $\Rightarrow$ (4): Let $x \sqcap y \leqslant 0 \leqslant x \sqcup y$. Then $x - (x-y)^+ \leqslant 0 \leqslant y + (x-y)^+$. Putting $p := -x + (x-y)^+ \geqslant 0$ and $q := y + (x-y)^+ \geqslant 0$, we have

$$p \circ q = -x \circ y - x \circ (x-y)^+ + y \circ (x-y)^+ + (x-y)^+ \circ (x-y)^+, \quad (6)$$

and so

$$\begin{aligned} x \circ y + p \circ q &= -(x-y)\circ(x-y)^+ + (x-y)^+ \circ (x-y)^+ \\ &= (-(x-y)^+ + (x-y)^-)\circ(x-y)^+ + (x-y)^+ \circ (x-y)^+ \\ &= 0. \end{aligned}$$

From this we get

$$\langle x, y\rangle + \langle p, q\rangle = \langle x \circ y + p \circ q, e\rangle = 0. \quad (7)$$

Since $p \geqslant 0$ and $q \geqslant 0$, we have $\langle p, q\rangle \geqslant 0$ and $\langle x, y\rangle \leqslant 0$. $\square$

4. Order P, Jordan P, and P properties

In this section, we introduce the Euclidean Jordan algebra analogs of conditions (2) and (3) of the Introduction.

Definition 10. Consider a linear transformation $L : V \to V$. We say that L is/has

(1) *monotone* (*strictly* = *strongly monotone*) if $\langle L(x), x\rangle \geqslant 0$ (respectively, > 0) for all $0 \neq x \in V$;
(2) the *order* **P**- *property* if $x \sqcap L(x) \leqslant 0 \leqslant x \sqcup L(x) \Rightarrow x = 0$;
(3) the *Jordan* **P**- *property* if $x \circ L(x) \leqslant 0 \Rightarrow x = 0$;
(4) the **P**- *property* if

$$\left.\begin{array}{l} x \text{ and } L(x) \text{ operator commute} \\ x \circ L(x) \leqslant 0 \end{array}\right\} \Rightarrow x = 0.$$

Remarks. (1) In the case of $V = R^n$ and $K = R^n_+$, properties formulated in (2)–(4) coincide. We will see in Section 8 that the same is true if K is polyhedral.

(2) The above order **P**-property is similar to the one described by Borwein and Dempster [2] in the context of order linear complementarity problems over vector lattices. We may view the order **P**-property as a "noncommuting version" of the **P**-property: the commuting version of the order **P**-property, namely, the condition

$$\left.\begin{array}{l} x \text{ and } L(x) \text{ operator commute} \\ x \sqcap L(x) \leqslant 0 \leqslant x \sqcup L(x) \end{array}\right\} \Rightarrow x = 0$$

is exactly the **P**-property.

(3) In the context of $V = \mathscr{S}^n$ and $K = \mathscr{S}^n_+$, the Jordan **P**-property has previously been called the $\mathbf{P}_1$-property. We have adopted the terminology used in [5] for $\mathscr{S}^n$.

(4) In the context of $V = \mathscr{S}^n$ and $K = \mathscr{S}^n_+$, the **P**- and the Jordan **P**-properties were introduced by Gowda and Song [13]. It is known, see [12,13], that the Lyapunov transformation L_A defined by

$$L_A(X) = A \circ X := \tfrac{1}{2}(AX + XA^{\mathrm{T}})$$

has the **P**-property (Jordan **P**-property) if and only if A is positive stable (that is, all its eigenvalues lie in the open right-half plane), and S_A defined by

$$S_A(X) = X - AXA^{\mathrm{T}}$$

has the **P**-property if and only if A is Schur stable (that is, all its eigenvalues lie in the open unit disk). (Note: The definition of L_A given above conforms with the definition given in Section 2.2. However, in various literature, the definition $L_A(X) = AX + XA^{\mathrm{T}}$ is commonly used.)

The following result gives some interconnections between the above four concepts.

Theorem 11. *For a linear transformation* $L : V \to V$, *the following implications hold*:

Strong monotonicity $\Rightarrow$ *Order* **P** $\Rightarrow$ *Jordan* **P** $\Rightarrow$ **P**.

Moreover, *if* L *has the* **P**-*property*, *then every real eigenvalue of* L *is positive and the determinant of* L *is positive*.

Proof. The implications follow immediately from Lemma 9. Now suppose that L has the **P**-property. If λ is a real, nonpositive eigenvalue of L, then there exists a nonzero $x \in V$ such that $L(x) = \lambda x$. It follows that x and $L(x)$ operator commute and $x \circ L(x) = \lambda x^2 \leqslant 0$. We get a contradiction to the **P**-property. Hence all real eigenvalues of L are positive. It follows that the determinant of L (being the product of all eigenvalues) is also positive. □

5. Linear complementarity problems over symmetric cones and the GUS property

In this section, we address the Euclidean Jordan algebra analogs of implications (2) ⇔ (4) given in the Introduction. Toward this end, we define a linear complementarity problem over a symmetric cone. Related mixed LCP and geometric/horizontal LCP will be mentioned. However, we will not be dealing with applications and computational aspects of these problems.

Consider a closed convex set C in a finite dimensional real inner product space H. Given a function $f : H \to H$ and a vector $q \in H$, the *Variational Inequality Problem*, $\mathrm{VI}(f, C, q)$, is to find an $x \in C$ such that

$$\langle f(x) + q, y - x\rangle \geqslant 0 \quad \forall y \in C.$$

Corresponding to this problem, we define the so-called normal map

$$F(x) := f(\Pi_C(x)) + x - \Pi_C(x),$$

where Π_C denotes the projection map onto C. It is well known that $\mathrm{VI}(f, C, q)$ has a (unique) solution if and only if the equation $F(x) = -q$ has a (unique) solution, see Propositions 1.5.9 and 1.5.11 in [5].

Now suppose that C is a closed convex cone and $f = L$ is linear. Then $\mathrm{VI}(f, C, q)$ becomes cone-LCP(L,C,q): Find x such that

$$x \in C,\ L(x) + q \in C^* \quad \text{and} \quad \langle L(x) + q, x\rangle = 0,$$

where C^* is the dual of C. The recent book [5] by Facchinei and Pang contains extensive literature on variational inequality and complementarity problems.

5.1. Standard form LCP

We now assume that $H = V$ (a Euclidean Jordan algebra) and $C = K$ (the cone of squares in V). In this setting, the cone-LCP becomes a *standard form LCP on a symmetric cone*: Given a linear transformation $L : V \to V$, and a $q \in V$, $\mathrm{LCP}(L, K, q)$ is to find an $x \in V$ such that

$$x \in K,\ L(x) + q \in K \quad \text{and} \quad \langle x, L(x) + q\rangle = 0.$$

In view of Proposition 6 and the normal map formulation, we see that $\mathrm{LCP}(L, K, q)$ can be described by means of the following equivalent conditions: Find $x \in V$ such that

- $x \sqcap [L(x) + q] = 0$;
- $x \geqslant 0,\ L(x) + q \geqslant 0$ and $x \circ [L(x) + q] = 0$;
- $x + [L(x) + q] - \sqrt{x^2 + [L(x) + q]^2} = 0$;
- $L(x^+) - x^- = -q$.

We point out that if x is a solution of LCP(L, K, q), then x and $L(x) + q$ operator commute. Also, note that when $V = R^n$ and $K = R^n_+$, we get the standard LCP [3] and when $V = \mathscr{S}^n$ and $K = \mathscr{S}^n_+$, we get the Semidefinite LCP [13].

5.2. Mixed LCP

Let E and V be Euclidean Jordan algebras and K be the cone of squares in V. We put $H := E \times V$ and $C := E \times K$. Then C is a closed convex cone in H with $C^* = \{0\} \times K$. Given a linear transformation $L : H \to H$ defined by $L(y, x) = (Px + Qy, Rx + Sy)$ where P, Q, R, and S are appropriate linear transformations, and $q := (a, b) \in H$, we consider cone-LCP(L, C, q): Find $y \in E, x \in V$ such that

$$\begin{aligned} &Px + Qy + a = 0, \\ &x \geqslant 0, \quad Rx + Sy + b \geqslant 0, \\ &\langle x, Rx + Sy + b\rangle = 0. \end{aligned}$$

In view of Proposition 6, we may rewrite the last condition as $x \circ [Rx + Sy + b] = 0$. We call this cone-LCP a *mixed-LCP*, as one of the variables, namely, y is a free variable. It is easy to see that at least in theory, the mixed LCP and the symmetric cone-LCP are equivalent.

To give an example of a mixed LCP, let $E = R^n$, V be a Euclidean Jordan algebra, and K be the cone of squares in V. Corresponding to a linear transformation $A : V \to R^n$, define $L : R^n \times V \to R^n \times V$ by $L(y, x) := (Ax, -A^{\mathrm{T}}y)$. Then for $q = (-b, c)$, the above mixed-LCP becomes the problem of finding $y \in E, x \in V, s \in V$ such that

$$\begin{aligned} &Ax = b, \\ &A^{\mathrm{T}}y + s = c, \\ &x \circ s = 0, \\ &x \geqslant 0, \quad s \geqslant 0. \end{aligned}$$

We note that the above system is the primal-dual optimal solution system corresponding to the (following) primal-dual pair of linear programs over the symmetric cone K [32]:

$$(\textit{Primal}) \quad \min\left\{\langle c, x\rangle : Ax = b, x \in K\right\},$$

and

$$(\textit{Dual}) \quad \max\left\{\langle b, y\rangle : A^{\mathrm{T}}y + s = c, s \in K, y \in R^n\right\}.$$

When $V = \mathscr{S}^n$ and $K = \mathscr{S}^n_+$, the above pair becomes the primal–dual pair of semidefinite linear programs.

5.3. The geometric/horizontal LCP

Let V be a Euclidean Jordan algebra, and K be the cone of squares in V. Let $(a, b) \in V \times V$ and $Y \subset V \times V$ be a vector subspace such that $\dim(Y) = \dim(V)$. Then the *geometric LCP* is to find

$$(x, y) \in [(a, b) + Y] \cap (K \times K) \quad \text{such that} \quad \langle x, y\rangle = 0.$$

Under the assumption that $(u, v) \in Y \Rightarrow \langle u, v\rangle \geqslant 0$, this reduces to the monotone LCP on a symmetric cone considered in [6]. When $V = \mathscr{S}^n$ and $K = \mathscr{S}^n_+$, this reduces to the monotone semidefinite LCP studied in [19]. To see an equivalent formulation, we write $(a, b) + Y$ as $\{(x, y) : Ax + By = q\}$ for suitable linear transformations A and B on V and $q \in V$. (This can be done since $\dim(Y) = \dim(V)$.) Then the problem reads: Find x, y such that

$$x \geqslant 0,\ y \geqslant 0,\ Ax + By = q \quad \text{and} \quad \langle x, y\rangle = 0.$$

It is in this form, the horizontal LCPs are introduced in the standard LCP literature [5].

6. Complementary properties of P-transformations

A result of Karamardian [18] stated in the setting of the cone K in V says that if the two problems $\mathrm{LCP}(L, K, 0)$ and $\mathrm{LCP}(L, K, e)$ (where e is the unit element of V) have unique solutions, namely zero, then for all $q \in V$, $\mathrm{LCP}(L, K, q)$ will have a solution. We will use this result in the following theorem.

Theorem 12. *Suppose that $L : V \to V$ has the* **P***-property. Then for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a nonempty compact solution set.*

Proof. Suppose that $t \geqslant 0$ in R and let x be any solution of $\mathrm{LCP}(L, K, te)$. Then $x \geqslant 0$ and $y = L(x) + te \geqslant 0$ operator commute and $x \circ y = 0$. It follows that x and $L(x)$ operator commute and $x \circ L(x) = -tx \leqslant 0$. Since L has the **P**-property, we get $x = 0$. Thus, the problems $\mathrm{LCP}(L, K, 0)$ and $\mathrm{LCP}(L, K, e)$ have unique solutions. By the above mentioned result of Karamardian, we see that for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a solution. Clearly, the solution set of $\mathrm{LCP}(L, K, q)$ is closed. If the solution set is not bounded, we will have a sequence $x^{(k)}$ of solutions with $\|x^{(k)}\| \to \infty$. A subsequential limit, say, x, of the sequence $\frac{x^{(k)}}{\|x^{(k)}\|}$ will satisfy the conditions $x \circ L(x) \leqslant 0$, x and $L(x)$ operator commute, and $\|x\| = 1$. This clearly is a violation of the **P**-property of L. We thus have the compactness of the solution set. □

Definition 13. A linear transformation $L : V \to V$ is said to have the *Globally Uniquely Solvable* (**GUS**) property if for all $q \in V$, LCP(L, K, q) has a unique solution. It is said to have the *Cross Commutative* property if for any q and for any two solutions x_1 and x_2 of LCP(L, K, q), x_1 operator commutes with y_2 and x_2 operator commutes with y_1, where $y_i = L(x_i) + q$ $(i = 1, 2)$.

We note that in view of Karamardian's result mentioned previously, the **GUS**-property of L is equivalent to: for all $q \in V$, LCP(L, K, q) has at most one solution.

Theorem 14. *For a linear transformation* $L : V \to V$,

$$\mathbf{GUS} = \mathbf{P} + \mathbf{Cross\,Commutative}.$$

Proof. Suppose that L has the **GUS**-property. Let $x \in V$ such that x and $y := L(x)$ operator commute, and $x \circ L(x) \leqslant 0$. From Lemma X.2.2 in [4], we may write $x = \sum \lambda_i e_i$ and $y = \sum \mu_i e_i$, where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame. Then $x \circ L(x) \leqslant 0$ yields $\sum \lambda_i \mu_i e_i \leqslant 0$. It follows that $\lambda_i \mu_i \leqslant 0$ for all i. This implies that $\lambda_i^+ \mu_i^+ = \lambda_i^- \mu_i^- = 0$ for all i. From this we conclude that $x^+ \circ y^+ = x^- \circ y^- = 0$. Now define $q := [L(x)]^+ - L(x^+)$. We see that $q = [L(x)]^- - L(x^-)$ and that x^+ and x^- are two solutions of LCP(L, K, q). Thus $x^+ = x^-$ and $x = 0$. This proves the **P**-property of L. By the uniqueness of solution, the cross commutative property is obvious.

Now for the converse. Suppose L has the **P** and the cross commutative properties. For any q, let x_1 and x_2 be solutions of LCP(L, K, q) and $y_i = L(x_i) + q$ $(i = 1, 2)$. Since $x_1 \geqslant 0$ operator commutes with $y_2 \geqslant 0$, it follows that $x_1 \circ y_2 \geqslant 0$. Similarly, $x_2 \circ y_1 \geqslant 0$. Now $x := x_1 - x_2$ operator commutes with $L(x) = y_1 - y_2$ and $x \circ L(x) = -[x_1 \circ y_2 + x_2 \circ y_1] \leqslant 0$. By the **P**-property, $x = 0$ so $x_1 = x_2$. This argument shows that L has the **GUS**-property. □

Remarks. When $V = R^n$ and $K = R^n_+$, the **GUS** and **P** properties coincide. It has been shown [13] that in $\mathscr{S}^n$, $L_A(X) = \frac{1}{2}(AX + XA^{\mathrm{T}})$ (for $A \in R^{n\times n}$) has the **P** (**GUS**) property if and only if A is positive stable (respectively, positively stable and positive semidefinite). So the **GUS** and **P** are different in $\mathscr{S}^n$; see [13] for an explicit example.

7. Lipschitzian GUS-property

For a linear transformation $L : V \to V$, recall that the normal map is defined by

$$F(x) := L(x^+) + x - x^+.$$

It is well known, see [5, Proposition 1.5.11], that L has the **GUS**-property if and only if F is a homeomorphism of V. (In [5] the result is stated for R^n. The same proof is valid for V also.)

Definition 15. A linear transformation $L : V \to V$ is said to have the *Lipschitzian* **GUS**-*property* if the normal map $F(x) := L(x^+) + x - x^+$ is a homeomorphism of V and the inverse of F is Lipschitzian.

Note that the above property holds for L if and only if there exists a positive number α such that

$$\|F(x) - F(y)\| \geqslant \alpha\|x - y\| \quad (\forall x, y \in V).$$

Furthermore, it is well known [5] that the above property holds if and only if the function which takes $q \in V$ to the solution set of LCP(L, K, q) is a homeomorphism and a Lipschitz function.

In this section, we describe some necessary conditions for the Lipschitzian **GUS**-property of L, which become sufficient when the cone is polyhedral. These conditions are in terms of determinants of certain transformations associated with L.

We first recall certain concepts from nonsmooth analysis. Since Π_K is nonexpansive, both Π_K and F are Lipschitz functions on V. By Rademacher's theorem both are (Fréchet) differentiable almost everywhere and the Bouligand differentials

$$\partial_B \Pi_K(0) := \left\{\Lambda : \exists x^{(k)} \to 0,\ \Pi_K'(x^{(k)}) \to \Lambda\right\}$$

and

$$\partial_B F(0) := \left\{S : \exists x^{(k)} \to 0,\ F'(x^{(k)}) \to S\right\}$$

exist with 'prime' denoting the (Fréchet) derivative. Note that

$$\partial_B F(0) \supseteq \left\{L \circ \Lambda + I - \Lambda : \Lambda \in \partial_B \Pi_K(0)\right\}. \tag{8}$$

In what follows, we introduce the notion of "principal subtransformations" of a given linear transformation on V and show that the determinants of these are also positive under certain conditions.

In V, fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$, and define

$$V^{(l)} := V(e_1 + e_2 + \cdots + e_l, 1) := \{x \in V : x \circ (e_1 + e_2 + \cdots + e_l) = x\}$$

for $1 \leqslant l \leqslant r$. Corresponding to $V^{(l)}$, we consider the (orthogonal) projection $P^{(l)} : V \to V^{(l)}$. For a given linear transformation $L : V \to V$, the transformation $P^{(l)} \circ L : V^{(l)} \to V^{(l)}$ is a *principal subtransformation* of L corresponding to $\{e_1, e_2, \ldots, e_l\}$, and is denoted by

$$L_{\{e_1,e_2,\ldots,e_l\}}.$$

We call the determinant of $L_{\{e_1,e_2,\ldots,e_l\}}$ a *principal minor* of L. This is a modified version of the concept of principal minor of an element in a Euclidean Jordan algebra, see [4]. Note that for a given Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can permute the objects and select the first l objects (for any $1 \leqslant l \leqslant r$). Thus there are $2^r - 1$ principal subtransformations (minors) corresponding to a Jordan frame. Of course, by taking other Jordan frames, we generate other principal subtransformations (minors).

Definition 16. $L : V \to V$ is said to have the *positive principal minor* (positive **PM**) property if all principal minors of L are positive.

We illustrate this concept by means of the following examples.

Example 1.2. In $\mathscr{S}^n$, consider the Jordan frame $\{E_1, E_2, \ldots, E_n\}$ (defined in Example 1.0). Let $\alpha := \{1, 2, \ldots, l\}$ and $l = |\alpha|$. Then it is easily seen that $X \in W^{(l)} := \{X \in \mathscr{S}^n : X \circ (E_1 + E_2 + \cdots + E_l) = X\}$ if and only if

$$X = \begin{bmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{bmatrix},$$

where $X_{\alpha\alpha}$ is the principal submatrix of X corresponding to the index set α. Thus we may view $W^{(l)}$ as $S^{|\alpha|}$. Since the projection $Q^{(l)} : \mathscr{S}^n \to W^{(l)}$ is given by

$$Q^{(l)}(Y) = \begin{bmatrix} Y_{\alpha\alpha} & 0 \\ 0 & 0 \end{bmatrix}$$

for any $Y \in \mathscr{S}^n$, we see that the principal subtransformation $L_{\{E_1, E_2, \ldots, E_n\}}$ takes any $X_{\alpha\alpha}$ to the $\alpha\alpha$-submatrix of

$$L\left(\begin{bmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{bmatrix}\right).$$

As in [16], we denote this transformation by $L_{\alpha\alpha}$. Suppose $\{e_1, e_2, \ldots, e_n\}$ is any other Jordan frame in $\mathscr{S}^n$ with the corresponding eigenspace $V^{(l)}$ and the projection $P^{(l)}$. Then there is an automorphism Λ of $\mathscr{S}^n$ such that $E_i = \Lambda(e_i)$ for all $i = 1, 2, \ldots, n$ [4, p. 71]. (Conversely, every automorphism of $\mathscr{S}^n$ takes one Jordan frame into another Jordan frame.) Corresponding to this Λ, there exists a real orthogonal matrix U such that $\Lambda(Z) = UZU^{\mathrm{T}}$ for all $Z \in \mathscr{S}^n$. Since this Λ is an automorphism of $\mathscr{S}^n$ that preserves inner products, we easily verify the following:

$$V^{(l)} = \Lambda^{-1}(W^{(l)}), \quad \Lambda P^{(l)} = Q^{(l)}\Lambda, \quad \text{and}$$
$$\Lambda L_{\{e_1, e_2, \ldots, e_l\}}\Lambda^{-1} = (\widetilde{L})_{\{E_1, E_2, \ldots, E_l\}},$$

where

$$\widetilde{L}(Z) := (\Lambda\, L\, \Lambda^{-1})(Z) = UL(U^{\mathrm{T}}ZU)U^{\mathrm{T}}.$$

By considering the matrix representations of $L_{\{e_1, e_2, \ldots, e_l\}}$ with respect to a basis $\mathscr{B}$ in $W^{(l)}$ and that of $\Lambda L_{\{e_1, e_2, \ldots, e_l\}}\Lambda^{-1}$ with respect to the basis $\Lambda^{-1}\mathscr{B}$ in $V^{(l)} = \Lambda^{-1}(W^{(l)})$, we see that the determinant of $\Lambda L_{\{e_1, e_2, \ldots, e_l\}}\Lambda^{-1}$ is the same as that of $L_{\{e_1, e_2, \ldots, e_l\}}$. From this we see that L has the positive **PM**-property if and only if for any automorphism Λ of $\mathscr{S}^n$ and for any $\alpha = \{1, 2, \ldots, l\}$, the determinant of $(\Lambda\, L\, \Lambda^{-1})_{\alpha\alpha}$ is positive.

To further illustrate, consider a real $n \times n$ matrix A and the corresponding Lyapunov transformation $L_A : \mathscr{S}^n \to \mathscr{S}^n$ defined by $L_A(X) = \frac{1}{2}(AX + XA^{\mathrm{T}})$. It can be easily shown that $(L_A)_{\alpha\alpha} = L_{A_{\alpha\alpha}}$ where $A_{\alpha\alpha}$ is the principal submatrix of A corresponding to the index set α. Moreover, corresponding to the automorphism $\Lambda(Z) = UZU^{\mathrm{T}}$ (where U is an orthogonal matrix),

$$\widetilde{L_A} = L_B,$$

where $B = UAU^{\mathrm{T}}$. Now suppose L_A has the positive **PM**-property. Then for every orthogonal matrix U and for every $\alpha = \{1, 2, \ldots, l\}$, the determinant of $L_{B_{\alpha\alpha}}$ defined on $\mathscr{S}^{|\alpha|}$ is positive. This leads to, by taking $\alpha = \{1\}$, to the inequality $B_{11} > 0$ where B_{11} is the $(1, 1)$-entry of the matrix B. We conclude that

$$(UAU^{\mathrm{T}})_{11} > 0$$

for all orthogonal matrices U. Now, starting with any unit vector $u \in R^n$, we can create an orthogonal matrix U whose first column is u. Then $(UAU^{\mathrm{T}})_{11} = \langle Au, u\rangle$. Thus when L_A has the positive **PM**-property, we get

$$\langle Au, u\rangle > 0 \quad \text{for all unit vectors } u \in R^n$$

which means that the matrix A is positive definite. In conclusion,

$$L_A \text{ has the positive } \mathbf{PM}\text{-property } \Rightarrow A \text{ is positive definite.}$$

(As a consequence of our next result, we even have the converse.)

Example 2.2. In $V = \mathscr{L}^n$, let $\{e_1, e_2\}$ be any Jordan frame (see Example 2.0). Then corresponding to $l = 2$ we have $V^{(l)} = \{X \in V : x \circ (e_1 + e_2) = x\} = V$ (because $e_1 + e_2 = e$) and corresponding to $l = 1$, we have $V^{(l)} = \operatorname{span}\{e_1\}$. Since the latter space is one dimensional, the orthogonal projection onto this space is easily described and we have the following: for any linear transformation $L : V \to V$:

$$L_{\{e_1,e_2\}} = L \quad \text{and} \quad L_{\{e_1\}}(\lambda e_1) = \frac{\langle L(e_1), e_1\rangle}{\|e_1\|^2}\lambda e_1.$$

Now e_1 belongs to the boundary of $\mathscr{L}^n_+$ and every nonzero element on the boundary of $\mathscr{L}^n_+$ is a multiple of some e_1. Hence,

L has the positive **PM**-property if and only if the determinant of L is positive and L is positive definite on the boundary of $\mathscr{L}^n_+$.

We now state the main result of this section.

Theorem 17. *Let $L : V \to V$ be linear. Consider the following statements*:

(i) *L is strongly monotone*.
(ii) *L has the Lipschitzian* **GUS**-*property*.
(iii) $\det(L \circ \Lambda + I - \Lambda) > 0$ *for all* $\Lambda \in \partial_{\mathrm{B}}\Pi_K(0)$.
(iv) *Every principal minor of L is positive*.

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

The proof of the above result is based on several lemmas. In our second lemma, we discuss the (Fréchet) differentiability of Π_K. Toward this, we fix an $\bar{x} \in V$ with a corresponding Jordan frame $\{e_1, e_2, \ldots, e_r\}$. Let

$$\bar{x} = \sum_{i=1}^{r} \overline{\lambda}_i e_i.$$

For any $h \in V$, we have the Peirce decomposition corresponding to $\{e_1, e_2, \ldots, e_r\}$:

$$h = \sum_{i=1}^{r} h_i e_i + \sum_{j<k} h_{jk},$$

where h_i are real numbers and $h_{jk} \in V_{jk}$.

Lemma 18 (Korányi [20]). *Let V be simple. Let (a, b) be an interval in R that contains $\overline{\lambda}_i$ for all i. Define $V(a, b)$ to be the set of all $z \in V$ whose eigenvalues lie in (a, b). Let $\phi : (a, b) \to R$ be a continuously differentiable function. Define $\widehat{\phi} : V(a, b) \to V$ by*

$$\widehat{\phi}(z) = \sum \phi(\mu_i) f_i$$

for any $z \in V(a, b)$ whose spectral decomposition is given by $z = \sum \mu_i f_i$. Then $\widehat{\phi}$ is Fréchet differentiable at $\bar{x}$ and its derivative is given by

$$(\widehat{\phi})'(\bar{x})\, h = \sum_{i} [\overline{\lambda}_i, \overline{\lambda}_i] h_i e_i + \sum_{j<k} [\overline{\lambda}_j, \overline{\lambda}_k] h_{jk},$$

where

$$[\overline{\lambda}_i, \overline{\lambda}_j] := \begin{cases} \frac{\phi(\overline{\lambda}_i) - \phi(\overline{\lambda}_j)}{\overline{\lambda}_i - \overline{\lambda}_j} & \text{for } i \neq j, \\ \phi'(\overline{\lambda}_i) & \text{for } i = j. \end{cases}$$

Based on the above result, we prove the following.

Lemma 19. *The projection map Π_K is differentiable at $\bar{x}$ if and only if $\bar{x}$ is invertible. In particular, when V is simple and $\bar{x}$ is invertible, the derivative of Π_K at $\bar{x}$ is given by*

$$\Pi_K'(\bar{x})h = \begin{cases} h & \text{when } \overline{\lambda}_i > 0\ \forall i, \\ 0 & \text{when } \overline{\lambda}_i < 0\ \forall i, \\ \sum_1^l h_i e_i + \sum_{j<k} \theta_{jk} h_{jk} & \text{when } \overline{\lambda}_i > 0\ (i \leqslant l) \\ & \quad \text{and } \overline{\lambda}_i < 0\ (i > l), \end{cases} \tag{9}$$

where

$$\theta_{jk} = \begin{cases} 1 & \textit{for } j < k \leqslant l, \\ \frac{\overline{\lambda}_j}{\overline{\lambda}_j - \overline{\lambda}_k} & \textit{for } j \leqslant l < k, \\ 0 & \textit{for } l < j < k. \end{cases}$$

Proof. We will first prove (9). Assume that V is simple and $\bar{x}$ is invertible so $\overline{\lambda}_i \neq 0$ for all i.

When every $\overline{\lambda}_i > 0$, $\bar{x}$ is in the interior of K and hence Π_K coincides with the identity transformation in a neighborhood of $\bar{x}$. When every $\overline{\lambda}_i < 0$, $\bar{x}$ is the interior of $-K$ and hence Π_K coincides with the zero transformation in a neighborhood of $\bar{x}$. In these two cases, we have (9).

Now suppose that there is an index l such that $\overline{\lambda}_i > 0$ for $i \leqslant l$ and $\overline{\lambda}_i < 0$ for $i > l$. Pick a $\delta > 0$ such that the interval $(-\delta, \delta)$ does not contain any of the $\overline{\lambda}_i$'s. Let ϕ be a continuously differentiable function on $(-\infty, \infty)$ such that $\phi(t) = 0$ on $(-\infty, -\delta]$ and $\phi(t) = t$ on $[\delta, \infty)$. Then, if $\|h\|$ is sufficiently small and $\bar{x} + h$ has spectral decomposition given by $\sum_1^r \mu_i f_i$, we have

$$\widehat{\phi}(\bar{x} + h) = \widehat{\phi}\left(\sum_1^r \mu_i f_i\right) = \sum_1^l \mu_i f_i$$

and

$$\Pi_K(\bar{x} + h) = \sum_1^r \mu_i^+ f_i = \sum_1^l \mu_i f_i.$$

(Here we have used the fact that eigenvalues depend continuously on the element [20].) Thus $\widehat{\phi}$ coincides with Π_K in a neighborhood of $\bar{x}$. Now by the above mentioned result of Korányi [20], $\widehat{\phi}$ is differentiable at $\bar{x}$ and the derivative is given by

$$(\widehat{\phi})'(\bar{x})h = \sum_{i=1}^r [\overline{\lambda}_i, \overline{\lambda}_i] h_i e_i + \sum_{j<k} [\overline{\lambda}_j, \overline{\lambda}_k] h_{jk}.$$

Using the definition of ϕ and putting $\theta_{jk} = [\overline{\lambda}_j, \overline{\lambda}_k]$, we see that

$$\Pi'_K(\bar{x})h = \sum_1^l h_i e_i + \sum_{j<k} \theta_{jk} h_{jk},$$

where

$$\theta_{jk} = \begin{cases} 1 & \text{for } j < k \leqslant l, \\ \frac{\overline{\lambda}_j}{\overline{\lambda}_j - \overline{\lambda}_k} & \text{for } j \leqslant l < k, \\ 0 & \text{for } l < j < k. \end{cases}$$

We now prove the first part of the lemma. Suppose that V is a general Euclidean Jordan algebra. Then by Theorem 5, $V = V_1 \times V_2 \times \cdots \times V_n$ and $K = K_1 \times$

$K_2 \times \cdots \times K_n$ where each V_i is a simple Euclidean Jordan algebra and K_i is the corresponding cone of squares. Writing any $x \in V$ as $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in V_i$, we see that $\Pi_K(x) = (\Pi_{K_1}(x_1), \ldots, \Pi_{K_n}(x_n))$. Since x is invertible in V if and only if x_i is invertible in V_i for all $i = 1, 2, \ldots, n$ and Π_K is differentiable at x if and only if Π_{K_i} is differentiable at x_i for all i, we see that Π_K is differentiable at $\bar{x}$ when $\bar{x}$ is invertible. We now prove the converse. Suppose that Π_K is differentiable at $\bar{x}$ and zero is an eigenvalue of $\bar{x}$. Let

$$\bar{x} = 0e_1 + 0e_2 + \cdots + 0e_l + \overline{\lambda}_{l+1}e_{l+1} + \cdots + \overline{\lambda}_r e_r$$

be the spectral decomposition of $\bar{x}$ where $\overline{\lambda}_i \neq 0$ for all $i = l + 1, \ldots, r$. For $\varepsilon > 0$, let $h = e_1 + e_2 + \cdots + e_l$. Then it is easily seen that

$$\Pi_K(\bar{x} + \varepsilon h) - \Pi_K(\bar{x}) = \varepsilon h \quad \text{and} \quad \Pi_K(\bar{x} - \varepsilon h) - \Pi_K(\bar{x}) = 0.$$

It follows that

$$\lim_{\varepsilon \downarrow 0} \frac{\Pi_K(\bar{x} + \varepsilon h) - \Pi_K(\bar{x})}{\varepsilon} = h$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{\Pi_K(\bar{x} - \varepsilon h) - \Pi_K(\bar{x})}{\varepsilon} = 0.$$

Since the derivative $(\Pi_K)'(\bar{x})$ is linear, we must have $h = 0$, i.e., $e_1 + e_2 + \cdots + e_l = 0$. However this cannot happen as e_is are primitive (so nonzero) idempotents. This argument proves that $\bar{x}$ is necessarily invertible when Π_K is differentiable at $\bar{x}$. □

Our next lemma deals with the descriptions of $V^{(l)}$ and $P^{(l)}$.

Lemma 20. *Given a Jordan frame* $\{e_1, e_2, \ldots, e_r\}$ *and a subset* $\{e_1, e_2, \ldots, e_l\}$, *consider* $V^{(l)}$ *and the corresponding projection* $P^{(l)}$. *We have*

(a) $V^{(l)} = Re_1 + Re_2 + \cdots + Re_l + \sum_{j<k\leqslant l} V_{jk}$.
(b) *For any* $h = \sum_1^r h_i e_i + \sum_{j<k} h_{jk}$,

$$P^{(l)}h = \sum_1^l h_i e_i + \sum_{j<k\leqslant l} h_{jk}.$$

(c) $P^{(l)} \in \partial_B \Pi_K(0)$.

Proof. Recall that $V^{(l)} = \{x \in V : x \circ (e_1 + e_2 + \cdots + e_l) = x\}$. From $e_i \circ e_j = \delta_{ij} e_i$, it follows that $e_i \in V^{(l)}$ for all $i = 1, 2, \ldots, l$. From the definition of V_{jk} and its properties, it follows that, for $j < k \leqslant l$, $h_{jk} \circ (e_1 + e_2 + \cdots + e_l) = h_{jk} \circ$

$(e_j + e_k) = \frac{1}{2}h_{jk} + \frac{1}{2}h_{jk} = h_{jk}$. We conclude that $V^{(l)} \supset Re_1 + Re_2 + \cdots + Re_l + \sum_{j<k\leqslant l} V_{jk}$. To prove the reverse inclusion, let $h \in V^{(l)}$. Then $h \circ (e_1 + e_2 + \cdots + e_l) = h$. Writing $h = \sum_1^r h_i e_i + \sum_{j<k} h_{jk}$, we get

$$h = \sum_1^r h_i e_i \circ (e_1 + e_2 + \cdots + e_l) + \left(\sum_{j<k\leqslant l} + \sum_{j\leqslant l<k} + \sum_{l<j<k} \right) h_{jk} \circ (e_1 + e_2 + \cdots + e_l).$$

The first sum reduces to $\sum_{1\leqslant i\leqslant l} h_i e_i$. Any term in the second sum reduces to $h_{jk} \circ (e_j + e_k) = h_{jk}$. Any term in the third sum reduces to $h_{jk} \circ (e_1 + e_2 + \cdots + e_j + \cdots + e_l) = \frac{1}{2}h_{jk}$. Any term in the fourth sum is zero. Thus,

$$\sum_1^r h_i e_i + \sum_{j<k} h_{jk} = \sum_{1\leqslant i\leqslant l} h_i e_i + \sum_{j<k\leqslant l} h_{jk} + \frac{1}{2} \sum_{j\leqslant l<k} h_{jk}.$$

Using the orthogonality of V_{ij}'s, we get $h_i = 0$ for $i > l$ and $\|h_{jk}\|^2 = \frac{1}{2}\|h_{jk}\|^2$ for $j \leqslant l < k$. It follows that $h = \sum_1^l h_i e_i + \sum_{j<k\leqslant l} h_{jk}$ proving the reverse inclusion. Thus we have (a). Item (b) follows from (a) since the Peirce decomposition of V is an orthogonal decomposition. To prove (c), we first assume that V is simple. Consider the sequence $\{x^{(m)}\}$ defined by

$$x^{(m)} := \frac{1}{m^2}(e_1 + e_2 + \cdots + e_l) - \frac{1}{m}(e_{l+1} + \cdots + e_r).$$

Then $x^{(m)} \to 0$, and Π_K is differentiable at $x^{(m)}$ with the derivative given by, for any $h = \sum_1^r h_i e_i + \sum_{j<k} h_{jk}$,

$$(\Pi_K)'(x^{(m)})h = \sum_1^l h_i e_i + \sum_{j\leqslant l<k} \frac{1/m^2}{1/m^2 + 1/m} h_{jk} + \sum_{j<k\leqslant l} h_{jk}.$$

Since $P^{(l)}h = \sum_1^l h_i e_i + \sum_{j<k\leqslant l} h_{jk}$, it follows that $P^{(l)}$ is the limit of $(\Pi_K)'(x^{(m)})$. Now consider a general $V = V_1 \times V_2 \times \cdots \times V_n$ where each V_i is simple. If $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ is an idempotent in V, then so are $x^{(i)}$ in V_i. Moreover, if x is also primitive, then each $x^{(i)}$ is either zero or primitive. It can be easily verified that $V^{(l)}$ is a product of $V_i^{(l_i)}$ where $i = 1, 2, \ldots, n$, $0 \leqslant l_i \leqslant l$ and $V_i^{(0)} := \{0\}$, so the projection $P^{(l)} : V \to V^{(l)}$ is the product of projections $P_i^{(l_i)} : V_i \to V_i^{(l_i)}$. From our earlier analysis, $P_i^{(l_i)}$ belongs to $\partial_B \Pi_{K_i}(0)$ for each i. Since $\partial_B \Pi_K(0)$ is the product of $\partial_B \Pi_{K_i}(0)$, it follows that $P^{(l)} \in \partial_B \Pi_K(0)$. This proves (c). □

Proof of Theorem 17. The implication (i) $\Rightarrow$ (ii) follows from Proposition 2.3.11 of [5].

(ii) $\Rightarrow$ (iii): The imposed assumption on L implies that F is locally invertible at zero with a Lipschitzian inverse. In this setting, it is known that $\partial_B F(0)$ is coherently oriented [11, Theorem 3], that is, all transformations in $\partial_B F(0)$ have the same nonzero determinantal sign. By taking the sequence $\{x^{(k)}\}$ (that appears in the definition of $\partial_B \Pi_K(0)$) inside $-\text{int}\, K$, we see that the zero transformation belongs to $\partial_B \Pi_K(0)$, and hence the identity transformation belongs to $\partial_B F(0)$. Thus, every transformation in $\partial_B F(0)$ has positive determinant. The result follows from (8).

(iii) $\Rightarrow$ (iv): Consider any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and a subset $\{e_1, e_2, \ldots, e_l\}$ $(1 \leqslant l \leqslant r)$. Correspondingly, consider the principal subtransformation $L_{\{e_1,e_2,\ldots,e_l\}} : V^{(l)} \to V^{(l)}$. For simplicity, let us write X for $V^{(l)}$, Y for the orthogonal complement of X in V, M for $L \circ P^{(l)} + I - P^{(l)}$, A for $L_{\{e_1,e_2,\ldots,e_l\}}$ $(= P^{(l)} \circ L$ restricted to $X)$ and B for $(I - P^{(l)}) \circ L$ restricted to X. For $x \in X$ and $y \in Y$, we may write

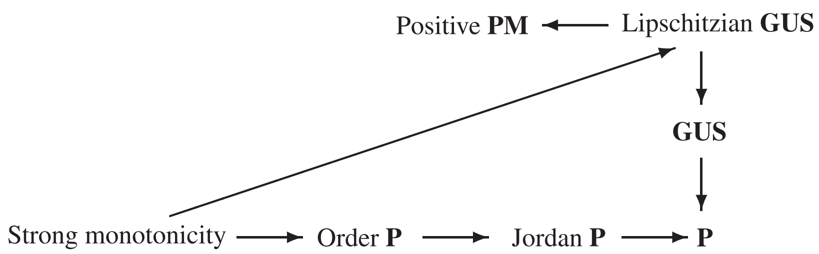
$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax \\ Bx + y \end{bmatrix}.$$

By considering M as a block matrix

$$\begin{bmatrix} A & 0 \\ B & I \end{bmatrix},$$

we see that $\det M = \det A$. From (iii), $\det M > 0$ and so $\det A > 0$. Thus every principal minor of L is positive. This proves (iii) $\Rightarrow$ (iv). $\square$

We summarize the results proved so far in the following diagram:



8. Some special cases

Theorem 21. *When L is self-adjoint,*

$$\textit{Strong monotonicity} = \textit{Order}\ \mathbf{P} = \textit{Jordan}\ \mathbf{P} = \mathbf{P} = \mathbf{GUS} = \textit{Lip}.\ \mathbf{GUS}.$$

Proof. Since the **P**-property is implied by all other properties, we assume that L has the **P**-property and self-adjoint. It follows from Theorem 11 that all eigenvalues (which are real) are positive. Hence L is strongly monotone. This being the strongest of all other properties, we get the desired equivalences. □

Theorem 22. *When L is monotone*,

$$\textit{Order}\ \mathbf{P} = \textit{Jordan}\ \mathbf{P} \quad \textit{and} \quad \mathbf{P} = \mathbf{GUS}.$$

Proof. Since Order **P**-property implies the Jordan **P**-property always, we prove the reverse implication. Assume that L has the Jordan **P**-property (in addition to the monotonicity property). Suppose Let $x \sqcap y \leqslant 0 \leqslant x \sqcup y$ where $y = L(x)$. Then as in the proof of the implication (3) $\Rightarrow$ (4) of Lemma 9, we get $\langle x, y\rangle + \langle p, q\rangle = 0$ where $p := -x + (x - y)^+ \geqslant 0$ and $q := y + (x - y)^+ \geqslant 0$. Since L is monotone, we have $\langle x, y\rangle = 0 = \langle p, q\rangle$. Now $x + y = x \sqcup y + x \sqcap y = q - p$ and $p, q \geqslant 0$, $\langle p, q\rangle = 0$ imply that $q = (x + y)^+$ and $p = (x + y)^-$. Thus, $|x + y| = p + q = (y - x) + 2(x - y)^+ = (y - x) + (x - y) + |x - y| = |x - y|$. Upon squaring, we get $x \circ y = 0$. From the Jordan **P**-property, we get $x = 0$. Thus we have the Order **P**-property. Now for the equality **P = GUS.** In view of Theorem 14, it is enough to show that monotonicity implies the cross commutative property. To this end, for any q, let x_1 and x_2 be two solutions of LCP(L, K, q). Letting $x := x_1 - x_2$ and $y_i = L(x_i) + q$ $(i = 1, 2)$, we see that $x \circ L(x) = -[x_1 \circ y_2 + x_2 \circ y_1]$. Since x_i and y_i are in K, from the monotonicity of L,

$$\begin{aligned} 0 &\leqslant \langle x, L(x)\rangle = \langle x \circ L(x), e\rangle \\ &= \langle -[x_1 \circ y_2 + x_2 \circ y_1], e\rangle \\ &= -[\langle x_1, y_2\rangle + \langle x_2, y_1\rangle] \leqslant 0. \end{aligned}$$

It follows that $\langle x_1, y_2\rangle = 0 = \langle x_2, y_1\rangle$. By Proposition 6, x_1 (x_2) operator commutes with y_2 (respectively, y_1). □

Theorem 23. *When K is polyhedral*,

$$\textit{Order}\ \mathbf{P} = \textit{Jordan}\ \mathbf{P} = \mathbf{P} = \mathbf{GUS} = \textit{Lipschitzian}\ \mathbf{GUS} = \textit{positive}\ \mathbf{PM}.$$

Proof. Since K is polyhedral, it follows easily from Theorem 5, that there is an invertible transformation $Q : R^n \to V$ with $n = \dim(V)$ such that $K = Q(R^n_+)$, $Q(r * s) = Q(r) \circ Q(s)$, where $r, s \in R^n$ and $r * s$ denotes the usual componentwise product of vectors. It is easily seen that $L : V \to V$ has the **P**-property (order **P**-property, Lipschitzian GUS property, positive PM property) if and only if $Q^{-1}LQ : R^n \to R^n$ satisfies condition (2) (respectively, (3), (6), (1)) in the Introduction. Since conditions (1)–(6) of the Introduction are equivalent, we get the desired statement of the theorem. □

Example 1.3. Let $A \in R^{n\times n}$ and consider the Lyapunov transformation L_A defined in Example 1.2. As noted in that example, L_A has the positive **PM**-property if and only if A is positive definite. It is also known (see Theorem 9 in [13]) that L_A has the **GUS**-property if and only if A is positive semidefinite and positive stable. It is easy to construct an example of a matrix A that is positive semidefinite and positive stable but not positive definite: Take

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

We conclude that the **GUS**-property need not imply the positive **PM**-property. In particular, for a general linear transformation,

the **GUS**-property does not imply the Lipschitzian **GUS**-property.

In terms of a normal map $F(x) = L(x^+) + x - x^+$, the above statement says that the homeomorphism property of F does not imply Lipschitzian homeomorphism property of F.

It has been observed (see [12, Remark 1]) that L_A has the Jordan **P**-property if A is positive stable. So when A is the 2×2 matrix given above, L_A has the Jordan **P**-property and monotone. By Theorem 22, L_A has the Order **P**-property. However, L_A is not strongly monotone as A is not positive definite. We conclude that, in general,

Order **P**-property does not imply the strong monotonicity property.

Example 2.3. Consider $\mathscr{L}^n$ with $n(>1)$ odd. Define the transformation L by

$$L\left(\begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}\right) = \begin{bmatrix} 2x_0 \\ -\bar{x} \end{bmatrix}.$$

(The transformation L is induced on R^n by the diagonal matrix diag$(2, -1, -1, \ldots, -1)$.) Then for $e_1 = \frac{1}{2}[1 \;\; u]^{\mathrm{T}}$ with $u \in R^{(n-1)}$ and $\|u\| = 1$, we have

$$\langle L(e_1), e_1\rangle = \tfrac{1}{4}.$$

Since any nonzero vector on the boundary of $\mathscr{L}^n_+$ is a multiple of e_1 for a suitable u, we see that L is positive definite on the boundary of $\mathscr{L}^n_+$. In addition, the determinant of L is positive. We conclude (see Example 2.2) that L has the positive **PM**-property. Since L is self-adjoint with negative eigenvalues, it cannot be strongly monotone and hence cannot have the **P**-property (by Theorem 21). Thus we conclude that even when L is self-adjoint,

the positive **PM**-property does not imply the **P**-property.

Concluding remarks

In this article, we introduced some generalizations of the **P**-matrix concept for a linear transformation defined on a Euclidean Jordan algebra. Some interconnections between these generalized concepts were studied. In a subsequent paper, we hope to study the analogs of the $\mathbf{P}_0$-property of a matrix, and some **P**-properties that are induced by the automorphism groups of a Euclidean Jordan algebra and the corresponding symmetric cone.

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