SPLITTING OFF EDGES WITHIN A SPECIFIED SUBSET PRESERVING THE EDGE-CONNECTIVITY OF THE GRAPH

Jørgen Bang-Jensen* Tibor Jordán†

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Abstract

Splitting off a pair su, sv of edges in a graph G means the operation that deletes su and sv and adds a new edge uv. Given a graph $G = (V + s, E)$ which is k-edge-connected ($k \geq 2$) between vertices of V and a specified subset $R \subseteq V$, first we consider the problem of finding a longest possible sequence of disjoint pairs (splittings) of edges sx, sy, $x, y \in R$ which can be split off preserving k-edge-connectivity in V. If $R = V$ and $d(s)$ is even then the well-known splitting off theorem of Lovász asserts that a complete R-splitting exists, that is, all the edges connecting s to R can be split off in pairs. This is not the case in general. We characterize the graphs possessing a complete R-splitting and give a formula for the length of a longest R-splitting sequence.

The main result of our paper is a solution for the following optimization problem: given G and R as above, find a smallest set F of new edges incident to s such that $G' = (V + s, E + F)$ has a complete R-splitting. We give a min-max formula for $|F|$ as well as a polynomial algorithm to find a smallest F.

The motivation of our research is the well-known strong connection between splitting off results and connectivity augmentation problems. Here we propose a general framework for solving edge-connectivity augmentation problems where the set of new edges has to satisfy some extra property. This shows that our main result may be an important step towards the solution of the following open problem, raised in [2]: given a graph $H = (V, E)$, an integer $k \geq 2$ and a set $R \subseteq V$, find a smallest set F' of new edges for which $H' = (V, E + F')$ is k-edge-connected and no edge of F' crosses R.

1 Introduction

Splitting off two incident edges su, sv in a graph means deleting su, sv and adding a new edge uv. This operation is a fundamental tool in several problems involving connectivity of graphs.

*Department of Mathematics and Computer Science, Odense University, DK-5230 Odense, Denmark.

†Department of Mathematics and Computer Science, Odense University, DK-5230 Odense, Denmark. Supported by the Danish Natural Science Research Council grant no. 28808.

The existence of a pair of edges which can be split off preserving certain connectivity properties of the graph often leads to inductive proofs and/or efficient algorithms for optimization problems. Typical examples are the disjoint paths problem [8] and the connectivity augmentation problem [5], [6], [9].

Lovász introduced this operation in 1974 [15] and proved the following basic result.

Theorem 1.1 *[17] Let $G = (V + s, E)$ be a graph with a designated vertex s of even degree and suppose that G is k-edge-connected in V, that is, there are at least k edge-disjoint paths between every pair of vertices of V for some $k \geq 2$. Then for every edge st there exists an edge su such that the graph obtained by splitting off the pair st, su is k-edge-connected in V.* □

Clearly, a splitting operation may decrease (and cannot increase) the edge-connectivity. Theorem 1.1 shows that by choosing an appropriate ("admissible") pair st, su incident to s we can preserve the edge-connectivity in V by splitting off st and su. By repeated applications of the theorem we obtain that there exists a *complete admissible splitting* at vertex s, that is, the edges incident to s can be paired in such a way that splitting off all the pairs (and removing s) results in a k-edge-connected graph on V.

Somewhat later Mader [18] gave a powerful extension of Theorem 1.1 concerning splittings preserving the local edge-connectivities in G. Mader [19] proved the directed counterpart of Theorem 1.1, too. This latter result was refined later by Frank [7] and Jackson [13]. These theorems have become standard tools in connectivity problems.

In the late 80's a new application of the splitting off operation was discovered. Cai and Sun [5] gave an algorithm for solving the *k-edge-connectivity augmentation problem* based on Mader's splitting off theorem. In this optimization problem a graph G and a target integer k are given and the goal is to find a smallest set F of new edges for which $G + F$ is k-edge-connected. Frank [9] improved and extended the results of [5]. These results and subsequent work on more general augmentation problems led to the investigation of possible extensions of the basic splitting off theorems. For example, some extensions to mixed graphs are given in [1]. Another way of generalizing the splitting off results is to consider problems where not only the k-edge-connectivity must be preserved but the split edges have to satisfy some additional property $\mathcal{P}$, as well. Examples of this type include the problems of finding complete admissible splittings which preserve the bipartiteness [2], simplicity [3], or planarity [20] of the graph. Note that the problem of deciding whether a complete admissible splitting exists turned out to be polynomially solvable in the first case (bipartiteness) and NP-hard in the second case (simplicity). The third problem (planarity) is still open. For a survey of this area and applications see [10] and [12].

In this paper we introduce a new class of optimization problems involving the splitting off operation. Several previous splitting results as well as open problems can be formulated to fit this new framework. By the close relationship between splitting off and augmentation, problems in this class have direct applications in the connecivity augmentation problem.

In detail, in our problem we are not only interested in characterizing the existence of a complete admissible splitting at s which preserves k-edge-connectivity and for which the split edges satisfy some extra property $\mathcal{P}$ but more generally, our goal is to find a smallest set of new edges

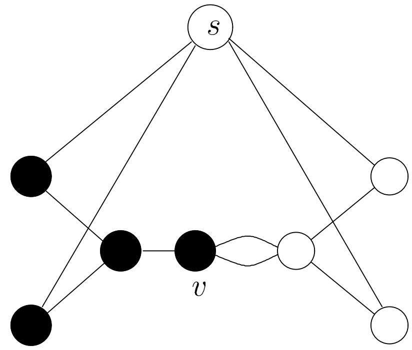

Figure 1: A graph $G = (V + s, E)$ with a specified subset R (consisting of the black vertices) satisfying (1) with respect to $k = 2$. There is no complete R-splitting in G but adding two parallel copies of the edge sv results in a graph possessing a complete R-splitting. The set F of these two edges is an optimal solution to the R-split completion problem.

incident to s to be added in order to produce a graph where a complete admissible splitting satisfying $\mathcal{P}$ exists. We call this the *split completion problem (with respect to property $\mathcal{P}$)*.

Let us illustrate this idea by a specific problem of this type – the one we solve in this paper. Suppose that we are given $G = (V + s, E)$, $k \geq 2$ as in Theorem 1.1 and, in addition, we are also given a non-empty subset $R \subseteq V$. The goal is to split off the edges incident to s in such a way that the k-edge-connectivity of G (within V) is preserved and no split edge crosses R, that is, no new edge has one endvertex in R and the other in $V - R$. A splitting satisfying this condition is an *R-splitting*. Equivalently, we want to find a complete admissible splitting where the edges from s to R are paired with each other. In some cases such a complete splitting does not exist, see Figure 1. Thus we consider two more general questions: we want to find a longest sequence of admissible R-splittings and a smallest set F of new edges incident to s for which $G' = (V + s, E + F)$ has the required complete admissible splitting. We call this latter problem the *R-split completion problem*.

We give a necessary and sufficient condition for the existence of a complete admissible R-splitting and a formula for the length of a longest admissible R-splitting sequence. Our main result (Theorem 3.3) is the solution of the R-split completion problem. The proofs are algorithmic and lead to polynomial algorithms for the above optimization problems.

As we remarked, the motivation of our research was the close connection between splittings and augmenting sets. The edge-connectivity augmentation problem corresponding to our R-splitting problem is the following: given a graph $G = (V, E)$, a bipartition $V = R \cup T$ of its vertices and a target connectivity $k \geq 2$, make G k-edge-connected by adding a smallest set of new edges in a such a way that each new edge lies within R or within T. This problem was introduced in [2], where it was shown that replacing the bipartition by a general t-partition (t is not fixed) makes this augmentation problem NP-hard. The complexity of the problem with bipartition constraints was left open. Our results suggest that this problem is polynomially solvable. In fact, the solution of the R-split completion problem seems to be an important step of such a polynomial algorithm. We discuss this connection in more detail in Section 5, where we propose a general framework for solving edge-connectivity augmentation problems where the set of new edges has to satisfy some extra property $\mathcal{P}$. In this framework an algorithm for the corresponding $\mathcal{P}$-split completion problem is a subroutine of the augmentation algorithm. (The reader may find it useful to read Section 5 right after the introduction to see this general framework before seeing our results on the R-splitting problem.)

The organization of the paper is as follows. Section 2 contains the notation and some preliminary results. Section 3 gives the characterization of the existence of a complete admissible R-splitting and the length of a longest admissible R-splitting sequence. The algorithm which finds an optimal solution for the R-split completion problem is also described there. Section 4 contains a brief discussion about the algorithmic aspects. The above mentioned connections to the augmentation problem and the general framework are discussed in Section 5. The last section contains some concluding remarks.

2 Terminology and some basic results

Graphs in this paper are undirected and may contain multiple edges but not loops. We will often consider a graph G with a designated vertex s. Such a graph will be denoted by $G = (V + s, E)$ or by $G = (V + s, E + F)$, when the set F of edges incident to s is distinguished. For two subsets $X, Y \subseteq V$, $d(X, Y)$ denotes the number of edges with one endvertex in $X - Y$ and the other in $Y - X$. We define the degree of a subset X as $d(X) := d(X, V - X)$. A set consisting of a single vertex v is simply denoted by v. Thus $d(v)$ stands for the degree of v. The degree-function of a graph G' will be denoted by d'. An edge connecting the vertices x and y will be denoted by xy. Sometimes xy will refer to an arbitrary copy of the parallel edges between x and y but this will not cause any confusion. Adding or deleting an edge e (or a set of vertices, etc.) to/from a graph G is denoted by $G + e$ and $G - e$, respectively. The subgraph of G induced by a subset X of vertices is denoted by $G[X]$. For a vertex v we use $N(v)$ to denote the set of vertices adjacent to v. We write $N(v, W)$ to specify the set of vertices from W which are adjacent to v for some $W \subset V - \{v\}$. A *subpartition* of V is a collection of pairwise disjoint subsets of V. Containment and proper containment are denoted by $\subseteq$ and $\subset$, respectively.

The operation *splitting off* a pair vs, st of edges from a vertex s means that we replace the edges vs, st by a new edge vt. If $v = t$ then the resulting loop is deleted from the graph. We use the notation G_{vt} to denote the graph obtained after splitting off the edges sv, st in G, if the vertex s is clear from the context. By a *sequence of splittings* $\mathcal{S} = (sx_1, sy_1), \ldots, (sx_r, sy_r)$ we mean a sequence of splitting operations involving the pairs of $\mathcal{S}$, executed in the given order. Thus, sx_2, sy_2 is split off in $G_{x_1y_1}$, and so on. A *complete splitting* from a vertex s (with even degree) is a sequence of $d(s)/2$ splittings of pairs of edges incident to s.

A graph $G = (V', E)$ is *k-edge-connected in* $V \subseteq V'$ if

$$d(X) \geq k \quad \text{for all } \emptyset \neq X \subset V. \tag{1}$$

The *edge-connectivity* of $G = (V, E)$ is the largest integer k for which G is k-edge-connected in V. The following two equalities are well-known. The first one shows that the degree-function d

of a graph is *submodular*.

Proposition 2.1 *Let $H = (V, E)$ be a graph. For arbitrary subsets $X, Y \subseteq V$:*

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X - Y, Y - X), \tag{2}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V - (X \cup Y)). \tag{3}$$

In the rest of this section we let s be a specified vertex of a graph $G = (V + s, E)$ for which (1) holds, that is, G is k-edge-connected in V.

A subset $\emptyset \neq X \subset V$ is called *dangerous* if $d(X) \leq k + 1$ and *critical* if $d(X) = k$. We say that $X, Y \subset V$ are *intersecting* if none of the sets $X - Y, Y - X$ and $X \cap Y$ is empty. If X, Y are intersecting and $V - (X \cup Y) \neq \emptyset$, then X, Y are *crossing*.

We say that a pair of edges vs, st is an *admissible pair* if (1) holds in G_{vt}. Otherwise vs, st form a *non-admissible* pair. An *admissible complete splitting* is a complete splitting sequence $\mathcal{S}$ for which splitting all the pairs in $\mathcal{S}$ preserves (1). Note that performing an admissible complete splitting and then deleting s gives a k-edge-connected graph. It is easy to verify that if $\mathcal{S} = (sx_1, sy_1), \ldots (sx_r, sy_r)$ is an admissible splitting sequence then these splittings form an admissible splitting sequence in every possible order.

It is easy to see that vs and st are non-admissible if and only if there exists a dangerous set $X \subset V$ such that $t, v \in X$. The following lemma was implicitely proved in [9] in the case when $d(s)$ is even. We give the proof here to show that it holds even if $d(s)$ is odd. Note that in this section $d(s)$ may be arbitrary.

Lemma 2.2 *Let X, Y and Z be maximal dangerous sets which are pairwise crossing and let $k \geq 2$. Then $d(s, X \cap Y \cap Z) = 0$.*

Proof: Suppose that $d(s, X \cap Y \cap Z) \geq 1$ holds. Without loss of generality we may assume that $|X \cap Y| \geq |X \cap Z|, |Z \cap Y|$. Let $M := X \cap Y$. By (2) and the maximality of X, we get that $d(M) = k$. Since Z is maximal dangerous we have $Z - M \neq \emptyset$ and it follows from (2) that $M \subset Z$. (Using the fact that $M \cup Z \neq V$, which is clear since X and Z are crossing.) Now it follows from the choice of X, Y that $X \cap Z = Y \cap Z = X \cap Y = M$. Applying (3) to each of the pairs $(X, Y), (X, Z), (Y, Z)$ we get that $d(X \cap Y, V + s - (X \cup Y)), d(X \cap Z, V + s - (X \cup Z)), d(Y \cap Z, V + s - (Y \cup Z)) \leq 1$ and this together with the existence of an edge from s to $X \cap Y \cap Z$ implies that $d(M) = 1$, contradicting (1). $\square$

Lemma 2.3 *If $k \geq 2$, then there exist no three pairwise crossing maximal dangerous sets X, Y, Z such that $d(s, X \cap Y), d(s, X \cap Z), d(s, Y \cap Z) \geq 1$.*

Proof: Suppose that such sets X, Y, Z do exist. Then it follows from the maximality of X, Y, Z and (2) that $d(X) = d(Y) = d(Z) = k+1$, $d(X, Y) = d(X, Z) = d(Y, Z) = 0$ and $d(X \cap Y) = d(X \cap Z) = d(Y \cap Z) = k$. It follows from Lemma 2.2 that s does not have a neighbour in

$X \cap Y \cap Z$. Thus s has a neighbour in each of the sets $(X-Y) \cap Z$, $(X-Z) \cap Y$, $(Y-X) \cap Z$. It follows from (3) that $d(X-Y) = d(X-Z) = d(Y-X) = d(Y-Z) = d(Z-X) = d(Z-Y) = k$. Now it follows from (3) that $X - Y = Z - Y$, $X - Z = Y - Z$ and $Y - X = Z - X$ (for example $d(X-Y) = k$ so $X - Y$ and Z cannot be intersecting, implying that $X - Y \subset Z - Y$, similarly $Z - Y \subset X - Y$). Since $X \cap Y \not\subset Z$ and $d(X \cap Y) = k$ it follows from the maximality of Z and (2) that $X \cap Y \cap Z = \emptyset$. Let $A = X - Y, B = X - Z, C = Y - X$, then we have $X = A \cup B, Y = B \cup C, Z = A \cup C$ and it follows from the remarks above that $d(A, B) = d(A, C) = d(B, C) = 0$ and $d(A) = d(B) = d(C) = k$. This shows $W := V - (X \cup Y \cup Z) \neq \emptyset$. Now $d(A, W + s) = d(B, W + s) = d(C, W + s) = k$. On the other hand, it follows from (3) applied to X, Y that $d(B, W + s) = 1$, a contradiction since $k \geq 2$. □

In the next two lemmas $T \subseteq V$ is a specified subset. A T*-split* is a splitting of a pair (sx, sy) where $x, y \in T$.

Lemma 2.4 *If $d(s,T) \geq 2$ and there is no admissible T-split involving the edge st, $t \in T$, then either there exists a unique maximal dangerous set X containing all neighbours of s in T, or there exist two maximal dangerous sets X, Y covering $N(s,T)$ such that $t \in X \cap Y$ and $d(s, X \cap Y) = 1$.*

Proof: Suppose one needs at least three maximal dangerous sets to cover $N(s,T)$ in such a way that each of these sets contains t. Then these sets are pairwise crossing and contradict Lemma 2.2. Thus we can cover $N(s,T)$ with either one dangerous set (in which case the maximal one is unique by (3), since $d(s,T) \geq 2$) or by two dangerous sets X, Y in which case it follows from (3) and $t \in X \cap Y$ that $d(s, X \cap Y) = 1$. □

Lemma 2.5 *If there is no admissible T-split, then either $d(s,T) = 1$, or there exists a unique maximal dangerous set X containing all neighbours of s in T, or $d(s) = d(s,T) = 3$ and there exist two maximal dangerous sets X, Y such that $d(s, X \cap Y) = 1$ and $X \cup Y = V$.*

Proof: Suppose $d(s,T) \geq 3$ since otherwise either the first alternative holds, or it follows from the argument above that the second alternative holds. Let t be a neighbour of s in T and suppose there is no maximal dangerous set containing all neighbours of s in T. Since there is no admissible T-split involving the edge st, it follows from Lemma 2.4 that there exist maximal dangerous sets X, Y that cover $N(s,T)$ and $d(s, X \cap Y) = 1$. Choose $u \in N(s,T) \cap (X - Y)$ and $v \in N(s,T) \cap (Y - X)$ arbitrarily. Since the pair su, sv is not admissible there is some maximal dangerous set Z with $u, v \in Z$. By (3) and the maximality of Y, $t \notin Z$ and hence X, Z and Y, Z are intersecting pairs. Now it follows from Lemma 2.3 that $X \cup Y \cup Z = V$ and it follows as in the proof of Lemma 2.3 that $X - (Z \cup Y) = Y - (X \cup Z) = Z - (X \cup Y) = \emptyset$. Suppose $X \cap Y \cap Z \neq \emptyset$, then it follows from the fact that $X \not\subset Z$ that $X \cap Y$ and Z are intersecting and by (3), $d(s, X \cap Y \cap Z) = 0$. Applying (3) to each of the pairs X, Y, X, Z and Y, Z we see that $d(s) = d(s,T) = 3$. It also follows from our arguments above that $X \cup Y = V$

and that $d(s, X \cap Y) = 1$. □

The following lemma plays an important role in our main proof. Suppose that after repeatedly splitting off some admissible T-splits we get stuck, that is, no more admissible T-splits exist in the current graph. The next lemma implies that in some cases we can find a longer sequence of admissible T-splits by first "lifting back" a pair of edges and then choosing two other pairs to split instead.

Lemma 2.6 *Let $k \geq 2$ and suppose $G = (V + s, E)$ satisfies (1). Let (su, sv) be an admisible splitting in G and let $\tilde{G}$ be the graph we obtain after performing that split. Suppose that there exists a maximal dangerous set X in $\tilde{G}$ such that $\{u, v\} \cap X = \emptyset$ and $d_{\tilde{G}}(s, X) \geq 2$. Then for every choice of $p, q \in N_G(s, X)$, either the pair (su, sp) is an admissible splitting in G and (sv, sq) is admissible for splitting in the graph G_{up}, or the pair (su, sq) is an admissible splitting in G and the pair (sv, sp) is admissible for splitting in the graph G_{uq}.*

Proof: Let $T = N_G(s, X) \cup \{u, v\}$ and choose $p, q \in N_G(s, X)$ arbitrarily. Since (su, sv) is admissible in G, there is no dangerous set in G which contains both u and v in G. Furthermore observe that X is also maximal dangerous in G. Since $d(s, T) \geq 4$ it follows from Lemma 2.4 (applied to the edge sp), (3) and the fact that X is dangerous that one of the pairs $(sp, sv), (sp, su)$ is admissible in G. W.l.o.g (sp, su) is an admissible splitting in G. Suppose that (sq, sv) is not admissible in G_{up} and let Y be a maximal dangerous set containing q, v in G_{up}. We claim that neither u nor p are in Y. If $p \notin Y$ and $u \in Y$, then $d_{\tilde{G}}(Y) \leq k - 1$, contradicting (1). If $p \in Y, u \notin Y$, then Y is dangerous in G and $d_G(s, X \cap Y) \geq 2$, so by (3) we have $X \subset Y$, contradicting the maximality of X in G. If $p, u \in Y$, then Y is dangerous in $\tilde{G}$ and $d(s, X \cap Y) \geq 2$, and so again we obtain a contradiction on the maximaility of X, this time in $\tilde{G}$. Hence neither p nor u belong to Y, in particular $p \neq q$. Suppose (su, sq) is not admissible in G and let W be a maximal dangerous set containing u and q. Note that $v \notin W$, because (su, sv) is admissible in G and $p \notin W$ because $(sp, su)$is admissible in G. Now the three sets X, Y, W are crossing and contradict Lemma 2.2. Hence we get that (su, sq) is admissible in G. Now we can argue similarly as above that if (sv, sp) is not admissible in the graph G_{uq} that we obtain by splitting the pair (su, sq) in G, then there exists a dangerous set Z in G'' with $Z \cap \{u, v, p, q\} = \{v, p\}$. Note that each of the sets Y, Z are dangerous in G as well, their intersection contains the neighbour v of s and $Y \cup Z \neq V$, since $u \in V - (Y \cup Z)$. Now the existence of X, Y, Z contradicts Lemma 2.3. Hence the pair (sv, sp) is admissible in G_{uq} and the proof is complete. □

3 The R-split completion problem

In this section we solve the R-split completion problem. First we show that if the graph contains a subset of vertices satisfying a certain degree-inequality then there is no complete R-splitting. These "obstacles" define a lower bound for the size of an optimal solution of the R-split completion problem, as well. Then we describe our algorithm which finds a solution for the R-split completion problem. We will verify that the size of the solution equals the lower bound defined by the obstacles, showing that the solution is optimal.

Let $G = (V + s, E + F)$, R and $k \geq 2$ be a given instance of the R-split completion problem. Thus G satisfies (1) with respect to k, $R \subseteq V$, and F denotes the set of edges incident to the designated vertex s. An R-*splitting* is a splitting of a pair sr, sr' with $r, r' \in R$. A sequence $\mathcal{S}$ of splittings is a *complete* R-splitting if every split in $\mathcal{S}$ is an R-splitting and after performing the splittings in $\mathcal{S}$, there are no edges from s to R. Our goal is to find a smallest set F^* of new edges incident to s for which the graph $G^* = (V + s, E + F + F^*)$ has an admissible complete R-splitting. In what follows by a complete R-splitting we mean an admissible complete R-splitting.

Definition 3.1 *Let $G = (V + s, E + F)$ satisfy (1) and let $R \subseteq V$. For a subset $\emptyset \neq X \subset V$ we define*

$$c_G(X) = k - (d_{G-s}(X) + d_G(s, X - R) + d_G(s, R - X)). \tag{4}$$

*An R-*obstacle *is a subset $X \subset V$ with $c_G(X) > 0$.*

The set $X = R - v$ is an obstacle in the graph G of Figure 1 with $c_G(X) = 1$. The following lemma shows that if G contains an obstacle then no complete R-splitting exists. Note that for an obstacle X we have $X \cap R \neq \emptyset$ by (1).

Lemma 3.2 *If $c_G(X) = \delta \geq 0$ for some $X \subset V$, then for every sequence $\mathcal{S}$ of admissible R-splittings we have $d_{G'}(s, R \cap X) \geq \delta$ in the graph G' which we obtain after performing the splittings in $\mathcal{S}$. Furthermore, $d_{G'}(s, R \cap X) = \delta$ can hold only if $d_G(X)$ is even.*

Proof: Let $d_G(X) = k + \beta$, where $\beta \geq 0$. Since $c_G(X) = \delta$ we have

$$d_{G-s}(X) + d_G(s, X - R) + d_G(s, R - X) = k - \delta \tag{5}$$

$$d_G(X) = d_{G-s}(X) + d_G(s, X - R) + d_G(s, R \cap X) = k + \beta \tag{6}$$

Combining (5) and (6) we get

$$d_G(s, R \cap X) = d_G(s, R - X) + \beta + \delta. \tag{7}$$

Let m be the number of splittings in $\mathcal{S}$ that pair edges from s to $R \cap X$ with edges from s to $R - X$. Clearly $m \leq d_G(s, R - X)$. The only other type of R-splittings involving edges from s to $R \cap X$ are those of the kind (sr, sr'), where $r, r' \in R \cap X$. Such a split reduces the degree of X (as well as the number of edges from s to R) by 2 and hence we have at most $\lfloor \frac{\beta}{2} \rfloor$ splittings of this kind. Now the claim follows easily. $\square$

We define three parameters of G as follows.

$$\rho(G) = \max\{c_G(X) : X \text{ is an } R\text{-obstacle in } G\}, \tag{8}$$

$$\rho_o(G) = \max\{c_G(Y) : Y \text{ is an } R\text{-obstacle in } G \text{ with } R \subset Y\}, \tag{9}$$

$$\rho_i(G) = \max\{c_G(W) : W \text{ is an } R\text{-obstacle in } G \text{ with } W \subset R\}. \tag{10}$$

If no such obstacle X, Y or W exists then the corresponding parameter ($\rho(G), \rho_o(G)$ or $\rho_i(G)$, respectively) is defined to be zero. An R-obstacle Z is called *tight* (or *semi-tight*) if $c_G(Z) = \rho(G)$ (respectively, $c_G(Z) = \rho(G) - 1$). Let

$$\phi(G) = \max\{\rho(G), \rho_i(G) + \rho_o(G)\}. \tag{11}$$

Note that if R is an R-obstacle, or equivalently $d_{G-s}(R) < k$, then clearly there is no feasible solution for the R-split completion problem. (On the other hand, it is easy to see that there exists a solution when $d_{G-s}(R) \geq k$.) From now on we assume that R is not an R-obstacle. The following min-max equality is the main result of the paper.

Theorem 3.3 *Let $H = (V+s, E+F)$ satisfy (1) with respect to $k \geq 2$ and suppose $c_H(R) \leq 0$. Define $\psi(H), \psi_i(H)$ and $\psi_o(H)$ as follows:*

- *$\psi(H) = 1$ if $d_H(s) + \rho(H)$ is odd and zero otherwise,*
- *$\psi_i(H) = 1$ if $d_H(s, R) + \rho_i(H)$ is odd and zero otherwise,*
- *$\psi_o(H) = 1$ if $d_H(s, V-R) + \rho_o(H)$ is odd and zero otherwise.*

The minimum number $\gamma(H)$ of new edges incident to s to be added to H in order to obtain a new graph $G^* = (V+s, E+F+F^*)$ in which $d_{G^*}(s)$ is even and which has a complete R-splitting is equal to

$$\phi'(H) := \max\{\rho(H) + \psi(H), \rho_i(H) + \rho_o(H) + \psi_i(H) + \psi_o(H)\}. \tag{12}$$

Proof: We first prove that $\gamma(H) \geq \phi'(H)$. Note that by (4) and (8,9,10), adding an arbitrary edge sq from s to V will decrease ρ and $\rho_i + \rho_o$ by at most one. It follows from Lemma 3.2 that a graph G'' has a complete R-splitting only if $\rho(G'') = \rho_i(G'') = \rho_o(G'') = 0$. This implies $\gamma(H) \geq \phi(H)$. Clearly, G'' has a complete R-splitting only if $d_{G''}(s, R)$ is even. Since $d_{G''}(s)$ must be even, this shows that $d_{G''}(s, V-R)$ must also be even. Therefore, if $\rho_i(H) + d_H(s, R)$ is odd, then we must add at least $\rho_i(H) + 1$ edges from s to R to have a complete R-splitting. Similarly, if $\rho_o(H) + d_H(s, V-R)$ is odd, then we must add at least $\rho_o(H) + 1$ edges from s to $V - R$. Finally, if $\rho(H) + d_H(s)$ is odd, then we must add at least $\rho(H) + 1$ new edges to H to obtain a graph G^* which has a complete R-splitting and for which $d_{G^*}(s)$ is even. This shows that $\gamma(H) \geq \phi'(H)$.

We give an algorithmic proof which shows that there exists a set F^* of $\phi'(H)$ new edges from s to V such that there is a complete R-splitting in the graph $G^* = (V+s, E+F+F^*)$ and $d_{G^*}(s)$ is even. This will imply $\gamma(H) \leq \phi'(H)$.

In what follows we describe algorithm $\mathcal{A}$ and simultaneously we prove that $\mathcal{A}$ is a well-defined finite algorithm which finds a set F^* satisfying the requirements. The optimality of F^* is verified at the end of the proof.

$\mathcal{A}$ has two main steps, a splitting step and an adding step. We start by the description of the splitting step $\mathcal{P}$. Note that the splitting routine is also used as a subroutine in the adding step.

Splitting routine $\mathcal{P}$. (Input: a graph $G = (V + s, E + F)$ with a designated vertex s, satisfying (1) with respect to $k \geq 2$, and a subset $R \subseteq V$.)

The routine $\mathcal{P}$ tries to find a complete R-splitting by performing arbitrary R-splittings as long as possible. Let $\mathcal{S} = (sx_1, sy_1), \ldots, (sx_r, sy_r)$ be a maximal sequence of admissible R-splittings (that is, the graph G' obtained from G by performing the splittings in $\mathcal{S}$ satisfies (1) and there is no admissible R-splitting in G').

If $\mathcal{S}$ is a complete R-splitting, then $\mathcal{P}$ halts. Otherwise $d_{G'}(s, R) \geq 1$. By the maximality of $\mathcal{S}$ it follows that in G', for each pair of edges sx, sy where $x, y \in R$ there exists a dangerous set $Z_{x,y}$ containing x and y. Lemma 2.5 implies the following property of G'.

Claim 3.4 *In G' one of the following holds:*

(a) There exists a unique maximal dangerous set X containing all neighbours of s in R.

(b) $d_{G'}(s) = d_{G'}(s, R) = 3$ and there exist two maximal dangerous sets X, Y such that $X \cup Y = V$, $d_{G'}(X) = d_{G'}(Y) = k+1$ and $d_{G'}(s, X \cap Y) = d_{G'}(X - Y) = d_{G'}(Y - X) = 1$.

(c) $d_{G'}(s, R) = 1$. □

If Claim 3.4(b) or (c) holds in G' then $\mathcal{P}$ halts. Assume that $\mathcal{P}$ has found the maximal dangerous set X in G' containing all the remaining neighbours of s in R. Now we have

$$d_{G'}(s, R) \geq 2. \tag{13}$$

We consider two cases depending on the positions of the edges split by $\mathcal{P}$.

Case 1: In the current maximal splitting sequence $\mathcal{S}$ some splitting added an edge $x_i y_i$ where $x_i, y_i \in R - X$.

We can assume that $i = r$, that is (sx_i, sy_i) was the last split in $\mathcal{S}$. Let $\hat{G}$ denote the graph obtained from G by performing the splittings in $\mathcal{S} - (sx_r, sy_r)$. Note that X is maximal dangerous in $\hat{G}$ (if a set is dangerous in $\hat{G}$, then it is also dangerous in G'). By Lemma 2.6 and (13), we can find a new sequence of admissible R-splittings $\mathcal{S}' = \mathcal{S} - (sx_r, sy_r) + (sx_r, sp) + (sy_r, sq)$ with $r + 1$ splittings, where p, q are neighbours of s in R in the graph $\hat{G}$.

Now $\mathcal{P}$ tries to continue splitting admissible R-splittings until it either finds a complete R-splitting, or identifies a new maximal sequence of splittings which we again denote by $\mathcal{S}$. As before, if Claim 3.4 (b) or (c) applies in the graph where $\mathcal{P}$ gets stuck, $\mathcal{P}$ halts. Otherwise, if Case 1 applies, then as above, $\mathcal{P}$ can find a longer splitting sequence, so we may assume that Case 2 occurs eventually.

Case 2: In the current maximal splitting sequence $\mathcal{S}$ every splitting added an edge with at least one endvertex in X.

As before, we denote by G' the graph obtained from G by performing all the splittings in $\mathcal{S}$. Since we are in Case 2, it is easy to see from Definition 3.1 that $c_{G'}(X) = c_G(X)$. Since X contains all neighbours of s in R in G', we have $d_{G'}(s, R - X) = 0$. From this we deduce that

$$\begin{aligned} k - c_{G'}(X) &= d_{G'-s}(X) + d_{G'}(s, X - R) \\ &= d_{G'}(X) - d_{G'}(s, R \cap X) \\ &\leq k + 1 - d_{G'}(s, R \cap X). \end{aligned} \tag{14}$$

From this and Lemma 3.2 we get

$$\begin{aligned} \rho(G) \geq c_G(X) = c_{G'}(X) &\geq d_{G'}(s, R \cap X) - 1 \\ &= d_{G'}(s, R) - 1 (\geq 1) \\ &\geq \rho(G) - 1. \end{aligned} \tag{15}$$

Hence X is an R-obstacle (since $d_{G'}(s, R) \geq 2$ by (13)) and $c_G(X) \geq \rho(G) - 1$. Thus X is either tight or semi-tight. Furthermore, Lemma 3.2 and (15) show that $\mathcal{S}$ is a maximum length admissible R-splitting sequence in G. Now $\mathcal{P}$ stops and returns the set X.

This description of $\mathcal{P}$ shows that there are four possible outputs of $\mathcal{P}$. Either it finds a complete R-splitting or it halts with a graph G' for which Claim 3.4 (a), (b) or (c) holds. If (a) holds then Case 2 applies and $\mathcal{P}$ outputs the unique maximal dangerous set X, as well.

Our calculations show that $\mathcal{P}$ finds a complete R-splitting, if there is any. In fact, $\mathcal{P}$ finds a longest possible admissible R-splitting sequence in G provided that when $\mathcal{P}$ halts Claim 3.4 (b) does not hold. If Claim 3.4 (b) holds then a one longer admissible R-splitting sequence may exist in G. It is easy to characterize the case when this may happen. We omit these details. Summarizing these observations we get:

Corollary 3.5 *There exists a complete R-splitting in G if and only if $d_G(s, R)$ is even and G contains no R-obstacles. The length of a longest admissible R-splitting sequence equals $\lfloor (d_G(s, R) - \rho(G))/2 \rfloor$ unless $d_G(s) =_G (s, R)$ holds and $d_G(s)$ is odd. In this latter case the length of a longest admissible R-splitting sequence is either $\lfloor d_G(s)/2 \rfloor$ or every edge incident to s in G is critical with respect to (1), in which case the length of a longest admissible R-splitting sequence is $\lfloor d_G(s)/2 \rfloor - 1$.* □

Note that the first part of the above corrollary (the characterization of the existence of a complete R-splitting) has a shorter proof using Mader's splitting off theorem. This proof can be found in Section 6.

Before continuing the description of $\mathcal{A}$ we derive some useful properties of the set X that is returned by $\mathcal{P}$ if Case 2 applies. Recall that G' denotes the graph obtained by performing the splitting sequence $\mathcal{S}$.

Lemma 3.6 *Let X be the maximal dangerous set returned by $\mathcal{P}$ when Case 2 applies. For every tight or semi-tight R-obstacle Y of G the following holds:*

(a) $X \subseteq Y$ *or* $Y \subseteq X$.

(b) If Y *is tight, then*

(b1) $d_{G'}(s, R \cap Y) = d_{G'}(s, R \cap X)$.

(b2) $d_{G'}(Y) \leq k + 1$, *i.e.* Y *is dangerous in* G'.

Proof: Recall that $c_G(Y) \geq 1$ since Y is an R-obstacle. Let $\delta = c_G(Y) \geq \rho(G) - 1$ and $\epsilon = c_G(X) \geq \rho(G) - 1$. Let p be the number of splits (sx_i, sy_i) in $\mathcal{S}$ for which $x_i, y_i \in R - Y$. Then it follows from (4) that in G' we have $c_{G'}(Y) = \delta + 2p$. Similarly, since every split in $\mathcal{S}$ adds an edge incident with X, we have $c_{G'}(X) = \epsilon$. Define $h'(U) = d_{G'-s}(U) + d_{G'}(s, U - R)$ for all $U \subset V$ and choose $m \geq 0$ such that $d_{G'}(s, R \cap X) = d_{G'}(s, R \cap Y) + m$. Hence $m = d_{G'}(s, R - Y)$, since $d_{G'}(s, R) = d_{G'}(s, R \cap X)$. Then we get

$$h'(X) + d_{G'}(s, R - X) = k - c_{G'}(X) \quad = \quad k - \epsilon \tag{16}$$

$$h'(Y) + d_{G'}(s, R - Y) = k - c_{G'}(Y) \quad = \quad k - \delta - 2p. \tag{17}$$

Combining (16) and (17) we get

$$h'(Y) = h'(X) + (\epsilon - \delta - 2p) - m \tag{18}$$

Furthermore, since X is dangerous in G' we have

$$\begin{aligned} k + 1 &\geq d_{G'}(X) \\ &= h'(X) + d_{G'}(s, R \cap X). \end{aligned} \tag{19}$$

Using that $d_{G'}(s, R - X) = 0$, it follows from (18) and (19) that

$$\begin{aligned} d_{G'}(Y) &= h'(Y) + d_{G'}(s, R \cap Y) \\ &= h'(X) + (\epsilon - \delta - 2p) - m + d_{G'}(s, R \cap Y) \\ &= d_{G'}(X) - 2m + (\epsilon - \delta - 2p) \\ &\leq k + 1 + (\epsilon - \delta) - 2(m + p). \end{aligned} \tag{20}$$

If Y is tight, then $\delta \geq \epsilon$ and hence $d_{G'}(Y) \geq k$ and (20) implies $m = p = 0$, and (b1) and (b2) follow.

To prove (a) we may assume that $X - Y$ and $Y - X$ are both non-empty. Observe that

$$d_{G'}(s, X \cap Y) \geq d_{G'}(s, R \cap Y) \geq \delta \tag{21}$$

by Lemma 3.2. Thus $X \cap Y \neq \emptyset$. It follows from (20) and $\delta \geq \rho(G) - 1$ that $d_{G'}(Y) \leq k + 2$. If $d_{G'}(Y) \geq k+1$ then $m = 0$ follows from (20). Thus $d_{G'}(s, X \cap Y) \geq 2$ by (13). Now (3), applied to X and Y gives $k+1+k+2 \geq d_{G'}(X)+d_{G'}(Y) \geq d_{G'}(X \cap Y)+d_{G'}(X \cup Y)+2d_{G'}(s, X \cap Y) \geq k + k + 4$, using that (1) holds in G'. This contradiction shows $d_{G'}(Y) = k$. Applying (3) and

(21) we get a similar contradiction in this case, too. This proves (a). $\square$

Note that the maximality of X was not used in the previous proof. The proof works for every dangerous set X which contains all the neighbours of s in R (in G') and for which $\mathcal{S}$ added no new (split) edges with both endvertices in $R - X$.

Lemma 3.7 *Let X be the maximal dangerous set returned by $\mathcal{P}$. The tight R-obstacles are all contained in X and they form a chain. All the tight R-obstacles have the same degree in G'. Furthermore, if X is semi-tight, then $d_{G'}(Y) = k$ for every tight R-obstacle Y.*

Proof: By Lemma 3.6(b) every tight R-obstacle Y is dangerous in G' and since X is maximal dangerous it follows from Lemma 3.6(a) that $Y \subseteq X$. If Y, Y' are distinct tight R-obstacles, then by Lemma 3.6(b1,b2) each of Y, Y' is dangerous in G' and $d_{G'}(s, Y \cap Y') \geq d_{G'}(s, X \cap R) \geq 2$ by (13). Now it follows from (3) that Y and Y' cannot be intersecting. Hence the tight R-obstacles form a chain. Suppose that Y and Z are tight R-obstacles such that $d_{G'}(Z) = k$ and $d_{G'}(Y) = k + 1$. As in Lemma 3.6, we can see that $m = p = 0$ for the corresponding values m and p of Y and Z. Thus replacing X by Z and using (20) we get $k+1 = d_{G'}(Y) \leq d_{G'}(Z) = k$, a contradiction. Thus all tight R-obstacles have the same degree in G', namely either k or $k + 1$. Finally, assume that X is semi-tight. Then it follows from (20) that $d_{G'}(Y) = k$ for every tight Y. $\square$

Lemma 3.8 *Let X be the maximal dangerous set returned by $\mathcal{P}$ when Case 2 applies and suppose that $R - X \neq \emptyset$. If Z is an R-obstacle with $R \subset Z$, then $X \subset Z$.*

Proof: Let $\delta = c_G(Z)$ and $\epsilon = c_G(X)$. Suppose that $X - Z \neq \emptyset$. By (20), $d_{G'}(Z) \leq k + 1 + (\epsilon - \delta)$, since Z contains R. Now using (3) we get

$$\begin{aligned}(k+1) + (k+1+(\epsilon-\delta)) &\geq d_{G'}(X) + d_{G'}(Z) \\ &= d_{G'}(X - Z) + d_{G'}(Z - X) + 2d_{G'}(X \cap Z, (V+s) - (X \cup Z)) \\ &\geq k + k + 2d_{G'}(s, R \cap X) \\ &\geq k + k + 2\rho(G), \qquad (22)\end{aligned}$$

since $d_{G'}(s, R) = d_{G'}(s, R \cap X)$ and $d_{G'}(s, R) \geq \rho(G)$, by Lemma 3.2. Using that $1 \leq \epsilon \leq \rho(G)$ and $\delta \geq 1$ we get that $\delta = \epsilon = \rho(G) = 1$ and by the calculation above we also get $d_{G'}(s, R) = 1$. By (13) this cannot happen, hence we must have $X \subset Z$. $\square$

Let us still focus on the case when $\mathcal{P}$ halts in Case 2 and returns the maximal dangerous set X. The next lemma shows that adding a properly chosen edge incident to s to the input graph decreases its ϕ value by one. The adding step of $\mathcal{A}$ will be based on this lemma.

Lemma 3.9 *Let $G = (V + s, E + F)$ denote the input graph of the splitting routine $\mathcal{P}$. Suppose $\mathcal{P}$ halts with a graph G' satisfying $d_{G'}(s, R) \geq 2$ and let X be the maximal dangerous set containing all the neighbours of s in R in the graph G'. Then a vertex $q \in V$ can be chosen with the following properties. Let $\tilde{G} := G' + sq$.*

(i) $\phi(\tilde{G}) = \phi(G') - 1$*,*

(ii) $\rho(\tilde{G}) = \rho(G') - 1$ *unless* $\rho(G') = \rho_i(G') = \rho_o(G')$ *holds,*

(iii) if $q \in R$*, then* $\rho(G') = \rho_i(G')$ *and* $\rho_i(\tilde{G}) = \rho_i(G') - 1$*,*

(iv) if $q \in V - R$*, then* $\rho_o(\tilde{G}) = \rho_o(G') - 1$ *unless* $\rho_o(G') = 0$*.*

Proof: We distinguish three cases depending on the values $\rho(G')$ and $\rho_i(G') + \rho_o(G')$.

Case A: $\rho(G') > \rho_i(G') + \rho_o(G')$.

Let W be the unique minimal tight R-obstacle. W exists and $W \subseteq X$ by Lemma 3.7. Since $\rho_i(G') < \rho(G')$ we have $W - R \neq \emptyset$. Let q be an arbitrary vertex of $W - R$. Since $q \in Y \cap R$ for every tight set Y, adding sq decreases $\rho(G')$ (and hence $\phi(G')$) by one. By Lemma 3.8 $\rho_o(\tilde{G}) = \rho_o(G') - 1$ also holds unless $\rho_o(G') = 0$.

Case B: $\rho(G') < \rho_i(G') + \rho_o(G')$.

Suppose first that $\rho(G') = \rho_i(G')$. Let U be the unique maximal tight R-obstacle with respect to $R - U \neq \emptyset$. Since the tight sets form a chain by Lemma 3.7, this U is indeed unique. Let q be an arbitrary vertex of $R - U$. It follows from the choice of q and (4) that adding the edge sq decreases $\rho_i(G')$ (and hence $\phi(G')$) by one. If $\rho_o(G') < \rho(G')$, then no tight R-obstacle contains R and U is maximal among all tight R-obstacles. Thus adding the edge sq decreases $\rho(G')$ by one.

Suppose next that $\rho(G') > \rho_i(G')$. Then $\rho_o(G') > 0$ and no tight R-obstacle is contained in R. Let W' be the unique minimal tight R-obstacle and let $q \in W' - R$ be arbitrary. Since $W' \subset X$ and $X \subset Z$ for every R-obstacle containing R by Lemma 3.8, it follows that sq decreases $\rho_o(G')$ (and hence $\phi(G')$) by one and it follows from the minimality of W' and Lemma 3.7 that sq also decreases $\rho(G')$ by one (q is contained in every tight R-obstacle).

Case C: $\rho(G') = \rho_i(G') + \rho_o(G')$.

If $\rho_o(G') = 0$, then $\rho(G') = \rho_i(G')$ and we can decrease $\rho(G')$ and $\rho_i(G')$ as we did in the first part of Case B. If $\rho_i(G') = 0$, then $\rho(G') = \rho_o(G')$ and we can decrease $\rho(G')$ and $\rho_i(G')$ as we did in the last part of Case B. Suppose now that $\rho_i(G'), \rho_o(G') \geq 1$. Since $\rho(G') > \rho_i(G')$, no tight R-obstacle is contained in R. Let W' be the unique minimal tight R-obstacle and let $q \in W' - R$ be arbitrary. As we argued in the last part of Case B, the edge sq decreases $\rho(G')$ and $\rho_o(G')$ by one. In each of these cases $\phi(G')$ is also decreased by one. □

The following corollary can be deduced from the proof above.

Corollary 3.10 *Suppose that we iteratively decrease* ϕ *by adding edges following the choice of Lemma 3.9. If at some point of this procedure we have* $\rho(G') = \rho_i(G') = \rho_o(G')$ *in the current graph* G' *then from that point on we will have* $\rho = \rho_o$*.* □

Now we give an (informal but complete) description of $\mathcal{A}$.

Algorithm $\mathcal{A}$ (Input: $H = (V + s, E + F), R \subseteq V, k \geq 2$.)

First $\mathcal{A}$ executes $\mathcal{P}$. Every time when the splitting routine $\mathcal{P}$ halts $\mathcal{A}$ goes to the 'branching step', where there are four possible ways to continue, depending on which situation (complete R-splitting or Claim 3.4(a),(b) or (c)) occurs when $\mathcal{P}$ halts. If at any point the splitting routine $\mathcal{P}$ finds a complete R-splitting then $\mathcal{A}$ halts. Otherwise it follows the steps of one of the three cases described below. Let G' be the graph output by $\mathcal{P}$, that is, G' is obtained from H by executing an admissible splitting sequence $\mathcal{S}$ (and possibly adding some new edges incident to s at some points).

Case I: $\mathcal{P}$ returns with a maximal dangerous set X in G'. (Recall that in this case none of Claim 3.4(b) and (c) occurs and the current maximal splitting sequence adds no edges with both endvertices in $R - X$.)

In this case $\mathcal{A}$ first calculates the chain $T_1 \subset T_2 \subset \ldots \subset T_f$ of tight R-obstacles in G'. Then $\mathcal{A}$ adds a new edge sq following the rule of Lemma 3.9. (In detail, first $\mathcal{A}$ decides whether $\rho_i(G') = \rho(G')$. If this is the case then it finds the unique maximal T_i such that $R - T_i \neq \emptyset$ and adds a new edge sq with $q \in R - T_i$. If $\rho_i(G') < \rho(G')$, then $\mathcal{A}$ adds a new edge sq with $q \in T_1 - R$.) This way the current ϕ' value of G' is decreased by one, as it was verified in Lemma 3.9.

Now $\mathcal{A}$ executes $\mathcal{P}$ on the graph obtained from G' by adding the edge sq. Then $\mathcal{A}$ continues from the branching step. (Note that by Lemma 3.2 $\mathcal{P}$ will perform at most one new R-split before it halts and $\mathcal{A}$ returns to the branching step.)

Case II: $\mathcal{P}$ halts when $d_{G'}(s) = d_{G'}(s, R) = 3$ and there exist two maximal dangerous sets X, Y such that $X \cup Y = V$ and $d_{G'}(X) = d_{G'}(Y) = k + 1$.

Now $\mathcal{A}$ adds a new edge sx with $x \in R$. In the new graph G'' obtained this way $d_{G''}(s)$ is even and all neighbours of s are in R. Thus by Theorem 1.1 there is a complete R-splitting in G''. Now $\mathcal{A}$ finds a complete R-splitting sequence and halts.

Case III: $\mathcal{P}$ halts when $d_{G'}(s, R) = 1$.

Here $\mathcal{A}$ has two possible ways to continue depending on which of the following two subcases occurs:

Case IIIa: $d_{G'}(s, V - R)$ is even.

Let st denote the only edge from s to R in G'. Let $W \subset V$ be a maximal set not containing R with $t \in W$ and $d_{G'}(W) = k$. If there is no such W then let $W = \emptyset$. Let $W' \subset V$ be a minimal set containing R with $d_{G'}(W') = k$. If there is no such set then let $W' = \emptyset$. Now $\mathcal{A}$ adds an edge st' with $t' \in R - W$ and if $W' \neq \emptyset$, then $\mathcal{A}$ adds an edge st'' with $t'' \in W' - R$. Let G'' denote the graph obtained this way. We claim that the pair (st, st') is admissible for splitting in G''. For suppose Z is a dangerous set in G'' containg t and t'. Then $d_{G'}(Z) = k$ must hold. Thus either $R \subset Z$ or $W \neq \emptyset$ and by the choice of t' the sets Z and W are intersecting. In the latter case (3), applied to Z and W in G', leads to a contradiction. In the former case Z and W' are intersecting and $t'' \notin Z$, thus a similar application of (3) gives a contradiction. Thus the pair

(st, st') is indeed admissible. Now $\mathcal{A}$ performs this admissible R-splitting (st, st'). If $d(s)$ is odd in the resulting graph, then $\mathcal{A}$ adds a new edge $s\tilde{t}$ with $\tilde{t} \in V - R$. Then $\mathcal{A}$ halts with the resulting graph $\tilde{G}$. Thus in this case $\mathcal{A}$ has found a complete R-splitting sequence and it returns a graph $\tilde{G}$ in which $d_{\tilde{G}}(s)$ is even.

Case IIIb: $d_{G'}(s, V - R)$ is odd.

This is the only subcase where $\mathcal{A}$ may need to backtrack, that is, to 'lift back' some previously split edges and delete some previously added new edges in order to find a solution of smaller size and reach optimality. This is due to the fact that in one particular case the adding rule of Case IIIa may result in adding more than $\phi'(H)$ new edges in total. Namely, to make the parities of $d(s, R)$ and $d(s, V - R)$ even, we may be forced to add two new edges even if the parities could be made even (preserving the connectivity condition) just by 'moving' the endvertex of an edge incident to s from $V - R$ to R. This moving step is not always possible. The extra subroutine $\mathcal{B}$ desribed below detects whether such a moving is possible. If it is possible (this happens, roughly speaking, if and only if $\rho_o(G')$ was already zero when $\mathcal{A}$ added the last edge from s to $V - R$), $\mathcal{B}$ moves one edge and then find a complete R-splitting following a special rule. If no moving is possible, $\mathcal{A}$ returns to the position where it called $\mathcal{B}$ and continues like in Case IIIa.

In detail, the following steps are made in Case IIIb. If $\mathcal{A}$ has added no new edges at all so far from s to $V - R$ then $\mathcal{A}$ continues following the steps of Case IIIa. If this is not the case and $\mathcal{A}$ has added some new edges from s to $V - R$ then $\mathcal{A}$ needs to backtrack to the point where it was just about to add the last new edge sq from s to $V - R$. (There may be edges added from s to R later.) Let $\check{G}$ denote the current graph at this point. Let $G^{\#}$ denote the graph obtained from $\check{G}$ by adding the edge sq. Since this is the last edge that $\mathcal{A}$ added from s to $V - R$, it follows from the description of Case I of $\mathcal{A}$ that

$$\rho(\check{G}) = \rho_i(\check{G}) + 1 \text{ and } \rho(G^{\#}) = \rho_i(G^{\#}). \tag{23}$$

Furthermore it follows from Lemma 3.6 and (23) that $(*)$ *in $\check{G}$ every R-obstacle W with $W \subset R$ and $c_{\check{G}}(W) = \rho_i(\check{G})$ is contained in every tight R-obstacle*.

Let U denote the maximal tight R-obstacle in $\check{G}$. Note that by the description of Case I of $\mathcal{A}$ it follows that $q \in U - R$. If $R \subset U$, then $\rho_o(\check{G}) = \rho(\check{G}) > 0$ and again $\mathcal{A}$ goes back to G' and continues following the steps of Case IIIa. Otherwise $R - U \neq \emptyset$. Let $p \in R - U$ be arbitrary. Now $\mathcal{A}$ adds the edge sp to $\check{G}$ (instead of sq). Let H' denote the resulting graph obtained by adding sp to $\check{G}$. Note that

$$\rho(H') = \rho(G^{\#}). \tag{24}$$

Furthermore it follows from (23) and $(*)$ that

$$\rho(H') = \rho_i(H') + 1. \tag{25}$$

Now $\mathcal{A}$ executes the following subroutine $\mathcal{B}$ on H':

Subroutine $\mathcal{B}$

$\mathcal{B}$ runs the same way as $\mathcal{A}$ does except the following modifications. In Case I $\mathcal{B}$ always adds the new edge so that it goes from s to R and still decreases ρ of the current graph. If at some point this is not possible, then it follows from the proof of Lemma 3.9 that the current graph contains a tight R-obstacle U with $R \subset U$. If this is the case, then $\mathcal{B}$ stops.

If $\mathcal{B}$ is always able to decrease ρ of the current graph, then eventually $\mathcal{P}$ will either find a complete R-splitting (after being called from $\mathcal{B}$), or one of the two last cases of Claim 3.4 will occur. If $\mathcal{P}$ returns with $d(s) = 3$ in the current graph, then $\mathcal{B}$ adds a new edge from s to R and finds a complete R-splitting, as $\mathcal{A}$ does in Case II. If $d_{H''}(s, R) = 1$ in the current graph H'' (the one returned by $\mathcal{P}$), then $\mathcal{B}$ tries to find a set U with $d_{H''}(U) = k$ and $R \subset U$. If such a set exists, then $\mathcal{B}$ stops. Thus we may assume that no set U exists. Let $s\bar{t}$ denote the only edge from s to R in H''. Let W be a maximal set not containing R such that the edge st enters W and $d_{H''}(W) = k$. If there is no such W then let $W = \emptyset$. Now $\mathcal{B}$ adds an edge st' to $R - W$. We can argue just as in Case IIIa that the pair $(s\bar{t}, st')$ is admissible for splitting in the graph that we obtain from H'' by adding the edge st'. Now $\mathcal{B}$ performs that splitting and stops with a complete R-splitting sequence and a graph where $d(s, V - R)$ is even. This completes the description of $\mathcal{B}$.

If $\mathcal{B}$ stops without finding a complete R-splitting sequence, then $\mathcal{A}$ returns to G' and continues following the steps of Case IIIa. This completes the desrciption of $\mathcal{A}$.

Figure 3 illustrates the steps of $\mathcal{A}$. It is easy to see that $\mathcal{A}$ is a well-defined finite algorithm which finds a set F^* of new edges for which $(V + s, E + F + F^*)$ has a complete R-splitting. In the rest of the proof we prove that F^* is an optimal solution of the R-split completion problem by showing that $|F^*| \leq \phi'(H)$. To see this we will show that every new edge added by $\mathcal{A}$ either decreases the current ϕ value by one or it is necessary to add by the parity conditions. We will show that the number of these 'parity edges' equals $\phi'(H) - \phi(H)$. (In Section 4 we show that $\mathcal{A}$ can be easily implemented as a polynomial time algorithm.)

Consider first an arbitrary iteration of $\mathcal{A}$ in Case I where a new edge sq is added and $\mathcal{P}$ is executed again. It follows from Lemma 3.9 that the edge sq decreases $\phi(G')$ by one. Hence it remains to prove that the edges added in Case II or Case III have the required property. Note that if $\mathcal{A}$ enters Case II or Case III then it adds at most three edges before terminating with a feasible solution.

Let G' be the current graph when $\mathcal{A}$ enters Case II or Case III. Since the new edges added so far have been chosen by the rule of Lemma 3.9, the following properties hold for G. (These can be seen by Lemma 3.9 (iii) and (iv) and by the fact that splitting off a pair of edges does not change the parity of $c(X)$ and $d(s, R)$):

(A) $d_{G'}(s, R) + \rho_i(G') \equiv d_H(s, R) + \rho_i(H)$ modulo 2.

(B) If $\rho_o(G') > 0$ then $d_{G'}(s, V - R) + \rho_o(G') \equiv d_H(s, V - R) + \rho_o(H)$ modulo 2.

Now we verify optimality in the three different cases $\mathcal{A}$ may terminate in. Consider first Case II, when $d_{G'}(s, R) = d_{G'}(s) = 3$ and there are maximal dangerous sets X, Y covering V such that $d_{G'}(X) = d_{G'}(Y) = k + 1$ and $d_{G'}(s, X \cap Y) = 1$. We claim that $\rho(G') = 0$.

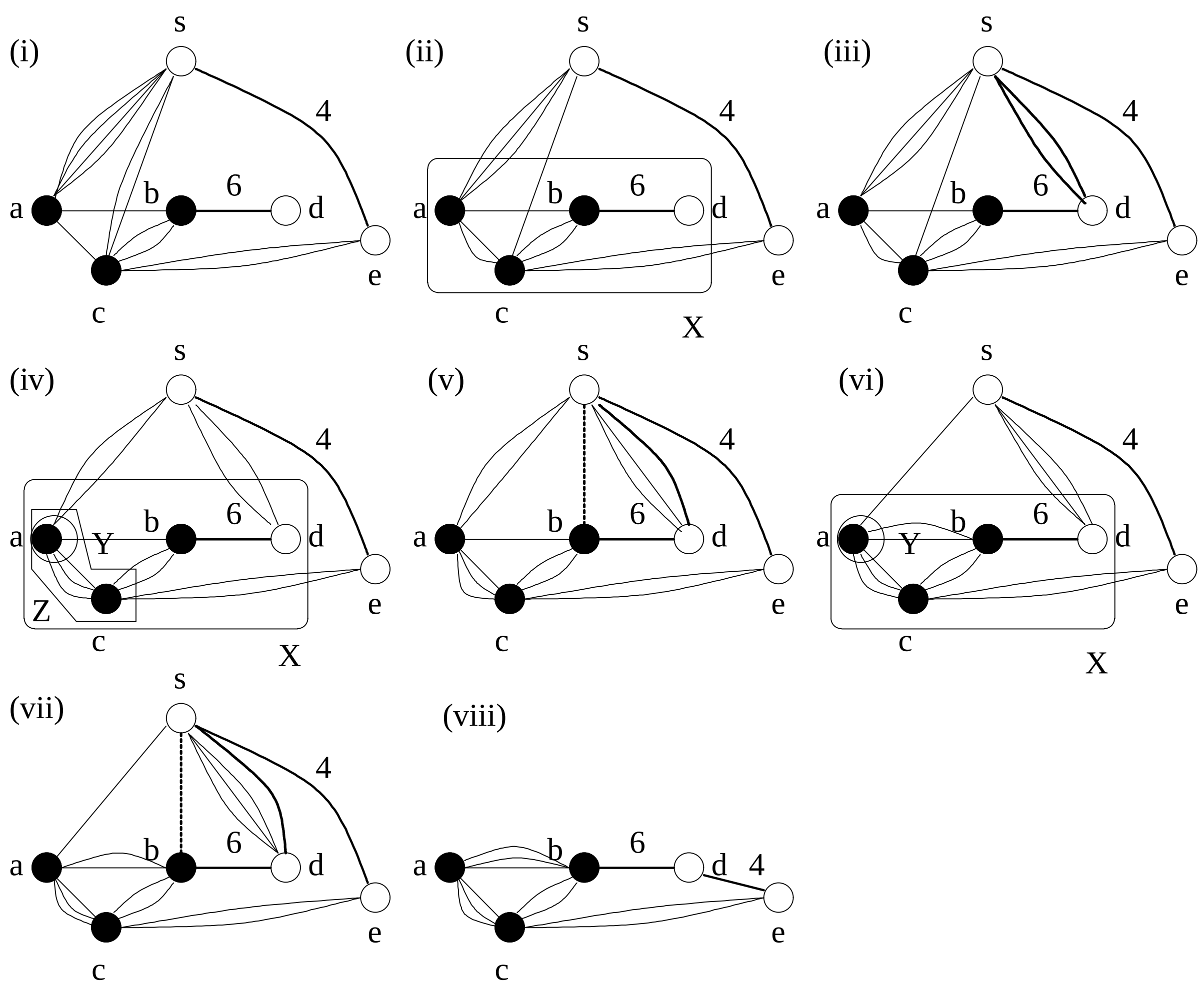

Figure 2: The steps of $\mathcal{A}$ made on the input graph $H = (V + s, E + F)$ of (i). R is formed by the black vertices and $k = 6$. Bold edges with numbers 4 and 6 represent four and six copies of parallel edges, respectively. First $\mathcal{A}$ runs the splitting routine $\mathcal{P}$ which halts with the maximal dangerous set X (ii). Then $\mathcal{A}$ adds a new edge sd. The next call of $\mathcal{P}$ finds X again and hence $\mathcal{A}$ adds another copy of sd (iii). Then $\mathcal{A}$ splits off the pair (sa, sc) and identifies the chain $Y \subset X$ of tight R-obstacles (iv). The next step is to add a new edge sb and then another one sd (v). Then $\mathcal{A}$ splits the pair (sa, sb) and identifies the chain $Y \subset X$ of tight R-obstacles again (vi). Adding edges sb and sd follows (vii) and then a new split (sa, sb) is possible. This yields a complete R-splitting and a graph with $d(s)$ even. Splitting off the remaining edges from s to $V - R$ gives a 6-edge-connected graph (viii).

Suppose this is not the case and let Z be an R-obstacle. Then $c_{G'}(Z) = 1$ by Lemma 3.2 and by the fact that we can find a complete R-splitting by adding one more edge to R (this can be seen directly from Definition 3.1 and (1), too). It is easy to see from (4) that this implies that $d_{G'}(s, R \cap Z) = 2$, $d_{G'}(s, R - Z) = 1$ and $d_{G'}(Z) = k$. Since Z is intersecting with either X or Y, we obtain a contradiction by (3). As we saw in the description of Case II, $\mathcal{A}$ adds a new 'parity' edge from s to R and completes the R-splitting sequence. Clearly, $\psi_o(H) = \rho_o(H) = 0$ must hold and hence $\rho(H) + \psi(H) \geq \rho_i(H) + \psi_i(H)$ and $\phi'(H) = \rho(H) + \psi(H)$. Furthermore, $\rho(G') + d_{G'}(s, R) \equiv \rho(H) + d_H(s, R) \equiv 1$ and hence $\psi(H) = 1$ follows. Thus $\phi'(H) = \rho(H) + 1$, showing that adding one parity edge preserves optimality.

Consider now Case IIIa when $d_{G'}(s, R) = 1$ and $d_{G'}(s, V - R)$ is even. By Lemma 3.2 we have $\rho(G') \leq 1$. Note that after adding the edge st' to G' $d(s)$ is even and hence $\mathcal{A}$ either adds one or 3 edges in Case IIIa (such that one edge connects s to R and either no edge or 2 edges are added from s to $V - R$). It follows from (A) that $\psi_i(H) = 1 - \rho_i(G')$ and hence by Lemma 3.9(iii) $\mathcal{A}$ adds $\rho_i(H) + \psi_i(H)$ edges from s to R altogether. Suppose first that $\mathcal{A}$ adds one more edge only. If every previously added new edge from s to $V - R$ decreased the current ρ_o value then $\rho_o(H)$ edges were added from s to $V - R$ and hence altogether $\mathcal{A}$ adds $\rho_i(H) + \psi_i(H) + \rho_o(H) \leq \phi'(H)$ edges. If $\mathcal{A}$ previously added an edge e from s to $V - R$ when the current ρ_o value was already zero then, since $\rho \geq 1$ when e was added, Lemma 3.9(ii) and Corollary 3.10 shows that every edge which was added previously by $\mathcal{A}$ decreased the current ρ value by one. This and the fact that $d_{G'}(s)$ is odd implies that $\psi(H) = 1 - \rho(G')$ and hence altogether $\mathcal{A}$ adds $\rho(H) + \psi(H) \leq \phi'(H)$ edges.

Suppose that $\mathcal{A}$ adds 3 edges to G' in Case IIIa. Then $\rho_o(G') = 1$ by the existence of W' and hence by (B) and the fact that $d_{G'}(s, V - R)$ is even it follows that $\psi_o(H) = 1$. Thus $\rho_o(H) + \psi_o(H)$ edges are added from s to $V - R$ and $\rho_i(H) + \psi_i(H)$ edges are added from s to R, and hence $|F^*| \leq \phi'(H)$ follows.

Finally consider Case IIIb. The first case is when $\mathcal{A}$ never calls $\mathcal{B}$. This happens only if no edge was added previously from s to $V - R$. Then $\rho_o(G') = \rho_o(H)$ and the fact that $d_{G'}(s, V - R)$ is odd implies that $\psi_o(H) = 1 - \rho_o(H)$ and as we argued in the proof for Case IIIa, we also have $\psi_i(H) = 1 - \rho_i(G')$. Now $\mathcal{A}$ adds 2 more edges (one from s to R and one from s to $V - R$) and hence altogether $\rho_i(H) + \psi_i(H) + \rho_o(H) + \psi_o(H)$ edges. Thus $|F^*| \leq \phi'(H)$.

Now consider the case when $\mathcal{A}$ executes $\mathcal{B}$. There are several possible ways $\mathcal{B}$ (and $\mathcal{A}$) can terminate in this case. $\mathcal{B}$ can itself find a solution by either finding a complete R-splitting in one of the splitting routines called by $\mathcal{B}$ or by adding one more edge when $\mathcal{P}$ halts with a configuration of Claim 3.4 (b) or (c). In some other cases $\mathcal{B}$ returns control to $\mathcal{A}$ and then $\mathcal{A}$ finds a solution starting from G' again, following the steps of Case IIIa.

Suppose first that $\mathcal{B}$ finds a complete R-splitting in one of the calls of the splitting routine. By (25) and Lemma 3.6 it follows that as long as ρ of the current graph is at least 2, the new edge added by $\mathcal{B}$ will decrease both ρ and ρ_i by one. Then it follows from (24) that in total $\mathcal{A}$ (including the call to $\mathcal{B}$) has added at most $\rho(H) \leq \phi'(H)$ new edges. Suppose next that Claim 3.4 (b) holds when $\mathcal{P}$ returns last time. Let K be the current graph at this point. Since $\rho > \rho_o$ at the time when $\mathcal{A}$ started executing $\mathcal{B}$, every new edge added so far by $\mathcal{A}$ and $\mathcal{B}$ has decreased ρ by Lemma 3.9 and hence $\psi(H) = 1 - \rho(K)$ (since $d_K(s)$ is odd). Now $\mathcal{B}$ either adds one more edge and then terminates with a solution or (if $\rho(K) = \rho_o(K) = 1$) returns to $\mathcal{A}$. In the former case the total number of new edges added equals $\rho(H) + \psi(H) \leq \phi'(H)$. In the latter case $\mathcal{A}$ adds two more edges and our arguments show that $\rho_i(H) + \psi_i(H) + \rho_o(H) + \psi_o(H) \leq \phi'(H)$ edges are added.

Finally, suppose that Claim 3.4 (c) holds when $\mathcal{P}$ returns last time. As we saw earlier, this configuration implies $\rho(K) = 0$, where K is the current graph as above. Since all the previously added edges decreased ρ and $d_K(s)$ is odd, the total number of edges added is $\rho(H) + \phi(H) \leq \phi'(H)$ in this case.

Suppose now that $\mathcal{B}$ halts (and returns control to $\mathcal{A}$) because it cannot find a new edge from s to R whose addition decreases ρ. By the description of $\mathcal{B}$ this happens if $\mathcal{B}$ has found an R-obstacle U containing R. This shows $\rho_o(\check{G}) = 1$. Now we get from (B) and the fact that $d_{\check{G}}(s, V - R)$ is even that $\psi_o(H) = 1$. Thus when $\mathcal{A}$ follows the steps of Case IIIa on G' and adds 2 more edges, $\rho_o(H) + \psi_o(H)$ edges are added from s to $V - R$. As we saw above, the number of edges added from s to R is $\rho_i(H) + \psi_i(H)$. Therefore $|F^*| = \rho_i(H) + \psi_i(H) + \rho_o(H) + \psi_o(H) \leq \phi'(H)$.

This completes the proof that $\mathcal{A}$ adds at most $\phi'(H)$ new edges when it constructs G^* which has a complete R-splitting and for which $d_{G^*}(s)$ is even. □

4 Algorithmic aspects

Given a graph $H = (V + s, E + F)$, $R \subseteq V$ and $k \geq 2$ with $d_{H-s}(R) \geq k$, Algorithm $\mathcal{A}$ of Section 3 finds an optimal solution for the R-split completion problem. Let $n := |V|$ and $m := |E + F|$. It is easy to construct an algorithm following the steps of $\mathcal{A}$ whose running time is a polynomial of n, m and k.

To avoid a long analysis we only sketch some essential points here and omit most of the details. The basic subroutine of $\mathcal{A}$ is the splitting routine $\mathcal{P}$. This routine first finds a maximal R-splitting sequence possibly after executing some change-operations when Case 1 applies. For this the polynomial splitting off algorithms can be used [9], [11], [23]. Then either $\mathcal{P}$ arrives at a case where after adding at most two edges properly an optimal solution can be found (these cases – Case II and Case IIIa – are easy to handle by, say, max flow computations) or it calls algorithm $\mathcal{B}$ (in Case IIIb) or it identifies a maximal dangerous set X as desribed in Case I of the analysis. Estimating the running time of $\mathcal{B}$ is similar to that of $\mathcal{A}$ and $\mathcal{B}$ gives an optimal solution. Thus we may focus on the other case. If Case I holds then $\mathcal{A}$ adds a new edge incident to s which decreases parameter ϕ and calls $\mathcal{P}$ again. To find such an edge to be added the chain of tight R-obstacles has to be found. This can be done by max flow computations, based on Lemma 3.6 and 3.7. Clearly, $\phi(H) \leq 2k$ and hence Case I occurs at most $2k$ times.

In the rest of this section we prove some further observations.

Lemma 4.1 $\phi(H) \leq k$.

Proof: By (4) it is obvious that $c(X) \leq k$ for every $\emptyset \neq X \subset V$ and hence $\rho_i(H), \rho_o(H)$ and $\rho(H)$ are at most k. We claim that $\rho_i(H) + \rho_o(H) \leq k$ holds, too. To see this suppose that Y and

Z are obstacles with $Y \subset R$ and $R \subset Z$. By definition $c(Y) = k - (d(Y, Z - Y) + d(Y, V - Z) + d(s, R - Y))$ and $c(Z) = k - (d(Y, V - Z) + d(Z - Y, V - Z) + d(s, Z - R))$. By (1) we have $d(Y, Z - Y) + d(Z - Y, V - Z) + d(s, Z - R) + d(s, R - Y) \geq k$, since $Z - Y \neq \emptyset$. Combining these we obtain $c(Y) + c(Z) \leq k$ and hence $\rho_i(H) + \rho_o(H) \leq k$ follows. □

Lemma 4.1 and Theorem 3.3 implies $\gamma(H) \leq k + 2$. Therefore $\mathcal{A}$ adds at most $k + 2$ edges.

It is easy to see that every solution for the R-split completion problem is a solution for the $(V\text{-}R)$-split completion problem. That is, if adding a set F^* of new edges results in a graph H^* where $d(s)$ is even and which has a complete R-splitting then H^* has a complete $(V\text{-}R)$-splitting, as well. The following lemma formulates this 'duality' property for the corresponding 'dual' parameters.

Lemma 4.2 *$\rho(H, R) = \rho(H, V - R)$, $\rho_i(H, R) = \rho_o(H, V - R)$, $\phi(H, R) = \phi(H, V - R)$ and $\phi'(H, R) = \phi'(H, V - R)$, where $\rho(H, X)$ denotes $\rho(H)$ with respect to the specified set $X \subseteq V$.*

Proof: Applying (4) to some $\emptyset \neq X \subset V$ we get $c_R(X) = k - (d_{H-s}(X) + d(s, X - R) + d(s, R - X)) = k - (d_{H-s}(V - X) + d(s, V - R - (V - X)) + d(s, V - X - (V - R)) = c_{V-R}(V - X)$, which shows $\rho(H, R) = \rho(H, V - R)$. Clearly, $\psi_i(H, R) = \psi_o(H, V - R)$. From these equalities the lemma follows esily. □

Lemma 3.8 and 4.2 give the following.

Lemma 4.3 *Let X be the maximal dangerous set returned by $\mathcal{P}$ when Case 2 applies and suppose that $X - R \neq \emptyset$. If Z is an R-obstacle with $Z \subset R$, then $Z \subset X$.* □

5 The augmentation problem

As we noted in the introduction, the main motivation to study the R-split completion problem (and other split-completion problems) is the close relationship with the corresponding edge-connectivity augmentation problems. This connection was discovered by Cai and Sun [5] who gave a polynomial algorithm for finding a smallest set of new edges which makes a given graph $G = (V, E)$ k-edge-connected ($k \geq 2$), based on splitting off. This algorithm was simplified (and extended) by Frank [9]. Frank's algorithm has three steps as follows:

(i) Add a new vertex s to V and a minimal set F of new edges incident to s so that $G' = (V + s, E + F)$ satisfies (1).

(ii) If $d'(s)$ is odd in G' then add a new edge sv for some $v \in V$.

(iii) Split off all the edges incident to s in pairs, maintaining (1). The resulting graph (after deleting s) is a k-edge-connected augmentation of G.

The required set F of edges in step (i) is easy to find by a greedy deletion procedure. The complete admissible splitting in step (iii) exists by Theorem 1.1. Frank [9] proved that every possible minimal set F of edges found in step (i) has the same cardinality and (hence) the resulting graph (the union of G and the split edges) is an *optimal* augmentation of G.

Now consider the following more general problem: given a graph $G = (V, E)$, an integer $k \geq 2$ and a property $\mathcal{P}$. Find a smallest set F' of new edges for which $G' = (V, E + F')$ is k-edge-connected and such that each edge of F' (or the set F' collectively) satisfy $\mathcal{P}$. For example, one may want to preserve the bipartititeness (simplicity, planarity, etc.) of G or one may want to avoid adding split edges from R to $V - R$ for a specified set $R \subset V$. These problems with the corresponding properties $\mathcal{P}$ all fit this framework.

We propose a general procedure to solve a problem of this type. Our method may not work for certain properties $\mathcal{P}$ but may lead to the solution of some others (see the examples below). Our "algorithm" has five steps as follows:

(o) Solve the split-completion problem for general graphs with respect to property $\mathcal{P}$. That is, find a function (a min-max formula) $f(H, \mathcal{P})$ which, given a graph $H = (V + s, E)$ satisfying (1), determines the minimum number of new edges incident to s to be added to H to obtain a graph which has a complete admissible $\mathcal{P}$-splitting sequence at s.

(i) Add a new vertex s to G and a minimal set F of edges incident to s for which $H = (V + s, E + F)$ satisfies (1).

(iia) By "rearranging" some edges of F in $H = (V+s, E+F)$ find a graph $H'' = (V+s, E+F'')$ for which (1) holds, $|F''| = |F|$ and $f(H'', \mathcal{P})$ is as small as possible.

(iib) Find an optimal solution F^* for the $\mathcal{P}$-split completion problem on H''.

(iii) Split off a complete admissible $\mathcal{P}$-splitting at s in $H^* = (V + s, E + F'' + F^*)$. The resulting graph (after deleting s) will be a k-edge-connected augmentation of G for which the set of new edges satisfies $\mathcal{P}$.

It is easy to see that the resulting graph is indeed a feasible solution to our problem. The following lemma verifies optimality.

Lemma 5.1 *Let I be the set of new edges added to G in step (iii) by the complete admissible $\mathcal{P}$-splitting and let $G' = (V, E + I)$. Then G' is an optimal k-edge-connected augmentation of G with respect to $\mathcal{P}$, that is, I is a smallest set of new edges satisfying property $\mathcal{P}$ which makes G k-edge-connected.*

Proof: For a contradiction suppose that $\tilde{G} = (V, E + I')$ is k-edge-connected for some I' with $|I'| < |I|$ and such that I' also satisfies $\mathcal{P}$. Let us "lift back" the edges of I' in $\tilde{G}$, that is, let us subdivide each edge of I' by a new vertex and then contract the set of these new vertices to one vertex s. (This corresponds to the inverse operation of a complete splitting at s.) The graph H obtained this way clearly satifies (1) and has a complete admissible $\mathcal{P}$-splitting at s. Let F'_H be the set of edges incident to s in H and let $F_H \subseteq F'_H$ be a minimal set of edges for which $H' = H - (F'_H - F_H) = (V + s, E + F_H)$ satisfies (1). Since H' is a possible output of step (i), by our remark after Frank's algorithm we can see that $|F| = |F_H|$ holds, where F is the set of new edges incident to s (added to G) at the end of step (iia). Let F'' be the set of edges added to our graph in step (iib). Then $F + F''$ is precisely the set of edges we obtain by lifing back I in G'. Now $|F + F''| > |F'_H|$ and hence $|F''| > |F'_H - F_H|$. Thus $f(H', \mathcal{P}) < f((V + s, E + F), \mathcal{P})$, contradicting the minimality of $f((V + s, E + F), \mathcal{P})$. □

Thus we have an efficient method to solve the augmentation problem provided we can solve step (0) and have efficient algorithms for steps (iia),(iib), (iii). Of course, these subproblems may be quite different for different properties $\mathcal{P}$. For example, if G is simple and the set of new edges must form a simple graph in the complement of G (that is, we want to preserve simplicity) then the augmentation problem is NP-hard and it is NP-complete to decide whether a complete admissible splitting preserving simplicity exists [14]. Thus one cannot expect to find a good function $f(H, \mathcal{P})$ and it is unlikely to have efficient methods for solving step (iib), say.

On the other hand, our method seems to work well for certain properties. We illustrate this by two examples. In the first example suppose that $\mathcal{P}$ gives no extra requirement, that is, simply a smallest augmenting set has to be found. This is the original edge-connectivity augmentation problem. Specializing our framework to this case we obtain Frank's algorithm. The solution of step (0) is simple: by Theorem 1.1 the value of the function $f(H, \mathcal{P})$ is either zero or one depending on whether $d(s)$ is even or odd. Step (i) is identical to step (i) of Frank's algorithm. In step (iia) the parity of $d(s)$ cannot be changed, hence this step has no role. The optimal solution in step (iib) is simply adding $f(H, \mathcal{P})$ new edges arbitrarily. This corresponds to step (ii) of Frank's algorithm. Step (iii) includes finding a complete admissible splitting (which can be done by – an efficient implementation of – Theorem 1.1).

The second example is the *partition-constrained k-edge-connectivity augmentation problem* in the special case when k is even. In this problem we are given $G = (V, E)$, $k \geq 2$ and a partition $\mathcal{X} = \{X_1, ..., X_t\}$ ($t \geq 2$) of V. The goal is to find a smallest set F of new edges whose addition makes G k-edge-connected and so that each edge of F connects two different classes of $\mathcal{X}$. This problem was solved in [2] for all values of k. Clearly, this kind of constrained augmentation problem belongs to the general problem we consider. Now we briefly sketch why the algorithm given in [2] (for the case when k is even) is a special instance of our general framework.

Let $\mathcal{P}$ be the property of satisfying the partition constraints. In [2] it was shown that a complete $\mathcal{P}$-splitting exists if and only if $d(s)$ is even and $d(s, X_i) \leq d(s)/2$ for every $1 \leq i \leq t$. This solves step (0): given $H = (V + s, E + F)$ we obtain $f(H, \mathcal{P}) = \max\{d(s)/2 - \lfloor d(s)/2 \rfloor, 2d(s, X_j) - |F|\}$, where X_j maximizes $d(s, X_j)$. The first step in the algorithm of [2] is identical to step (i). It was also shown how $f(H, \mathcal{P})$ can be minimized for some H by rearranging its edges incident to s (maintaining (1)). The optimal arrangement can be found by a "moving" procedure, which iteratively replaces some edge sx by another edge su [2]. This moving step correspond to step (iia) while the next step of the algorithm is identical to step (iib). Then a complete admissible $\mathcal{P}$-splitting has to be found. It produces an optimal augmenting set of edges. (This can be done by repeatedly choosing an edge st with $t \in X_j$ and applying a stronger form of Theorem 1.1, which guarentees the existence of an admissible pair st, sw with $w \notin X_j$.) This corresponds to step (iii).

In Section 3 we have solved step (0) and steps (iib) and (iii) for the R-splitting problem. The only remaining step to solve the corresponding augmentation problem (in our general framework) is to solve the problem arising in step (iia). We believe that a moving step similar to the one in [2] can be applied, although it may be necessary to "move" pairs of edges together (instead of moving the edges one by one).

6 Remarks

Recall that Corollary 3.5 gave a characterization of those graphs which have a complete admissible R-splitting. In this section first we show that there exists another simple and easy to prove characterization of these graphs. However, in the proof we rely on the following deep theorem due to Mader. The *local edge-connectivity* between vertices u, v in a graph H is $\lambda_H(u,v) := \min\{d_H(X) : u \in X, v \notin X\}$.

Theorem 6.1 *[18] Let $G = (V+s, E+F)$ be an undirected graph. Suppose that $d(s)$ is even and G contains no cut-edges incident to s. Then all the edges incident to s can be split off in pairs in such a way that the resulting graph $G^* = (V, E \cup F^*)$ satisfies $\lambda_{G^*}(u,v) = \lambda_G(u,v)$ for each pair $u, v \in V$.* □

Let $G = (V+s, E+F)$ satisfy (1) and let $R \subseteq V$, $k \geq 2$ be given. Let $G' = (V \cup \{s_R, s_{V-R}\}, E + F_R + F_{V-R})$ denote the graph that we obtain from G by splitting s into two new vertices s_R and s_{V-R} and partitioning F into two sets F_R, F_{V-R} in such a way that F_R (F_{V-R}) contains precisely those edges that go from s to R ($V-R$, respectively) in G.

Claim 6.2 *There exists a complete admissible R-splitting in G if and only if $d_G(s,R)$ is even and $\lambda(u,v) \geq k$ for every $u, v \in V$ in G'.*

Proof: Suppose G has a complete admissible R-splitting and let G^* be the k-edge-connected graph resulting from such a splitting sequence $\mathcal{S}$. Then G' can be obtained from G^* by replacing those edges uv that were added inside R by two edges us_R, vs_R. Since this "lifting back" operation does not decrease the local edge-connectivities in V and no splitting in $\mathcal{S}$ added an edge from R to $V-R$, it follows that the resulting graph is indeed G' and $\lambda_{G'}(u,v) \geq k$ for every $u, v \in V$.

Conversely suppose that G' is k-edge-connected in V. Note that since $k \geq 2$ there is no cut-edge incident to s_R in G'. Thus by Theorem 6.1 applied to the graph $G' = ((V \cup \{s_{V-R}\}) + s_R, (E \cup F_{V-R}) \cup F_R)$ all the edges incident to s_R can be split off in such a way that the resulting graph $G'' = (V + s_{V-R}, E \cup F_R^* \cup F_{V-R})$ is k-edge-connected in V. It follows easily from the way we performed the splittings above that G'' could also have been obtained from G by pairing the edges incident from s to R in the same way as their corresponding edges were paired above. Hence it follows that G has a complete admissible R-splitting. □

By repeated applications of Theorem 6.1 this statement can be extended easily to the case where a (sub)partition $\mathcal{X} = \{X_1, ..., X_r\}$ of V is given in G and we want to decide if there exists a complete admissible $\mathcal{X}$-splitting, that is, an admissible splitting sequence involving all the edges from s to members of $\mathcal{X}$ for which every edge st, $t \in X_i$ ($1 \leq i \leq r$) is split with an edge su, $u \in X_i$.

Claim 6.2 implies the first part of Corollary 3.5 as follows.

Corollary 6.3 *There exists a complete admissible R-splitting in G if and only if $d_G(s,R)$ is even and G contains no R-obstacle.*

Proof: Suppose first that G contains an R-obstacle $X \subset V$. Then it follows from the definition of G' that $d_{G'}(X \cup \{s_R\}) = d_G(X) - d_G(s, R \cap X) + d_G(s, R - X) = (d_{G-s}(X) + d_G(s, R \cap X) + d_G(s, X - R)) - d_G(s, R \cap X) + d_G(s, R - X) = (k - c_G(X) + d_G(s, R \cap X) - d_G(s, R - X)) - d_G(s, R \cap X) + d_G(s, R - X) = k - c_G(X) < k$, since $c_G(X) > 0$. Thus for two vertices u, v ($u \in X, v \in V - X$) of G' we have $\lambda_{G'}(u, v) < k$ and hence there is no complete admissible R-splitting in G by Claim 6.2.

Suppose now that there is no complete admissible R-splitting in G. Then, by Claim 6.2, either $d_G(s, R)$ is odd or G' has two vertices $u, v \in V$ with $\lambda_{G'}(u, v) < k$. Since G is k-edge-connected in V this implies that there exists a nonempty proper subset $Y \subset V$ such that the set $Y \cup \{s_R\}$ has degree less than k in G'. Let δ be chosen such that $d_{G'}(Y \cup \{s_R\}) = k - \delta$. Then by calculations similar to the ones above we can conclude that $c_G(Y) = \delta > 0$ and hence G contains an R-obstacle. □

Note that it follows from the argument above that we can find an R-obstacle Z in G with maximum value of $c(Z)$ by finding a subset $Z \cup \{s_R\}$ of $V \cup \{s_R\}$ which has the minimum degree among all such subsets. This is equivalent to computing the smallest $\lambda_{G'}(u, v)$ value for $u, v \in V$, which can be done by max-flow computations.

Finally we remark that the split-completion problem where the goal is to add edges incident to s in order to guarantee the existence of complete admissible splitting where each split edge connects two given nonempty subsets $R, Q \subset V$ ($R \cap Q = \emptyset$) has been solved (for even values of k) using similar techniques [4].

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