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FAIR ALLOCATION OF INDIVISIBLE GOODS AND CRITERIA OF JUSTICE

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A set of n objects and an amount M of money is to be distributed among m people. Example: the objects are tasks and the money is compensation from a fixed budget. An elementary argument via constrained optimization shows that for M sufficiently large the set of efficient, envy free allocations is nonempty and has a nice structure. In particular, various criteria of justice lead to unique best fair allocations which are well behaved with respect to changes of M. This is in sharp contrast to the usual fair division theory with divisible goods.

KEYWORDS: Fairness, envy-free, Pareto efficient, justice, Rawlsian justice.

1. INTRODUCTION

AS IS WELL KNOWN, competitive equilibrium theory runs into difficulties when one considers the case of indivisible rather than divisible goods. However, recent studies have shown that for the special case where there is a single divisible good, usually thought of as money, and when each agent gets at most one indivisible good, the situation becomes tractable and in fact it exhibits structural features which are not present in the usual theory with divisible goods. For example, Quinzii (1984) shows that the core coincides with the set of competitive equilibria and in Demange and Gale (1985) it is shown that the set of equilibria have the structure of a lattice.

The present work continues the study of this model in the context of the problem of so called fair allocation. There has been an extensive literature dealing with this problem starting with the introduction of the no-envy concept by Foley (1967) and the general formulation of the problem by Varian (1974). However, almost all of this work has treated the case of divisible goods (the exceptions will be noted in subsequent references). We here treat the discrete case.

Thus, a set of objects and an amount of money are to be distributed to a group of individuals in a manner which is (Pareto) efficient and envy free (everyone likes his own allocation at least as well as that of anyone else). This problem has been treated by Maskin (1982) for the case where all the objects are desirable and there are the same number of people as objects. He shows that if there is enough money, in a suitable sense, then such allocations will exist. We will here consider a somewhat more general model with particular emphasis on its structure (e.g. comparative statics) and especially the possibility of making (single valued) selections of especially desirable allocations from the point of view of justice from the set of all fair allocations. Briefly, here are some of the features of our presentation.

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A. *Generality*

We allow any number of people and objects. Further, the objects may be undesirable, for example, tasks which must be performed, and the amounts of money may be negative as, for example, costs to be shared by the people. Indeed the most natural real life examples seem to be of these types.

EXAMPLE I: A group of people, say the members of an academic department, are to be assigned various administrative tasks for which they are to be compensated from the department's fixed administrative budget. All members are assumed to be equally qualified for the jobs but they differ in the extent to which they like or dislike the jobs. Who should get which job and at what level of compensation? In this problem it is not unreasonable to consider the possibility of negative compensation to some of the people, the interpretation being that some of the department members are willing to pay others for doing the jobs they don't want to do themselves.

EXAMPLE II: A group of students share a house for which they pay a fixed rent. The rooms are of various qualities, e.g. size, convenience, quietness, etc. The fraction of the rent paid by each student is to depend in a fair way on the room he gets. Here in order to get the fair allocation model we must assume the total amount of money to be divided, that is, the rent for the house, is negative. In this example it is also natural to impose additional constraints. For example, no student should have to pay more for a room than he thinks it is worth (an "individual rationality" condition) and/or no student should pay a negative rent which would mean being paid by the others to occupy a room. We will consider such additional constraints in Section 4.

B. *Existence*

Simple necessary and sufficient conditions are given for the existence of fair allocations. The notable feature here is that the existence proof, unlike those of almost all other equilibrium existence theorems, is elementary in the sense that it does not use fixed point theorems but obtains the desired allocation by solving a constrained optimization problem. The proof is in fact constructive and can be used to give an algorithm for finding a fair allocation (Alkan (1989)). The only mathematical tool needed is the duality theorem for the optimal assignment problem which will be summarized briefly in Section 3. The main advantage to our proof, however, is that because of its constructive nature we are able to derive qualitative properties of fair allocations which do not seem to be obtainable by the usual fixed point methods (in fact it seems the fixed point methods will not work even for proving existence for the case when there are more objects than people) and we are able to show that the set of fair allocations has a strong connectedness property which is a nonlinear generalization of convexity.

C. *Comparative Statics*

The main result here shows that if there are at least as many people as objects and one has a fair allocation, then if the amount of money is increased, there is a new fair allocation which makes everyone strictly better off. Results of this sort typically fail to hold for the case of divisible goods. Even in the present case examples show that if there are more objects than people, then an increase in the supply of money may necessarily make one of the people worse off in any fair allocation.

Still on comparative statics, what happens if one adds new desirable objects? A simple example shows that this may make some of the people worse off. However, a surprising result shows that there is a unique fair allocation (later to be called the minmax money allocation) with the property that, if one adds just one new object, then there is a new fair allocation in which no one is made worse off.

D. *Selection*

The main results of this paper concern the relation of fair allocations and various notions of justice. In this connection it has been noted (see, e.g., Holcombe (1983)) that using the word fair for efficient and envy free does not correspond to the everyday notion of fairness. To see how unfair a fair allocation can be, consider the case of two people and two objects, say an apple and an orange, and some money. Person A prefers the apple to all the money and values the orange at zero (by the value of an object to a person we mean the amount of money he would give for the object) while for person B it is the other way around. Then a possible fair allocation would give the apple and all of the money to A and only the orange to B. This arrangement seems unjust, and a more equitable procedure would be, for example, to divide the money equally. A second possibility would be to divide the money so that both people receive equal values. Thus if the apple is worth 10 to A and the orange is worth 8 to B and there are 6 units of money, perhaps A should get 2 and B should get 4. In general, egalitarian considerations suggest that a just allocation should divide either money or value "as equally as possible" subject of course to the requirement of fairness. The question is then to decide on the definition of "as equally as possible" and a pleasant and, to us, unexpected result is that if one adopts the Rawlsian criterion of achieving the greatest satisfaction for the least well off person, this selects a *unique* allocation from the set of fair allocations. The Rawls criterion is in fact just one of a collection of criteria with this uniqueness property. A second is what we have called the *anti elitist* criterion in which the richest person is to be made as poor as possible. As a third rather extreme example, the object may be to maximize the satisfaction of one particular person. Again it turns out that there is only one fair allocation (up to indifferent redistribution of objects) which will achieve this. Further, all of these criteria lead to selections with strong monotonicity properties meaning that when the amount of money is increased all people are made strictly better off.[2]

It is worth pointing out that none of the above nice properties of our fair allocation problem hold for the traditional case in which all goods are divisible (see, e.g., Moulin and Thomson (1988)).

2. THE MODEL AND FAIR ALLOCATIONS

Let $\mathscr{P}$ be a set of m people with members $A, B, \ldots$ and let $\mathscr{O}$ be a set of n objects with members $\alpha, \beta, \ldots$. A *bundle* is a pair (α, x) consisting of an object α and x units of money. We will wish to consider bundles containing only money and no object, so for notational convenience we will introduce a *null* or worthless object denoted by $\varnothing$. Thus, a bundle containing only money will be denoted by $(\varnothing, x)$. Each A in $\mathscr{P}$ is assumed to have a preference ordering $\succcurlyeq_A$ on the set of bundles which is continuous and increasing in money. Thus

$$(1) \qquad x > y \quad \text{implies} \quad (\alpha, x) \underset{A}{\succ} (\alpha, y) \qquad \text{for all } A \text{ and } \alpha.$$

We also want to assume that no object is infinitely desirable or undesirable as compared with money, which is expressed as

$$(2) \qquad \begin{array}{l} \text{for any bundle } (\alpha, x) \text{ and person } A \text{ there is an} \\ \text{amount of money } v \text{ such that } (\alpha, x) \underset{A}{\sim} (\varnothing, v). \end{array}$$

From (1) the number v is unique and is called the *value* of the bundle (α, x) to person A. This defines the *value* function of person A by

$$v = \phi_A(\alpha, x).$$

The function ϕ_A should not be confused with utility functions which we do not use here. Value functions unlike utility functions are directly observable and are sometimes referred to as willingness to pay.

It is convenient to reduce the general problem to one in which there are the same number of people as objects. This is easily done for the case $m > n$ (objects are "scarce") by introducing $m - n$ null objects (e.g., worthless pieces of paper). If $m < n$ we introduce $n - m$ *fictitious* people who value only money. Thus if A is fictitious, then $\phi_A(\alpha, x) = x$ for all α. We denote real and fictitious people by $\mathscr{P}_R$ and $\mathscr{P}_F$ respectively.

An *allocation* $\mathscr{A}$ is an assignment of bundles to people where no two people receive the same object. Formally an allocation is a triple $(\sigma, \bar{v}, \bar{x})$ where σ is a bijection from $\mathscr{P}$ to $\mathscr{O}$, $\bar{x}$ is an $\mathscr{O}$-vector associating x_α dollars with the object α, and $\bar{v}$ is a $\mathscr{P}$-vector defined by $v_A = \phi_A(\sigma(A), x_{\sigma(A)})$. This is the value to A of his/her bundle.

[2] A rather different set of selection criteria are given in Tadenuma and Thomson (1989).

Our key definition is the following:
An allocation is *envy free* if

(3) $v_A \geqslant \phi_A(\alpha, x_\alpha)$ for all α and all A in $\mathscr{P}_R$

and it is *strongly envy free* if (3) holds also for A in $\mathscr{P}_F$.

An equivalent definition of strongly envy free which does not involve fictitious players is the following: a person not only prefers his bundle to that of anyone else but also prefers it to a bundle consisting of any unassigned object accompanied by y dollars where y is the maximum amount of money assigned to any person; thus $y = \max_{\mathscr{P}} x_{\sigma(A)}$. The reader should check that this is equivalent to the definition in terms of fictitious players. The notion of strongly envy-free is essential for the proof of existence in the case where there are more objects than people.

Suppose now the total amount of money is X. An allocation is called *X-feasible* if

(4) $$\sum_{A \in \mathscr{P}_R} x_{\sigma(A)} = X$$

which means simply that all the money is distributed to the people.

In the literature a feasible allocation has been called *fair* if it is Pareto efficient and envy-free. Svensson (1983) has noted that for the case $m = n$ the condition of efficiency is redundant since it is implied by envy-freeness. Our first result extends this observation.

THEOREM 1: *A feasible, strongly envy-free allocation is efficient.*

The proof uses the following basic lemma.

LEMMA 1: *Let* $(\sigma, \bar{v}, \bar{x})$ *be strongly envy free and let* $(\tau, \bar{w}, \bar{y})$ *be any allocation and let* $\tau(A) = \alpha$. *Then*

(5) $w_A > v_A$ *implies* $y_\alpha > x_\alpha$,

(6) $w_A \geqslant v_A$ *implies* $y_\alpha \geqslant x_\alpha$.

PROOF: By hypothesis, $\phi_A(\alpha, y_\alpha) = w_A > v_A$ and from (3), $v_A \geqslant \phi_A(\alpha, x_\alpha)$, so $\phi_A(\alpha, y_\alpha) > \phi_A(\alpha, x_\alpha)$, so $y_\alpha > x_a$ since ϕ_A is increasing. The proof of (6) is similar. *Q.E.D.*

PROOF OF THEOREM: Let $\mathscr{A} = (\sigma, \bar{v}, \bar{x})$ be feasible and strongly envy free. Then $A \in \mathscr{P}_F$, $B \in \mathscr{P}_R$ implies $x_{\sigma(A)} \geqslant x_{\sigma(B)}$ (because $\phi_A(\alpha, x) = x$). This means if S is any $|\mathscr{P}_R|$ element subset of $\mathscr{O}$, then

(7) $$\sum_{\alpha \in S} x_\alpha \geqslant X.$$

Now if $(\tau, \bar{w}, \bar{y})$ dominates $(\sigma, \bar{v}, \bar{x})$, then $w_A \geqslant v_A$ for all A and $w_{A_0} > v_{A_0}$ for some A_0, so from the lemma $y_{\tau(A)} \geqslant x_{\tau(A)}$ and $y_{\tau(A_0)} > x_{\tau(A_0)}$. So from (4) $\sum_{\alpha \in \tau(\mathscr{P}_R)} y_\alpha > \sum_{\alpha \in \tau(\mathscr{P}_R)} x_a = X$. So (τ, w, y) is not feasible. *Q.E.D.*

A strongly envy free feasible allocation will henceforth be called *strongly fair*. We remark that for the result above one needs the condition that the allocation is strongly envy free. As a simple example, suppose there is one person A and there are two objects α and β and A prefers (α, x) to (β, x) for all x, but he prefers (β, X) to $(\alpha, 0)$. Then allocating (β, X) to A is envy free but clearly not efficient as it is dominated by giving him the bundle (α, X).

3. EXISTENCE OF FAIR ALLOCATIONS

As mentioned in the introduction the basic tool needed in our analysis is the duality theorem for the optimal assignment problem which we now recall.

Given an $\mathscr{P} \times \mathscr{O}$ matrix $C = (c_{A\alpha})$, an assignment (bijection) σ from $\mathscr{P}$ to $\mathscr{O}$ is an *optimal assignment* if $\sum_{\mathscr{P}} c_{A\sigma(A)} \geqslant \sum_{\mathscr{P}} c_{A\tau(A)}$ where τ is any other assignment.

DUALITY THEOREM: *If σ is an optimal assignment, then there is a $\mathscr{P}$-vector $\bar{v}$ and an $\mathscr{O}$-vector $\bar{x}$ such that*

(8) $$v_A - x_\alpha \geqslant c_{A\alpha} \qquad \text{for all } A \text{ and } \alpha$$

and

(9) $$v_A - x_{\sigma(A)} = c_{A\sigma(A)}.$$

This result can be proved from linear programming duality or by a direct combinatorial argument (see, e.g., Gale (1960)). The result is also exploited by Shapley and Shubik (1972) in a related context.

A convenient equivalent formulation is given by eliminating v_A from (8) and (9), giving

(10) $$c_{A\sigma(A)} + x_{\sigma(A)} \geqslant c_{A\alpha} + x_\alpha \qquad \text{for all } A, \alpha.$$

An immediate consequence of (10) is the following multiplicative version which plays a key role in our existence proof.

LEMMA 2: *Let B be a nonnegative $\mathscr{P} \times \mathscr{O}$ matrix which contains a positive assignment, that is, there exists σ such that $b_{A\sigma(A)} > 0$ for all A in $\mathscr{P}$. Then there is an assignment τ and a positive $\mathscr{O}$-vector $\bar{d}$ such that*

(11) $$d_{\tau(A)} b_{A\tau(A)} \geqslant d_\alpha b_{A\alpha} \qquad \textit{for all } A, \alpha.$$

PROOF: Let $C = (C_{A\alpha})$ where

$$C_{A\alpha} = \begin{cases} \log(b_{A\alpha}) & \text{for } b_{A\alpha} > 0, \\ -\infty & \text{for } b_{A\alpha} = 0. \end{cases}$$

Since B contains a positive assignment the optimal assignment for C has a finite value. From (10) we have an $\mathscr{O}$-vector $\bar{x}$ such that $\log b_{A\sigma(A)} + x_{\sigma(A)} \geqslant \log b_{A\alpha} + x_\alpha$. Taking exponentials and letting $d_\alpha = e^{x\alpha}$ gives (11). *Q.E.D.*

The proof of existence proceeds in stages. We call the preference relation $\succcurlyeq_A$ *separable* if

$$(12) \qquad (\alpha, x) \succcurlyeq_A (\beta, y) \qquad \text{implies} \qquad (\alpha, x+\delta) \succcurlyeq (\beta, y+\delta)$$

for all α, x, y, δ. In words, separability means the marginal utility of money is independent of which object a person owns. In many situations this is quite an acceptable assumption. Thus in the student housing example the value of a dollar to a student probably does not depend on which room he occupies.

One easily sees that (12) is equivalent to assuming the value functions are given by $\phi_A(\alpha, x) = c_{A\alpha} + x$ where $c_{A\alpha} = \phi(\alpha, 0)$. The existence of envy-free allocations for this case now follows from optimal assignment duality since relation (10) is exactly (3), but also notice that these inequalities are invariant if a constant k is added to all x_α. By adding a suitable k one can therefore satisfy $\Sigma_{\mathscr{P}_R} x_{\sigma(A)} = X$ giving the desired fair allocation.

We will next prove the existence of fair allocations when the value functions are piecewise linear (i.e., the marginal utility of money is constant over a finite set of intervals). Since any continuous increasing function can be uniformly approximated by piecewise linear functions, the existence for the general case will then follow.

Let ϕ_A^+ denote the right-hand derivative of ϕ_A. Then piecewise linearity implies

$$(13) \qquad \phi_A(\alpha, x+d) = \phi_A(\alpha, x) + d\phi_A^+(\alpha, x)$$

for d positive and sufficiently small.

The key tool for everything which follows is the following comparative statics result.

PERTURBATION LEMMA: *If $(\sigma, \bar{v}, \bar{x})$ is envy free, then for $\varepsilon > 0$ there exists $(\tau, \bar{w}, \bar{y})$ envy free such that $\bar{y} \gg \bar{x}$ but $y_\alpha - x_\alpha < \varepsilon$ for all α.*

PROOF: Let D be all pairs (A, α) such that $\phi_A(\alpha, x_\alpha) = v_A$. Note that D contains the assigned pairs $(A, \sigma(A))$. Define the matrix B by

$$\begin{aligned} b_{A\alpha} &= \phi_A^+(\alpha, x_\alpha) \quad && \text{for } (\alpha, x_\alpha) \text{ in } D, \\ &= 0 && \text{otherwise.} \end{aligned}$$

From Lemma 2 there exists an assignment τ and positive vector $\bar{d}$ such that

$$(14) \qquad d_{\tau(A)} \phi_A^+(\tau(A), x_{\tau(A)}) \geqslant d_\alpha \phi_A^+(\alpha, x_\alpha).$$

We claim for $|\bar{d}|$ sufficiently small the allocation location $(\tau, \bar{w}, \bar{y})$ is envy free where $\bar{y} = \bar{x} + \bar{d}$ and where $|\bar{d}| = \max_{\alpha \in \mathscr{O}} d_\alpha$. We must show

$$(15) \qquad \phi_A\big(\tau(A), x_{\tau(A)} + d_{\tau(A)}\big) = \phi_A\big(\tau(A), x_{\tau(A)}\big) + d_{\tau(A)}\phi_A^+\big(\tau(A), d_{\tau(A)}\big)$$
$$= v_A + d_{\tau(A)}\phi^+\big(\tau(A), x_{\tau(A)}\big)$$
$$\geqslant \phi_A(\alpha, x_\alpha) + d_\alpha \phi_A^+(\alpha, x_\alpha).$$

There are two cases.

Case I: $(A, \alpha) \in D$. Then $\phi(\alpha, x_\alpha) = v_\alpha$ and (15) follows from (14).

Case II: $(a, \alpha) \notin D$. Then since $(\sigma, \bar{v}, \bar{x})$ is envy-free we have $\phi_A(\alpha, x_\alpha) < v_A$ so by choosing $|d|$ sufficiently small (15) is again satisfied. Finally $|d|$ can be made smaller if necessary so that $y_\alpha - x_\alpha = d_\alpha < \varepsilon$. *Q.E.D.*

Theorem 2: *There exists a strongly fair allocation.*

Proof: For the proof we will define the *altered value* functions as follows. Choose $M > X$ and such that

$$(16) \qquad \phi_A(\alpha, M) \geqslant \phi_A(\beta, 0) \qquad \text{for all } A, \alpha, \beta$$

and define $\hat{\phi}_A$ by

$$\hat{\phi}_A(\alpha, x) = \phi_A(\alpha, x) \qquad \text{for } x \leqslant M,$$
$$= \phi_A(\alpha, M) - M + X \quad \text{for } x > M,$$

and note that $\hat{\phi}_A$ is continuous, increasing, piecewise linear, and for the fictitious people $\hat{\phi}_A = \phi_A$. Let $c_{A\alpha} = \phi_A(\alpha, M) - M$ and let $(\sigma, \bar{v}, \bar{x})$ solve the separable case $\phi_A(\alpha, x) = c_{A\alpha} + x$. By adding a large constant to all x_α we may assume that $x_\alpha > M$ for all α; hence we have an envy-free allocation for the $\hat{\phi}_A$.

Now consider the set $\tilde{X}$ of all $\bar{x}$ such that there is an envy-free allocation for the $\hat{\phi}_A$ with

$$(17) \qquad \sum_{A \in \mathscr{P}_R} x_{\sigma(A)} \geqslant X.$$

This set is clearly closed and it is bounded below, for if for some A, $x_{\sigma(A)} < \min[0, X - mM]$, then from (17) there is some B such that $x_{\sigma(B)} > M$ so from (16) A would envy B. We can thus choose $(\sigma, \bar{v}, \bar{x})$ so that $\bar{x}$ is minimal in $\tilde{X}$. We claim that (17) is then satisfied as an equation, for if not, by the Perturbation Lemma one could find a smaller $\bar{x}$ in $\tilde{X}$ contradicting minimality of $\bar{x}$. Finally, we see that $(\sigma, \bar{v}, \bar{x})$ is fair for the original functions ϕ_A, that is, $\bar{x}_{\sigma(A)} \leqslant M$, for if $x_{\sigma(A)} > M$ for some A, then $x_{\sigma(B)} < 0$ for some B and B would envy A. *Q.E.D.*

4. CONSTRAINTS

In the existence proof of the preceding section it was essential that the money vector $\bar{x}$ could have positive or negative entries. In the usual fair allocation problem one requires that all quantities be nonnegative. If we insist on nonnegative $\bar{x}$, it is clear that envy-free allocations need not exist. For example, if there are m identical people and only one non-null object, then there must be at least enough money to compensate the people who don't get the object. Thus we must have $X \geqslant (m-1)v$ where v is the value (to everyone) of the object. For the general case where $\bar{x}$ is required to be nonnegative, the requirement is

$$(18) \qquad \phi_A(\beta, X/m-1) > \phi_A(\alpha, 0) \qquad \text{for all } \alpha, \beta, A,$$

which is a slight weakening of the sufficient condition given by Maskin. Still more generally there may be lower bounds on the $\bar{x}$ vector other than zero. Thus, if $c_{A\alpha}$ is the value of α to A, it is reasonable that A should not have to pay more than $c_{A\alpha}$ for α, so we would require

$$(19) \qquad x_{A\sigma(A)} \geqslant -c_{A\sigma(A)}.$$

An allocation satisfying (19) will be called *individually rational.*

THEOREM 3: *There exists an individually rational fair allocation provided X is sufficiently large so that*

$$(20) \qquad \phi_A\left(\beta, \frac{X + c_{A\alpha}}{m-1}\right) \geqslant \phi_A(\alpha, -c_{A\alpha}) \qquad \textit{for all } A, \alpha, \beta.$$

Let $(\sigma, \bar{v}, \bar{x})$ be a fair allocation and suppose

$$(21) \qquad x_{\sigma(\tilde{A})} < -c_{\tilde{A}(\sigma(\tilde{A})} \qquad \text{for some } \tilde{A}.$$

Then from feasibility,

$$X = \sum_{\mathscr{P}} x_{\alpha(A)} < -c_{\tilde{A}\sigma(\tilde{A})} + \sum_{A \neq \tilde{A}} x_{\sigma(A)}.$$

Hence for some B, $x_{\sigma(B)} > (X + c_{\tilde{A}(\sigma(\tilde{A})})/(m-1)$, so letting $\beta = \sigma(B)$ we have

$$\begin{aligned} \phi_A(\beta, x_\beta) &> \phi_A\big(\beta, \big(X + c_{\tilde{A}\sigma(\tilde{A})}\big)/(m-1)\big) \\ &\geqslant \phi_A\big(\sigma(A), -c_{A\sigma(A)}\big) \\ &\geqslant \phi\big(\sigma(A), x_{\sigma(A)}\big) \end{aligned}$$

from (20) and (21), so A would envy B. It follows that the allocation must be individually rational. *Q.E.D.*

It may also be natural to impose upper bounds on the x_α. In the house-renting example, for instance, it would be unusual to have any student paying a negative rent. An upper bound on the x_α requires an upper bound on X. In the house renting example the rent cannot be too low, for then again everyone might envy the person who gets the nicest room. An inequality like (20) with signs reversed is sufficient for existence in this case. We omit the details.

5. COMPARATIVE STATICS

The main result of this section is a strict monotonicity property of fair allocations. Namely, if $\mathscr{A}$ is strongly X-fair and $Y > X$, then there is a strongly Y-fair allocation which makes all people strictly better off. An example will show that the condition of strongness is needed. Namely there is an example with two people and three objects and an X-fair allocation such that if X is increased one of the people will be worse off in any new fair allocation.

We need the following basic fact. If $(\sigma, \bar{u}, \bar{x})$ and $(\tau, \bar{v}, \bar{y})$ are strongly envy free, then define the following sets

$$\mathscr{P}_u = \{A \mid u_A > v_A\}, \qquad \mathscr{O}_x = \{\alpha \mid x_\alpha > y_\alpha\},$$
$$\mathscr{P}_v = \{A \mid v_A > u_A\}, \qquad \mathscr{O}_y = \{\alpha \mid y_\alpha > x_\alpha\},$$
$$\mathscr{P}_0 = \{A \mid u_A = v_A\}, \qquad \mathscr{O}_0 = \{\alpha \mid x_\alpha = y_\alpha\}.$$

LEMMA 3 (Decomposition): *The mappings σ and τ are bijections between $\mathscr{P}_u$ and $\mathscr{O}_x$, $\mathscr{P}_v$ and $\mathscr{O}_y$, $\mathscr{P}_0$ and $\mathscr{O}_0$.*[3]

PROOF: Lemma 1 says precisely that σ maps $\mathscr{P}_u$ into $\mathscr{O}_x$ so $|\mathscr{P}_u| \leqslant |\mathscr{O}_x|$. Likewise τ maps $\mathscr{P}_v$ into $\mathscr{O}_y$ so $|\mathscr{P}_v| \leqq |\mathscr{O}_y|$ and σ and τ map $\mathscr{P}_0$ into $\mathscr{O}_0$ so $|\mathscr{P}_0| \leqq |\mathscr{O}_0|$ but since $|\mathscr{P}| = |\mathscr{O}|$ all of the above inequalities are equations, hence the mappings are bijections. *Q.E.D.*

A consequence of this lemma is the lattice property which will be used later. We introduce some standard notation. Given vectors $\bar{u}$ and $\bar{v}$, we define

$$\bar{w} = \bar{u} \wedge \bar{v} \quad \text{where} \quad w_A = \min(u_A, v_A),$$
$$\bar{w} = \bar{u} \vee \bar{v} \quad \text{where} \quad w = \max(u_A, v_A).$$

Referring to Lemma 3, define the bijection μ by

$$\mu(A) = \sigma(A) \quad \text{for } A \text{ in } \mathscr{P}_u,$$
$$= \tau(A) \quad \text{for } A \text{ in } \mathscr{P}_v \cup \mathscr{P}_0,$$

and let $\bar{w} = \bar{u} \wedge \bar{v}$, $\bar{z} = \bar{x} \wedge \bar{y}$.

LEMMA 4 (Lattice Property): *The allocation $(\mu, \bar{w}, \bar{z})$ is envy free.*

PROOF: We consider the following cases:

(i) $A \in \mathscr{P}_u$, $\alpha \in \mathscr{O}_x$. Then $u_A \geqslant \phi_A(\alpha, x_\alpha) = \phi_A(\alpha, z_\alpha)$ because σ is envy free.

(ii) $A \in \mathscr{P}_u$, $\alpha \in \mathscr{O}_0 \cup \mathscr{O}_y$. Then $\phi_A(\alpha, z_\alpha) = \phi_A(\alpha, y_\alpha) < v_A < u_A$ since τ is envy free.

[3] Proofs of Lemmas 3 and 4 are given in Demange and Gale (1985) and are included here to make the presentation self-contained.

(iii) $A \in \mathscr{P}_u \cup \mathscr{P}_0$, $\alpha \in \mathscr{O}_y \cup \mathscr{O}_0$, same argument as (i).
(iv) $A \in \mathscr{P}_v \cup \mathscr{P}_0$, $\alpha \in \mathscr{O}_x$, same argument as (ii). *Q.E.D.*

The same argument shows that if $(\sigma, \bar{u}, \bar{x})$ and $(\tau, \bar{v}, \bar{y})$ are envy free then there is a ν such that $(\nu, \bar{u} \wedge \bar{v}, \bar{x} \wedge \bar{y})$ is envy free.

THEOREM 4 (Strict Monotonicity): *If $(\sigma, \bar{u}, \bar{x})$ is strongly X-fair and $Y > X$, then there is a strongly Y-fair allocation $(\tau, \bar{v}, \bar{y})$ such that $\bar{v} \gg \bar{u}$.*

PROOF: The proof proceeds in two stages. We first consider again the case where the ϕ_A are piecewise linear. Let Z be the set of all $\bar{z}$ in $R^{\mathscr{O}}$ such that there is an envy free allocation $(\mu, \bar{w}, \bar{z})$ with $\bar{z} \geqslant \bar{x}$ and

$$(22) \qquad \sum_{A \in \mathscr{P}_R} z_{\mu(A)} \leqslant Y.$$

The set Z is closed and nonempty since it contains $\bar{x}$. Let $\bar{y}$ be a member of Z which maximizes the left side of (22). We claim $\sum_{A \in \mathscr{P}_R} y_{\tau(A)} = Y$, for if not, from the Perturbation Lemma there is $(\tau', \bar{v}', \bar{y}')$ with $\bar{y}' \gg \bar{y}$ in Z, contradicting maximality of $\bar{y}$. By uniform approximation this shows that for continuous ϕ_A there is a $(\tau, \bar{v}, \bar{y})$, $\bar{y} \geqslant \bar{x}$, but $\bar{y} = \bar{x}$ is impossible since $Y > X$. Hence $\bar{y} > \bar{x}$, so from Lemma 3, $\bar{v} > \bar{u}$.

We now wish to find an allocation with $\bar{v} \gg \bar{u}$, that is, such that $\mathscr{O}_0$ and $\mathscr{P}_0$ are empty. Suppose not. Now if $A \in \mathscr{P}_v$, then $v_A > u_A \geqslant \phi_A(\alpha, x_\alpha) = \phi_A(\alpha, y_\alpha)$ for α in $\mathscr{O}_0$ so no one in $\mathscr{P}_v$ desires an object in $\mathscr{O}_0$ under $(\tau, \bar{v}, \bar{y})$. Let $Y_0 = \sum_{A \in \mathscr{P}_0} Y_{\tau(A)}$ and let $Y_v = \sum_{A \in \mathscr{P}_R \cap \mathscr{P}_v} y_{\tau(A)}$. Then by weak monotonicity we can find strongly envy free allocation $(\tau^0, \bar{v}^0, \bar{y}^0)$ on $(\mathscr{P}_0, \mathscr{O}_0)$ with money $Y_0 + \varepsilon$ where $y_\alpha^0 \geqslant y_\alpha$ for $\alpha \in \mathscr{O}_0$ with strict inequality for some α. Similarly there is a strongly $(Y_v - \varepsilon)$-fair allocation on $(\mathscr{P}_v, \mathscr{O}_y)$ and by a previous remark, for ε small, the combined allocation will also be envy free, hence Y-fair. Thus, we have decreased the cardinality of $\mathscr{O}_0$ and this can be repeated until $\mathscr{O}_0$ is empty. *Q.E.D.*

Theorem 4 shows that increasing the amount of money can make everyone strictly better off. One might hope that the same would be the case if one increased the number of desirable objects. The following simple example shows that this is not true. There are three people A, B, C and one object α, say an apple, which A and B value at 6, C values at 1, and $X = 12$. For efficiency the object must go to A or B so if A gets the object and x dollars then B and C must each get $6 + x$ dollars, so the solution is $\bar{x} = (0, 6, 6)$ and $\bar{u} = (6, 6, 6)$. Now suppose there are two additional apples. Then everyone gets an apple and 4 dollars giving $\bar{x} = (4, 4, 4)$, $\bar{u} = (10, 10, 5)$, so C is worse off.

Note that in this example the phenomenon occurs when two additional objects are added. In Section 7 it will be shown that there is one particular allocation with the property that if a single new object is added there will always be a dominating fair assignment.

As mentioned, monotonicity with respect to money does not hold for fair allocations which are not strongly fair, as the following example shows.

There are people A, B, objects α, β, γ, and the value functions are

$$\phi_A(\alpha, x) = 4 + x, \qquad \phi_A(\beta, x) = x, \qquad \phi_A(\gamma, x) = 2 + x \quad \text{for } x \leqslant 6,$$
$$= 5 + \frac{x}{2} \quad \text{for } x \geqslant 6;$$

$$\phi_B(\alpha, x) = 6 + x \quad \text{for } x \leqslant 2, \qquad \phi_B(\beta, x) = 4 + x, \qquad \phi_B(\gamma, x) = x,$$
$$= 7 + \frac{x}{2} \quad \text{for } x \geqslant 2,$$

Let X be the amount of money and let σ assign A to α and B to β. Then for any distribution of X the total value $v_A + v_B$ will be $X + 8$. For $X \leqslant 8$ this total value can also be achieved by τ which assigns A to γ and B to α if and only if $x_\alpha \leqslant 2$ and $x_\gamma \leqslant 6$ as one sees by examining $\phi_A(\gamma, x)$ and $\phi_B(\alpha, x)$. It is also clear that no allocation can achieve total value greater than $X + 8$ and hence any efficient allocation must achieve this total value. From this and the preceding remark it follows that τ cannot belong to any efficient allocation for $X > 8$.

One now verifies directly that $(\tau, \bar{v}, \bar{x})$ is an 8-fair allocation where $\bar{v} = (8, 8)$ and $\bar{x} = (2, 0, 6)$. Suppose now the money supply is increased by an arbitrarily small amount to 8^+. Then from the above remarks any efficient allocation is of form $(\sigma, \bar{v}, \bar{x})$ where $x = (x_\alpha, x_\beta, 0)$. We will show that v_A drops from 8 to at most $7\frac{1}{3}$, for since B must not envy A we must have

$$\phi_B(\beta, x_\beta) = 4 + x_\beta = 4 + 8^+ - x_\alpha = 12^+ - x_\alpha \geqslant \phi_B(\alpha, x_\alpha) \geqslant 7 + x_\alpha/2$$

or $x_\alpha \leqslant (10/3)^+$ and hence $v_A \leqslant 7\frac{1}{3}^+$. Thus we see that an arbitrarily small increase in X has produced a decrease in value of $\frac{2}{3}$ for A. The example thus illustrates not only the failure of monotonicity but the possibility of sharp discontinuities of the set of fair allocations as a function of the supply of money.

6. GEOMETRY OF THE FAIR SET

From the Monotonicity Theorem we now show that the money and value vectors of the set of fair allocations have a strong connectedness property which may be thought of as a nonlinear generalization of convexity. We say a vector $\bar{x}$ in R^n is *between* $\bar{y}$ and $\bar{z}$, if for every i either $y_i \leqslant x_i \leqslant z_i$ or $z_i \leqslant x_i \leqslant y_i$ and $\bar{x} \neq \bar{y}$, $\bar{x} \neq \bar{z}$. A set S is a *B-set* if for any $y \neq z$ in S there is an $\bar{x}$ between $\bar{y}$ and $\bar{z}$. A monotone path in R^n is a function f from $(0, 1)$ to R^n such that each f_i is either nonincreasing or nondecreasing for $i = 1, \ldots, n$. A set S is an *M-set* if for any x, y in S there is a monotone path from x to y.

Clearly, convex sets are M-sets. Also it is a fairly standard exercise to show that any closed B-set is an M-set (see, e.g., Alkan and Gale (1989)). We now show the following theorem.

THEOREM 5: *For a given model* $(\mathscr{P}, \mathscr{O})$, *the set of all fair money and value vectors are M-sets.*

PROOF: In view of the remarks above it suffices to show that the set of fair money vectors is a B-set. Let $(\sigma, \bar{u}, \bar{x})$ and $(\tau, \bar{v}, \bar{y})$ be fair allocations and let $\mathscr{P}_u$, $\mathscr{O}_x$, etc., be as in Lemma 3. Now for A in $\mathscr{P}_u$ A strictly prefers his bundle under $(\sigma, \bar{u}, \bar{x})$ to any bundle (α, x_α) with $\alpha \in \mathscr{O}_0 \cup \mathscr{O}_y$ because $u_A > v_A \geqslant \phi_A(\alpha, y_\alpha) \geqslant \phi_A(\alpha, x_\alpha)$ since $y_\alpha \geqslant x_\alpha$. It follows that it is possible to slightly decrease the x_α for α in $\mathscr{O}_x$ and slightly increase the x_α for α in $\mathscr{O}_y$ without producing envy between members of $\mathscr{P}_u$ and those of $\mathscr{P}_v \cup \mathscr{P}_0$. Let $X_x = \sum_{\alpha \in \mathscr{O}_x} x_\alpha$, $X_y = \sum_{\alpha \in \mathscr{O}_y} x_\alpha$. For δ sufficiently small, by Theorem 4 there is an $(X_x - \delta)$-fair allocation on $(\mathscr{P}_u, \mathscr{O}_x)$ and an $(X_y + \delta)$-fair allocation on $(\mathscr{P}_v, \mathscr{O}_y)$ with x' between x_α and y_α and by the previous remark the combined allocation will be X-fair on $(\mathscr{P}, \mathscr{O})$ so the money vector of this allocation will be between $\bar{x}$ and $\bar{y}$. *Q.E.D.*

It is conjectured that M-sets share many of the properties of convex sets, such as contractibility. However, this has been proved only in R^2.

7. FAIRNESS AND JUSTICE

Except for degenerate situations there will always exist infinitely many fair allocations, so to choose among them one must introduce additional criteria. As mentioned in the introduction, the general philosophy is to divide things as equally as possible. This may be taken to mean either that money or value should be divided as equally as possible. In this section we consider only the case where there are the same number of (real) people as objects (though some of the objects may be fictitious).

For value vector $\bar{u}$ we write $u_m = \min_{A \in \mathscr{P}} u_A$ and $u_M = \max_{a \in \mathscr{P}} u_A$, and x_m and x_M are defined similarly. A fair allocation $(\sigma, \bar{u}, \bar{x})$ will be called *value Rawlsian* or *money Rawlsian* according to $u_m \geqslant v_m$ or $x_m \geqslant y_m$ for all fair $(\tau, \bar{v}, \bar{y})$. Thus we are considering maxmin allocations where the worst off person is as well off as possible. Analogously there are the minmax allocations which minimize either u_M or x_M, that is, they minimize the welfare of the best off person. Instead of working with value or money vector directly we could consider weighted value of money vectors replacing u_A by $f_A(u_A)$ (or x_α by $g_A(x_\alpha)$) where, for example $f_A(u_A)$ represents the "social utility" of allocating u_A units of value to A. Of course, we assume that f's and g's are continuous and increasing. One might wish, for example, to achieve a higher level of utility or income to certain disadvantaged groups in the society, as long as this was consistent with fairness. In order to keep notation simple we will carry out the following analysis only for the value Rawlsian case, but the arguments are precisely the same for the more general case.

A *social welfare function* is a function Φ from $R^{\mathscr{P}}$ to R. The Rawlsian social welfare function Φ_R is defined by $\Phi_R(\bar{u}) = u_m$. We may then redefine a

Rawlsian allocation as one which maximizes $\Phi_R(\bar{u})$ among all X-fair value vectors.

THEOREM 6: *For each X the value vector $\bar{u}$ of the Rawlsian allocation is unique, and if $\bar{v}$ is the value vector of an envy free allocation with $\Phi_R(\bar{v}) > \Phi_R(\bar{u})$, then $\bar{v} \gg \bar{u}$.*

An immediate consequence of this is the following corollary.

COROLLARY: *The value vector $\bar{u}$ of a Rawlsian allocation is a strictly increasing function of X.*

PROOF: Suppose $Y > X$. By Theorem 4 there is a Y-fair allocation $(\tau, \bar{v}, \bar{y})$ with $\bar{v} \gg \bar{u}$ and hence $v_m > u_m$ so $\Phi_R(\bar{v}) > \Phi_R(\bar{u})$, so $\Phi_R(\bar{w}) > \Phi_R(\bar{u})$ where $\bar{w}$ is the Rawlsian value, so $\bar{w} \gg \bar{u}$.

PROOF OF THEOREM: Let $\mathscr{A} = (\sigma, \bar{u}, \bar{x})$. The uniqueness is a consequence of Lemma 4. If $\bar{u}$ and $\bar{v}$ are both Rawlsian, then $u_m = v_m = (u \wedge v)_m$ but if $\bar{u} \neq \bar{v}$, then $\bar{w} = \bar{u} \wedge \bar{v} < \bar{u}$ and $\bar{z} = \bar{x} \wedge \bar{y} < \bar{x}$, but then in the envy free allocation $(\mu, \bar{w}, \bar{z})$ we would have $\Sigma z_\alpha < X$ so by Theorem 4 there is a X-fair allocation $(\sigma', \bar{u}', \bar{x}')$ with $\bar{u}' \gg \bar{u}$, hence $u'_m > u_m$, contradicting the assumption that $(\sigma, \bar{u}, \bar{x})$ is Rawlsian.

Next suppose for $(\tau, \bar{v}, \bar{y})$ we have $v_m > u_m$. We must show that $\mathscr{P}_v = \mathscr{P}$. Suppose not and let $\tilde{\mathscr{P}} = \mathscr{P} - \mathscr{P}_v$ and $\tilde{\mathscr{O}} = \mathscr{O} - \mathscr{O}_y$. Then for $A \in \tilde{\mathscr{P}}$, $\alpha \in \mathscr{O}_y$, we have

$$(23) \qquad u_A \geqslant v_A \geqslant v_m > u_m$$

and

$$(24) \qquad u_A \geqslant v_A \geqslant \phi_A(\alpha, y_A) > \phi_A(\alpha, x_\alpha).$$

From Theorem 4 there is an envy-free allocation $\mathscr{A}'$ restricted to $(\tilde{\mathscr{P}}, \tilde{\mathscr{O}})$ with $x_\alpha - \delta < x'_\alpha < x_\alpha$ and $u_\alpha - \delta < u'_\alpha < u_\alpha$, and for δ sufficiently small, (23) and (24) will still hold. Then combining $\mathscr{A}'$ on $(\tilde{\mathscr{P}}, \tilde{\mathscr{O}})$ with $\mathscr{A}$ restricted to $(\mathscr{P}_v, \mathscr{O}_y)$ gives an envy-free allocation (because of (24)) with total money $X' = \Sigma_{\mathscr{O}} x'_\alpha < X$. Applying Theorem 4 again gives an X-fair allocation $\mathscr{A}''$ with $u''_m > u_m$ contradicting that $\mathscr{A}$ is Rawlsian. *Q.E.D.*

8. EXTENDABILITY

In this section we again restrict ourselves to the case of an equal number people and objects. We say that a fair allocation $\mathscr{A}$ is *universally extendable* if whenever a new object is introduced there is a new fair allocation in which no one is made worse off. We show that among all fair allocations there is *exactly one* allocation which is universally extendable. It is the minmax money allocation, that is, the allocation which makes the richest person as poor as possible.

The result is sharp in the sense that for any other allocation it is possible to introduce a new object such that at least one person will be made strictly worse off under any fair allocation. We limit ourselves here to the case of separable preferences, as the argument in the general case is much more complicated (see Alkan (1990)). We first show that any allocation which is not minmax money is not extendable.

Terminology: If $\bar{x}$ is a money vector we call α a *max object* if $x_\alpha = x_M$.

LEMMA 5: *Let* $\mathscr{A}^* = (\sigma^*, \bar{v}^*, \bar{x}^*)$ *be the minmax money allocation and let* $\mathscr{A} = (\sigma, \bar{v}, \bar{x})$ *be any other allocation. Then there is some max object, say* α_1, *of* $\mathscr{A}^*$ *such that* $x_{\alpha_1} > x_M^*$.

PROOF: In the separable case the set of fair money vectors is convex, so for any λ the vector $\bar{x}(\lambda) = (1-\lambda)\bar{x}^* + \lambda \bar{x}$ is a fair money vector. Suppose, say, $\alpha_1, \dots, \alpha_k$ are the max objects for $\mathscr{A}^*$ and $x_{\alpha_i} \leqslant x_{\alpha_i}^*$ for $i \leqslant k$. Then for λ sufficiently small, $x(\lambda)_M \leqslant x_M^*$, contradicting that x^* is the unique minmax money vector. *Q.E.D.*

Let us order people and objects so that $\sigma^*(A_i) = \alpha_i$ and let α_1 be as in the lemma. Now introduce a new object α_{n+1} such that

$$\phi_{A_1}(\alpha_{n+1}, 0) = c_{11} + \varepsilon \qquad \text{where} \quad \varepsilon < x_1 - x_1^*,$$

$$\phi_{A_i}(\alpha_{n+1}, 0) = c_{ii} \qquad \text{for} \quad i \neq 1.$$

THEOREM 6: *Let* $\mathscr{A} = (\sigma, \bar{v}, \bar{x}) \neq \mathscr{A}^*$ *for* $(\mathscr{P}, \mathscr{O})$ *and let* $(\tau, \bar{w}, \bar{y})$ *be any fair allocation for* $(\mathscr{P}, \mathscr{O} \cup \alpha_{n+1})$. *Then* $w_1 < v_1$ *(so* A_1 *is worse off).*

First note that the allocation which assigns to A_1 the bundle (α_{n+1}, x_1^*) and assigns the other A_i by $\mathscr{A}^*$ is strongly envy free, and the sum of the values of this allocation is $\sum_{i=1}^n c_{ii} + \varepsilon + X$, so this must be the sum of the values for any fair allocation, but this can only be achieved by assigning A_1 to α_{n+1} since otherwise the sum of the values would not depend on ε.

We will show that $w_1 < v_1$, for if not then $w_1 = c_{11} + \varepsilon + y_{n+1} \geqslant v_1 \geqslant c_{11} + x_1$ (since $\mathscr{A}$ is envy free), so $y_{n+1} > x_1 - \varepsilon > x_1^*$ since $\varepsilon < x_1 - x_1^*$. Hence by feasibility for some i, say $i = 2$, $y_2 < x_2^*$. Suppose $\tau(A_K) = \alpha_2$. We will show that A_K envies A_1 because

$$w_k = c_{k2} + y_2 < c_{k2} + x_2^* \leqslant c_{kk} + x_k^* \leqslant c_{kk} + x_1^* < c_{kk} + y_{n+1},$$

so $(\tau, \bar{w}, \bar{y})$ is not envy free. *Q.E.D.*

It remains to show that the minmax money allocation is universally extendable. We will need some terminology. We say that A *likes* α under allocation $\mathscr{A}$ if $\phi_A(\alpha, x_\alpha) = v_A$.

DEFINITION: The bipartite *graph* $\Gamma_{\mathscr{A}}$ of $\mathscr{A}$ has vertices $\mathscr{P} \cup \mathscr{O}$ and (A, α) is an edge if and only if A likes α.

An *alternating path* in $\Gamma_{\mathscr{A}}$ is a sequence $A_1, \sigma(A_1), \ldots, A_K, \sigma(A_K)$ such that A_i likes $\sigma(A_{i+1})$ for $1 \leqslant i < k$.

LEMMA 6: *In the minmax money allocation every person and object lies on an alternating path from some max object.* (*The converse is also true but will not be needed here.*)

Let $\tilde{\mathscr{P}}, \tilde{\mathscr{O}}$ be all people and objects connected by an alternating path to some max object and suppose $\mathscr{O} - \tilde{\mathscr{O}}$ is not empty. Then no one in $\tilde{\mathscr{P}}$ likes an object in $\mathscr{O} - \tilde{\mathscr{O}}$. By Theorem 4 one can then slightly increase the x_α for all α in $\mathscr{O} - \tilde{\mathscr{O}}$ and decrease them for α in $\tilde{\mathscr{O}}$ while maintaining fairness, but this would decrease x_M contrary to the assumption that $\mathscr{A}$ was minmax. *Q.E.D.*

THEOREM 7: *The minmax money allocation is universally extendable.*

Let β be the added object and choose x_β as small as possible so that at least one person B likes β.

Case I: $x_\beta \geqslant x_M^*$. Then the minmax allocation is still strongly envy free.

Case II: $x_\beta < \bar{x}_M^*$. Now B lies on an alternating path from some max object $\alpha = \sigma(A_1), A_1, \sigma(A_1), A_2, \sigma(A_2), \ldots, B, \sigma(B)$, so we may re-assign A_1 to $\sigma(A_2)$, A_2 to $\sigma(A_3)$, B to β, and since the v_A are unchanged the allocation is still strongly envy-free, but note that the total amount of money is now $X - x_M^* + x_\beta < X$, so by Theorem 4 there is a new fair allocation making everyone strictly better off. *Q.E.D.*

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APPENDIX: MANIPULATION

The results of the previous section suggest that using the various minmax or maxmin allocations has a number of desirable properties from the point of view of normative economics. However, if it is to be used successfully, say, in the task assignment problem, it is essential that the people involved give their true value functions for, as we will show, there is *no* mechanism, Rawlsian or otherwise, which is not susceptible to *manipulation*, meaning that, except for degenerate cases, there will always be at least one person who can make himself better off by falsifying his utility. We illustrate this for the case of two people A and B, two objects α and β, and one unit of money.

Let x_A have the property that $\phi_A(\alpha, x_A) = \phi_A(\beta, 1 - x_A)$, i.e. at prices $(x_A, 1 - x_A)$, A is indifferent between the two bundles, and let x_B be the corresponding number for B. The condition of enough money insures that x_A and x_B are on the unit interval. Except for degenerate cases $x_A \neq x_B$, say $x_A < x_B$. Then in any fair allocation one easily verifies that A gets α, B gets β, and x_α must be on the interval (x_A, x_B), but if the allocation mechanism chooses $x_\alpha < x_B$ then it will pay A to falsify his x_A so that $x_\alpha < x_A < x_B$ because now the mechanism must choose some $x'_\alpha \geqslant x_A > x_\alpha$. On the other hand, if $x_\alpha = x_B$, then B should falsify to decrease x_B, thus increasing $l - x_B$ and increasing his utility.

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