DEFINITIONS OF COMPACTNESS AND THE AXIOM OF CHOICE

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Abstract. We study the relationships between definitions of compactness in topological spaces and the roll the axiom of choice plays in these relationships.

1. Introduction and Definitions

We have found several definitions of compactness which have been used in the literature and our goal is to find the relationships between these definitions in various types of topological spaces. In this article we use three basic forms of set theory: ZF, which uses the axioms of Zermelo-Fraenkel set theory (and does not include AC); ZF^0, which is a weaker version of ZF in which the axiom of extensionality is modified to allow the presence of *atoms*, objects which have no elements but are not equal to each other or to the empty set; and ZFC which is Zermelo-Fraenkel set theory with the addition of the Axiom of Choice. ZF^0 is relevant in our study because there is a well-known method for producing models of $\mathrm{ZF}^0 + \neg\mathrm{AC}$, namely, Fraenkel-Mostowski permutation models.

The notion of compactness is meant to capture, in the context of general topology, the properties that characterize spaces which are closed and bounded subsets of a Euclidean space (see [th]). However, several topological properties that characterize closed and bounded subsets of Euclidean spaces (for example: the Bolzano-Weierstrass theorem, (Every infinite bounded subset (sequence) of $\mathbb{R}^n$ has a limit point.); the Heine-Borel theorem (Every open cover of a closed set in $\mathbb{R}$ has a finite subcover.); Cantor's theorem on nested closed sets, and others turn out to be not always equivalent for general topological spaces. Several of these properties were proposed as definitions of compactness, mainly by Fréchet [fr] 1906, Riesz [ri] 1908, Alexandroff [al] 1924 and Urysohn [ur] 1924; the last two introduced the notion of *bicompactness*, which was eventually taken as the definition of compactness, based on the Heine-Borel theorem (see Group A, version 1, below).

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In 1924, Alexandroff and Urysohn established the relationship of their definition with other notions of compactness proposed before. However, they also introduced new properties, like the one in Group F (see below), which is a strengthening of the Bolzano-Weierstrass property and was proved by them to be equivalent to Group A. However, they used the axiom of choice (AC) quite freely (see [mo], p. 237). In 1930, Tychonoff [ty] gave a proof of the Product Theorem for the closed unit interval $[0,1]$, but his proof was easily generalized to arbitrary compact spaces, a product of compact spaces is compact. C̃ech [c], in 1937, extended the theorem to arbitrary compact spaces. Both Tychonoff and C̃ech used the notion of a complete accumulation point in their proofs. A few years later, two generalizations of the notion of convergence were introduced independently in the United States and in France: using nets (Birkhoff, building on work by E. H. Moore and H. L. Smith), and filters (Cartan), respectively. These definitions made it possible to establish the equivalence of Group A with properties obtained by modifying the Bolzano-Weierstrass property using nets, filters, and ultrafilters instead of sequences (see forms 3, 4, and 5 in Group A, and all forms in Group B). However, some of these results again relied heavily on AC.

In this article we study the relations between several of the properties mentioned above when AC is not assumed; also, we study how these relations are affected when we restrict our attention to some particular classes of topological spaces: first countable, second countable, and Lindelöf spaces, and subsets of $\mathbb{R}$ spaces. Also we study how much can be proved when we assume the Boolean Prime Ideal Theorem, which is a strictly weaker consequence of AC but is strong enough to obtain several equivalences that cannot be obtained in the general case. Similar studies have been carried out before in [ho] and [he], and we build on their results. We also include many well known results, especially for the case when AC holds. For the sake of completeness of the presentation; references will be made whenever possible, but some of these will be to secondary sources, since they are more readily available.

The properties studied here include those that appeared during the development of the notion of compactness, and which appear in many later topology textbooks; in particular, we include a few properties which have been known to be non-equivalent to the others even when AC is assumed.. However, the relations involving these properties become non-trivial when we consider classes of topological spaces with particular properties. That all these properties somehow capture at least part of the notion of compactness is underlined by the fact that all of them are equivalent for first countable T_1 spaces, if AC is assumed.

On the other hand, we do not include other notions which were introduced later, and are not, in general, equivalent to topological compactness even under AC. These notions include paracompact (A space X is *paracompact* if it is T_2 and every open cover of X has an open locally finite refinement.); pseudocompact (X is *pseudocompact* if every continuous function from X to $\mathbb{R}$ is bounded.); real compact (X can be embedded as a closed subset of a product of copies of the real line.); Comfort compact (X can be embedded into a closed subset of $[0,1]^k$.); and precompact or totally bounded for pseudometric spaces (X is *precompact* if every net in X has a Cauchy subnet.). Of course, the study of these notions and their relations when AC is not assumed is also interesting, but is a matter for a different paper.

The definitions we shall be using include the following:

Definitions. Assume (X, T) is a topological space.

1. $\mathcal{B}$ is a *base* of the topology T if every $U \in T$ is a union of elements of $\mathcal{B}$.
2. $\mathcal{S}$ is a *subbase* of the topology T if the set of all finite intersections of elements of $\mathcal{S}$ is a base for T.
3. A *net* $(x_\lambda)_{\lambda\in\Lambda}$ in X is a function P from a directed set, Λ to X.
4. A *subnet* of a net $P : \Lambda \to X$ is the composition $P \circ \phi$, where ϕ is an increasing cofinal function from a directed set M to Λ.
5. The net $(x_\lambda)_{\lambda\in\Lambda}$ *converges* to a point $x \in X$ if for each neighborhood U of x there is a $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $x_\lambda \in U$.
6. x is an *accumulation point (cluster point)* of a net $(x_\lambda)_{\lambda\in\Lambda}$ if for each neighborhood U of x and for each $\lambda_0 \in \Lambda$ there is a $\lambda \geq \lambda_0$ such the $x_\lambda \in U$.
7. A filter $\mathcal{F}$ on X *converges to* x if every neighborhood of x is in $\mathcal{F}$.
8. A filter $\mathcal{F}$ on X has an *accumulation point (cluster point)* x if each F in $\mathcal{F}$ intersects each neighborhood of x.
9. A point $x_0 \in X$ is called an *accumulation point (cluster point) of* $A \subseteq X$ if every neighborhood of x_0 contains some point of A other than x_0.
10. If E is an infinite subset of X, a point $x_0 \in X$ is called a *a complete accumulation point of* E if for every open neighborhood U of x_0, $\mathrm{card}(E \cap U) = \mathrm{card}(E)$.
11. A *nest* is a set which is linearly ordered by inclusion, $\subseteq$.

The relationship between nets and filters is given by the following. (See [w].)

Definition. If $(x_\lambda)_\lambda \in \Lambda$ is a net then the filter which has the base $\{B_\mu = \{x_\lambda : \lambda \geq \mu\} : \mu \in \Lambda\}$ is called the *filter generated by the net* (x_λ).

Definition. If $\mathcal{F}$ is a filter and $\Lambda = \{(x, F) : x \in F \in \mathcal{F}\}$, then Λ is directed by $(x_1, F_1) \leq (x_2, F_2)$ if $F_1 \supseteq F_2$. Then the map $\Lambda \to X$ defined by $(x, F) \to x$ is a net and is called the *net based on* $\mathcal{F}$.

Theorem 1.

(a) A filter $\mathcal{F}$ converges to x iff the net based on $\mathcal{F}$ converges to x.
(b) A net (x_λ) converges to x iff the filter generated by (x_λ) converges to x.
(c) A filter $\mathcal{F}$ has x as an accumulation point iff the net based on $\mathcal{F}$ has x as an accumulation point.
(d) A net (x_λ) has x as an accumulation point iff the filter generated by (x_λ) has x as an accumulation point.

The statements in each of the following groups are equivalent in ZF^0.

GROUP A: Heine-Borel Compact or H-B Compact

1. Every open covering has a finite subcovering.
2. Every family of closed sets with the finite intersection property has a non-empty intersection.
3. Every net has a convergent subnet.
4. Every net has an accumulation point.
5. Every filter has an accumulation point.
6. There is a base for the topology such that every open covering by elements of the base has a finite subcovering.

GROUP B: Bourbaki Compact or B Compact

1. Every ultrafilter has an accumulation point.
2. Every ultranet has an accumulation point.
3. Every ultrafilter converges.
4. Every ultranet converges.

GROUP C: Subbase Compact or S-B Compact

1. There is a subbase for the topology such that every open covering by elements of the subbase has a finite subcovering.

GROUP D: Linearly Compact or L Compact

1. Every covering by elements of a nest of open sets has a finite subcovering.
2. Every nest of non-empty closed sets has a non-empty intersection.

GROUP E: Sequentially Compact or SEQ Compact

1. Every sequence has a convergent subsequence.

GROUP F: Alexandroff-Urysohn Compact or A-U Compact

1. Every infinite subset of X has a complete accumulation point.

GROUP G: Weierstrass Compact or W Compact

1. Every infinite subset of X has an accumulation point.

GROUP H: Countably Compact

1. Every countable open covering has a finite subcovering.
2. Every countable family of closed sets with the finite intersection property has a non-empty intersection.

2. General Topological Spaces

In this section we do not put any restriction on the topological space.

Theorem 2. *In ZF*^{0}, $A \to C \to B$, $A \to D \to H$, $A \to G$, *and* $F \to G$.

Proof. All the implications are clear except for $C \to B$ and we will give a proof of that here.

Let $\mathcal{S}$ be a subbase for a topology on X such that every open cover of X by elements of $\mathcal{S}$ has a finite subcover and let $\mathcal{F}$ be an ultrafilter on X which doesn't converge. Then for every $x \in X$ there is a neighborhood of x which is not in $\mathcal{F}$ and therefore, there is a finite subset S' of S such that $x \in \bigcap S' \notin \mathcal{F}$. This implies that for each $x \in X$, there is $U \in S'$ such that $x \in U$ and $U \notin \mathcal{F}$. Therefore, $\{U \in S : U \notin \mathcal{F}\}$ is an open cover of X by elements of S. Let $\{U_1, \dots, U_n\}$ be a finite subcover. Since $\mathcal{F}$ is an ultrafilter and $U_i \notin \mathcal{F}$, $X \setminus U_i \in \mathcal{F}$ for $i = 1, \dots, n$. But this is a contradiction because

$$\emptyset = X \setminus \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} (X \setminus U_i) \in \mathcal{F}. \quad \square$$

In the next lemma we state some of the known independence results.

Lemma 1.
(a) In ZF, $D \not\to C$ *(*[ho]*, model N1 in* [hr] *and the Jech/Sochor transfer theorem, see* [j]*.)*
(b) In ZF, $F \not\to A$*, and* $F \not\to C$ *(*[ho]*, model N56 in* [hr] *and the Jech/Sochor transfer theorem, see* [j]*.)*
(c) In ZF, $X \not\to F$*, for* $X = A, B, C, D, E$*, or* G *(*[ho]*, If* $A \to F$*, then AC and similarly for the others.)*
(d) In ZF, $C \not\to A$ *(*[p], [wo]*, If* $C \to A$*, then the Boolean Prime Ideal Theorem (BPI) holds.)*

In our next theorem, we shall prove additional independence results.

Theorem 3.
(a) In a model of ZFC, the space $X = 2^{2^\omega}$ *with the product topology (the discrete topology on* 2*) is not compact E, but it is compact A, B, C, D, F, G, and H.*
(b) In Feferman's model ($\mathcal{M}2$ *in* [hr]*),* ω *with the discrete topology is compact B, but not compact A, C, D, E, F, G, or H.*
(c) In the second Fraenkel model ($\mathcal{N}2$ *in* [hr]*),* 2^A *with the product topology (the discrete topology on* 2*) is compact C, but not compact D, G, or H.*
(d) In the basic Fraenkel model ($\mathcal{N}1$ *in* [hr]*) the set of atoms with the discrete topology, is compact D, E, and H, but not compact B, F, or G.*
(e) In a model of ZFC, ω_1 *with the order topology is compact E, G, and H but not compact A, B, C, D, or F.*
(f) BPI implies that compact A, B, and C are equivalent.

(g) In the ordered Mostowski model, ($\mathcal{N}3$ in [hr]*), the set of atoms with the order topology is compact E, G, and H but not compact A, B, C, D or F. The set of atoms with the discrete topology is compact E and H, but not compact G.*

(h) There is a topological space which is compact G, but not compact A, B, C, D, E, F, or H. (We are assuming BPI.)

(i) In $\mathcal{M}1$, the basic Cohen model (see [hr]*), there is a set that is compact E and G, but not compact A, B, C, D or H.*

Proof. (a) To show X is not compact E, we shall define a sequence of functions in X that has no convergent subsequence. For each $i \in \omega$, and for each $s \in 2^\omega$ let $f_i(s) = s(i)$. Let $\{f_{i_n} : n \in \omega\}$ be any subsequence of the sequence $\{f_i : i \in \omega\}$ and let $f \in 2^{2^\omega}$. We will argue that the subsequence does not converge to f. Define $s \in 2^\omega$ by

$$s(j) = \begin{cases} 1 & (\exists n \in \omega)(n \text{ is even and } j = i_n) \\ 0 & (\exists n \in \omega)(n \text{ is odd and } j = i_n) \\ 1 & \text{otherwise} \end{cases} .$$

Then the neighborhood $\{g \in 2^{2^\omega} : g(s) = f(s)\}$ of f contains every other point of the subsequence $\{f_{i_n} : n \in \omega\}$.

The Boolean Prime Ideal Theorem (BPI) implies that X is compact A. (See [14 Z] in [hr]. The set 2 with the discrete topology is a compact Hausdorf space.) Since compact A implies compact B, C, D, G, and H, it remains to prove that X is compact F. The proof of this is given in Theorem 5 where we show that in ZFC, compact A implies compact F.

(b) It is shown in [tr] that there are no non-principal ultrafilters on ω in $\mathcal{M}2$. This implies that ω is compact B. However, ω with the discrete topology is clearly not compact A, D, E, F, G, or H.

We claim that ω is not compact C in $\mathcal{M}2$. Let $\mathcal{S}$ be a subbase for ω. We will use dependent choice, which holds in $\mathcal{M}2$, to show that $\mathcal{S}$ has an open cover with no finite subcover and thus that ω is not C compact.

Let $n_0 = 0$. Since $\mathcal{S}$ is a subbase there exists some $S_0 \in \mathcal{S}$ so that $n_0 \in S_0$ and $\omega \backslash S_0$ is infinite. (If $\omega \backslash S$ was finite for every $S \in \mathcal{S}$ such that $n_0 \in S$, then $\{n_0\}$ would not be a finite intersection of basic sets.) Suppose we have found $n_0, n_1, \dots, n_k$ and $S_0, S_1, \dots, S_k$ so that for $j = 0, 1, \dots k$, $n_j \in S_j \setminus (S_0 \cup S_1 \cup \dots \cup S_{j-1})$ and

$$(*) \qquad \omega \backslash (S_0 \cup S_1 \cup \cdots S_k), \text{ is infinite} .$$

Let $n_{k+1} = \inf\{\omega \backslash (S_0 \cup \dots \cup S_k)\}$. Since $\mathcal{S}$ is a subbase and $\{n_{k+1}\}$ is open, there is a finite collection $\{T_0, T_1, ..., T_m\} \subseteq \mathcal{S}$ so that $\bigcap_{0 \le i \le m} T_i = \{n_{k+1}\}$.

Consider the following

$$\bigcup_{0 \le i \le m} (\omega \backslash (S_0 \cup ... \cup S_k \cup T_i)) = \omega \backslash (S_0 \cup ... \cup S_k \cup \bigcap_{0 \le i \le m} T_i) = \omega \backslash (S_0 \cup ... \cup S_k \cup \{n_{k+1}\}).$$

It follows from (*) that $\omega \backslash (S_0 \cup ... \cup S_k \cup \{n_{k+1}\})$ is infinite. Since m is finite, it follows that for some i, $0 \leq i \leq m$, $\omega \backslash (S_0 \cup ... \cup S_k \cup T_i)$ is infinite, say $i = p$. Thus, $\omega \backslash (S_0 \cup ... \cup S_k \cup T_p)$ is infinite and we can take $S_{k+1} = T_p$. It follows by induction that the set $\{S_0, S_1, \cdots ,\}$ is a cover of ω by subbasic sets that has no finite subcover. Therefore, ω is not C compact.

(c) We shall show in ZF that if X is any finite set then X^A is compact C if we use the discrete topology on X. (It follows then that it is true in $\mathcal{N}2$.) For each $a \in A$ and each $x \in X$, let $S_{a,x} = \{f \in X^A : f(a) = x\}$. $\mathcal{S} = \{S_{a,x} : a \in A \text{ and } x \in X\}$ is a subbasis for the product topology on X^A. Further, we claim that any cover of X^A by elements of $\mathcal{S}$ has a finite subcover. Let $\mathcal{C}$ be such a cover and let $<$ be a linear ordering of X. For each $a \in A$, let $D_a = \{x : S_{a,x} \in \mathcal{C}\}$. For some $a \in A$, $D_a = X$. Otherwise the function $f \in X^A$ defined by $f(a) =$ the $<$-least element of X not in D_a, is not in $\bigcup \mathcal{C}$. But if $D_a = X$, then $\{S_{a,x} : x \in X\}$ is a finite subcover of $\mathcal{C}$.

Next, we shall show that if A is the set of atoms in $\mathcal{N}2$, then 2^A is neither compact D, G nor H. In $\mathcal{N}2$, $A = \bigcup_{i \in \omega} \{a_i, b_i\}$. (The group determining the model is the group of all permutations ϕ of A such that $(\forall i \in \omega)(\phi(\{a_i, b_i\}) = \{a_i, b_i\})$ and supports are finite subsets of A. See [hr] p 178.) To show that D and H compactness fails, let $\mathcal{C} = \{C_i : i \in \omega\}$ where $C_i = \{f \in 2^A : (\exists j \leq i)(f(a_j) = f(b_j))\}$. $\mathcal{C}$ is a chain no element of which covers 2^A. But in $\mathcal{N}2$, for every $f \in 2^A$, there is a $k \in \omega$ such that for all $j \geq k$, $f(a_j) = f(b_j)$. Consequently, $f \in C_k$, So $\mathcal{C}$ is a cover.

To show that G compactness fails, let $B = \{f \in 2^A : (\exists k \in \omega)(\forall j \leq k)(f(a_j) \neq f(b_j)) \text{ and } (\forall j > k)(f(a_j) = f(b_j) = 1)\}$. B has empty support and is therefore in $\mathcal{N}2$. Also, note that for every $f \in B$, $f(a_0) \neq f(b_0)$. B has no accumulation point for if g is an accumulation point of B, then for no $i \in \omega$ is $g(a_i) = g(b_i) = 0$. (If $g(a_i) = g(b_i) = 0$ then the neighborhood $N = \{f \in 2^A : f(a_i) = f(b_i) = 0\}$ of g excludes all points of B.) Therefore if k is the least element of ω such that for every $j \geq k$, $g(a_j) = g(b_j)$, then for every $j \geq k$, $g(a_j) = g(b_j) = 1$. We now consider the neighborhood $N = \{f \in 2^A : (\forall j \leq k)(f(a_j) = g(a_j) \text{ and } f(b_j) = g(b_j))\}$ of g. If $k = 0$, then $g(a_0) = g(b_0) = 1$ and N excludes all points of B. If $k > 0$, then any element $f \in B \cap N$ must agree with g on a_i and b_i for $0 \leq i \leq k$. Therefore, since $g(a_k) = g(b_k) = 1$, $f(a_k) = f(b_k) = 1$ and hence, $f(a_j) = f(b_j) = 1$ for all $j > k$. So f agrees with g on a_j and b_j for all $j > k$. Therefore, $f = g$. So in the case where $k > 0$, the only element of $B \cap N$ is g. Consequently, g cannot be an accumulation point of B.

(d) The only subsets of A in $\mathcal{N}1$ are the finite and cofinite sets. In addition, there is no linear ordering of A in $\mathcal{N}1$. Consequently, A has no countably infinite open cover, there are no infinite nests covering A, and there is no countably infinite subset of A. It follows that A is compact D, E, and H. The set of cofinite subsets of A is an ultrafilter with no accumulation point so A is not compact B. Moreover, no infinite set with the discrete topology is compact G and compact F implies compact G.

(e) It follows from AC that ω_1 has no cofinal sequence. Thus, every sequence is bounded and therefore has a convergent subsequence. In addition, every infinite subset of ω_1 has a countable limit ordinal as an accumulation point. It follows that ω_1 is compact E and G.

For each $\alpha \in \omega_1$, let $X_\alpha = \{\beta \in \omega_1 : \beta < \alpha\}$. Then $\{X_\alpha : \alpha \in \omega_1\}$ is an open cover of ω_1 both by a nest and by elements of a subbase, which has no finite subcover. Therefore, ω_1 is neither compact C nor D.

It is compact H because any countable cover must contain a set of the form $S = \{\beta \in \omega_1 : \beta > \alpha\}$ were α is a countable ordinal. It is easy to see that $\omega_1 - S$ is countably compact.

ω_1 is not compact F, because for each $\alpha \in \omega_1$, $\{\beta \in \omega_1 : \beta < \alpha + 1\}$ is a neighborhood of α of cardinality $\aleph_0 < \aleph_1 = \text{card}(\omega_1)$.

(f) Use the fact that compact A implies compact C implies compact B and see the web page for [hr], form [14 CO], and [he] to obtain the fact that BPI implies that compact B implies compact A.

(g) BPI holds in $\mathcal{N}3$, so compact A, B and C are equivalent. The set of atoms, A, has no countably infinite subset, and neither does its power set. It follows that $(A, \mathcal{T})$ is both compact E and compact H, where $\mathcal{T}$ is any topology.

$(A, \leq)$, the set of atoms with the order topology, is compact G because every infinite subset contains an open interval and every point of such an interval is an accumulation point. However, $(A, \leq)$ is not compact D because the nest of open sets $\{(-\infty, a) : a \in A\}$ covers A, but does not have a finite subcover. If $(A, \leq)$ were compact F, it would have an accumulation point x such that if U is an open interval containing x, then there would be a $1-1$ function f mapping U onto A. But such a function f does not have a finite support if $U \neq A$, and, therefore, cannot be in the model.

Clearly $(A, \mathcal{D})$, the set of atoms with the discrete topology, is not compact G, as no discrete space is compact G.

(h) Let $\mathbb{Z}$ be the set of integers and take the topology to be $\mathcal{T} = \{(x, \infty) : x \in \mathbb{Z}\}$. Suppose Y is any infinite subset of $\mathbb{Z}$. Then if $x \in Y$, there is a $y \in \mathbb{Z}$ such that $y < x$. It follows that y is an accumulation point of Y. Therefore, $(\mathbb{Z}, \mathcal{T})$ is compact G. However, $(\mathbb{Z}, \mathcal{T})$ is not compact E because the sequence $\langle -1, -2, -3, \dots \rangle$ has no convergent subsequence. Since $\mathcal{T}$ is a countable nest and has no finite subcover, it is not compact D or H. It is clear that it is not compact F because the set $\{-1, -2, -3, \dots\}$ has no complete accumulation point. Since, BPI holds, it follows from (f) that compact A, B, and C are equivalent, A implies D and D implies H. Thus, $(\mathbb{Z}, \mathcal{T})$ is compact G, but not compact A, B, C, D, E, F, or H.

(i) It follows from the results of Cohen that in $\mathcal{M}1$, the basic Cohen model, there is a set $K \subset \mathbb{R}$ with no countable infinite subset which is dense in $\mathbb{R}$ and second countable, but is not Lindelöf. Since K has no countable infinite subset it is compact E and since it is dense in $\mathbb{R}$, it is not compact A, B, C, D, or H, but it is compact G. Also see Lemma 1(c),(d) and Theorem 3(f). (BPI is true in $\mathcal{M}1$.) □

Theorem 4. *There is a model of ZF + BPI in which there is a topological space that is compact D and F, but not compact A, B, or C.*

Proof. We shall use Cohen's basic model $\mathcal{M}1$ in [hr]. $\mathcal{M}1$ is obtained by extending a model of ZFC using the forcing notion

$$\mathbb{P} = \{p : p \text{ is a finite partial function from } \omega \times \omega \text{ to } 2\}$$

and then taking a symmetric submodel of the extension, using an appropriate group of automorphisms of $\mathbb{P}$ (obtained from the group of all permutations of ω) and a normal filter obtained using finite subsets of ω as supports. See [j] for the details of the construction.

It is well known that in $\mathcal{M}1$, AC is false and BPI is true. Also, there is a set $K = \{x_n : n \in \omega\}$ of reals (i.e. subsets of ω) which is infinite but Dedekind finite. (The enumeration $n \mapsto x_n$ is not in the model.) (See Theorem 3(i).) Another property that we will use is inherited from the generic extension from which $\mathcal{M}1$ is obtained as a symmetric submodel; namely, that the elements of K are independent in the Boolean sense; that is, $y_1 \cap \cdots \cap y_r \subset x_1 \cup \cdots \cup x_s$ only if $\{y_1, \ldots, y_r\} \cap \{x_1, \ldots, x_s\} \neq \emptyset$, for all $y_1, \ldots, y_r, x_1, \ldots, x_s \in K$. In particular, no finite union of elements of K is equal to ω, and no finite intersection is empty.

We consider the topological space (ω, τ), where τ is the topology generated by K as a subbase. (A base for the topology τ is the set K^* of all finite intersections of elements from K.) It is clear that (ω, τ) is not compact A, since K is an open cover of ω with no finite subcover. Since compact A, B, and C are equivalent in $\mathcal{M}1$, (ω, τ) is not compact B or C either. We will show in Lemmas 3 and 4 that (ω, τ) is compact D and compact F. The proofs rely heavily on Lemma 2, which characterizes the open sets of (ω, τ).

Notice that (ω, τ) is not Hausdorff, since the intersection of any two non-empty basic neighborhoods is non-empty. However, the space is T_1, since for any pair $a, b \in \omega$ there exists $x \in K$ such that $a \in x$ and $b \notin x$.

Lemma 2. *Every non-empty open set in τ is either:*
(a) A cofinite subset of ω or
(b) A finite union of basic neighborhoods $\setminus F$, where F is a finite subset of ω.

Proof of Lemma 2. Let $\bigcup S$ be a non-empty open set in τ, where S is a set of basic neighborhoods. We can assume, without loss of generality, that if $U \in S$ and $V \in K^*$ such that $V \subset U$, then $V \in S$.

Let $\dot{S}$ be a symmetric name for S, and let $E \subset \omega$ be a support for $\dot{S}$. Consider the finite set $\{x_n : n \in E\} \subset K$. We have two cases:
Case 1. $\bigcup S \subset \bigcup_{n \in E} x_n$. Let H be the $\subset$-minimal positive Boolean combination of elements of $\{x_n : n \in E\}$ that contains $\bigcup S$, and, using the disjunctive normal form, express H as a finite union of finite intersections:

$$H = (y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s). \tag{1}$$

By the independence of the elements of K, we have that the expression above is uniquely determined (up to reordering), and consequently it has no redundancy (that is, if we delete any part of the expression we obtain a different set). If every finite intersection $y_1^j \cap \cdots \cap y_{r_s}^j$, for $j = 1, \ldots, s$, is an element of S, then $\bigcup S = H$ and we are done.

Suppose then that $y_1^j \cap \cdots \cap y_{r_s}^j$ is not a member of S. We will prove that $y_1^j \cap \cdots \cap y_{r_s}^j$ is contained in $\bigcup S$, except possibly for finitely many points. This way we will conclude that $\bigcup S$ equals H minus, possibly, finitely many points.

Take $u = y_1^j \cap \cdots \cap y_{r_s}^j \cap z_1 \cap \cdots \cap z_t \in S$. Such a set u exists, because we assumed S to be closed downwards and $(\bigcup S) \cap y_1^j \cap \cdots \cap y_{r_s}^j \neq \emptyset$. Also, $z_1, \ldots, z_t$ can be taken such that they are not in $\{x_n : n \in E\}$, since otherwise H is not the minimal Boolean combination that contains $\bigcup S$, or (1) is redundant.

Let p be a forcing condition in the generic set G that forces $\dot{u} \in \dot{S}$. Then there is at most a finite set $d \subset \omega$ of elements that p forces *not to be* in at least one of the sets $\dot{z}_1, \ldots, \dot{z}_t$. Let $k \in y_1^j \cap \cdots \cap y_{r_s}^j \setminus d$, and let $p' \in G$ be an extension of p that forces $\check{k} \in \dot{y}_1^j \cap \cdots \cap \dot{y}_{r_s}^j \setminus \check{d}$; we will prove that p' forces the fact that there exists $u' \in S$ such that $k \in u'$. We will do that by showing that every extension q of p' has an extension q' that forces the required properties.

Let q be an extension of p'. Let π be a permutation of ω that interchanges the indices corresponding to the sets $z_1, \ldots, z_t$ with indices $l_1, \ldots, l_t \notin E$ such that $\mathrm{dom}(q) \cap \{(n, m) : n = l_1, \ldots, l_t\} = \emptyset$, while fixing all elements of E. Let p'' be the condition obtained by adding to p' the information to force that k is in each of the sets named by $\pi\dot{z}_1, \ldots, \pi\dot{z}_t$. We still have that p'' is compatible with q, so $q' = q \cup p''$ is an extension of q that forces

$$\check{k} \in \dot{y}_1^j \cap \cdots \cap \dot{y}_{r_s}^j \cap \pi\dot{z}_1 \cap \cdots \cap \pi\dot{z}_t$$

and

$$\dot{y}_1^j \cap \cdots \cap \dot{y}_{r_s}^j \cap \pi\dot{z}_1 \cap \cdots \cap \pi\dot{z}_t \in \pi\dot{S} = \dot{S}.$$

This proves the fact that $y_1^j \cap \cdots \cap y_{r_s}^j \setminus d$ is contained in $\bigcup S$.

Case 2. $\bigcup S \not\subset \bigcup_{n \in E} x_n$. In this case there exists $u = z_1 \cap \cdots \cap z_t \in S$ such that $\{x_n : n \in E\} \cap \{z_1, \ldots, z_t\} = \emptyset$. If we take a condition p that forces $\dot{u} \in \dot{S}$, and $d \subset \omega$ is the set of elements that p forces not to be in at least one of $\dot{z}_1, \ldots, \dot{z}_t$, we can repeat an argument similar to that of Case 1 to obtain now that $\omega \setminus d \subset \bigcup S$.

This finishes the proof of Lemma 2.

Lemma 3. (ω, τ) *is compact D.*

Proof of Lemma 3. Let T be a nest of open sets that covers ω. If there is in T an open set which is a cofinite subset of ω, then there are only finitely many sets above it in T, and those form a finite subcover of ω. Therefore, by Lemma 2, the remaining possibility is that all the elements of T are finite unions of basic neighborhoods, possibly with some points deleted. We will show that this case leads to a contradiction.

We will define a one-to-one ω-sequence $(z_m : m \in \omega)$ in K, contradicting the fact that K is Dedekind finite.

First, choose $k_0 = \min(\omega \setminus \bigcap T)$ (of course, if $\bigcap T = \omega$ then $T = \{\omega\}$ and we are in the previous case).

If k_m is defined and $k_m \notin \bigcap T$, let $a_m = \bigcup\{u \in T : k_m \notin u\}$. Since it is a union of open sets, a_m is an open set, and it is non-empty. If a_m is equal to ω minus finitely many points, there must be an open set $u \in T$ such that $a_m \subset u$ (since $k_m \notin a_m$ and $k_m \in \bigcup T$), and such u must be equal to ω minus finitely many points, contradicting our assumption. Then, by Lemma 2, and using the disjunctive normal form, we can write uniquely

$$a_m = (y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s) \setminus F,$$

where $y_i^j \in K$, $1 \leq i \leq r_j$, $1 \leq j \leq s$, and $F \subset \omega$ is finite. Now we define $z_m \in K$ as the least element (in the order of the real numbers) of the set $\{y_i^j : k_m \in y_i^j, 1 \leq i \leq r_s, 1 \leq j \leq s\}$ which is not in $\{z_i : i < m\}$; we assume as an induction hypothesis that it exists. Finally, to complete the induction, define k_{m+1} as the minimum $k \in \omega$ for which there exists $u \in T$ such that $u \not\subset \bigcup_{i<m+1} z_i$ and $k \notin u$.

Notice that the construction guarantees that the sequence $(z_m : m \in \omega)$ is one-to-one. This finishes the proof of the fact that (ω, τ) is compact D.

Lemma 4. *(ω, τ) is compact F.*

Proof of lemma 4. Let $X \subset \omega$ be infinite. If for every open set v in (ω, τ) we have that $v \cap X$ is infinite, then every point in ω is a complete accumulation point. Assume then that there are non-empty open sets that contain only finitely many points of X; this implies that there are open sets that are disjoint from X.

Let $u = \bigcup\{v : v \text{ is open and } v \cap X = \emptyset\}$. Since $u \neq \emptyset$, and $\omega \setminus u$ contains the infinite set X, Lemma 2 implies that

$$u = (y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s) \setminus F,$$

where $y_i^j \in A$, $1 \leq i \leq r_j$, $1 \leq j \leq s$, and $F \subset \omega$ is finite.

We claim that every point in the complement of $(y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s)$ is a complete accumulation point of X. Indeed, if a is in the complement of $(y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s)$ and $z_1 \cap \cdots \cap z_t$ is a basic neighborhood that contains a, then the set

$$z_1 \cap \cdots \cap z_t \setminus \left[(y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s)\right]$$

must be infinite, since a finite Boolean combination of elements of K must be either empty or infinite. This means that $(z_1 \cap \cdots \cap z_t) \cap X$ is infinite: otherwise, the set $(z_1 \cap \cdots \cap z_t) \setminus X$ would be an open set disjoint from X, which would imply that $(z_1 \cap \cdots \cap z_t) \setminus X$ is contained in $(y_1^1 \cap \cdots \cap y_{r_1}^1) \cup \cdots \cup (y_1^s \cap \cdots \cap y_{r_s}^s)$. Therefore, being both countably infinite sets, $(z_1 \cap \cdots \cap z_t) \cap X$ and X have the same cardinality.

This completes the proof of Theorem 4. $\square$

The relations between the definitions are shown in the following implication diagram.

$$\begin{array}{ccccccccc} H & \longleftarrow & D & \longleftarrow & A & \longrightarrow & C & \longrightarrow & B \\ & & & & \big\downarrow & & & & \\ & & F & \longrightarrow & G & & & & \end{array}$$

The matrix below shows the relationships between the definitions. An arrow means that the row entry implies the column entry. The other entries are models in which the row entry is true and the column entry is false.

	A	B	C	D	E	F	G	H
A	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	2^{2^ω}	$(A \rightarrow F)$ $\rightarrow AC$ [ho]	$\rightarrow$	$\rightarrow$
B	ω in $M2$ [tr]	$\rightarrow$	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]
C	2^A in $N2$	$\rightarrow$	$\rightarrow$	2^A in $N2$	2^{2^ω}	$(C \rightarrow F)$ $\rightarrow AC$ [ho]	2^A in $N2$	2^A in $N2$
D	(ω, τ) in $M1$	(ω, τ) in $M1$	(ω, τ) in $M1$	$\rightarrow$	2^{2^ω}	A in $N1$	A in $N1$	$\rightarrow$
E	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$\rightarrow$	$(\omega_1, \leq)$	A in $N1$	$K \subset \mathbb{R}$ in $M1$
F	(ω, τ) in $M1$	(ω, τ) in $M1$	(ω, τ) in $M1$		2^{2^ω}	$\rightarrow$	$\rightarrow$	
G	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	2^{2^ω}	$(\omega_1, \leq)$	$\rightarrow$	$(\mathbb{Z}, \mathcal{T})$
H	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	2^{2^ω}	$(\omega_1, \leq)$	A in $N1$	$\rightarrow$

Finally, in this section, we discuss the relationships between the definitions of compactness in ZFC.

Theorem 5. *In ZFC, compact A, B, C, D, and F are all equivalent, compact E implies compact H, and compact H implies compact G.*

Proof. To prove that compact A, B, C, D, and F are equivalent it is sufficient to prove, B $\rightarrow$ A, D $\rightarrow$ A, A $\rightarrow$ F, and F $\rightarrow$ A. The remaining implications follow from Theorem 2.

A proof that compact B implies compact A can be found in [w] p 118. The proof that compact A and F are equivalent is due to Alexandroff [al] and Urysohn [ur]. An outline of the proof can be found in [k] p163.

Suppose X is compact D and not compact A. Let $\mathcal{U}$ be an open cover of X which fails to have a finite subcover. Well order $\mathcal{U}$ as $\{U_{\alpha,0} : \alpha < \beta_0\}$. Since X is compact D, there is an $\alpha_0 < \beta_0$ such that $X \subseteq \bigcup_{\alpha<\alpha_0} U_{\alpha,0}$. It follows that $\mathcal{U}_0 = \{U_{\alpha,0} : \alpha < \alpha_0\}$ is an open cover of X. Since $\mathcal{U}_0 \subseteq \mathcal{U}$, it has no finite subcover, so $\alpha_0 \geq \omega$. We can well order $\mathcal{U}_0$ as $\{U_{\alpha,1} : \alpha < \beta_1\}$, where β_1 is a limit ordinal $\leq \alpha_0$. Since X is compact D, there is an $\alpha_1 < \beta_1$ such that $X \subseteq \bigcup_{\alpha<\alpha_1} U_{\alpha,1}$. We then continue by induction and construct a strictly decreasing sequence of ordinals $\{\alpha_i : i \in \omega\}$, which is a contradiction.

The fact that compact E implies compact H is an exercise in [k] p162. The fact that compact H implies compact G follows easily from the next lemma. □

Lemma 5. *In ZF^0, any sequence in an H compact space has an accumulation point.*

(We are using accumulation point here in the sense of an accumulation point of a net, not of a set. To say that any sequence has an accumulation point in this sense is the same as saying that any countable set has a complete accumulation point. See the definitions in the introduction.)

Proof. Suppose $\langle s_i : i \in \omega\rangle$ is a sequence in X with no accumulation point; we shall show that X is not compact H. Let S be $\{s_i : i \in \omega\}$. There is no loss of generality in assuming that S is infinite because if it were finite than it easy to construct a convergent subsequence whose limit is an accumulation point of the sequence.

We will now construct a countable open cover of X with no finite subcover. For each finite subset $T \subset S$ (including $T = \emptyset$), let U_T be the union of all open sets which contain T but do not contain any points of $S \setminus T$ as members. (If X is not T_1, then U_T may be empty for some T, but the fact that the sequence has no accumulation point ensures that every $x \in X$ is in some U_T.) Clearly no finite union of U_T's covers all of X, since each contains only finitely many members of S. Also, there are only countably many finite subsets of the countable set S, so the U_T's form a countable open cover of X with no finite subcover. □

In ZFC, we have the following relations between the forms.

$$\boxed{A,\ B,\ C,\ D,\ F}$$
$$\downarrow$$
$$E \longrightarrow H \longrightarrow G$$

3. First Countable Topological Spaces

A topological space is first countable if every point has a countable neighborhood base. All the implications shown in the last section hold for first countable spaces and, in addition, compact H implies compact E.

Theorem 6. *For first countable spaces, compact H implies compact E.*

Proof. By Lemma 5, any sequence in a compact H space X has an accumulation point. In a first countable space, if p is a accumulation point of a sequence, then some subsequence of S converges to p. (See [k], pp 72-3. It is easy to see that the axiom of choice is not required in the proof in [k].) Thus, if X is H compact, it is also E compact. □

The topological space $(\mathbb{Z}, \mathcal{T})$ described in Theorem 3(h) is a first countable topological space (in fact it is second countable) which is compact G but not compact E or H.

If we look at the models we used in section 2 for the independence results, we see that ω in $\mathcal{M}2$ is second countable, and therefore, first countable. The set of atoms, A in $\mathcal{N}1$, is first countable because for each $a \in A$, $\{\{a\}\}$ is a neighborhood base for a. The power set of the atoms, 2^A in $\mathcal{N}2$, is first countable. To show this, let $f \in 2^A$ and for each $n \in \omega$, let $U_n = \{g \in 2^A : (\forall i \leq n)(g(a_i) = f(a_i) \wedge g(b_i) = f(b_i))\}$. Then $\{U_n : n \in \omega\}$ is a countable neighborhood base at f. In addition, $(\omega_1, \leq)$ is first countable because if $\alpha \in \omega_1$, then $\{(\beta, \alpha+1) : \beta < \alpha\}$ is a neighborhood base at α. The set $K \subset \mathbb{R}$ defined in Theorem 3(i) is second countable. However, $(A, \leq)$ in $\mathcal{N}3$ and 2^{2^ω} are not first countable.

A matrix and implication diagram which shows the implication and independence relationships is given below.

	A	B	C	D	E	F	G	H
A	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$(A \rightarrow F)$ $\rightarrow AC$ [ho]	$\rightarrow$	$\rightarrow$
B	ω in $M2$ [tr]	$\rightarrow$	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]
C	2^A in $N2$	$\rightarrow$	$\rightarrow$	2^A in $N2$		$(C \rightarrow F)$ $\rightarrow AC$ [ho]	2^A in $N2$	2^A in $N2$
D	A in $N1$ [ho]	A in $N1$	A in $N1$ [ho]	$\rightarrow$	$\rightarrow$	A in $N1$	A in $N1$	$\rightarrow$
E	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$\rightarrow$	$(\omega_1, \leq)$	A in $N1$	$K \subset \mathbb{R}$ in $M1$
F						$\rightarrow$	$\rightarrow$	
G	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\mathbb{Z}, T)$	$(\omega_1, \leq)$	$\rightarrow$	$(\mathbb{Z}, T)$
H	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$(\omega_1, \leq)$	$\rightarrow$	$(\omega_1, \leq)$	A in $N1$	$\rightarrow$

$$\begin{array}{ccccccccccc} E & \longleftarrow & H & \longleftarrow & D & \longleftarrow & A & \longrightarrow & C & \longrightarrow & B \\ & & & & & & \downarrow & & & & \\ & & & & F & \longrightarrow & G & & & & \end{array}$$

Assuming AC, we obtain that compact E and H are equivalent. (See [k] p162.) Thus, the diagram in this case becomes:

$$\boxed{A,\ B,\ C,\ D,\ F} \longrightarrow \boxed{E,\ H} \longrightarrow G$$

(It is also interesting to note that if the space is first countable and T_1, then compact G implies compact E. In this case compact E, H, and G are equivalent. See Lemma 7 below.)

4. Second Countable Topological Spaces

A second countable space is one that has a countable base for the topology, therefore, it is also first countable. For these spaces we have the following additional results.

Theorem 7. *For second countable spaces, compact A, D and H are equivalent and compact E implies compact G.*

Proof. Compact D implies compact H from Theorem 2. It is clear that for second countable spaces, compact H implies compact A. (See 6 in the list of statements in Group A.)

Assume X is an infinite set which is compact E and let $\mathcal{B} = \{U_n : n \in \omega\}$ be a countable open base for X. Suppose Y is an infinite subset of X with no accumulation point. Therefore, for each $y \in Y$ there is there is an $n \in \omega$ such that $U_n \cap Y = \{y\}$. Let n_y be the smallest natural number such that $U_{n_y} \cap Y = \{y\}$. Clearly, if $y \neq z$, $n_y \neq n_z$. Therefore, it follows that Y has a countably infinite subset W and, since X is compact E, W has a convergent subsequence. S. The limit point of S is an accumulation point of Y. This contradiction proves the theorem. □

To prove independence results, we note that ω with the discrete topology in $\mathcal{M}2$ is second countable so the results we had previously for the independence of compact B are true here. The set $K \subset \mathbb{R}$ defined in Theorem 3(i) is second countable. In addition, the space $(\mathbb{Z}, \mathcal{T})$, described in Theorem 3(h), is second countable. Our results are below.

$$F \longrightarrow G \longleftarrow E \longleftarrow \boxed{A, D, H} \longrightarrow C \longrightarrow B$$

	A, D, H	B	C	E	F	G
A, D, H	$\to$	$\to$	$\to$	$\to$	$(A \to F)$ $\to AC$ [ho]	$\to$
B	ω in $M2$ [tr]	$\to$	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]
C	$(C \to A)$ $\to$ BPI [p] [wo]	$\to$	$\to$		$(C \to F)$ $\to AC$ [ho]	
E	$K \subset \mathbb{R}$ in $M1$	$K \subset \mathbb{R}$ in $M1$	$K \subset \mathbb{R}$ in $M1$	$\to$	$(E \to F)$ $\to AC$ [ho]	
F					$\to$	$\to$
G	$K \subset \mathbb{R}$ in $M1$	$K \subset \mathbb{R}$ in $M1$	$K \subset \mathbb{R}$ in $M1$	$(\mathbb{Z}, \mathcal{T})$	$(G \to F)$ $\to AC$ [ho]	$\to$

In ZFC, the additional result we have is that that compact E implies compact A. (See [k] p138.) Thus, for second countable spaces in ZFC, we have the following implication diagram.

$$\boxed{A, B, C, D, E, F, H}$$
$$\downarrow$$
$$G$$

In addition, if a space is second countable and T_1 then compact G implies compact E, so all these forms of compactness are equivalent. (See Lemma 7 below.)

5. Lindelöf Spaces

Lindelöf spaces have the property that every open cover has a countable subcover. They have many of the same properties as second countable spaces.

It is interesting to note that the axiom of choice for a countable number of subsets of $\mathbb{R}$ is equivalent to each of the following statements in ZF:

1. ω with the discrete topology is Lindelöf.
2. $\mathbb{R}$, the set of real numbers with the order topology, is Lindelöf.
3. $\mathbb{Q}$, the set of rational numbers with the order topology, is Lindelöf.
4. Every second countable topological space is Lindelöf.

See form 94 in [hr] for additional equivalent statements.

Theorem 8. *For Lindelöf spaces, compact A, D and H are equivalent and compact E implies compact G.*

The proof is similar to the proof of Theorem 7.

Theorem 9. *For Lindelöf spaces, if compact E implies compact A in ZF, then the axiom of choice for a countable number of sets (CAC) holds.*

Proof. Suppose compact E implies compact A and let $\mathcal{A}$ be a countable family of non-empty, pairwise disjoint sets. We write $\mathcal{A}$ as $\{A_k : k \in \mathbb{Z}\}$. Define a topological space $(X, \mathcal{T})$ as follows: $X = \bigcup \mathcal{A}$ and

$$\mathcal{T} = \{\bigcup_{i \geq k} A_i : k \in \mathbb{Z}\} \cup \{\emptyset, X\}.$$

The space $(X, \mathcal{T})$ is second countable and Lindelöf (there are only countably many open sets). It is clear that $(X, \mathcal{T})$ is not compact A, and therefore, by hypothesis, not compact E. Let $\langle x_n : n \in \omega \rangle$ be a sequence with no convergent subsequence. We shall show that $S_x \cap A_k$ is finite for each $k \in \mathbb{Z}$, where $S_x = \{x_n : n \in \omega\}$. Fix $k \in \mathbb{Z}$ and let $\langle y_n : n \in \omega \rangle$ be a sequence contained in A_k. Let U be any open set containing y_0. Clearly,

$$\{y_n : n \in \omega\} \subset A_k \subset \bigcup_{i \geq k} A_i \subset U.$$

Thus, $\langle y_n : n \in \omega \rangle$ converges to y_0. Since $\langle x_n : n \in \omega \rangle$ has no convergent subsequence,S_x can only have a finite intersection with each A_k.

Let $F = \{k \in \mathbb{Z} : S_x \cap A_k \neq \emptyset\}$. Since S_x intersects each A_k at most in a finite set, F must be infinite. For each $k \in F$ define $n(k) \in \omega$ to be least natural number such that $x_{n(k)} \in A_k$. Then $\{x_{n(k)} : k \in F\}$ is a choice set for $\{A_k : k \in F\}$. Therefore, it follows from [m] that CAC holds. □

In the model $\mathcal{M}2$ in [hr], ω with the discrete topology is Lindelöf (see [94 A] in [hr]) and compact B (because there is no ultrafilter on ω in $\mathcal{M}2$), but it is not compact A, C, D, E, F, G, or H. The space $(\mathbb{Z}, \mathcal{T})$ defined in the proof of Theorem 3(h) is also Lindelöf. Also, by Theorem 3(f), if BPI holds, then compact A, B, and C are equivalent.

The arrow diagram and the matrix for Lindelöf spaces are below.

$$\begin{array}{ccccccc} & & \boxed{A, D, H} & \longrightarrow & C & \longrightarrow & B \\ & & \downarrow & & & & \\ F & \longrightarrow & G & \longleftarrow & E & & \end{array}$$

	A, D, H	B	C	E	F	G
A, D, H	$\rightarrow$	$\rightarrow$	$\rightarrow$		$(A \rightarrow F)$ $\rightarrow AC$ [ho]	$\rightarrow$
B	ω in $M2$ [tr]	$\rightarrow$	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]
C	$(C \rightarrow A)$ $\rightarrow BPI$ [p],[wo]	$\rightarrow$	$\rightarrow$		$(C \rightarrow F)$ $\rightarrow AC$ [ho]	
E	$(E \rightarrow A)$ $\rightarrow CAC$	BPI + $(E \rightarrow B)$ $\rightarrow CAC$	BPI + $(E \rightarrow C)$ $\rightarrow CAC$	$\rightarrow$	$(E \rightarrow F)$ $\rightarrow AC$ [ho]	$\rightarrow$
F					$\rightarrow$	$\rightarrow$
G	$(\mathbb{Z}, \mathcal{T})$	$(\mathbb{Z}, \mathcal{T})$	$(\mathbb{Z}, \mathcal{T})$	$(\mathbb{Z}, \mathcal{T})$	$(G \rightarrow F)$ $\rightarrow AC$ [ho]	$\rightarrow$

Moreover, using a proof similar to the proof in [k] that compact E implies compact A, we can show using the countable axiom of choice that if X is Lindelöf and compact E, then it is compact A. Thus, in ZFC, the diagram for Lindelöf spaces is the same as that for second countable spaces.

Another interesting result is given in the following Theorem.

Theorem 10. *In $\mathcal{N}2$, 2^A is linearly Lindelöf (every nest of open sets that covers 2^A has a countable subcover), but not compact A or D and not Lindelöf.*

Proof. See the proof of Theorem 3 (c) for a description of $\mathcal{N}2$. Let A be the set of atoms. For $n \in \omega$ define four basic open sets B_n^j, $j = 1, 2, 3, 4$ in 2^A as follows:
1. $B_n^1 = \{f \in 2^A : f(a_{2n}) = f(b_{2n})$ and $f(a_{2n+1}) = 0\}$,
2. $B_n^2 = \{f \in 2^A : f(a_{2n}) = f(b_{2n})$ and $f(a_{2n+1}) = 1\}$,
3. $B_n^3 = \{f \in 2^A : f(a_{2n}) = f(b_{2n})$ and $f(b_{2n+1}) = 0\}$, and
4. $B_n^4 = \{f \in 2^A : f(a_{2n}) = f(b_{2n})$ and $f(b_{2n+1}) = 1\}$.

Then for any permutation ϕ in the group determining the model, if $\phi(a_{2n+1}) = b_{2n+1}$ then ϕ interchanges B_n^1 and B_n^3 and interchanges B_n^2 and B_n^4. Otherwise ϕ fixes each of B_n^j, $j = 1, 2, 3$, and 4. It follows that the set $\mathcal{B} = \{B_n^j : n \in \omega \wedge j \in \{1, 2, 3, 4\}\}$ is in the model. Also, $\mathcal{B}$ is a cover of 2^A because if $f \in 2^A$ with support $\{a_0, b_0\} \cup \{a_1, b_1\} \cup \cdots \cup \{a_{2k+1}, b_{2k+1}\}$, then f must have the property that $f(a_i) = f(b_i)$ for $i > 2k+1$ and, therefore, $f \in B_{k+1}^1 \cup B_{k+1}^2$. We claim that for each $k \in \omega$, $\mathcal{B}_k = \{B_n^j : n \leq k \wedge j \in \{1, 2, 3, 4\}\}$ fails to cover 2^A. This follows because any $f \in 2^A$ such that $f(a_{2n}) \neq f(b_{2n})$ for $n = 1, 2, \ldots k$ fails to be in the union of $\mathcal{B}_k$. For each $n \in \omega$ let $C_n = B_n^1 \cup B_n^2 \cup B_n^3 \cup B_n^4$. Let $\mathcal{C} = \{C_n : n \in \omega\}$ and $\mathcal{N} = \{C_0, C_0 \cup C_1, C_0 \cup C_1 \cup C_2, \ldots\}$, then $\mathcal{C}$ is an open cover

of 2^A with no countable subcover and $\mathcal{N}$ is a nest of open sets which covers 2^A, but has no finite subcover. Therefore, 2^A is not compact A or D and is not Lindelöf.
To complete the proof we shall show that 2^A is linearly Lindelöf. We prove first that every linearly ordered set in $\mathcal{N}2$ is well orderable. Assume Y is linearly ordered by $\leq$ and that the pair $(Y, \leq)$ has finite support E. It suffices to show that every element of Y has support E. If some element x of Y does not have support E, then there is a $\phi \in \text{fix}_G(E)$. ($\text{fix}_G(E)$ is the set of permutations in the group determining the model which fixes E pointwise.) such that $\phi(x) \neq x$. Since ϕ fixes Y, $\phi(x) \in Y$. Since $\leq$ is a linear ordering, either $x < \phi(x)$ or $\phi(x) < x$, but not both. Assume $x < \phi(x)$. Then since ϕ^2 is the identity on A, and ϕ fixes $\leq$, $\phi(x < \phi(x))$ is $\phi(x) < x$, which is a contradiction. Thus, Y so is well orderable.

Let N in $\mathcal{N}2$ be a nest of open sets which covers 2^A. By the preceding result, N is well orderable in $\mathcal{N}2$. This implies that every subset of N is in $\mathcal{N}2$ and that every subset of N which is countable (in the ground model) is countable in $N2$. For finite $E \subseteq A$ and $f : E \to 2$, let $B_{E,f} = \{g \in 2^A : (\forall t \in E)(g(t) = f(t))\}$. Then since A is countable in the ground model, the basis $\mathcal{B} = \{B_{E,f} : E \text{ is a finite subset of } A \text{ and } f : E \to 2\}$ for 2^A is countable in the ground model. For each $B \in \mathcal{B}$ let $F(B)$ be some element of N containing B if there is such an element of N. F is in the ground model and the range of F is a countable subcovering of N which is in the ground model and, therefore, in $\mathcal{N}2$ and which is countable in $\mathcal{N}2$. □

6. The Boolean Prime Ideal Theorem

In this section, we assume BPI holds. Thus, it follows that compact A, B, and C are equivalent. (See Theorem 3 (f).) Also, see Theorem 4 for properties of spaces when BPI is true.

In Theorem 3(a) we have shown that the space 2^{2^ω} is compact D and F, but not compact E so we shall use that in the matrix below.

We get the following arrow diagram and matrix if BPI holds. As we have mentioned above, BPI is true in models $\mathcal{M}1$ and $\mathcal{N}3$. In the matrix below "$K \subseteq \mathbb{R}$ in $\mathcal{M}1$" could be replaced by "$(A, \leq)$ in $\mathcal{N}3$".

$$\begin{array}{l} \boxed{A, \;\; B, \;\; C} \longrightarrow D \longrightarrow H \\ \qquad \downarrow \\ F \rightarrow G \end{array}$$

	A, B, C	D	E	F	G	H
A, B, C	$\to$	$\to$	2^{2^ω}	$(A \to F)$ $\to AC$ [ho]	$\to$	$\to$
D	(ω, τ) in $M1$	$\to$	2^{2^ω}	$(D \to F)$ $\to AC$ [ho]		$\to$
E	$K \subseteq \mathbb{R}$ in $M1$	$K \subseteq \mathbb{R}$ in $M1$	$\to$	$(E \to F)$ $\to AC$ [ho]	$(A, \mathcal{D})$ in $N3$	$K \subseteq \mathbb{R}$ in $M1$
F	(ω, τ) in $M1$		2^{2^ω}	$\to$	$\to$	
G	$K \subseteq \mathbb{R}$ in $M1$	$K \subseteq \mathbb{R}$ in $M1$	2^{2^ω}	$(G \to F)$ $\to AC$ [ho]	$\to$	$K \subseteq \mathbb{R}$ in $M1$
H	$(\omega_1, \leq)$	$(\omega_1, \leq)$	2^{2^ω}	$(\omega_1, \leq)$	$(A, \mathcal{D})$ in $N3$	$\to$

If we assume AC, we get the same arrow diagram as we did for general topological spaces in section 2. Compact A, B, C, D, and F are equivalent and imply compact G and H, and compact E implies compact G and H.

7. Sets of Real Numbers with the Order Topology

The set of real numbers with the order topology is second countable and separable. In the basic Cohen model, $\mathcal{M}1$, the set K mentioned above is an infinite unbounded set of Dedekind finite real numbers that is not separable. However, K is compact E because it has no countably infinite subset. It is also compact G, because if an infinite subset had no accumulation point, then we can prove it has a countably infinite subset. (See [j] page 141.) Thus, K is compact E and G, but not compact A, C or D.

Lemma 6. *In $\mathcal{M}1$, $[0,1]$ is compact A, B, C, D, E, G, and H, but not compact F.*

Proof. It is clear that $[0,1]$ is compact A, and, therefore, compact B, C, D, E, and G. Let K' be an infinite Dedekind finite set of reals contained in $[0,1]$. Then, K' has no complete accumulation point because no proper subset of a Dedekind finite set has the same cardinality as the set. Consequently, no accumulation point of K' is complete. □

Lemma 7. *For first countable T_1 spaces, compact G implies compact E.*

Proof. Let X be a space that satisfies the hypothesis and let s be a sequence in X which has x has an accumulation point. By hypothesis, x has a countable open neighborhood base. Then using this base and the fact that the space is T_1, it is easy to construct a subsequence of s that converges to x. (Also see Theorem 16.1 in [th] p120.) □

Since any set of reals with the order topology is first countable and T_1, (in fact it is second countable and T_2), for the real numbers, compact G implies compact E.

The arrow diagram and the matrix for the reals is below.

$$F \longrightarrow \boxed{G, E} \longleftarrow \boxed{A, D, H} \longrightarrow C \longrightarrow B$$

	A, D, H	B	C	E, G	F
A, D, H	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$[0, 1]$ in $M1$
B	ω in $M2$ [tr]	$\rightarrow$	ω in $M2$ [tr]	ω in $M2$ [tr]	ω in $M2$ [tr]
C		$\rightarrow$	$\rightarrow$		$[0, 1]$ in $M1$
E, G	$K \subseteq \mathbb{R}$ in $M1$	$K \subseteq \mathbb{R}$ in $M1$	$K \subseteq \mathbb{R}$ in $M1$	$\rightarrow$	$[0, 1]$ in $M1$
F				$\rightarrow$	$\rightarrow$

In ZFC, for subsets of the reals, compact A–H, are all equivalent.

BIBLIOGRAPHY

[al] P. Alexandroff, *Über die Metrisation der Kleinen kompakten topologischen Räume* Math. Ann. **92** (1924) 294–301.

[c] E. C̃ech, *On bicompact spaces* Annals of Math., bf 38 (1937) 823-844.

[co] W. W. Comfort,*A theorem of Stone-C̃ech Type, and a theorem of Tychonoff Type, without the axiom of choice and their real compact analogues*, Fund. Math, bf 43 (1968) 97–110.

[fr] M. Fréchet, *Sur quelque points du calcul fonctionnel* Rend. Palermo, **22** (1906) 1–74.

[gj] L. Gillman and M. Jerrison, *Rings of Continuous Functions*, Van Nostrand, 1960.

[he] H. Herrlich, *Compactness and the axiom of choice*, Appl. Categ. Structures, **4** (1996) 1-14.

[ho] P. Howard, *Definitions of compactness*, J. Symbolic Logic **55** (1990) 645–655.

[hr] P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, AMS, Math. Surveys and Monographs Vol. 59, 1998. (http://www.math.purdue.edu/ ˜ jer/cgi-bin/conseq.html)

[j] T. J. Jech, *The Axiom of Choice*, North-Holland Publ. Co., 1973.

[k] J. L. Kelley, *General Topology.* Springer-Verlag. second edition 1985.

[m] G. P. Monroe, *Independence results comcerning Dedekind finite sets*, J. Austral. Math. Soc. **19** (1975) 35-46.

[mo] G. H. Moore, *Zermelo's Axiom of Choice*, Springer- Verlag, 1982.

[p] I. I. Parovičenko, *Topological equivalents of the Tihonov theorem*, Math. Doklady, **10** (1969) 33-34.

[ri] F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti IV Congr. Intern. Mat. Roma, II (1908) 18–24.

[s] W. Sierpinski, *General Topology*, University of Toronto Press, 1956.

[th] W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, 1966.

[tr] J. K. Truss, *The axiom of choice for linearly ordered families*, Fund. Math. **99** (1978) 122-139.

[ty] A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Annalen 102, (1930) 544-561.

[ur] P. Urysohn, *Über die Metrisation der kompakten topologischen Räume*, Math. Ann. **92** (1924) 275–293.

[w] S. Willard, *General Topology*, Addison-Wesley Publ. Co., 1968.

[we] K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen reeller Argumente*, S. B. Deutsch Akad. Wiss. Berlin KL. Math. Phys. Tech. (1885) 633-639, 789-805.

[wo] E. S. Wolk *On theorems of Tychonoff, Alexander, and R. Rado*, Proc. AMS **18** (1967) 113–115.

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