Journal of Combinatorial Theory, Series A **90**, 27–48 (2000)
doi:10.1006/jcta.1999.3018, available online at http://www.idealibrary.com on IDEAL®

Primitive Commutative Association Schemes with a Non-symmetric Relation of Valency 3[1]

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Communicated by A. A. Ivanov

Received June 5, 1998

Let $(X, \{R_i\}_{0\leqslant i\leqslant d})$ be a primitive commutative association scheme. If there is a non-symmetric relation R_i with valency 3, then the cardinality of X is equal to either p or p^2 where p is an odd prime. Moreover, if $|X|=p$ then $(X, \{R_i\}_{0\leqslant i\leqslant d})$ is isomorphic to a cyclotomic scheme.

1. INTRODUCTION

Let X be a finite set and $\{R_i\}_{0\leqslant i\leqslant d}$ be a partition of $X\times X$ which does not contain the empty set. Following [3] the pair $(X, \{R_i\}_{0\leqslant i\leqslant d})$ is called an *association scheme* (or simply, a *scheme*) if it satisfies the following conditions:

(i) $R_0=\{(x, x)\mid x\in X\}$;

(ii) For each $i\in\{0, 1, ..., d\}$ there exists $i'\in\{0, 1, ..., d\}$ such that $R_i^t:=\{(x, y)\mid (y, x)\in R_i\}=R_{i'}$;

(iii) For all $i, j, k\in\{0, 1, ..., d\}$ $p_{ij}^k(x, y):=|\{z\in X\mid (x, z)\in R_i, (y, z)\in R_j\}|$ is constant whenever $(x, y)\in R_k$. We shall write p_{ij}^k instead of $p_{ij}^k(x, y)$ and $\{p_{ij}^k\}_{0\leqslant i,j,k\leqslant d}$ is called the *intersection numbers* of $(X, \{R_i\}_{0\leqslant i\leqslant d})$. In particular, $k_i:=p_{ii'}^0$ is called the valency of R_i for each i.

An association scheme $(X, \{R_i\}_{0\leqslant i\leqslant d})$ is called *commutative* if $p_{ij}^k=p_{ji}^k$ for all $i, j, k\in\{0, 1, ..., d\}$.

Note that (X, R_i) is a regular digraph for each i. It is an interesting problem to find all regular graphs which might be a relation of an association scheme under certain hypotheses about intersection numbers or an induced subgraph of the graph, and to determine the whole structure of

[1] The author was supported by the Japan Society for Promotion of Science. He makes a grateful acknowledgment of their support and expresses his deepest gratitude to Professor Bannai and Professor Munemasa for their advice and encouragement.

0097-3165/00 $35.00

$(X, \{R_i\}_{0 \leqslant i \leqslant d})$. The motivation of this paper comes from researches about association schemes with a prime number of points. If $|X|$ is an odd prime then the digraph (X, R_i) is connected for each $i \neq 0$ (see [3]), that is called *primitive*, and there is no symmetric relation of odd valency since $|R_i| = |X| k_i$ is even if $R_i = R_i^t$.

The following is a typical example of such schemes:

EXAMPLE 1.1 [5, p. 66]. Let F_q be a finite field with q elements where q is a prime power, and K be a subgroup of the multiplicative group of F_q. Then we can define the *cyclotomic scheme* on $X := F_q$ by $R_i := \{(x, y) \mid y - x \in a_i K\}$ $(1 \leqslant i \leqslant d)$ where $\{a_i\}_{1 \leqslant i \leqslant d}$ is a transversal of $F_q - \{0\}$ by K and $d\,|K| = q - 1$. We denote it by $\mathsf{Cyc}(q, |K|)$.

Cyclotomic schemes are the only known example of association scheme such that $|X|$ is a prime and $d \neq 2$. For $d = 2$ it is known that there exists an association scheme which are not isomorphic to each cyclotomic scheme, and each association scheme with a prime number of points and $d = 2$ has the same intersection numbers as a cyclotomic scheme (see [7, p. 182]).

We shall introduce an example of a more general class of cyclotomic schemes

DEFINITION 1.2 [5, p. 65]. We call an association scheme $(X, \{R_i\}_{0 \leqslant i \leqslant d})$, where the underlying set X has the structure of an abelian group, *a translation scheme* if, for all classes R_i,

$$(x, y) \in R_i \Leftrightarrow (x + z, y + z) \in R_i \qquad \text{for all} \quad z \in X.$$

When the number of points is prime, the only translation schemes are cyclotomic schemes.

THEOREM 1.3 [5, p. 66]. *A translation scheme with a prime number of points is a cyclotomic scheme.*

There are association schemes with a prime number of points which are not translation schemes. An example appears when $d = 2$ and $|X| = 19$ (see [6, Theorem 2.6.6; 8]) and it is conjectured that there are quite a few isomorphism classes for large $|X|$. Thus, given an association scheme with a prime number of points, it is nontrivial to show that it is a translation scheme (hence a cyclotomic scheme). In the following special cases, the result is obtained rather easily.

PROPOSITION 1.4. *Let* $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ *be a primitive commutative association scheme. Then the following hold*:

(i) *If there exists a non-diagonal relation with valency* 1 *then* $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ *is isomorphic to* $\mathsf{Cyc}(|X|, 1)$;

(ii) *If there exists a relation with valency* 2 *then* $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ *is isomorphic to* $\mathsf{Cyc}(|X|, 2)$.

Our main theorem of this paper is a result analogous to Proposition 1.4 when there exists a relation of valency 3. The proof of our main theorem is considerably complicated in contrast to the (almost obvious) proof of Proposition 1.4.

THEOREM 1.5 (Main Theorem). *Let* $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ *be a primitive commutative association scheme. If there is a non-symmetric relation* R_i *with valency* 3, *then the cardinality of* X *is equal to either* p *or* p^2 *where* p *is an odd prime. Moreover, if* $|X| = p$ *then* $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ *is isomorphic to* $\mathsf{Cyc}(p, 3)$, *if* $|X| = p^2$ *then there exists a relation isomorphic to a non-diagonal relation of* $\mathsf{Cyc}(p^2, 3)$,

The outline of the proof is that we determine the graph (X, R_i), and show that $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ is a translation scheme if $|X|$ is a prime, so that we can prove our main theorem by using Theorem 1.3.

In order to prove our main theorem we prepare some basic notations and lemmas.

Let $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ be an association scheme. Following [3] we define the *adjacency matrix* with respect to R_i as

$$(A_i)_{xy} := \begin{cases} 1 & \text{if} \quad (x, y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

Then the third condition of the definition of association schemes can be expressed as

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \qquad \text{for all } i, j.$$

We set $R_i(x) := \{ y \mid (x, y) \in R_i \}$ for each i with $0 \leqslant i \leqslant d$ and each $x \in X$, so that $|R_i(x)| = k_i$. Now we give basic properties of intersection numbers (see [5, p. 44]).

LEMMA 1.6. *Let* $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ *be a association scheme. Then, for all* h, i, j, $(0 \leqslant h, i, j, \leqslant d)$ *we have the following*:

(i) $k_0 = 1$, $k_{i'} = k_i$, $|X| = k_0 + k_1 + \cdots + k_d$;

(ii) $p_{i0}^j = \delta_{ij}$, $p_{0j}^i = \delta_{ji}$, $p_{ij'}^0 = \delta_{ij} k_i$;

(iii) $k_h p_{ij}^h = k_i p_{hj'}^i = k_j p_{i'h}^j$;

(iv) $k_i k_j = \sum_{l=0}^d p_{ij}^l k_l$.

We investigate some intersection numbers about local structure in Section 2, give a proof of our main theorem in Section 3, and describe the related topics in Section 4.

2. ANALYSIS OF LOCAL STRUCTURES

Let $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a primitive commutative association scheme with a non-symmetric relation of valency 3 throughout this section.

Since all non-diagonal relations of each cyclotomic scheme have the same valency, the existence of a non-symmetric relation with valency 3 implies

$$\min_{1 \leqslant i \leqslant d} k_i = 3 \tag{1}$$

in view of Proposition 1.4, and $|X| \geqslant 7$ by Lemma 1.6(i).

LEMMA 2.1. *If* $k_j = 3$ *then* $\max_{1 \leqslant i \leqslant d} p_{jj'}^i = 1$.

Proof. Assume the contrary, i.e., $p_{jj'}^i \geqslant 2$ for some $i \neq 0$. Since $9 = k_j k_{j'} = \sum_{l=0}^d p_{jj'}^l k_l = 3 + \sum_{l=1}^d p_{jj'}^l k_l$ by Lemma 1.6(ii), (iv), we obtain from (2.1) the inequality

$$6 = \sum_{i=1}^{d} p_{jj'}^i k_i \geqslant \sum_{i=1}^{d} 3 p_{jj'}^i. \tag{2}$$

It follows from (1) that $p_{jj'}^i = 2$ and $k_i = 3$, i.e.,

$$A_j A_{j'} = 3A_0 + 2A_i.$$

Note that $R_i = R_i^t$ since $A_j A_j^t - 3A_0$ is symmetric, the induced subgraph of (X, R_i) by $R_i(x)$, $x \in X$, is the complete graph of degree 3 and $p_{ii}^i \geqslant 1$. Since $k_i = 3$, there exists a unique $y \in X - R_i(x)$ such that $(y, z) \in R_i$ for some $z \in R_i(x)$. Since $1 \leqslant p_{ii}^i = p_{ii}^i(y, z)$, we have $(y, w) \in R_i$ for some $w \in R_i(x) - \{z\}$. This implies $p_{ii}^i \geqslant 2$, and hence the induced subgraph of (X, R_i) by $R_i(x) \cup \{y\}$ is the complete graph of degree 4, contradicting $|X| \geqslant 7$ by primitivity. ∎

The following equations are a direct consequence of the Eq. (1) and Lemma 2.1:

$$A_j A_j^t = 3A_0 + A_\alpha + A_\alpha^t, \qquad \text{where} \quad k_\alpha = 3, \quad A_\alpha \neq A_\alpha^t \quad \text{or} \tag{3}$$

$$A_j A_j^t = 3A_0 + A_\beta, \qquad \text{where} \quad k_\beta = 6, \quad A_\beta = A_\beta^t. \tag{4}$$

LEMMA 2.2. *If* $k_i = k_j = 3$ *then we have* $\max_{1 \leqslant l \leqslant d} p_{ij}^m \leqslant 2$.

Proof. Assume the contrary, i.e., $A_i A_j = 3A_m$ for some $m \neq 0$ with $k_m = 3$ by (2.1) and Lemma 1.6(iv). Multiplying A_j^t on both sides of $A_i A_j = 3A_m$, we have either $A_i(3A_0 + A_\alpha + A_\alpha^t) = 3A_m A_j^t$ or $A_i(3A_0 + A_\beta) = 3A_m A_j^t$ where $k_\alpha = 3$, $k_\beta = 6$ by (3) and (4). We see that the coefficient of A_i on the right hand side equal to 9 since $p_{mj'}^i = p_{ij}^m k_m / k_i = 3$ by Lemma 1.6(iii).

On the other hand, that of the left hand side is less than 9, because we have $p_{i\alpha}^i + p_{i\alpha'}^i = p_{ii'}^\alpha + p_{ii'}^\alpha \leqslant 2$ or $p_{i\beta}^i = k_\beta p_{ii'}^\beta / k_i \leqslant 2$ by Lemma 2.1. This is a contradiction. ∎

Following [1] we define $\mathsf{Sup}(A_i A_j) := \{A_k \mid p_{ij}^k \neq 0\}$ for all adjacency matrices A_i, A_j of an association scheme.

LEMMA 2.3. *If* $k_i = k_j = 3$ *and* $p_{ij}^m \geqslant 2$ *for some* m *then* $A_i A_i^t = A_j A_j^t$.

Proof. The assumption implies that there exist distinct two elements z_1, $z_2 \in R_i(x) \cap R_{j'}(y)$ with $(x, y) \in m$. Write R_γ as the relation containing (z_1, z_2). Then $A_\gamma \in \mathsf{Sup}(A_i^t A_i) \cap \mathsf{Sup}(A_j A_j^t)$. In view of (3) and (4), we obtain from commutativity that $A_i A_i^t = A_j A_j^t$. ∎

LEMMA 2.4. *If* $k_j = 3$ *and* $p_{jj'}^\gamma > 0$ *for some* $\gamma \neq 0$ *then the induced subgraph of* (X, R_γ) *by* $R_j(x)$, $x \in X$, *is either of a directed cycle or the complete graph of degree* 3.

Proof. Since the induced subgraph of (X, R_γ) by $R_j(x)$ is regular with three points, it is obvious in view of (3) and (4). ∎

LEMMA 2.5. *Let* $x \in X$. *If* $k_i = k_j = 3$, $i \neq j'$ *and* $A_i A_i^t = A_j A_j^t$, *then there exsists a unique* $w_{yz} \in R_j(y) \cap R_j(z)$ *for each* $\{y, z\} \subset R_i(x)$ *with* $y \neq z$. *Furthermore* $|\{x\} \cup R_i(x) \cup \{w_{yz}\}_{y, z \in R_i(x)}| = 7$

Proof. Let $\{y, z\} \subset R_i(x)$ and $(y, z) \in R_\gamma$. Since $p_{i'i}^\gamma = p_{ii'}^\gamma = p_{jj'}^\gamma(y, z) = 1$ by commutativity and the assumption, the fist statement holds.

It is obvious that $\{x\} \cup R_i(x)$ are distinct. We claim that $\{x\} \cup R_i(x)$ does not intersect with $\{w_{yz}\}_{y, z \in R_i(x)}$. Assume the contrary, i.e., $w_{yz} = u$ for some $u \in R_i(x) - \{y, z\}$ since $i \neq j'$. It follows from Lemma 2.4 that $R_i = R_i^t$, contradicting (3) and (4).

We claim that $|\{w_{yz}\}_{y,\, z \in R_i(x)}| = 3$. Assume the contrary, i.e., $w_{yz} = w_{yu}$ for some $y \in R_i(x)$ where $R_i(x) = \{u, y, z\}$. Then $u, y, z \in R_i(x) \cap R_{j'}(w_{yz})$, contradicting Lemma 2.2.

These above claims completes the proof. ∎

Applying Lemma 2.5 with $i = j$, we obtain the following proposition.

PROPOSITION 2.6. *Then $|X| = 7$ if and only if $p^j_{jj} > 0$ for some j with $k_j = 3$.*

Proof. If $|X| = 7$ then $d = 2$ by the existence of non-symmetric relation of valency 3. Hence $p^1_{11} = p^1_{1'1} > 0$ by (3) and (4), as desired.

Conversely, suppose $p^j_{jj} > 0$ for some j with $k_j = 3$. Fix any element $x_0 \in X$ and $R_j(x) := \{x_1, x_2, x_3\}$. We may assume $(x_1, x_2) \in R_j$ without loss of generality. We have $(x_2, x_3) \in R_j$ and $(x_3, x_1) \in R_j$ by Lemma 2.4. By Lemma 2.5, there exists a point set $\{x_3, x_4, x_5, x_6\}$ such that $\{x_4\} = R_j(x_1) \cap R_j(x_2)$, $\{x_5\} = R_1(x_2) \cap R_j(x_3)$ and $\{x_6\} = R_j(x_3) \cap R_j(x_1)$.

Note $\{x_3, x_4, x_5\} = R_j(x_2)$ and $(x_3, x_5) \in R_j$ by construction. It follows that (x_5, x_4), $(x_4, x_3) \in R_j$ by Lemma 2.4. Note $\{x_1, x_6, x_5\} = R_j(x_3)$ and $(x_1, x_6) \in R_j$ by construction. It follows that $(x_6, x_5), (x_5, x_1) \in R_j$ by Lemma 2.4. Note $\{x_2, x_4, x_6\} = R_j(x_1)$ and $(x_2, x_4) \in R_j$ by construction. It follows that $(x_4, x_6), (x_6, x_2) \in R_j$ by Lemma 2.4. Note $\{x_0, x_2, x_4\} = R^t_j(x_3)$ and $(x_0, x_2) \in R_j$. It follows that $(x_4, x_0) \in R_j$ by Lemma 2.4 applied to the relation $R_{j'}$. Similarly we have $(x_5, x_0), (x_6, x_0) \in R_j$. Hence we see that $\{x_i \mid i = 0, 1, ..., 6\}$ is a connected component with respect to R_j since we have $R_j(x_l) \subset \{x_i \mid i = 0, 1, ..., 6\}$ for each $l \in \{0, 1, ..., 6\}$. It follows from the primitivity of $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ that $\{x_i \mid i = 0, 1, ..., 6\} = X$, as desired. ∎

We assume $|X| > 7$ for the rest of this section. Then we have $p^j_{jj} = 0$ if $k_j = 3$ by Proposition 2.6. Without loss of generality we may assume $k_1 = 3$.

DEFINITION 2.7. For each $x \in X$ we call the sets $R_1(x)$, $R^t_1(x)$ *the outer triangle*, *the inner triangle* of x, respectively. A triangle means either an outer triangle or an inner triangle. We say that two triangles are *adjacent* if they have exactly two points in common.

LEMMA 2.8. *Given an inner triangle Δ and its edge $\{y_1, y_2\}$, there exists a unique outer triangle adjacent to Δ sharing $\{y_1, y_2\}$. The same statement holds if we switch "inner triangle" and "outer triangle."*

Proof. Let $\Delta = R^t_1(x_1) = \{x_0, y_1, y_2\}$ be an inner triangle. Applying Lemma 2.5 with $R_i = R_j = R^t_1$ there exists a unique element $x_2 \in R^t_1(y_1) \cap R^t_1(y_2)$, different from x_0, x_1, y_1, y_2. Let $x_3 \in R_1(x_2) - \{y_1, y_2\}$. Then we have $x_3 \neq x_1$, for otherwise $2 \leqslant p^1_{11}(x_2, x_1) = p^1_{11'}$, contradicting

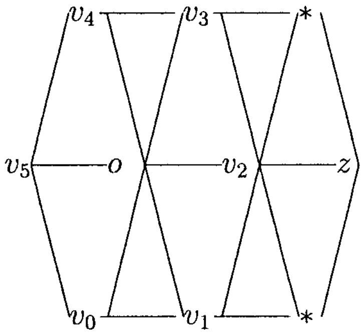

FIGURE 1

Lemma 2.1. We have $x_3 \neq x_0$., for otherwise $\{y_1, y_2, x_0 = x_3\} = R_1(x_2) \cap R_1^t(x_1)$, contradicting Lemma 2.2 (see Fig. 1).

This proves the first part of this lemma. The second part can be proved similarly if we replace R_1^t by R_1. ∎

Remark 2.9. It follows from Lemma 2.1 that one of any two adjacent triangles is inner and the other is outer.

LEMMA 2.10. *If* (4) *holds, i.e.,* $A_1 A_1^t = 3A_0 + A_\beta$ *where* $k_\beta = 6$, $A_\beta = A_\beta^t$. *There exists a relation* R_ε *of valency* 3 *such that for any inner triangle* $R_1^t(x_1) = \{x_0, y_1, y_2\}$ *and any outer triangle* $R_1(x_2) = \{x_3, y_1, y_2\}$ *sharing an edge* $\{y_1, y_2\}$, *the pair* (x_0, x_3) *belongs to* R_ε.

Proof. We set $(x_2, x_1) \in R_h$. Then we have $p_{11}^h = 2$ and $k_h = 3$ by Lemma 2.2. Note that the relation R_h is uniquely determined by Lemma 1.6(iv), independent of the choice of adjacent triangles. Thus we have $p_{1h'}^{1'} = p_{11}^h = 2$, so $A_1 A_h^t = 2A_1^t + A_m$ for some m with $k_m = 3$. Hence we have $(x_0, x_2) \in R_m$. Moreover, we have $A_1 A_m = A_\beta + A_\varepsilon$ for some ε with $k_\varepsilon = 3$ by observing $x_2 \in R_m(x_0) \cap R_1^t(y_1)$, $(x_0, y_1) \in R_\beta$. We claim $(x_0, x_3) \in R_\varepsilon$. Assume the contrary, i.e., we have y_1, y_2, $x_3 \in R_\beta(x_0) \cap R_1(x_2)$, which implies $3 = p_{\beta 1'}^m = 2p_{m1}^\beta$. This is a contradiction. ∎

PROPOSITION 2.11. *There exists* R_i *with* $k_i = 3$ *such that*

$$A_i^2 = 2A_i^t + A_j \qquad \textit{for some } j$$

where $A_j \notin \{A_i, A_i^t\}$.

Proof. By (3) and (4), we divide the proof into two cases according to the expansion of $A_1 A_1^t$.

First Case. $A_1 A_1^t = 3A_0 + A_\alpha + A_\alpha^t$. We use the same notation as in Fig. 1. Note $(x_0, y_1) \in R_\alpha \cup R_\alpha^t$. We may assume $(x_0, y_1) \in R_\alpha$ without loss of generality. By Lemma 2.4, we have $(y_1, y_2) \in R_\alpha$, $(y_2, x_0) \in R_\alpha$, $(y_2, x_3) \in R_\alpha$ and $(x_3, y_1) \in R_\alpha$. Since $x_0,\ x_3 \in R_\alpha(y_2) \cap R^t_\alpha(y_1)$, we see $p^{\alpha'}_{\alpha\alpha} \geqslant 2$. By Lemma 2.2, we have $p^{\alpha'}_{\alpha\alpha} = 2$. This implies $A^2_\alpha = 2A^t_\alpha + A_j$ for some j with $A_j \neq A^t_\alpha$. Note $A_j \neq A_\alpha$ by our assumption and Proposition 2.6.

Second Case. $A_1 A^t_1 = 3A_0 + A_\beta$. Let $\{o, v_0, v_1\}$ be an inner triangle. By Lemma 2.8, there exists a unique outer triangle adjacent to $\{o, v_0, v_1\}$ sharing $\{o, v_1\}$. By Remark 2.9, we can construct an alternating sequence of inner and outer triangles containing the vertex o. Since any edge of a triangle belongs to the relation R_β, the period of this sequence is bounded by $|R_\beta(o)| = 6$. The period is not two by Lemma 2.10 and is even by Remark 2.9. We claim that the period is six. Assume the contray, i.e., the only possible period is four. In this case there exists a sequence of four adjacent triangles $(R^t_1(x_1), R_1(x_2), R^t_1(x_3), R_1(x_4))$ containing o such that $R_1(x_4)$ is adjacent to $R^t_1(x_1)$. If $(x_2, x_1) \in R_h$, then R_h is the only relation with $p^h_{11} = 2$, and its valency is 3 by Lemma 2.2. Hence (x_2, x_3), (x_4, x_3), (x_4, x_1) belong to R_h as well. This implies $x_1,\ x_3 \in R_h(x_2) \cap R_h(x_4)$, contradicting Lemma 2.1.

We have shown that the neighbourhood of a vertex o in the R_β-graph consists of six vertices $v_0, ..., v_5$ such that o, v_t, v_{t+1} form a triangle for each $t = 0, ..., 5$, where the indices are read modulo 6. Let Δ denote the subgraph of the R_β-graph induced by $\{v_0, ..., v_5\}$. Note that by Lemma 2.10, we see $(v_t, v_{t+2}) \notin R_\beta$ for each t. Since Δ is a regular graph of valency $p^\beta_{\beta\beta}$, $(v_t, v_{t+3}) \in R_\beta$ for some t implies $(v_t, v_{t+3}) \in R_\beta$ for all t, and hence the Δ is locally complete bipartite $K_{3,3}$. It follows easily (see [5, Proposition 1.1.5]) that the R_β-graph is isomorphic to $K_{3,3,3}$. By the primitivity,

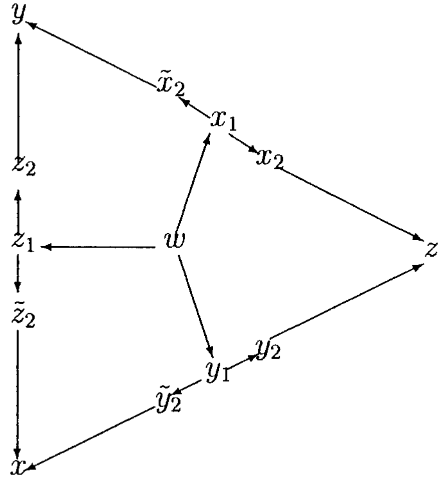

FIGURE 2

we have $|X|=9$, but this is impossible since $|X|=k_0+k_1+k_{1'}+k_\beta+\cdots\geqslant 1+3+3+6$ by Lemma 1.6(i). Hence Δ is a hexagon, so the R_β-graph itself is locally hexagon.

We consider the subgraph of the R_β-graph induced by $R_\beta(o)\cup R_\beta(v_2)$ (see Fig. 2). Since the R_β-graph is locally hexagon, we have $|R_\beta(o)\cap R_\beta(v_2)|=2$, and ten vertices shown in Fig. 2 are all distinct. Let z be the unique vertex of $R_\beta(v_2)$ which is at distance 3 from o in the hexagon $R_\beta(v_2)$. Now by Lemma 2.10, we have $(v_3, v_1)\in R_\varepsilon$, and $z, v_5\in R_\varepsilon(v_1)\cap R_\varepsilon{}^t(v_3)$.

Hence $p^{\varepsilon'}_{\varepsilon\varepsilon}\geqslant 2$. By Lemma 2.2, we have $p^{\varepsilon'}_{\varepsilon\varepsilon}=2$, so that $A_\varepsilon^2=2A_\varepsilon^t+A_j$ for some j with $A_j\neq A_\varepsilon^t$. Note $A_j\neq A_\varepsilon$ by our assumption and Proposition 2.6. This completes the proof of Proposition 2.11. ∎

3. PROOF OF THE MAIN THEOREM

Let $(X, \{R_i\}_{0\leqslant i\leqslant d})$ be a primitive commutative association scheme with $k_1=3$ and $R_1\neq R_1^t$ throughout this section. By Proposition 2.11, we may assume

$$A_1^2=2A_1^t+A_2, \tag{5}$$

where $A_2\notin\{A_1, A_1^t\}$ and $k_1=k_2=3$. In Fig. 1, the assumption (5) implies that

$$(x_1, x_2)\in R_1, \qquad (x_0, x_2)\in R_2, \qquad \text{and} \qquad (x_1, x_3)\in R_2.$$

If (3) occurs, then we may assume $(x_0, y_1)\in R_\alpha$ without loss of generality, whence $(x_0, y_2)\in R_\alpha^t$. Since $x_2\in R_2(x_0)\cap R_1^t(y_i)$ for $i=1, 2$, we have $p^\alpha_{21}>0$, $p^{\alpha'}_{21}>0$. If $(x_0, x_3)\in R_\alpha\cup R_\alpha^t$, then we have $p^\alpha_{\alpha\alpha}>0$, contradicting our assumption of $|X|>7$ by Proposition 2.6. Thus,

$$A_1A_1^t=3A_0+A_\alpha+A_\alpha^t, \qquad A_1A_2=A_\alpha+A_\alpha^t+A_\varepsilon, \quad \text{and} \quad A_\varepsilon\notin\{A_\alpha, A_\alpha^t\}. \tag{6}$$

If (4) occurs, then $(x_0, y_1)\in R_\beta$. Since $x_2\in R_2(x_0)\cap R_1^t(y_1)$, we have $p^\beta_{21}>0$. Thus,

$$A_1A_1^t=3A_0+A_\beta, \qquad A_1A_2=A_\beta+A_\varepsilon, \qquad A_\varepsilon\neq A_\beta. \tag{7}$$

DEFINITION 3.1. A sequence $(x_0, x_1, ..., x_n)$ of elements of X is called *a chain* of length n if $(x_j, x_{j+1})\in R_1$ for each $j\in\{0, 1, ..., n-1\}$ and $(x_j, x_{j+2})\in R_2$ for each $j\in\{0, 1, ..., n-2\}$.

Remark 3.2. Let $(x_0, x_1, ..., x_n)$, $(y_0, y_1, ..., y_n)$ be two chains of length n. If $x_0 = y_0$ and $x_1 = y_1$, then $x_i = y_i$ for each i with $0 \leqslant i \leqslant n$. If $x_n = y_n$ and $x_{n-1} = y_{n-1}$, then $x_i = y_i$ for each i with $0 \leqslant i \leqslant n$. These statements are obvious since $p^1_{21'} = p^2_{11} = 1$.

LEMMA 3.3. *For each $\gamma \neq 0$ we have $p^{\gamma}_{11'} p^{\gamma}_{22'} = 0$, or equivalently, $A_1 A_1^t \neq A_2 A_2^t$.*

Proof. Assume the contrary, i.e., $(x, y) \in R_\gamma \neq R_0$ and $p^{\gamma}_{11'} p^{\gamma}_{22'} \neq 0$. There exist two distinct elements $a, b \in X$ such that $a \in R_1^t(x) \cap R_1^t(y)$ and $b \in R_2(x) \cap R_2(y)$. Then we have $x, y \in R_1(a) \cap R_2^t(b)$, hence $p^m_{12} \geqslant 2$ where $(a, b) \in R_m$. This contradicts (6) or (7). ∎

DEFINITION 3.4 [1]. Let $\{B_n\}_{1 \leqslant n}$ be the sequence of matrices defined by the recurrence relation

$$\begin{aligned} &B_1 := A_1, \qquad B_2 := A_2, \quad B_3 := A_1 A_2 - A_1 A_1^t + 3A_0 \\ &B_n := A_1 B_{n-1} - A_1^t B_{n-2} + B_{n-3}, \qquad n \geqslant 4. \end{aligned} \tag{8}$$

Observe that the row sum of B_n is 3. Thus, if B_n is a (0, 1)-matrix, then B_n is the adjacency matrix of some relation of $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ having valency 3. If B_n is diagonal, then $B_n = 3A_0$.

Note $A_\varepsilon = B_3$ by (6) and (7). We denote by S_n the relation corresponding to B_n when B_n is a non-diagonal adjacency matrix. Note that $\{S_n\}_{1 \leqslant n}$ are not necessarily distinct. Indeed for Cyc(13, 3), we have $B_1 = B_3$. We shall interpret subscripts and superscripts of intersection numbers in an appropriate way. Subscripts of A defined in this section are $\{0, 1, 2, \alpha, \beta, \varepsilon\}$, those of B are $\{1, 2, ...\}$ with $A_1 = B_1$, $A_2 = B_2$. Hence we can distinguish subscripts of intersection numbers without confusions. For example, p^{α}_{1n} will mean the coefficient of A_α in the expansion of $A_1 B_n$, when n is a positive integer with $n \geqslant 3$.

PROPOSITION 3.5. *Suppose that B_i is a non-diagonal adjacency matrix for each i with $1 \leqslant i \leqslant n$. Then the following hold:*

(i) *We have $(x, y) \in S_n$ if and only if there exists a unique chain of length n from x to y;*

(ii) *B_{n+1} is either a non-diagonal adjacency matrix of $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ with valency 3 or $3A_0$;*

(iii) *The following are equivalent.*

(a) $B_{n+1} = 3A_0$;

(b) *If* $(x_0, x_1, x_2, ..., x_n, x_{n+1})$ *is a chain then* $x_0 = x_{n+1}$;

(c) *There is a simple closed chain of length* $n+1$.

Before we prove Proposition 3.5, we prepare some lemmas. We use induction on n for the proof of Proposition 3.5(i). We denote by L_n the statement of Proposition 3.5(i) with respect to length n.

LEMMA 3.6. *Suppose that* B_i *is a nondiagonal adjacency matrix for each* i *with* $1 \leqslant i \leqslant n$, *and* L_i *holds for each* i *with* $1 \leqslant i \leqslant n-1$. *Let* m *be a positive integer less than or equal to* n. *Then there exists at most one chain of length* m *from one vertex to another.*

Proof. Assume the contrary, i.e., there are two distinct chains

$$(w, x_1, x_2, ..., x_{m-1}, z) \qquad \text{and} \qquad (w, y_1, y_2, ..., y_{m-1}, z).$$

Clearly $m \geqslant 3$. Pick m to be minimal. Then $x_1 \neq y_1$ and $x_{m-1} \neq y_{m-1}$ by Remark 3.2. Note that if $m = 3$, then

$$w \in R_1^t(x_1) \cap R_1^t(y_1) \qquad \text{and} \qquad z \in R_2(x_1) \cap R_2(y_1).$$

Thus $p_{11'}^{\gamma} p_{22'}^{\gamma} > 0$, where $(x_1, y_1) \in R_\gamma$, but this contradicts Lemma 3.3. Suppose $m > 3$. The chains

$$(x_1, x_2, ..., x_{m-1}, z) \qquad \text{and} \qquad (y_1, y_2, ..., y_{m-1}, z)$$

are the unique ones from x_1 to z, and from y_1 to z, respectively, by the minimality of m. By L_{m-1}, we have $(x_1, z), (y_1, z) \in S_{m-1}$. It follows that $p_{1\,m-1}^{\delta} \geqslant 2$ where $(w, z) \in R_\delta$. We claim $w \neq z$. Assume the contrary, i.e., $w = z$. Then we have $B_{m-1} = A_1^t$ by $(x_1, w) = (x_1, z) \in S_{m-1}$. Since B_m and B_{m-3} are non-diagonal matrices by our assumption, we have $B_{m-2} = A_1$ by comparing the coefficients of A_0 in (8) with n replaced by m. It follows that

$$(w, x_2) = (z, x_2) \in R_2 \cap S_{m-2}^t = R_2 \cap R_1^t,$$

contradicting $A_2 \neq A_1^t$. Hence $w \neq z$, and we have $p_{1\,m-1}^{\delta} = 2$, $k_\delta = 3$ by Lemma 2.2. Let $R_1(w) = \{x_1, y_1, z_1\}$. Applying Lemma 2.5 with $(i, j, h) = (1, m-1, \delta)$, we find

$$y \in S_{m-1}(x_1) \cap S_{m-1}(z_1) \qquad \text{and} \qquad x \in S_{m-1}(z_1) \cap S_{m-1}(y_1).$$

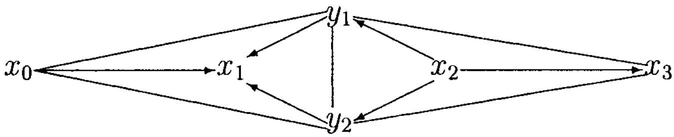

FIGURE 3

By L_{m-1}, there are four chains

$$(x_1, \tilde{x}_2, ..., \tilde{x}_{m-1}, y), \qquad (y_1, \tilde{y}_2, ..., \tilde{y}_{m-1}, x),$$

$$(z_1, z_2, ..., z_{m-1}, y) \qquad \text{and} \qquad (z_1, \tilde{z}_1, ..., \tilde{z}_{m-1}, x).$$

Note $x_2 \neq \tilde{x}_2$, $y_2 \neq \tilde{y}_2$, and $z_2 \neq \tilde{z}_2$ by Remark 3.2, and also $x_2 \neq y_2$, $\tilde{x}_2 \neq z_2$, and $\tilde{y}_2 \neq \tilde{z}_2$ by the minimality of m. Hence we can construct the diagram shown in Fig. 3.

By Remark 3.2, at most one of (w, z_1, z_2) or $(w, z_1, \tilde{z}_2)$ is a chain, so we may assume without loss of generality that $(w, z_1, \tilde{z}_2)$ is not a chain. In this case we have $(w, \tilde{z}_2) \in R_1^t$. Also by Remark 3.2, $(w, y_1, \tilde{y}_2)$ is not chain, since (w, y_1, y_2) is a chain. Thus we have $(w, \tilde{y}_2) \in R_1^t$. Therefore $\tilde{y}_2$, $\tilde{z}_2 \in R_1^t(w) \cap S_{m-2}^t(x)$, and $A_1 A_1^t = B_{m-2} B_{m-2}^t$ by Lemma 2.3. Since x_2, $y_2 \in R_2(w) \cap S_{m-2}^t(z)$, we have $A_2 A_2^t = B_{m-2} B_{m-2}^t$ by Lemma 2.3. However, this contradicts Lemma 3.3. This completes the proof. ∎

LEMMA 3.7. *Suppose that B_i is a nondiagonal adjacency matrix for each i with $1 \leqslant i \leqslant n$ and L_i holds for each i with $1 \leqslant i \leqslant n-1$. Let $(x_1, x_2, ..., x_{n-1}, x_n)$, $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}, \tilde{x}_n = x_n)$ be two chains such that $x_s \neq \tilde{x}_s$ for each s with $1 \leqslant s \leqslant n-1$. Then*

$$R_1^t(x_1) \cap R_1^t(\tilde{x}_1) \cap R_1(x_2) \cap R_1(\tilde{x}_2) \cap S_n^t(x_n) = \varnothing. \tag{9}$$

Proof. First, we prove that, for each m with $1 \leqslant m \leqslant n-3$, there is no pair (u, v) such that

$$u \in R_1^t(x_m) \cap R_1^t(\tilde{x}_m) \cap R_1(x_{m+1}) \cap R_1(\tilde{x}_{m+1}) \tag{10}$$

and

$$v \in R_1^t(x_{m+1}) \cap R_1^t(\tilde{x}_{m+1}) \cap R_1(x_{m+2}) \cap R_1(\tilde{x}_{m+2}). \tag{11}$$

Assume that there exists such a pair (u, v) and pick m to be maximal. Note $m < n-4$ by Lemma 3.3 since $x_n \in R_2(x_{n-2}) \cap R_2(\tilde{x}_2)$ and $v \in R_1(x_{m+2}) \cap R_1(\tilde{x}_{m+2})$. We claim $(\tilde{x}_{m+2}, x_{m+1}) \in R_2$. Assume the contrary, i.e., $(\tilde{x}_{m+2}, x_{m+1}) \in R_1^t$ by $v \in R_1(\tilde{x}_{m+2}) \cap R_1^t(x_{m+1})$ and (5). This implies that u, $\tilde{x}_{m+2} \in R_1(x_{m+1}) \cap R_1(\tilde{x}_{m+1})$. Since $u \neq \tilde{x}_{m+2}$ by $\tilde{x}_m \in R_1(u) \cap R_2^t(\tilde{x}_{m+2})$, this contradicts Lemma 2.1. Similarly, we have $(x_{m+2}, \tilde{x}_{m+1}) \in R_2$. Since $v \in R_1(x_{m+2}) \cap R_1(\tilde{x}_{m+2})$, we have either

$$(x_{m+2}, \tilde{x}_{m+2}) \in R_\alpha \cup R_\alpha^t \qquad \text{or} \qquad (x_{m+2}, \tilde{x}_{m+2}) \in R_\beta.$$

Hence there exists a unique element $w \in R_1^t(\tilde{x}_{m+2}) \cap R_1^t(x_{m+2})$ other than $\tilde{x}_{m+1}$, x_{m+1}, which implies $w \in R_1(x_{m+3}) \cap R_1(\tilde{x}_{m+3})$ by Remark 3.2. This contradicts the choice of m.

Now suppose that x_0 is an element of (9). It follows from $(x_0, x_n) \in S_n$ and (8) that

$$(B_n)_{x_0, x_n} = 1 = p^n_{1\, n-1} - p^n_{1'\, n-2} + \delta_{S_n, S_{n-3}}. \tag{12}$$

Note $p^n_{1\, n-1} \geqslant 2$ and $p^n_{1'\, n-2} \geqslant 2$ by the choice of x_0, and $p^n_{1\, n-1} = p^n_{1'\, n-2} = 2$ by Lemma 2.2. It follows from (12) that $S_n = S_{n-3}$.

We claim that $(\tilde{x}_2, x_1) \in R_2$ and $(x_2, \tilde{x}_1) \in R_2$. Assume the contrary. It suffices to show a contradiction if $(\tilde{x}_2, x_1) \in R_1^t$. Then we have $(x_1, \tilde{x}_3) \in R_1^t$ since $(\tilde{x}_1, \tilde{x}_3) \in R_2$ and $x_1 \neq \tilde{x}_1$. If $(\tilde{x}_3, x_2) \in R_1^t$ then x_2, $\tilde{x}_2 \in R_1^t(\tilde{x}_3) \cap R_1^t(x_0)$. Note $x_2 \neq \tilde{x}_2$ by the assumption, and $\tilde{x}_3 \neq x_0$ by $\tilde{x}_1 \in R_1(x_0) \cap R_2^t(\tilde{x}_3)$. But this contradicts Lemma 2.1. If $(\tilde{x}_3, x_2) \in R_2$ then $(\tilde{x}_3, x_1, x_2, x_3)$ is a chain of length 3. Hence we have $B_3 = A_\varepsilon \in \mathsf{Sup}(B_{n-3} B_{n-3}^t)$. Since $\tilde{x}_2 \in R_1^t(\tilde{x}_3) \cap R_1^t(x_0)$ and $(x_0, x_n) \in S_n = S_{n-3}$, we have

$$A_1 A_1^t = B_{n-3} B_{n-3}^t = 3A_0 + A_\varepsilon + A_\varepsilon^t$$

by Lemma 2.3, contradicting (6) or (7).

Since $x_0 \in R_1(x_2) \cap R_1(\tilde{x}_2)$, we have either

$$(x_2, \tilde{x}_2) \in R_\alpha \cup R_\alpha^t \qquad \text{or} \qquad (x_2, \tilde{x}_2) \in R_\beta.$$

Hence there exists a unique element $v \in R_1^t(\tilde{x}_2) \cap R_1^t(x_2)$ other than $\tilde{x}_1$, x_1 by the above claim. Note that $(v, x_3) \in R_1^t$ and $(v, \tilde{x}_3) \in R_1^t$ since (x_1, x_3), $(\tilde{x}_1, \tilde{x}_3) \in R_2$. Now the pair $(u = x_0, v)$ satisfies (10) and (11) with $m = 1$, which is impossible. This completes the proof. ∎

LEMMA 3.8. *Suppose that B_i is a non-diagonal adjacency matrix for each i with $1 \leqslant i \leqslant n$ and L_i holds for each i with $1 \leqslant i \leqslant n-1$. Then there is a chain of length n from x to w for each $(x, w) \in S_n$.*

Proof. If $n = 3$, then, as $B_3 = A_\varepsilon$, there exists $x_2 \in R_2(x) \cap R_1^t(w)$ by (6) or (7), and there exists $x_1 \in R_1(x) \cap R_1^t(x_2)$ by (5). If $(x_1, w) \in R_1^t$ then $x_1 \in R_1(x) \cap R_1(w)$. Since $(x, w) \in R_\varepsilon$ and $A_\varepsilon \notin \mathsf{Sup}(A_1 A_1^t)$ by (6) and (7), this is a contradiction. Thus $(x_1, w) \in R_2$. Hence (x, x_1, x_2, w) is a chain of length 3.

Assume $n \geqslant 4$ and $(x, w) \in S_n$. It follows from (8) that

$$(B_n)_{x,w} = 1 = p^n_{1\, n-1} - p^n_{1'\, n-2} + \delta_{S_n, S_{n-3}} \tag{13}$$

Note $-p^n_{1'\, n-2} + \delta_{S_n S_{n-3}} \leqslant 0$ since $p^{n-3}_{1'\, n-2} > 0$ by L_{n-2}, L_{n-3} and Lemma 3.6. Hence we have $p^n_{1\, n-1} > 0$ by (13). It follows from L_{n-1} that there exists an element $x_1 \in R_1(x) \cap S^t_{n-1}(w)$ and a chain $(x_1, x_2, ..., x_{n-1}, w)$ of length $n-1$. If $(x, x_2) \in R_2$ then we are done. Hence we may assume $(x, x_2) \in R^t_1$, which implies $p^n_{1'\, n-2} > 0$.

We claim $p^n_{1\, n-1} = 2$. The claim follows from (13) if $S_n \neq S_{n-3}$, since $p^n_{1'\, n-2} > 0$ and $0 < p^n_{1\, n-1} \leqslant 2$ by Lemma 2.2. If $S_n = S_{n-3}$ then we have

$$x, x_3 \in R_1(x_2) \cap S^t_{n-3}(w).$$

Note $x \neq x_3$ as $x_1 \in R_1(x) \cap R^t_2(x_3)$ and $R_1 \neq R^t_2$. Hence we have, by Lemma 2.2,

$$2 = p^{n-2}_{1\, n-3} = p^{n-3}_{1'\, n-2} = p^n_{1'\, n-2}.$$

It follows from (13) that $p^n_{1\, n-1} = 2$.

We conclude from the above claim and L_{n-1} that there exists $\tilde{x}_1 \in R_1(x) \cap S^t_{n-1}(w)$ other than x_1, and hence there exists a chain $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}, w)$ of length $n-1$. Note $x_s \neq \tilde{x}_s$ for each s with $1 \leqslant s \leqslant n-1$ by Lemma 3.6 and Remark 3.2. Then we have $(x, \tilde{x}_2) \in R_2$ by Lemma 3.7. This completes the proof. ▮

Lemma 3.9. *Let $(x_0, x_1, ..., x_n)$ be a chain of length $n \geqslant 3$. Then there is no chain $(y_2, y_3, ..., y_n = x_n)$ such that*

$$y_2 \in R^t_1(x_0) \cap R_1(x_1) \tag{14}$$

and

$$y_3 \in R^t_1(x_1) \cap R_1(x_2). \tag{15}$$

Proof. We use induction on n. If $n = 3$ then it is trivial by the definition of a chain. Suppose a chain $(y_2, y_3, ..., y_n = x_n)$ satisfying (14) and (15) exists for some $n \geqslant 4$.

We claim $(x_2, y_4) \in R^t_1$. Assume the contrary, i.e., $(x_2, y_4) \in R_2$ by $y_3 \in R_1(x_2) \cap R^t_1(y_4)$ and (5). This implies

$$x_2, y_2 \in R_1(x_1) \cap R^t_2(y_4).$$

Note $x_2 \neq y_2$ by $x_0 \in R_1(y_2) \cap R^t_2(x_2)$. This contradicts (6) or (7). Hence we have $(x_2, y_4) \in R^t_1$.

Since $x_1 \neq y_4$ by $y_2 \in R_1(x_1) \cap R_2^t(y_4)$, we have $y_4 \in R_1(x_3)$. This contradicts the assumption of the induction. ∎

LEMMA 3.10. *Suppose that B_i is a non-diagonal adjacency matrix for each i with $1 \leqslant i \leqslant n$ and L_i holds for each i with $1 \leqslant i \leqslant n-1$. Let $(x_0, x_1, ..., x_n)$ be a chain of length $n \geqslant 4$, with $(x_0, x_n) \notin S_n$. Then*

$$Y := R_1^t(x_0) \cap R_1(x_1) \cap S_{n-2}^t(x_n) = \emptyset. \tag{16}$$

Proof. Assume the contrary, i.e., $Y \neq \emptyset$ for some $n \geqslant 4$. By L_{n-2}, there exist y_2 in Y and a chain $(y_2, y_3, ..., y_{n-1}, x_n)$ of length $n-2$. Then we have $(x_1, y_3) \in R_1^t$, for otherwise $(x_1, y_3) \in R_2$ by $y_2 \in R_1(x_1) \cap R_1^t(y_3)$ and (5), there are two distinct chains

$$(x_1, x_2, ..., x_{n-1}, x_n), (x_1, y_2, y_3, ..., y_{n-1}, x_n),$$

contradicting Lemma 3.6.

Since $x_1 \in R_1(y_3) \cap R_1^t(x_2)$, we have $(y_3, x_2) \in R_2$ by Lemma 3.9. This implies $x_0 = y_3$ by $1 = p_{1'2}^1 = p_{1'2}^1(x_1, x_2)$, and hence we have $(x_0, x_n) \in S_{n-3}$ by L_{n-3} and Lemma 3.6. It follows from (8) that

$$(B_n)_{x_0, x_n} = 0 = p_{1\,n-1}^{n-3} - p_{1'\,n-2}^{n-3} + 1. \tag{17}$$

Note $p_{1\,n-1}^{n-3}$ since $x_1 \in R_1(x_0) \cap S_{n-1}^t(x_n)$. It follows from (17) that $p_{1'\,n-2}^{n-3} \geqslant 2$. Hence there exist $w_2 \in S_{n-2}^t(x_n) \cap R_1^t(x_0)$ other than y_2, and a unique chain $(w_2, ..., w_{n-1}, x_n)$.

Note $y_3 \neq w_3$ by $y_2 \neq w_2$ and Lemma 3.6. Note $(x_0, x_3) \in S_3 = R_\varepsilon$ and

$$x_n \in S_{n-3}(x_0) \cap S_{n-3}(x_3)$$

by L_{n-3} and Lemma 3.6. This implies $A_\varepsilon \in \mathsf{Sup}(B_{n-3} B_{n-3}^t)$. On the other hand, we have $x_0, w_3 \in R_1(w_2) \cap S_{n-3}^t(x_n)$. It follows from Lemma 2.3 that

$$B_{n-3} B_{n-3}^t = A_1 A_1^t.$$

This implies $A_\varepsilon \in \mathsf{Sup}(A_1 A_1^t)$, contradicting (6) or (7). This completes the proof. ∎

LEMMA 3.11. *Suppose that B_i is a non-diagonal adjacency matrix for each i with $1 \leqslant i \leqslant n$ and L_i holds for each i with $1 \leqslant i \leqslant n-1$. If $(x_0, x_1, ..., x_n)$ is a chain of length n, then we have $(x_0, x_n) \in S_n$.*

Proof. Suppose $n = 3$. Since $(x_1, x_2) \in R_1$, there exist two distinct elements $y_1, y_2 \in R_1^t(x_1) \cap R_1(x_2)$. Then y_1, y_2 are different from x_0, x_3 since $x_0 \in R_2^t(x_2)$, $x_3 \in R_2(x_1)$ and $A_1^t \neq A_2$. Thus $\{x_0, y_1, y_2\}$ is an inner triangle, and $\{x_3, y_1, y_2\}$ is an outer triangle.

If (6) occurs, then we have either $y_1 \in R_\alpha(x_0) \cap R_\alpha(x_3)$ or $y_1 \in R^t_\alpha(x_0) \cap R^t_\alpha(x_3)$ by Lemma 2.4. Note that $(x_0, x_3) \in R_\alpha \cup R^t_\alpha \cup R_\varepsilon$. Hence we have $(x_0, x_3) \in R_\varepsilon$, for otherwise it follows $p^\alpha_{\alpha\alpha} > 0$, contradicting the assumption of $|X| > 7$.

If (7) occurs, then we have $(x_0, x_3) \in R_\varepsilon = S_3$ by Lemma 2.10.

Assume $n \geqslant 4$ and the contrary, i.e., $(x_0, x_n) \in R_\mu \neq S_n$. It follows from (8) that

$$(B_n)_{x_0, x_n} = 0 = p^\mu_{1\,n-1} - p^\mu_{1'\,n-2} + \delta_{R_\mu, S_{n-3}}. \tag{18}$$

By L_{n-1} and Lemma 3.6, we have $(x_1, x_n) \in S_{n-1}$ hence $p^\mu_{1\,n-1} > 0$. It follows from (18) that $p^\mu_{1'\,n-2} > 0$. Hence there exist an element $y_2 \in R^t_1(x_0) \cap S^t_{n-2}(x_n)$ and a chain $(y_2, y_3, ..., y_{n-1}, x_n)$ by L_{n-2}. By Lemma 3.10, we have $(y_2, x_1) \in R_2$. Then there are exactly two elements $z_1, z_2 \in R^t_1(y_2) \cap R_1(x_0)$ other than x_1.

If $y_3 \neq x_0$ then we have either $(z_1, y_3) \in R_2$ or $(z_2, y_3) \in R_2$, for otherwise $z_1, z_2 \in R_1(x_0) \cap R_1(y_3)$, contradicting Lemma 2.1. We may assume $(z_1, y_3) \in R_2$ without loss of generality. Note that $(z_1, y_2, ..., y_{n-1}, x_0)$ is a chain of length $n-1$. By L_{n-1} and Lemma 3.6, we have $(z_1, x_n) \in S_{n-1}$. Hence we have $p^\mu_{1\,n-1} \geqslant 2$. It follows from (18) that $p^\mu_{1'\,n-2} = p^\mu_{1\,n-1} \geqslant 2$.

If $y_3 = x_0$ then we have $S_{n-3} = R_\mu$ by L_{n-3} and Lemma 3.6. It follows from (18) that $p^\mu_{1'\,n-2} \geqslant 2$.

Hence, in both cases, there exists $w_2 \in S^t_{n-2}(x_n) \cap R^t_1(x_0)$ other than y_2. Note $w_2 \in R_1(x_1)$ since y_2 is a unique element in $R^t_1(x_0) \cap R^t_2(x_1)$ by $p^1_{1'2} = p^2_{11} = 1$. But this contradicts Lemma 3.10. This completes the proof. ∎

Now we give a proof of Proposition 3.5.

Proof of Proposition 3.5. We use induction on n. It is clear that L_1 and L_2 hold. Assuming $L_1, ..., L_{n-1}$, we obtain L_n by Lemmas 3.6, 3.8, and 3.11. The following corollary is an immediate consequence of Proposition 3.5(i) and Lemma 3.6.

COROLLARY 3.12. *Suppose that B_i is a non-diagonal adjacency matrix for each i with $1 \leqslant i \leqslant n$. Then for each i with $1 \leqslant i \leqslant n$, we have $(x, y) \in S_i$ if and only if there exists a chain of length i from x to y.*

Next we prove (ii). Let $(x_0, x_1, ..., x_n)$ be a chain and $\{y_1, y_2, y_3\} = R_1(x_n)$. We may assume that $(x_{n-1}, y_1) \in R^t_1$, $(x_{n-1}, y_2) \in R^t_1$ and $(x_{n-1}, y_3) \in R_2$. It follows that

$$A_\tau, A_\sigma \in \mathsf{Sup}(A_1 B_n) \cap \mathsf{Sup}(A^t_1 B_{n-1}), \tag{19}$$

where $(x_0, y_1) \in R_\tau$, $(x_0, y_2) \in R_\sigma$ and A_τ, A_σ are not necessarily distinct.

We claim that

$$A_1 B_n - A_1^t B_{n-1} + B_{n-2}$$

has no negative entry. Assume the contrary, i.e., there exists some negative entry belonging to some relation R_μ. Then the following inequality holds:

$$p^\mu_{n\,1} - p^\mu_{n-1\,1'} + \delta_{R_\mu, S_{n-2}} < 0. \tag{20}$$

This implies that $p^\mu_{n-1\,1'} > 0$, or equivalently $R_\mu \in \mathsf{Sup}(A_1^t B_{n-1})$. Since $R_1^t(x_{n-1}) = \{x_{n-2}, y_1, y_2\}$, we have

$$\mathsf{Sup}(A_1^t B_{n-1}) = \{B_{n-2}, A_\tau, A_\sigma\}.$$

If $R_\mu = S_{n-2}$, then (20) implies $p^\mu_{n-1\,1'} = 2$ by Lemma 2.2, whence we have $R_\mu = S_{n-2} \in \{R_\tau, R_\sigma\}$ and (20) becomes $p^\mu_{n\,1} - 1 < 0$. But the former implies $p^\mu_{n\,1} > 0$ by (19), a contradiction.

If $R_\mu = R_\tau$, then $p^\mu_{n\,1} > 0$ by (19). This implies $p^\mu_{n-1\,1'} = 2$ by (20) and Lemma 2.2, whence we have $R_\tau \in \{R_\sigma, S_{n-2}\}$. We may assume that $R_\mu = R_\tau = R_\sigma \neq S_{n-2}$ by the same argument as above. By (20), we have $p^\tau_{n\,1} = 1$. Since $y_1, y_2 \in R_\tau(x_0) \cap R_1(x_n)$, we have

$$2 \leqslant p^n_{\tau 1'} = k_\tau p^\tau_{n1}/k_n = k_\tau/k_n.$$

This implies $k_\tau \geqslant 6$. But, by Lemma 1.6(iv),

$$9 = k_{n-1} k_{1'} = \sum_{l=0}^{d} p^l_{n-1\,1'} k_l \geqslant p^\tau_{n-1\,1'} k_\tau \geqslant 12.$$

This is a contradiction. Similarly, if $R_\mu = R_\sigma$, then we obtain a contradiction. Hence $A_1 B_n - A_1^t B_{n-1} + B_{n-2}$ has no negative entry.

In view of (8), we have shown that B_{n+1} has no negative entry. Since the row sum of B_{n+1} is 3, B_{n+1} is either $3A_0$ or a non-diagonal adjacency matrix with valency 3.

Finally, we prove (iii). Let $(x_0, x_1, ..., x_{n+1})$ be a closed chain. Then $B_n = A_1^t$ and $B_{n-1} = A_2^t$. Since $(x_{n-2}, x_{n-1}, x_n, x_0)$ is a chain of length 3, we have $(x_0, x_{n-2}) \in S_{n-2} \cap S_3^t$. Hence we have

$$B_{n+1} = A_1 B_n - A_1^t B_{n-1} + B_{n-2} = A_1 A_1^t - A_1^t A_2^t + B_3^t = 3A_0.$$

If $B_{n+1} = 3A_0$ then we have $B_n = A_1^t$ by (8). Let $(x_0, x_1, ..., x_{n+1})$ be a chain. Since $x_n \in S_n(x_0) \cap R_1(x_{n-1})$, we have $B_{n-1} \in \mathsf{Sup}((A_1^t)^2) = \{A_1, A_2^t\}$. We claim $B_{n-1} = A_2^t$. Assume the contrary, i.e.,

$$3A_0 = B_{n+1} = A_1 A_1^t - A_1^t A_1 + B_{n-2} = B_{n-2}$$

by (8), which contradicts the assumption that B_{n-2} is an adjacency matrix. Hence we have $B_{n-1}=A_2^t$ and $x_0\in R_2(x_{n-1})$. Since $p_{21'}^1(x_{n-1},x_n)=p_{21'}^1=1$, we have $x_{n+1}=x_0$.

It remains to show that there exists a simple closed chain of length $n+1$ if $B_{n+1}=3A_0$. We claim that each chain of length n is simple. Suppose $(x_0,...,x_n)$ is a chain with $x_i=x_j$ for some i, j with $0\leqslant i<j\leqslant n$. By Corollary 3.12, we have $S_{j-i}=R_0$, contradicting the assumption that B_{j-i} is a nondiagonal adjacency matrix. Now let x_{n+1} be a unique element of $R_1(x_n)\cap R_2(x_{n-1})$. Then, as shown in the previous paragraph, $B_{n+1}=3A_0$ implies $x_{n+1}=x_0$. Therefore $(x_0,...,x_{n+1})$ is a simple closed chain. This completes the proof. ∎

LEMMA 3.13. *We have the following*:

(i) *All simple closed chains have the same length*;

(ii) *Let n be the length of a simple closed chain. Then $S_{n-i}=S_i^t$ for each i with $1\leqslant i\leqslant n-1$.*

Proof. The length of any simple closed chain is equal to the minimal number $n>0$ such that $B_n=3A_0$ by Proposition 3.5(iii). Let $(x_0,x_1,...,x_n=x_0)$ ba a simple closed chain and fix i with $1\leqslant i\leqslant n-1$. By Corollary 3.12, we have $(x_0,x_i)\in S_i$ and $(x_i,x_n)\in S_{n-i}$. Hence we have $S_{n-i}=S_i^t$. This completes the proof. ∎

We set $(x_{00},x_{10},...,x_{n-1\,0},x_{00})$ and $(x_{00},x_{01},...,x_{0\,n-1},x_{00})$ to be two simple closed chains with $x_{10}\neq x_{01}$, so that $(x_{n-1\,0},x_{10})$, $(x_{0\,n-1},x_{01})\in R_2$ by Corollary 3.12 and Lemma 3.13(ii). We set x_{11} to be a unique element in $R_1(x_{10})\cap R_1(x_{01})$, so that $(x_{00},x_{11})\in R_1^t$. Hence we have $x_{11}\neq x_{20}$ since $x_{00}\in R_1(x_{11})\cap R_2^t(x_{20})$. Thus, we define inductively x_{j1} to be a unique element in $R_1(x_{j0})\cap R_1(x_{j-1\,1})$ for each j with $2\leqslant j\leqslant n-1$. Then we have $(x_{j0},x_{j+1\,1})\in R_1^t$ for each j with $0\leqslant j\leqslant n-2$.

We claim that $(x_{01},x_{11},...,x_{n-1\,1})$ is a chain. If we show $(x_{01},x_{21})\in R_2$ then it can be proved similarly that $(x_{i1},x_{i+2\,1})\in R_2$ for each i with $0\leqslant i\leqslant n-3$. Assume the contrary, i.e., $(x_{01},x_{21})\in R_1^t$. Since $x_{10}\neq x_{01}$ by the assumption and $x_{00}\neq x_{21}$ by $x_{20}\in R_2(x_{00})\cap R_1^t(x_{21})$, $\{x_{00},x_{10},x_{01},x_{21}\}$ are distinct. This contradicts Lemma 2.1 since

$$x_{01},x_{10}\in R_1(x_{00})\cap R_1(x_{21}).$$

Hence the claim holds, and we have $(x_{n-1\,1},x_{01})\in R_1$, $(x_{n-2\,1},x_{01})\in R_2$ by Corollary 3.12 and Lemma 3.13(ii).

Next we define inductively x_{1k} to be a unique element in $R_1(x_{0k})\cap R_1(x_{1\,k-1})$ for each k with $2\leqslant k\leqslant n-1$, moreover, we define inductively

x_{jk} to be a unique element in $R_1(x_{j\,k-1}) \cap R_1(x_{j-1\,k})$ for all j, k with $2 \leqslant j$, $k \leqslant n-1$ by using the same argument as above many times. We claim that

$$(x_{j0}, x_{j1}, ..., x_{j\,n-1}, x_{j0})$$

is also a simple closed chain for each j with $0 \leqslant j \leqslant n-1$ by symmetry and construction.

Remark that $\{x_{jk} \mid 0 \leqslant j, k \leqslant n-1\}$ are not necessarily distinct. We claim that $(x_{n-1\,n-1}, x_{n-2\,n-2}, ..., x_{11}, x_{00})$ is a chain. If we show $(x_{00}, x_{22}) \in R_2^t$ then it can be proved similarly that $(x_{ii}, x_{i+2\,i+2}) \in R_2^t$ for each i with $0 \leqslant i \leqslant n-3$. Assume the contrary, i.e., $(x_{00}, x_{22}) \in R_1$. Since $x_{10} \neq x_{22}$ by $x_{20} \in R_1(x_{10}) \cap R_2^t(x_{22})$ and $x_{00} \neq x_{21}$ by $x_{20} \in R_2(x_{00}) \cap R_1^t(x_{21})$, $\{x_{00}, x_{10}, x_{21}, x_{22}\}$ are distinct This contradicts Lemma 2.1 since $x_{01}, x_{22} \in R_1(x_{00}) \cap R_1(x_{21})$. Therefore, for each i with $1 \leqslant i \leqslant n-1$ we conclude from Corollary 3.12 that, x_{ii}, $x_{0\,n-i} \in R_{i'}(x_{00}) \cap R_i(x_{i0})$, and $x_{ii} \neq x_{0\,n-i}$ by Proposition 2.6(i). This implies that $p^i_{i'i'} = p^{i'}_{ii} = 2$ by Lemma 2.2, and hence $S_i \neq S_i^t$ by Lemma 2.1. Thus, we obtain from Lemma 3.13(ii) the following lemma:

LEMMA 3.14. *Let n be the length of a simple closed chain. For each i with* $1 \leqslant i \leqslant n-1$, *we have*

$$B_i^2 = 2B_i^t + B_t, \qquad B_t \notin \{B_i, B_i^t\},$$

where t is equal to $2i$ *if* $2i < n$, $2i - n$ *if* $2i > n$.

DEFINITION 3.15. Let n be the length of a simple closed chain. For each i with $1 \leqslant i \leqslant n-1$, we define S_i, S_t to be the relations given in Lemma 3.14. A sequence $(x_0, x_1, ..., x_m)$ of elements of X is called *a chain with respect to* S_i of length m if $(x_j, x_{j+1}) \in S_i$ for each $j \in \{0, 1, ..., m-1\}$ and $(x_j, x_{j+2}) \in S_t$ for each $j \in \{0, 1, ..., m-2\}$.

If we replace (R_1, R_2) by (S_i, S_t) where (S_i, S_t) is given in Definition 3.15, then similar statements as all lemmas and propositions in this section hold for chains with respect to S_i.

LEMMA 3.16. *Let* $(x_0, x_1, ..., x_{n-1}, x_0)$ *be a simple closed chain and* $P := \{x_i \mid 0 \leqslant i \leqslant n-1\}$. *Then* $|P \cap R_1(x_0)| \neq 2$.

Proof. Assume the contrary, i.e., there exists a unique element $x_k \in P \cap R_1(x_0)$ with $x_1 \neq x_k$. We consider the relation of (x_0, x_{k+1}). Since $(x_{k-1}, x_{k+1}) \in R_2$, we have $(x_0, x_{k+1}) \in R_1^t$. Since $(x_{k+1}, x_{k+2}, ..., x_0)$ is a chain of length $n-k-1$, we have $(x_0, x_{n-k-1}) \in R_1$. This implies that $n-k-1 = 1$ or k by the assumption $|P \cap R_1(x_0)| = 2$. If $n = k+2$ then $(x_k, x_0) \in R_2 \cap R_1^t$, contradicting (5). If $n = 2k+1$ then $(x_0, x_1, ..., x_{n-1})$ is a chain of length $n-1=2k$. This implies that $(x_0, x_{2k}) \in S_{2k} \cap R_1^t$. Hence we have $S_{2k}=R_1^t=S_k^t$. This implies $B_{2k}=B_k^t$, contradicting Lemma 3.14. ∎

Proof of the Main Theorem. Let $(x_{00}, x_{10}, ..., x_{p-1\,0}, x_{00})$ be a simple closed chain. Let i be a divisor of p such that p/i is a prime. Then, by Lemma 3.14 we can construct a chain with respect to S_i of prime length. Renumbering the relations, we may assume that p is a prime without loss of generality. Note $(x_{p-1\,0}, x_{10}) \in R_2$ since $S_{p-2}=R_2^t$ by Corollary 3.12.

We define $P:=\{x_{i0} \mid 0 \leqslant i \leqslant p-1\}$. Starting from (x_{00}, x_{j0}), we obtain a simple closed chain $(x_{00}, x_{j0}, ..., x_{pj-j\,0}, x_{00})$, with respect to S_j, where the subscripts of x are read modulo p. Observe that the set of elements of this chain coincides with P since p is a prime. Applying Corollary 3.12 by replacing R_1 by S_j, we can show $|P \cap S_j(x_0)| \neq 2$ for each j with $1 \leqslant j \leqslant p-1$ by the same argument as the proof of Lemma 3.16.

If $|P \cap S_j(x_{00})|=3$ for some j then we have $|P \cap S_j(x_t)|=3$ for each t with $0 \leqslant t \leqslant p-1$ by Corollary 3.12. It follows from primitivity that $P=X$. This implies that we can identify elements of X with elements of a cyclic group by Corollary 3.12. Moreover, $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ is isomorphic to $\mathsf{Cyc}(p, 3)$ by Theorem 1.3.

If $|P \cap S_j(x_{00})|=1$ for each j with $1 \leqslant j \leqslant p-1$ then we have $|P \cap R_1(x_t)|=1$ for each t with $0 \leqslant t \leqslant p-1$ by Corollary 3.12(i). Then there exists $x_{01} \in R_1(x_{00})$ not contained in P. We construct a simple closed chain $(x_{00}, x_{01}, ..., x_{0\,p-1}, x_{00})$ by starting from (x_{00}, x_{01}). Let $\{x_{jk} \mid 0 \leqslant j, k \leqslant p-1\}$ be defined in the proof of Lemma 3.14 replaced n by p.

For all j, k, $\{x_{j+1\,k}, x_{j\,k+1}, x_{j-1\,k-1}\}$ are distinct by construction, and we have $R_1(x_{jk})=\{x_{j+1\,k}, x_{j\,k+1}, x_{j-1\,k-1}\}$ where where the subscripts of x are read modulo p. Hence $\{x_{jk}\}_{0 \leqslant j, k \leqslant p-1}$ is the connected component containing x_{00} with respect to R_1.

We claim that the p^2 elements $\{x_{jk}\}_{0 \leqslant j, k \leqslant p-1}$ are all distinct. Assume the contrary. It suffices to show a contradiction if $x_{00}=x_{jk}$. Since $(x_{00}=x_{jk}, x_{j0}) \in S_j \cap S_{p-k}$, we have $S_j=S_{p-k}$. We claim $j=p-k$. Assume the contrary, i.e.,

$$x_{j0}, x_{p-k\,0} \in S_j(x_{00}) \cap P,$$

contradicting the assumption $|P \cap S_j(x_{00})|=1$. Hence

$$(x_{00}, x_{10}, ..., x_{j0}) \qquad \text{and} \qquad (x_{jk}, x_{j\,k+1}, ..., x_{j0})$$

are two distinct chains of length j since $x_{j\,p-1} \neq x_{j-1\,0}$ by construction. This contradicts Lemma 3.6.

By the above claim, we have

$$\{x_{jk}\}_{0 \leqslant j, k \leqslant p-1}=X \qquad \text{with} \quad |X|=p^2.$$

It can be easily verified that (X, R_1) is isomorphic to a relation of $\mathsf{Cyc}(3, p^2)$, mapping x_{jk} to $j + k\omega$ in $GF(p^2)$ for all j, k with $0 \leqslant j, k \leqslant p-1$, where ω is a primitive 3rd root of unity in $GF(p^2)$. This proves our main theorem.

Remark 3.17. If $|X| = p^2$ where p is an odd prime then we have shown that there are at least $p-1$ relations of valency 3. However, not all non-diagonal relations have valency 3 in general. For instance, $\overline{\mathsf{Cyc}}(5^2, 3)$, whose vertex set is the same as that of $\mathsf{Cyc}(5^2, 3) = (F_{25}, \{R_i\}_{0 \leqslant i \leqslant 8})$, whose relations are defined to be

$$\{R_0, R_1 \cup R_5, R_2, R_3 \cup R_7, R_4, R_6, R_8\},$$

is such an association scheme.

Remark 3.18. Let $\{x_{jk} \mid 0 \leqslant j, k \leqslant p-1\}$ be as in the proof of our main theorem. For all j, k with $0 \leqslant j, k \leqslant p-1$, the relation containing (x_{00}, x_{jk}) is in $\mathsf{Sup}(B_i B_j)$ since $x_{i0} \in S_i(x_{00}) \cap S_j^t(x_{ij})$ by construction. This implies that, for any relation R_a, there exist two relations R_b, R_c such that $A_a \in \mathsf{Sup}(A_b A_c)$ and $k_b = k_c = 3$, which gives an upper bound of valency.

4. RELATED TOPICS

We list some open problems on the characterization of cyclotomic association schemes.

(i) Is there an association scheme of class greater than 2 with a prime number of points which is not a translation association scheme?

(ii) Can we drop the condition "commutative" in Theorem 1.5?

(iii) Under what conditions can one characterize the cyclotomic association scheme $\mathsf{Cyc}(q, k)$?

(iv) Is there a noncommutative association scheme with a prime number of points?

The problem of determining the intersection numbers of association schemes is very much related to the classification of integral table algebras (see [1]). An integral table algebra is a Z-algebra with some basis with respect to which the structure constants are nonnegative integers, studied by Z. Arad, E. Fisman, V. Miloslavsky, M. Muzychuk, and H. I. Blau (see [1, 2, 4]). They classified homogeneous antisymmetric integral table algebras of degree 3 generated by only one base element, and integral table algebras of degree 2 containing a faithful base element. The Bose–Mesner algebra of an association scheme is an integral table algebra. The

Bose–Mesner algebra of an association scheme with $k_1 = k_2 = \cdots = k_d = 3$ and $A_i^t \neq A_i$ for each $i \in \{1, ..., d\}$ is a homogeneous antisymmetric integral table algebras of degree 3. In particular, the Bose–Mesner algebra of $\mathsf{Cyc}(q, 3)$ is a homogeneous antisymmetric integral table algebra of degree 3 if q is odd. Their works gave the author much imagination.

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