Multiple scattering by particles embedded in an absorbing medium. 1. Foldy–Lax equations, order-of-scattering expansion, and coherent field

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Abstract: This paper presents a systematic analysis of the problem of multiple scattering by a finite group of arbitrarily sized, shaped, and oriented particles embedded in an absorbing, homogeneous, isotropic, and unbounded medium. The volume integral equation is used to derive generalized Foldy–Lax equations and their order-of-scattering form. The far-field version of the Foldy–Lax equations is used to derive the transport equation for the so-called coherent field generated by a large group of sparsely, randomly, and uniformly distributed particles. The differences between the generalized equations and their counterparts describing multiple scattering by particles embedded in a non-absorbing medium are highlighted and discussed.

OCIS codes: (030.5620) Radiative transfer; (290.5825) Scattering theory; (290.5850) Scattering, particles; (290.5855) Scattering, polarization

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1. Introduction

Multiple scattering of electromagnetic waves by particles is an important discipline which has been the subject of numerous publications over the past few decades (see, e.g., [1–7] and references therein). The conventional theories of multiple scattering have explicitly relied on the assumption that the host medium surrounding the particles is non-absorbing. The important general case of an absorbing host medium has largely been ignored, a paper by Yang *et al*. [8] and two recent papers by Durant *et al*. [6, 9] being rare exceptions.

The objective of this series of papers is to perform a systematic analysis of the problem of multiple scattering by particles imbedded in an absorbing host medium by generalizing the results summarized in [3, 7]. The requisite study of the problem of single scattering has recently been published [10] (see Appendix below for errata). Like in [10], my goal here is to perform as general an analysis as possible without providing a detailed microphysical specification of the scattering particles. In particular, the particles are allowed to have arbitrary sizes, shapes, and orientations.

In this first part of the series, the focus is on such fundamental ingredients of the multiple-scattering theory as the vector Foldy–Lax equations, their order-of-scattering form, and the average (coherent) field. In order to save space and minimize redundancy, I assume that the reader has access to [3, 10] and use the same terminology and notation.

2. Vector Foldy–Lax equations

Consider electromagnetic scattering by a fixed group of N finite particles collectively occupying the interior region

$$V_{\mathrm{INT}} = \bigcup_{i=1}^{N} V_i, \tag{1}$$

where V_i is the volume occupied by the ith particle (Fig. 1). The host medium can be absorbing, but otherwise it is assumed to be infinite, homogeneous, linear, and isotropic. The particles are assumed to have the same constant permeability, but are allowed to have different and spatially varying permittivities. The general volume integral equation describing the total electric field everywhere in space [10] can be re-written in the following form:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{inc}}(\mathbf{r}) + \int_{\Re^3} \mathrm{d}\mathbf{r}' \, \overleftrightarrow{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') U(\mathbf{r}'), \qquad \mathbf{r} \in \Re^3, \tag{2}$$

where $\mathbf{E}^{\mathrm{inc}}(\mathbf{r})$ is the incident field, the integration is performed over the entire space,

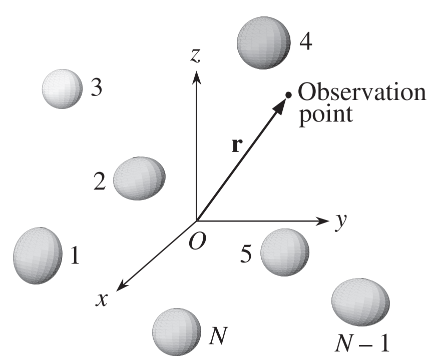

Fig. 1. Scattering by a fixed group of N finite particles.

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$$\overleftrightarrow{G}(\mathbf{r},\mathbf{r}') = \left(\overleftrightarrow{I} + \frac{1}{k_1^2}\nabla\otimes\nabla\right)\frac{\exp(\mathrm{i}k_1|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{3}$$

is the dyadic Green's function, $k_1 = k_1' + \mathrm{i}k_1''$ is the (complex) wave number of the host medium, $\overleftrightarrow{I}$ is the identity dyadic,

$$U(\mathbf{r}) = \sum_{i=1}^{N} U_i(\mathbf{r}), \qquad \mathbf{r}\in\Re^3 \tag{4}$$

is the potential function, and $U_i(\mathbf{r})$ is the ith-particle potential function. The latter is given by

$$U_i(\mathbf{r}) = \begin{cases} 0, & \mathbf{r}\notin V_i, \\ k_1^2[m_i^2(\mathbf{r})-1], & \mathbf{r}\in V_i, \end{cases} \tag{5}$$

where $m_i(\mathbf{r}) = k_{2i}(\mathbf{r})/k_1$ is the refractive index of particle i relative to that of the host medium. Importantly, all position vectors originate at the origin O of the laboratory coordinate system, Fig. 1.

By following step-by-step the derivation outlined in Section 4.1 of [3], it is straightforward to show that the solution of Eq. (2) can be expressed as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{inc}}(\mathbf{r}) + \sum_{i=1}^{N}\int_{V_i}\mathrm{d}\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\int_{V_i}\mathrm{d}\mathbf{r}''\,\overleftrightarrow{T}_i(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_i(\mathbf{r}''), \qquad \mathbf{r}\in\Re^3, \tag{6}$$

where the electric field $\mathbf{E}_i(\mathbf{r})$ exciting particle i is given by

$$\mathbf{E}_i(\mathbf{r}) = \mathbf{E}^{\mathrm{inc}}(\mathbf{r}) + \sum_{j(\neq i)=1}^{N}\mathbf{E}_{ij}^{\mathrm{exc}}(\mathbf{r}), \tag{7}$$

the $\mathbf{E}_{ij}^{\mathrm{exc}}(\mathbf{r})$ are partial exciting fields given by

$$\mathbf{E}_{ij}^{\mathrm{exc}}(\mathbf{r}) = \int_{V_j}\mathrm{d}\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\int_{V_j}\mathrm{d}\mathbf{r}''\,\overleftrightarrow{T}_j(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_j(\mathbf{r}''), \qquad \mathbf{r}\in V_i, \tag{8}$$

and $\overleftrightarrow{T}_i$ is the solution of the integral equation

$$\overleftrightarrow{T}_i(\mathbf{r},\mathbf{r}') = U_i(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}')\,\overleftrightarrow{I} + U_i(\mathbf{r})\int_{V_i}\mathrm{d}\mathbf{r}''\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}'')\cdot\overleftrightarrow{T}_i(\mathbf{r}'',\mathbf{r}'), \qquad \mathbf{r},\mathbf{r}'\in V_i \tag{9}$$

and thus the ith-particle dyadic transition operator with respect to the laboratory coordinate system.

Equations (6)–(8) do not differ mathematically from Eqs. (4.1.6)–(4.1.8) of [3] derived for the case of a non-absorbing host medium. They represent the generalized vector form of the Foldy–Lax equations (FLEs) and describe rigorously electromagnetic scattering by the fixed group of N particles embedded in an absorbing medium. A fundamental property of the FLEs is that $\overleftrightarrow{T}_i$ is the dyadic transition operator of particle i in the absence of all the other particles (cf. Eq. (9) above and Eq. (10) in [10]).

3. Multiple scattering

In general, the FLEs (6)–(9) are equivalent to Eqs. (9) and (10) of [10]. However, the fact that $\overleftrightarrow{T}_i$ for each i is an individual property of the ith particle computed as if this particle were alone allows one to introduce the mathematical concept of multiple scattering. Let us rewrite Eqs. (6)–(8) in a compact operator form:

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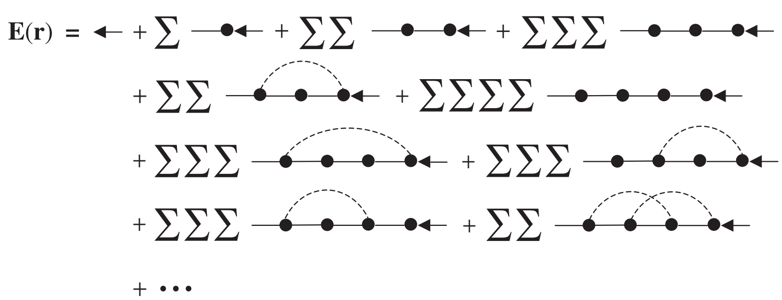

Fig. 2. Diagrammatic representation of Eq. (14).

$$E = E^{\text{inc}} + \sum_{i=1}^{N} \hat{G}\hat{T}_i E_i, \tag{10}$$

$$E_i = E^{\text{inc}} + \sum_{j(\neq i)=1}^{N} \hat{G}\hat{T}_j E_j, \tag{11}$$

where

$$\hat{G}\hat{T}_j E_j = \int_{V_j} \mathrm{d}\mathbf{r}' \overleftrightarrow{G}(\mathbf{r}, \mathbf{r}') \cdot \int_{V_j} \mathrm{d}\mathbf{r}'' \overleftrightarrow{T}_j(\mathbf{r}', \mathbf{r}'') \cdot \mathbf{E}_j(\mathbf{r}''). \tag{12}$$

Iterating Eq. (11) gives

$$E_i = E^{\text{inc}} + \sum_{j(\neq i)=1}^{N} \hat{G}\hat{T}_j E^{\text{inc}} + \sum_{\substack{j(\neq i)=1\\ l(\neq j)=1}}^{N} \hat{G}\hat{T}_j \hat{G}\hat{T}_l E^{\text{inc}} + \sum_{\substack{j(\neq i)=1\\ l(\neq j)=1\\ m(\neq l)=1}}^{N} \hat{G}\hat{T}_j \hat{G}\hat{T}_l \hat{G}\hat{T}_m E^{\text{inc}} + \cdots, \tag{13}$$

whereas the substitution of Eq. (13) in Eq. (10) results in the order-of-scattering expansion of the total electric field:

$$E = E^{\text{inc}} + \sum_{i=1}^{N} \hat{G}\hat{T}_i E^{\text{inc}} + \sum_{\substack{i=1\\ j(\neq i)=1}}^{N} \hat{G}\hat{T}_i \hat{G}\hat{T}_j E^{\text{inc}} + \sum_{\substack{i=1\\ j(\neq i)=1\\ l(\neq j)=1}}^{N} \hat{G}\hat{T}_i \hat{G}\hat{T}_j \hat{G}\hat{T}_l E^{\text{inc}} + \cdots. \tag{14}$$

The first term on the right-hand side of Eq. (14) represents the unscattered incident field, the second term is the sum of all "single-scattering" contributions, the third term is the sum of all "double-scattering" contributions, etc. The order-of-scattering interpretation of Eq. (14) is illustrated in Fig. 2. The arrows denote the incident field, the symbol represents the "multiplication" of a field by a $\hat{G}\hat{T}$ dyadic according to Eq. (12), and the dashed curve indicates that both "scattering centers" are represented by the same particle.

Equation (14) constitutes a very clear and fruitful way of re-writing the original FLEs. It is important to remember, however, that the concept of multiple scattering does not represent an actual time-sequential process in the framework of frequency-domain electromagnetics [7].

The mathematical structure of Eq. (14) is exactly the same as that in the case of a non-absorbing host medium [7], except now the wave number of the host medium is allowed to be complex-valued.

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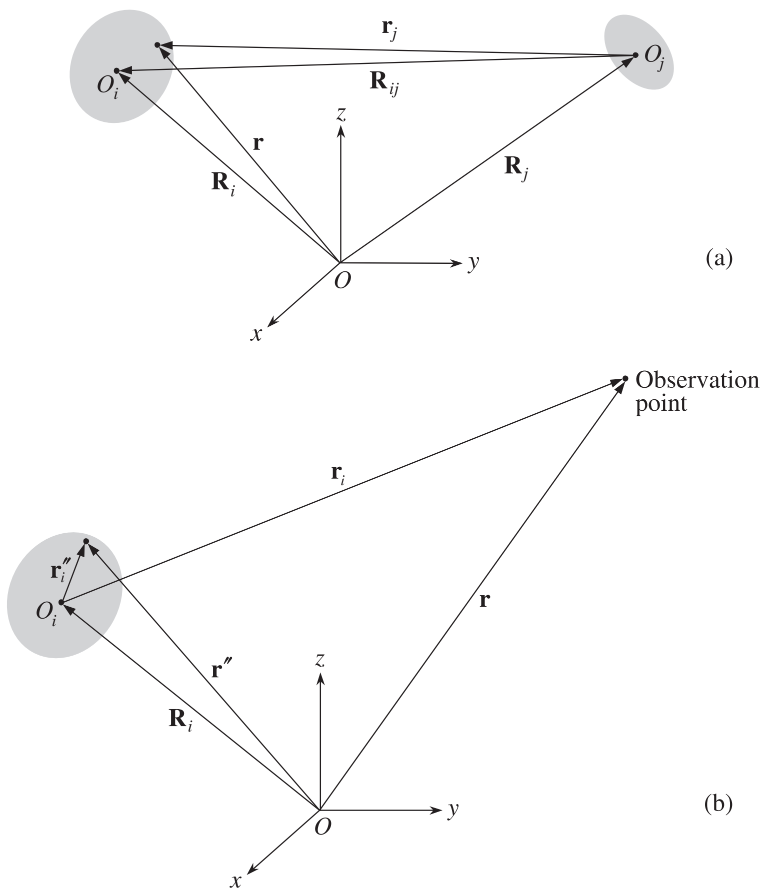

Fig. 3. Scattering by widely separated particles. The local origins O_i and O_j are chosen arbitrarily inside particles i and j, respectively.

4. Far-field Flody–Lax equations

Let us now make the following two simplifying assumptions:

- Each particle from the group is located in the far-field zones of all the other particles.
- The observation point is located in the far-field zone of any particle from the group.

Let us also chose for each particle an individual local origin positioned close to the particle's geometrical center, Fig. 3(a). Comparison with Eq. (9) of [10] shows that the right-hand side of Eq. (8) is the field scattered by particle j in response to the incident field represented by $\mathbf{E}_j(\mathbf{r})$. Since the resulting scattered field at any point is independent of the choice of coordinate system, it is convenient to evaluate the right-hand side of Eq. (8) in the far-field zone of particle j using the local coordinate system centered at O_j. This means that now the dyadic Green's function, the dyadic transition operator, and the incident field $\mathbf{E}_j$ are specified with respect to the jth local particle coordinate system. According to Section 3 of [10], the result of scattering is an outgoing spherical wavelet centered at O_j:

$$\mathbf{E}_{ij}^{\text{exc}}(\mathbf{r}) \approx G(r_j)\,\mathbf{E}_{1ij}(\hat{\mathbf{r}}_j) \tag{15}$$

$$\approx \exp(-\mathrm{i}k_1\hat{\mathbf{R}}_{ij}\cdot\mathbf{R}_i)\,\mathbf{E}_{ij}\exp(\mathrm{i}k_1\hat{\mathbf{R}}_{ij}\cdot\mathbf{r}), \qquad \mathbf{r}\in V_i, \tag{16}$$

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where

$$G(r) = \frac{\exp(\mathrm{i}k_1 r)}{r}, \tag{17}$$

$$\mathbf{E}_{ij} = G(R_{ij})\mathbf{E}_{1ij}(\hat{\mathbf{R}}_{ij}), \qquad \mathbf{E}_{ij} \cdot \hat{\mathbf{R}}_{ij} = 0, \tag{18}$$

$$\hat{\mathbf{r}}_j = \frac{\mathbf{r}_j}{r_j}, \qquad \hat{\mathbf{R}}_{ij} = \frac{\mathbf{R}_{ij}}{R_{ij}}, \tag{19}$$

$$r_j = |\mathbf{R}_{ij} + \mathbf{r} - \mathbf{R}_i| \approx R_{ij} + \hat{\mathbf{R}}_{ij} \cdot (\mathbf{r} - \mathbf{R}_i) + \frac{|\mathbf{r} - \mathbf{R}_i|^2}{2R_{ij}}, \tag{20}$$

and the vectors $\mathbf{r}$, $\mathbf{r}_j$, $\mathbf{R}_i$, $\mathbf{R}_j$, and $\mathbf{R}_{ij}$ are shown in Fig. 3(a). Note that we use a caret above a vector to denote a unit vector in the corresponding direction.

It is rather obvious that $\mathbf{E}_{ij}$ in Eq. (16) is the value of the exciting field caused by particle *j* at the origin of particle *i*. Since the radius of curvature of the exciting wavelet generated by particle *j* is much greater than the size of particle *i*, Eqs. (7) and (16) show that each particle is excited by the external incident field and the superposition of *locally* plane homogeneous waves with amplitudes $\exp(-\mathrm{i}k_1\hat{\mathbf{R}}_{ij} \cdot \mathbf{R}_i)\mathbf{E}_{ij}$ and propagation directions $\hat{\mathbf{R}}_{ij}$:

$$\mathbf{E}_i(\mathbf{r}) \approx \mathbf{E}_0^{\mathrm{inc}} \exp(\mathrm{i}k_1 \hat{\mathbf{n}}^{\mathrm{inc}} \cdot \mathbf{r}) + \sum_{j(\neq i)=1}^{N} \exp(-\mathrm{i}k_1\hat{\mathbf{R}}_{ij} \cdot \mathbf{R}_i)\mathbf{E}_{ij} \exp(\mathrm{i}k_1\hat{\mathbf{R}}_{ij} \cdot \mathbf{r}), \qquad \mathbf{r} \in V_i, \tag{21}$$

where we assume, as usual, that $\mathbf{E}^{\mathrm{inc}}(\mathbf{r})$ is a homogeneous plane electromagnetic wave propagating in the direction of the unit vector $\hat{\mathbf{n}}^{\mathrm{inc}}$:

$$\mathbf{E}^{\mathrm{inc}}(\mathbf{r}) = \mathbf{E}_0^{\mathrm{inc}} \exp(\mathrm{i}k_1 \hat{\mathbf{n}}^{\mathrm{inc}} \cdot \mathbf{r}), \qquad \mathbf{E}_0^{\mathrm{inc}} \cdot \hat{\mathbf{n}}^{\mathrm{inc}} = 0. \tag{22}$$

According to Eqs. (16) and (19) of [10], the outgoing spherical wave generated by the *j*th particle in response to a plane-wave excitation of the form $\mathbf{E}_0^{\mathrm{inc}} \exp(\mathrm{i}k_1 \hat{\mathbf{n}}^{\mathrm{inc}} \cdot \mathbf{r}_j)$ is given by $G(r_j)\overset{\leftrightarrow}{A}_j(\hat{\mathbf{r}}_j, \hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \mathbf{E}_0^{\mathrm{inc}}$, where $\mathbf{r}_j$ originates at O_j, $\mathbf{E}_0^{\mathrm{inc}}$ is the incident field at O_j, and $\overset{\leftrightarrow}{A}_j(\hat{\mathbf{r}}_j, \hat{\mathbf{n}}^{\mathrm{inc}})$ is the *j*th particle scattering dyadic centered at O_j. To make use of this fact, we rewrite Eq. (21) for particle *j* with respect to the *j*th-particle coordinate system centered at O_j, Fig. 3(a). Since $\mathbf{r} = \mathbf{r}_j + \mathbf{R}_j$, we obtain

$$\mathbf{E}_j(\mathbf{r}) \approx \mathbf{E}^{\mathrm{inc}}(\mathbf{R}_j) \exp(\mathrm{i}k_1 \hat{\mathbf{n}}^{\mathrm{inc}} \cdot \mathbf{r}_j) + \sum_{l(\neq j)=1}^{N} \mathbf{E}_{jl} \exp(\mathrm{i}k_1 \hat{\mathbf{R}}_{jl} \cdot \mathbf{r}_j), \qquad \mathbf{r} \in V_j. \tag{23}$$

The electric field at O_i caused by particle *j* in response to this excitation is given by

$$G(R_{ij})\overset{\leftrightarrow}{A}_j(\hat{\mathbf{R}}_{ij}, \hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{R}_j) + G(R_{ij}) \sum_{l(\neq j)=1}^{N} \overset{\leftrightarrow}{A}_j(\hat{\mathbf{R}}_{ij}, \hat{\mathbf{R}}_{jl}) \cdot \mathbf{E}_{jl}. \tag{24}$$

By equating Eq. (24) with the right-hand side of Eq. (16) corresponding to $\mathbf{r} = \mathbf{R}_i$, we finally obtain a system of linear algebraic equations for the partial exciting fields $\mathbf{E}_{ij}$:

$$\mathbf{E}_{ij} = G(R_{ij})\overset{\leftrightarrow}{A}_j(\hat{\mathbf{R}}_{ij}, \hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{R}_j) + G(R_{ij}) \sum_{l(\neq j)=1}^{N} \overset{\leftrightarrow}{A}_j(\hat{\mathbf{R}}_{ij}, \hat{\mathbf{R}}_{jl}) \cdot \mathbf{E}_{jl}, \qquad i, j = 1, \ldots, N, \quad j \neq i. \tag{25}$$

Obviously, this system is much simpler than the original integral FLEs.

The solution of the system (25) yields the electric field exciting each particle as well as the total field. Indeed, we have from Eq. (21) for a point $\mathbf{r}'' \in V_i$:

$$\mathbf{E}_i(\mathbf{r}'') \approx \mathbf{E}^{\mathrm{inc}}(\mathbf{R}_i)\exp(\mathrm{i}k_1\hat{\mathbf{n}}^{\mathrm{inc}}\cdot\mathbf{r}''_i) + \sum_{j(\neq i)=1}^{N}\mathbf{E}_{ij}\exp(\mathrm{i}k_1\hat{\mathbf{R}}_{ij}\cdot\mathbf{r}''_i) \tag{26}$$

[see Fig. 3(b)], which is a vector superposition of locally plane homogeneous waves. Substituting this formula in Eq. (6) and recalling the expression for the far-field electromagnetic response of a particle to a plane-wave excitation yields the total electric field:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{inc}}(\mathbf{r}) + \sum_{i=1}^{N} G(r_i)\overleftrightarrow{A}_i(\hat{\mathbf{r}}_i, \hat{\mathbf{n}}^{\mathrm{inc}})\cdot\mathbf{E}^{\mathrm{inc}}(\mathbf{R}_i) + \sum_{i=1}^{N} G(r_i)\sum_{j(\neq i)=1}^{N}\overleftrightarrow{A}_i(\hat{\mathbf{r}}_i, \hat{\mathbf{R}}_{ij})\cdot\mathbf{E}_{ij}, \tag{27}$$

where the observation point **r**, Fig. 3(b), is located in the far-field zone of any particle forming the group.

The expression for the order-of-scattering expansion of the total field also becomes much simpler under the assumption of far-field scattering. Indeed, iterating Eq. (25) and substituting the result in Eq. (27) yields

$$\mathbf{E} = \mathbf{E}^{\mathrm{inc}} + \sum_{i=1}^{N}\overleftrightarrow{B}_{ri0}\cdot\mathbf{E}_i^{\mathrm{inc}} + \sum_{i=1}^{N}\sum_{j(\neq i)=1}^{N}\overleftrightarrow{B}_{rij}\cdot\overleftrightarrow{B}_{ij0}\cdot\mathbf{E}_j^{\mathrm{inc}} + \sum_{i=1}^{N}\sum_{j(\neq i)=1}^{N}\sum_{l(\neq j)=1}^{N}\overleftrightarrow{B}_{rij}\cdot\overleftrightarrow{B}_{ijl}\cdot\overleftrightarrow{B}_{jl0}\cdot\mathbf{E}_l^{\mathrm{inc}} + \cdots, \tag{28}$$

where

$$\mathbf{E} = \mathbf{E}(\mathbf{r}), \qquad \mathbf{E}^{\mathrm{inc}} = \mathbf{E}^{\mathrm{inc}}(\mathbf{r}), \qquad \mathbf{E}_i^{\mathrm{inc}} = \mathbf{E}^{\mathrm{inc}}(\mathbf{R}_i), \tag{29}$$

$$\overleftrightarrow{B}_{ri0} = G(r_i)\overleftrightarrow{A}_i(\hat{\mathbf{r}}_i, \hat{\mathbf{n}}^{\mathrm{inc}}), \tag{30}$$

$$\overleftrightarrow{B}_{rij} = G(r_i)\overleftrightarrow{A}_i(\hat{\mathbf{r}}_i, \hat{\mathbf{R}}_{ij}), \tag{31}$$

$$\overleftrightarrow{B}_{ij0} = G(R_{ij})\overleftrightarrow{A}_j(\hat{\mathbf{R}}_{ij}, \hat{\mathbf{n}}^{\mathrm{inc}}), \tag{32}$$

$$\overleftrightarrow{B}_{ijl} = G(R_{ij})\overleftrightarrow{A}_j(\hat{\mathbf{R}}_{ij}, \hat{\mathbf{R}}_{jl}). \tag{33}$$

The diagrammatic formula shown in Fig. 2 can also represent equation (28) provided that the symbol —● is now interpreted as the multiplication of a field by a $\overleftrightarrow{B}$ dyadic.

Neither the derivation of Eqs. (25), (27), and (28) nor their formal mathematical structure differ from those in the case of a non-absorbing host medium [3].

5. The Twersky approximation

Let us consider electromagnetic scattering by a large group of N particles sparsely distributed throughout a finite macroscopic volume V, Fig. 4. Assuming that N is very large, we can keep in the far-field order-of-scattering expansion (28) only the terms corresponding to scattering paths going through a particle only once (so-called self-avoiding paths) [11]:

$$\mathbf{E} \approx \mathbf{E}^{\mathrm{inc}} + \sum_{i=1}^{N}\overleftrightarrow{B}_{ri0}\cdot\mathbf{E}_i^{\mathrm{inc}} + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\overleftrightarrow{B}_{rij}\cdot\overleftrightarrow{B}_{ij0}\cdot\mathbf{E}_j^{\mathrm{inc}} + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{\substack{l=1\\ l\neq i\\ l\neq j}}^{N}\overleftrightarrow{B}_{rij}\cdot\overleftrightarrow{B}_{ijl}\cdot\overleftrightarrow{B}_{jl0}\cdot\mathbf{E}_l^{\mathrm{inc}} + \cdots. \tag{34}$$

This formula represents the far-field Twersky approximation and is depicted diagrammatically in Fig. 5. Comparison with Fig. 2 illustrates the types of diagrams neglected in the Twersky expansion.

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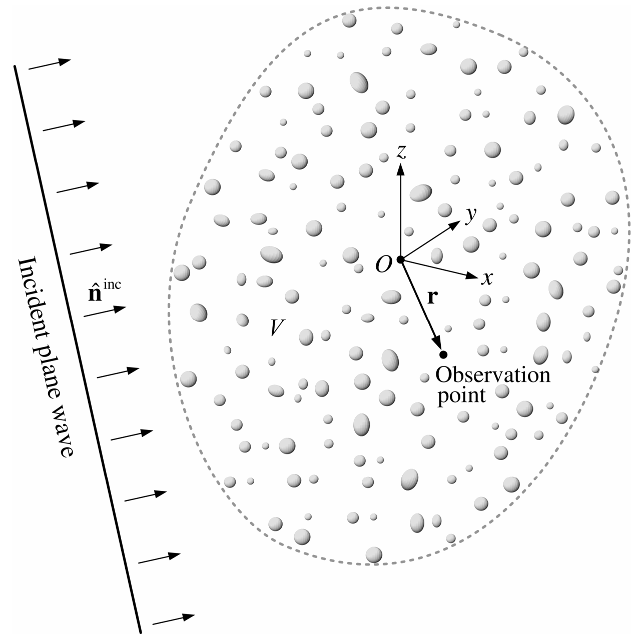

Fig. 4. Electromagnetic scattering by a large group of particles sparsely distributed throughout a macroscopic volume V.

6. Coherent field

Let us now assume that the N particles are randomly moving and decompose the field $\mathbf{E}(\mathbf{r})$ at an internal point $\mathbf{r} \in V$ into the average (or coherent), $\mathbf{E}_{\mathrm{c}}(\mathbf{r})$, and fluctuating, $\mathbf{E}_{\mathrm{f}}(\mathbf{r})$, parts:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\mathrm{c}}(\mathbf{r}) + \mathbf{E}_{\mathrm{f}}(\mathbf{r}). \tag{35}$$

Assuming also the full ergodicity of the particle ensemble and replacing time averaging by averaging over particle positions (subscript $\mathbf{R}$) and states (subscript ξ), we have

$$\mathbf{E}_{\mathrm{c}}(\mathbf{r}) = \langle \mathbf{E}(\mathbf{r}) \rangle_t = \langle \mathbf{E}(\mathbf{r}) \rangle_{\mathbf{R},\xi}, \tag{36}$$

$$\langle \mathbf{E}_{\mathrm{f}}(\mathbf{r}) \rangle_t = \langle \mathbf{E}_{\mathrm{f}}(\mathbf{r}) \rangle_{\mathbf{R},\xi} = \mathbf{0}, \tag{37}$$

where $\mathbf{0}$ is a zero vector. Furthermore, if all particles have the same statistical characteristics and the state and coordinates of each particle are independent of each other then we have from Eq. (34):

$$\mathbf{E}_{\mathrm{c}} = \mathbf{E}^{\mathrm{inc}} + \sum_{i=1}^{N} \langle \vec{\vec{B}}_{ri0} \cdot \mathbf{E}_i^{\mathrm{inc}} \rangle_{\mathbf{R},\xi} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \langle \vec{\vec{B}}_{rij} \cdot \vec{\vec{B}}_{ij0} \cdot \mathbf{E}_j^{\mathrm{inc}} \rangle_{\mathbf{R},\xi}$$

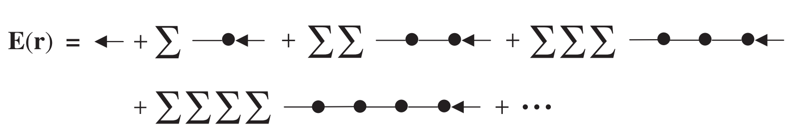

Fig. 5. Diagrammatic representation of the Twersky expansion.

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$$+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N}\langle\vec{\vec{B}}_{rij}\cdot\vec{\vec{B}}_{ijl}\cdot\vec{\vec{B}}_{jl0}\cdot\mathbf{E}_l^{\mathrm{inc}}\rangle_{\mathbf{R},\xi}+\cdots$$

$$=\mathbf{E}^{\mathrm{inc}}+\sum_{i=1}^{N}\int \mathrm{d}\mathbf{R}_i\,\mathrm{d}\xi_i\,p_{\mathbf{R}}(\mathbf{R}_i)p_{\xi}(\xi_i)\,\vec{\vec{B}}_{ri0}\cdot\mathbf{E}_i^{\mathrm{inc}}$$

$$+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\int \mathrm{d}\mathbf{R}_i\,\mathrm{d}\xi_i\,\mathrm{d}\mathbf{R}_j\,\mathrm{d}\xi_j\,p_{\mathbf{R}}(\mathbf{R}_i)p_{\xi}(\xi_i)p_{\mathbf{R}}(\mathbf{R}_j)p_{\xi}(\xi_j)\,\vec{\vec{B}}_{rij}\cdot\vec{\vec{B}}_{ij0}\cdot\mathbf{E}_j^{\mathrm{inc}}$$

$$+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N}\int \mathrm{d}\mathbf{R}_i\,\mathrm{d}\xi_i\,\mathrm{d}\mathbf{R}_j\,\mathrm{d}\xi_j\,\mathrm{d}\mathbf{R}_l\,\mathrm{d}\xi_l\,p_{\mathbf{R}}(\mathbf{R}_i)p_{\xi}(\xi_i)p_{\mathbf{R}}(\mathbf{R}_j)$$

$$\times p_{\xi}(\xi_j)p_{\mathbf{R}}(\mathbf{R}_l)p_{\xi}(\xi_l)\,\vec{\vec{B}}_{rij}\cdot\vec{\vec{B}}_{ijl}\cdot\vec{\vec{B}}_{jl0}\cdot\mathbf{E}_l^{\mathrm{inc}}$$

$$+\cdots, \tag{38}$$

where $p_{\mathbf{R}}(\mathbf{R})$ and $p_{\xi}(\xi)$ are the corresponding probability density functions, and the spatial integrations are performed over the entire volume V. Substituting Eqs. (30)–(33) yields

$$\mathbf{E}_{\mathrm{c}}=\mathbf{E}^{\mathrm{inc}}+\sum_{i=1}^{N}\int_V \mathrm{d}\mathbf{R}_i\,p_{\mathbf{R}}(\mathbf{R}_i)G(r_i)\langle\vec{\vec{A}}(\hat{\mathbf{r}}_i,\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_{\xi}\cdot\mathbf{E}_i^{\mathrm{inc}}$$

$$+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\int_V \mathrm{d}\mathbf{R}_i\,\mathrm{d}\mathbf{R}_j\,p_{\mathbf{R}}(\mathbf{R}_i)p_{\mathbf{R}}(\mathbf{R}_j)G(r_i)G(R_{ij})\langle\vec{\vec{A}}(\hat{\mathbf{r}}_i,\hat{\mathbf{R}}_{ij})\rangle_{\xi}\cdot\langle\vec{\vec{A}}(\hat{\mathbf{R}}_{ij},\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_{\xi}\cdot\mathbf{E}_j^{\mathrm{inc}}$$

$$+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N}\int_V \mathrm{d}\mathbf{R}_i\,\mathrm{d}\mathbf{R}_j\,\mathrm{d}\mathbf{R}_l\,p_{\mathbf{R}}(\mathbf{R}_i)p_{\mathbf{R}}(\mathbf{R}_j)p_{\mathbf{R}}(\mathbf{R}_l)G(r_i)G(R_{ij})G(R_{jl})\langle\vec{\vec{A}}(\hat{\mathbf{r}}_i,\hat{\mathbf{R}}_{ij})\rangle_{\xi}$$

$$\cdot\langle\vec{\vec{A}}(\hat{\mathbf{R}}_{ij},\hat{\mathbf{R}}_{jl})\rangle_{\xi}\cdot\langle\vec{\vec{A}}(\hat{\mathbf{R}}_{jl},\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_{\xi}\cdot\mathbf{E}_l^{\mathrm{inc}}$$

$$+\cdots, \tag{39}$$

where $\langle\vec{\vec{A}}(\hat{\mathbf{m}},\hat{\mathbf{n}})\rangle_{\xi}$ is the average of the single-particle scattering dyadic over the particle states. Taking into account that $p_{\mathbf{R}}(\mathbf{R})=n_0(\mathbf{R})/N$, where $n_0(\mathbf{R})$ is the number of particles per unit volume, we finally derive in the limit $N\to\infty$:

$$\mathbf{E}_{\mathrm{c}}\underset{N\to\infty}{=}\mathbf{E}^{\mathrm{inc}}+\int_V \mathrm{d}\mathbf{R}_i\,n_0(\mathbf{R}_i)G(r_i)\langle\vec{\vec{A}}(\hat{\mathbf{r}}_i,\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_{\xi}\cdot\mathbf{E}_i^{\mathrm{inc}}$$

$$+\int_V \mathrm{d}\mathbf{R}_i\,\mathrm{d}\mathbf{R}_j\,n_0(\mathbf{R}_i)n_0(\mathbf{R}_j)G(r_i)G(R_{ij})\langle\vec{\vec{A}}(\hat{\mathbf{r}}_i,\hat{\mathbf{R}}_{ij})\rangle_{\xi}\cdot\langle\vec{\vec{A}}(\hat{\mathbf{R}}_{ij},\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_{\xi}\cdot\mathbf{E}_j^{\mathrm{inc}}$$

$$+\int_V \mathrm{d}\mathbf{R}_i\,\mathrm{d}\mathbf{R}_j\,\mathrm{d}\mathbf{R}_l\,n_0(\mathbf{R}_i)n_0(\mathbf{R}_j)n_0(\mathbf{R}_l)G(r_i)G(R_{ij})G(R_{jl})$$

$$\times\langle\vec{\vec{A}}(\hat{\mathbf{r}}_i,\hat{\mathbf{R}}_{ij})\rangle_{\xi}\cdot\langle\vec{\vec{A}}(\hat{\mathbf{R}}_{ij},\hat{\mathbf{R}}_{jl})\rangle_{\xi}\cdot\langle\vec{\vec{A}}(\hat{\mathbf{R}}_{jl},\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_{\xi}\cdot\mathbf{E}_l^{\mathrm{inc}}$$

$$+\cdots. \tag{40}$$

Let us now assume that the distribution of the particles throughout the volume V is

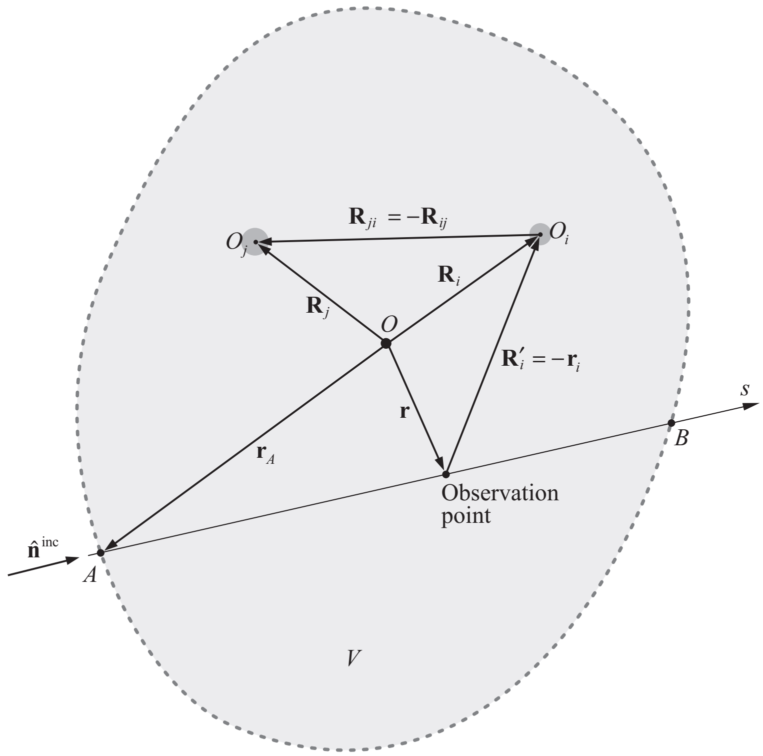

Fig. 6. Computation of the coherent field.

statistically uniform and introduce an s-axis parallel to the incidence direction and going through the observation point. This axis enters the volume V at a point A such that $s(A) = 0$ and exits it at a point B (Fig. 6). Let us evaluate the first integral on the right-hand side of Eq. (40):

$$\begin{aligned}\mathbf{I}_1 &= n_0 \int_V \mathrm{d}\mathbf{R}_i \, G(r_i) \langle \ddot{A}(\hat{\mathbf{r}}_i, \hat{\mathbf{n}}^{\text{inc}}) \rangle_\xi \cdot \mathbf{E}_i^{\text{inc}} \\ &= n_0 \int_V \mathrm{d}\mathbf{R}'_i \exp(\mathrm{i}k_1 \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{R}'_i) \frac{\exp(\mathrm{i}k_1 R'_i)}{R'_i} \langle \ddot{A}(-\hat{\mathbf{R}}'_i, \hat{\mathbf{n}}^{\text{inc}}) \rangle_\xi \cdot \mathbf{E}^{\text{inc}}(\mathbf{r}), \end{aligned} \tag{41}$$

where $n_0 = N/V$. The observation point is assumed to be in the far-field zone of any particle, which allows the use of the Saxon asymptotic expansion of a plane wave in spherical waves:

$$\exp(\mathrm{i}k_1 \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{R}'_i) \underset{k'_1 R'_i \to \infty}{=} \exp(-k''_1 \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{R}'_i) \frac{\mathrm{i}2\pi}{k'_1 R'_i} [\delta(\hat{\mathbf{n}}^{\text{inc}} + \hat{\mathbf{R}}'_i) \exp(-\mathrm{i}k'_1 R'_i) - \delta(\hat{\mathbf{n}}^{\text{inc}} - \hat{\mathbf{R}}'_i) \exp(\mathrm{i}k'_1 R'_i)]. \tag{42}$$

It is convenient to evaluate the integral (41) using a spherical polar coordinate system with origin at the observation point and with the z-axis directed along the s-axis. This gives

$$\mathbf{I}_1 = \frac{\mathrm{i}2\pi n_0}{k'_1} \int_{4\pi} \mathrm{d}\hat{\mathbf{R}}'_i \int \mathrm{d}R'_i \langle \ddot{A}(-\hat{\mathbf{R}}'_i, \hat{\mathbf{n}}^{\text{inc}}) \rangle_\xi \cdot \mathbf{E}^{\text{inc}}(\mathbf{r}) [\delta(\hat{\mathbf{n}}^{\text{inc}} + \hat{\mathbf{R}}'_i) - \delta(\hat{\mathbf{n}}^{\text{inc}} - \hat{\mathbf{R}}'_i) \exp(2\mathrm{i}k_1 R'_i)]$$

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$$= \frac{\mathrm{i}2\pi n_0}{k_1'} s(\mathbf{r})\langle \ddot{A}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{r}) - \frac{\pi n_0}{k_1 k_1'} \exp\{\mathrm{i}2k_1[s(B)-s(\mathbf{r})]\}\langle \ddot{A}(-\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{r}). \tag{43}$$

Notice now that the boundaries of the scattering volume V are not perfectly fixed and can be expected to fluctuate during the time interval necessary to compute the coherent field according to Eq. (36). Averaging over these fluctuations does not affect the first term on the right-hand side of Eq. (43), but effectively extinguishes the second term proportional to the rapidly oscillating exponential $\exp\{\mathrm{i}2k_1[s(B)-s(\mathbf{r})]\}$. Taking the average of the coherent field over a small volume element centered at the observation point $\mathbf{r}$ would have the same effect. Hence,

$$\mathbf{I}_1 = \frac{\mathrm{i}2\pi n_0}{k_1'} s(\mathbf{r})\langle \ddot{A}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{r}). \tag{44}$$

Analogously, since $\mathbf{R}_j = \mathbf{r} + \mathbf{R}_i' + \mathbf{R}_{ji}$, we have for the second integral on the right-hand side of Eq. (40):

$$\mathbf{I}_2 = n_0^2 \int \mathrm{d}R_i' R_i'^2 G(R_i') \int_{4\pi} \mathrm{d}\hat{\mathbf{R}}_i' \int \mathrm{d}R_{ji} R_{ji}^2 G(R_{ji}) \int_{4\pi} \mathrm{d}\hat{\mathbf{R}}_{ji} \times \langle \ddot{A}(-\hat{\mathbf{R}}_i', -\hat{\mathbf{R}}_{ji})\rangle_\xi \cdot \langle \ddot{A}(-\hat{\mathbf{R}}_{ji}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \cdot \mathbf{E}_j^{\mathrm{inc}}, \tag{45}$$

where

$$\mathbf{E}_j^{\mathrm{inc}} = \left(\frac{\mathrm{i}2\pi}{k_1'}\right)^2 \frac{1}{R_i'} \exp(-k_1''\hat{\mathbf{n}}^{\mathrm{inc}} \cdot \mathbf{R}_i')[\delta(\hat{\mathbf{n}}^{\mathrm{inc}} + \hat{\mathbf{R}}_i')\exp(-\mathrm{i}k_1' R_i') - \delta(\hat{\mathbf{n}}^{\mathrm{inc}} - \hat{\mathbf{R}}_i')\exp(\mathrm{i}k_1' R_i')] \times \frac{1}{R_{ji}} \exp(-k_1''\hat{\mathbf{n}}^{\mathrm{inc}} \cdot \mathbf{R}_{ji})[\delta(\hat{\mathbf{n}}^{\mathrm{inc}} + \hat{\mathbf{R}}_{ji})\exp(-\mathrm{i}k_1' R_{ji}) - \delta(\hat{\mathbf{n}}^{\mathrm{inc}} - \hat{\mathbf{R}}_{ji})\exp(\mathrm{i}k_1' R_{ji})]\mathbf{E}^{\mathrm{inc}}(\mathbf{r}). \tag{46}$$

This means that only particles with origins on the s-axis contribute to $\mathbf{I}_2$. Consequently,

$$\mathbf{I}_2 = \frac{1}{2}\left[\frac{\mathrm{i}2\pi n_0}{k_1'} s(\mathbf{r})\right]^2 \langle \ddot{A}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \cdot \langle \ddot{A}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{r}). \tag{47}$$

The computation of the remaining integrals in Eq. (40) is quite similar. We thus finally have:

$$\mathbf{E}_{\mathrm{c}}(\mathbf{r}) = \exp\left[\frac{\mathrm{i}2\pi n_0}{k_1'} s(\hat{\mathbf{n}}^{\mathrm{inc}})\langle \ddot{A}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi\right] \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{r}). \tag{48}$$

Since $\mathbf{r} = \mathbf{r}_A + s(\mathbf{r})\hat{\mathbf{n}}^{\mathrm{inc}}$ (Fig. 6), Eq. (48) yeilds

$$\mathbf{E}_{\mathrm{c}}(\mathbf{r}) = \exp[\mathrm{i}\ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}})s(\mathbf{r})] \cdot \mathbf{E}^{\mathrm{inc}}(\mathbf{r}_A) \tag{49}$$

or

$$\mathbf{E}_{\mathrm{c}}(s) = \ddot{\eta}(\hat{\mathbf{n}}^{\mathrm{inc}}, s) \cdot \mathbf{E}_{\mathrm{c}}(s=0), \tag{50}$$

where

$$\ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}}) = k_1 \ddot{I} + \frac{2\pi n_0}{k_1'} \langle \ddot{A}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \tag{51}$$

is the dyadic propagation constant,

$$\ddot{\eta}(\hat{\mathbf{n}}^{\mathrm{inc}}, s) = \exp[\mathrm{i}\ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}})s] \tag{52}$$

is the coherent transmission dyadic, and $\mathbf{E}_{\mathrm{c}}(s=0) = \mathbf{E}^{\mathrm{inc}}(\mathbf{r}_A)$ is the boundary value of the

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coherent field. Another form of Eq. (48) is

$$\frac{\mathrm{d}\mathbf{E}_{\mathrm{c}}(\mathbf{r})}{\mathrm{d}s} = \mathrm{i}\ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \mathbf{E}_{\mathrm{c}}(\mathbf{r}). \tag{53}$$

It is quite obvious that the coherent field is transverse: $\mathbf{E}_{\mathrm{c}}(\mathbf{r}) \cdot \hat{\mathbf{n}}^{\mathrm{inc}} = 0$. Therefore, the electric vector of the coherent field can be written as the vector sum of the corresponding θ- and φ-components in the local coordinate system centered at the observation point:

$$\mathbf{E}_{\mathrm{c}}(\mathbf{r}) = E_{\mathrm{c}\theta}(\mathbf{r})\hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}^{\mathrm{inc}}) + E_{\mathrm{c}\varphi}(\mathbf{r})\hat{\boldsymbol{\varphi}}(\hat{\mathbf{n}}^{\mathrm{inc}}), \qquad \hat{\mathbf{n}}^{\mathrm{inc}} = \hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}^{\mathrm{inc}}) \times \hat{\boldsymbol{\varphi}}(\hat{\mathbf{n}}^{\mathrm{inc}}). \tag{54}$$

Introducing the two-component electric column vector of the coherent field according to

$$\mathsf{E}_{\mathrm{c}}(\mathbf{r}) = \begin{bmatrix} E_{\mathrm{c}\theta}(\mathbf{r}) \\ E_{\mathrm{c}\varphi}(\mathbf{r}) \end{bmatrix}, \tag{55}$$

we have

$$\frac{\mathrm{d}\mathsf{E}_{\mathrm{c}}(\mathbf{r})}{\mathrm{d}s} = \mathrm{i}\mathsf{k}(\hat{\mathbf{n}}^{\mathrm{inc}})\mathsf{E}_{\mathrm{c}}(\mathbf{r}), \tag{56}$$

where $\mathsf{k}(\hat{\mathbf{n}}^{\mathrm{inc}})$ is the 2×2 matrix propagation constant with elements

$$k_{11}(\hat{\mathbf{n}}^{\mathrm{inc}}) = \hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}^{\mathrm{inc}}), \tag{57}$$

$$k_{12}(\hat{\mathbf{n}}^{\mathrm{inc}}) = \hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \hat{\boldsymbol{\varphi}}(\hat{\mathbf{n}}^{\mathrm{inc}}), \tag{58}$$

$$k_{21}(\hat{\mathbf{n}}^{\mathrm{inc}}) = \hat{\boldsymbol{\varphi}}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}^{\mathrm{inc}}), \tag{59}$$

$$k_{22}(\hat{\mathbf{n}}^{\mathrm{inc}}) = \hat{\boldsymbol{\varphi}}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \ddot{\kappa}(\hat{\mathbf{n}}^{\mathrm{inc}}) \cdot \hat{\boldsymbol{\varphi}}(\hat{\mathbf{n}}^{\mathrm{inc}}). \tag{60}$$

This implies that

$$\mathsf{k}(\hat{\mathbf{n}}^{\mathrm{inc}}) = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2\pi n_0}{k_1'} \langle \mathsf{S}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}}) \rangle_{\xi}, \tag{61}$$

where $\langle \mathsf{S}(\hat{\mathbf{n}}^{\mathrm{inc}}, \hat{\mathbf{n}}^{\mathrm{inc}}) \rangle_{\xi}$ is the forward-scattering amplitude matrix averaged over the particle states.

Equation (56) can be re-written as

$$\mathsf{E}_{\mathrm{c}}(s) = \mathsf{h}(\hat{\mathbf{n}}^{\mathrm{inc}}, s)\mathsf{E}_{\mathrm{c}}(s=0), \tag{62}$$

where

$$\mathsf{h}(\hat{\mathbf{n}}^{\mathrm{inc}}, s) = \exp[\mathrm{i}s\mathsf{k}(\hat{\mathbf{n}}^{\mathrm{inc}})] \tag{63}$$

is the 2×2 coherent transmission amplitude matrix. The reciprocity relations (79) and (80) of [10] imply the following reciprocity relations for the coherent transmission dyadic and the coherent transmission amplitude matrix:

$$\ddot{\eta}(-\hat{\mathbf{n}}^{\mathrm{inc}}, s) = [\ddot{\eta}(\hat{\mathbf{n}}^{\mathrm{inc}}, s)]^{\mathrm{T}}, \tag{64}$$

$$\mathsf{h}(-\hat{\mathbf{n}}^{\mathrm{inc}}, s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} [\mathsf{h}(\hat{\mathbf{n}}^{\mathrm{inc}}, s)]^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{65}$$

where T denotes the transpose of a dyadic or a matrix.

7. Transfer equation for the coherent field

We will now switch to potentially observable quantities having the dimension of monochromatic energy flux. The coherency column vector of the coherent field is defined as

$$\mathbf{J}_{\mathrm{c}} = \mathrm{Re}\left(\frac{k_1}{2\omega\mu_1}\right)\begin{bmatrix} E_{\mathrm{c}\theta} E_{\mathrm{c}\theta}^* \\ E_{\mathrm{c}\theta} E_{\mathrm{c}\varphi}^* \\ E_{\mathrm{c}\varphi} E_{\mathrm{c}\theta}^* \\ E_{\mathrm{c}\varphi} E_{\mathrm{c}\varphi}^* \end{bmatrix}, \tag{66}$$

where ω is the angular frequency and μ_1 is the permeability of the host medium. As follows from Eqs. (56) and (61), $\mathbf{J}_{\mathrm{c}}$ satisfies the transfer equation

$$\frac{\mathrm{d}\mathbf{J}_{\mathrm{c}}(\mathbf{r})}{\mathrm{d}s} = -2k_1''\mathbf{J}_{\mathrm{c}}(\mathbf{r}) - n_0\langle \mathbf{K}^J(\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \mathbf{J}_{\mathrm{c}}(\mathbf{r}), \tag{67}$$

where $\mathbf{K}^J$ is the coherency extinction matrix given by Eq. (68) of [10]. In the Stokes-vector representation,

$$\mathbf{I}_{\mathrm{c}} = \mathbf{D}\,\mathbf{J}_{\mathrm{c}} = \mathrm{Re}\left(\frac{k_1}{2\omega\mu_1}\right)\begin{bmatrix} E_{\mathrm{c}\theta} E_{\mathrm{c}\theta}^* + E_{\mathrm{c}\varphi} E_{\mathrm{c}\varphi}^* \\ E_{\mathrm{c}\theta} E_{\mathrm{c}\theta}^* - E_{\mathrm{c}\varphi} E_{\mathrm{c}\varphi}^* \\ -2\,\mathrm{Re}(E_{\mathrm{c}\theta} E_{\mathrm{c}\varphi}^*) \\ 2\,\mathrm{Im}(E_{\mathrm{c}\theta} E_{\mathrm{c}\varphi}^*) \end{bmatrix} \tag{68}$$

and

$$\frac{\mathrm{d}\mathbf{I}_{\mathrm{c}}(\mathbf{r})}{\mathrm{d}s} = -2k_1''\mathbf{I}_{\mathrm{c}}(\mathbf{r}) - n_0\langle \mathbf{K}(\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi \mathbf{I}_{\mathrm{c}}(\mathbf{r}), \tag{69}$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \end{bmatrix}$$

and $\mathbf{K}$ is the Stokes extinction matrix given by Eqs. (71)–(78) of [10]. The formal solution of Eq. (69) is given by

$$\mathbf{I}_{\mathrm{c}}(\mathbf{r}) = \mathbf{H}[\hat{\mathbf{n}}^{\mathrm{inc}}, s(\mathbf{r})]\mathbf{I}_{\mathrm{c}}(\mathbf{r}_A), \tag{70}$$

where

$$\mathbf{H}(\hat{\mathbf{n}}^{\mathrm{inc}}, s) = \exp[-2k_1''s - n_0 s\langle \mathbf{K}(\hat{\mathbf{n}}^{\mathrm{inc}})\rangle_\xi] \tag{71}$$

is the 4×4 coherent transmission Stokes matrix. Equation (82) of [10] implies the following reciprocity relation:

$$\mathbf{H}(-\hat{\mathbf{n}}^{\mathrm{inc}}, s) = \mathbf{\Delta}_3[\mathbf{H}(\hat{\mathbf{n}}^{\mathrm{inc}}, s)]^{\mathrm{T}}\mathbf{\Delta}_3, \tag{72}$$

where $\mathbf{\Delta}_3 = \mathrm{diag}[1, 1, -1, 1]$.

8. Discussion

The results of Sections 6 and 7 generalize the Foldy approximation [12] to the case of electromagnetic scattering by particles imbedded in an absorbing host medium. Unlike in [6], our results are applicable to particles of any size, shape, orientation, and polydispersity.

Although the vector Foldy–Lax equations (Section 2) and their order-of-scattering expansion (Section 3) as well as their far-field versions (Section 4) fully preserve their original mathematical structure, a non-vanishing absorptivity of the host medium leads to explicit changes in formulas of Sections 6 and 7. Specifically, Eqs. (48), (51), and (61) differ from their counterparts in [3] in that k_1 is replaced by k_1'. Furthermore, Eqs. (67), (69), and

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(71) contain additional, intuitively obvious terms which are proportional to k_1'' and describe additional exponential attenuation due to true absorption of electromagnetic energy by the host medium. Importantly, these results have been derived directly from the Maxwell equations and involve no phenomenological assumptions or hypotheses.

Equation (71) shows that the effect of a non-zero k_1'' on the coherent propagation of an electromagnetic wave through a turbid medium is two-fold. First, it modifies the numerical values of the ensemble-averaged extinction matrix elements. Second, it causes an additional exponential-attenuation factor $\exp(-2k_1''s)$. There is no doubt that the second manifestation of a non-vanishing absorptivity of the host medium is much more important than the first one since it affects directly the long-range transport of electromagnetic energy.

An essential ingredient of our derivation has been the explicit representation of exponentials of the type $\exp(\mathrm{i}k_1\alpha)$ (with a real-valued α) as a product of a real-valued exponential $\exp(-k_1''\alpha)$ and a "purely complex" exponential $\exp(\mathrm{i}k_1'\alpha)$ with a real-valued $k_1'\alpha$. It is important to remember that mathematical results such as the Jones lemma, the method of stationary phase, or the Saxon expansion of a plane wave in spherical waves [3, 13] are applicable only to situations involving purely complex exponentials of the type $\exp(\mathrm{i}k_1'\alpha)$ with a real-valued $k_1'\alpha$. We note in this regard that Eq. (61), when applied to the case of spherical particles, appears to be inconsistent with Eq. (55) of [6] in that the denominator in the second term of the former contains k_1' rather than k_1. Although the origin of this discrepancy is not immediately obvious, it is likely to be the same as that of the discrepancy discussed in the penultimate paragraph of [10].

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Appendix

Two typos have been identified in [10]. First, "independence" on the 5th line following Eq. (38) should read "dependence". Second, Eq. (64) should read

$$\textbf{Signal 2} \approx S\,\mathbf{I}^{\mathrm{sca}}(r\hat{\mathbf{n}}^{\mathrm{sca}}) = S\,\frac{\exp(-2k_1''r)}{r^2}\,\mathbf{Z}(\hat{\mathbf{n}}^{\mathrm{sca}},\hat{\mathbf{n}}^{\mathrm{inc}})\,\mathbf{I}^{\mathrm{inc}}.$$