Capacity of Multiple-Antenna Systems With Both Receiver and Transmitter Channel State Information

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***Abstract*—The capacity of multiple-antenna systems operating in Rayleigh flat fading is considered under the assumptions that channel state information (CSI) is available at both transmitter and receiver, and that the transmitter is subjected to an average power constraint. First, the capacity of such systems is derived for the special case of multiple transmit antennas and a single receive antenna. The optimal power-allocation scheme for such a system is shown to be a water-filling algorithm, and the corresponding capacity is seen to be the same as that of a system having multiple receive antennas (with a single transmitter antenna) whose outputs are combined via maximal ratio combining. A suboptimal adaptive transmission technique that transmits only over the antenna having the best channel is also proposed for this special case. It is shown that the capacity of such a system under the proposed suboptimal adaptive transmission scheme is the same as the capacity of a system having multiple receiver antennas (with a single transmitter antenna) combined via selection combining.**

Next, the capacity of a general system of multiple transmitter and receiver antennas is derived together with an equation that determines the cutoff value for such a system. The optimal power allocation scheme for such a multiple-antenna system is given by a matrix water-filling algorithm. In order to eliminate the need for cumbersome numerical techniques in solving the cutoff equation, approximate expressions for the cutoff transmission value are also provided. It is shown that, compared to the case in which there is only receiver CSI, large capacity gains are available with optimal power and rate adaptation schemes. The increased capacity is shown to come at the price of channel outage, and bounds are derived for this outage probability.

***Index Terms*—Adaptive transmission, channel capacity, matrix water-filling, multiple-antenna systems, outage probability, Wishart distribution.**

I. Introduction

The capacity of fading channels varies depending on the assumptions one makes about fading statistics and the knowledge of fading coefficients. Over the years, the capacity of single-antenna systems (where both transmitter and receiver are equipped with only one antenna each) has been considered for various assumptions on knowledge of fading coefficients. For example, [1], [2] have treated the case where the receiver has access to channel state information (CSI), [3], [4] have considered the capacity under the assumption that both transmitter and receiver have access to CSI, and [5]–[9] have all treated the case when neither the transmitter nor the receiver knows the channel fading coefficients.

Recently, there has been a surge of interest in multiple-antenna communications systems. Naturally, this has led to capacity investigation of fading channels with multiple antennas either at the receiver or at the transmitter or at both ends of the communication link. For example, [10]–[12] considered the capacity of multiple transmit and receiver antenna systems when CSI is available only at the receiver, and [13] investigated the capacity of such systems when neither the transmitter nor the receiver knows the channel coefficients. The capacity of multiple receiver antenna systems (with a single transmit antenna) when CSI is available at both transmitter and receiver has also been previously considered in [14].

When both transmitter and receiver have access to the CSI, the optimal strategies would make use of this information at both ends of the link. Intuitively, one would expect the transmitter to adjust its power and rate depending on the instantaneous value of the observed CSI. This results in adaptive transmission techniques. However, such an optimal scheme could easily become too complicated to implement, for example, when the fading is correlated. In order to overcome this possible transmitter complexity, it is also of interest to investigate low-complexity adaptive transmission techniques and determine the capacities under such suboptimal adaptive transmission techniques. As mentioned earlier, this problem has been treated previously in [3] for the case of single-antenna systems and in [14] for the case of receiver diversity. With recent interest in multiple transmit antenna systems for wireless communications, it is also of interest to consider this problem in the context of multiple antennas at both transmitter and receiver. In this correspondence, we investigate the capacity of such systems under adaptive transmission techniques.

First, we obtain the capacity of the optimal power and rate allocation scheme for a system having multiple transmit antennas but one receiver antenna and, not surprisingly, this is seen to be identical to the capacity of a receiver diversity scheme with maximal ratio combining. We also derive the capacity of a multiple transmit antenna system with a suboptimal adaptive transmission technique which is seen to be mathematically equivalent to a receiver diversity system with selection combining.

Next we consider a general system with multiple antennas at both the receiver and the transmitter. The capacity of the optimal power and rate allocation scheme for such a system is derived and this capacity is evaluated for several representative situations. As we will show, the capacity of such systems could be much larger than corresponding systems with only receiver channel state information. The increased capacity comes at the price of channel outage which we characterize in terms of the outage probability. We also derive simple upper bounds for this outage probability.

In all these situations, we also provide approximate expressions for the capacity, which are easy to evaluate and thus eliminate the need for any numerical integration or root finding techniques that might be required otherwise.

The rest of this correspondence is organized as follows. In Section II, we outline our system model and the assumptions; Section III considers the special case of capacity of multiple transmit antenna systems with a single receiver antenna. Next, in Section IV, we treat the capacity of a general system having multiple antennas at both transmitter and receiver. We obtain the capacity of such systems under optimal power adaptation as well as the cutoff equation associated with the optimal transmission scheme. In Section IV, we also derive simple upper bounds for the outage probability of the optimal adaptive transmission scheme for a multiple-antenna system. Finally, in Section V, we give some concluding remarks.

II. System Model Description

We consider a single-user flat-fading communications link in which the transmitter and receiver are equipped with N_T and N_R antennas,

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respectively. The discrete-time received signal in such a system can be written in matrix form as

$$\boldsymbol{y}(i) = \boldsymbol{H}(i)\boldsymbol{x}(i) + \boldsymbol{n}(i) \tag{1}$$

where $\boldsymbol{y}(i)$ is the complex N_R vector of received signals at the N_R receive antennas at symbol time i, $\boldsymbol{x}(i)$ is the (possibly) complex N_T vector of transmit signals on the N_T transmit antennas at time i and $\boldsymbol{n}(i)$ is the complex N_R vector of receiver noise at time i. The components of $\boldsymbol{n}(i)$ are zero mean, circularly symmetric, complex Gaussian with independent real and imaginary parts having equal variance. It is assumed that noise on each receiver antenna is independent of that on all others and thus, $E\{\boldsymbol{n}(i)\boldsymbol{n}(i)^H\} = N_0\boldsymbol{I}_{N_R}$, where $\boldsymbol{I}_{N_R}$ denotes the $N_R \times N_R$ identity matrix. We also assume that $\boldsymbol{n}(i)$ is a sequence of uncorrelated (and thus independent) random vectors.

The matrix $\boldsymbol{H}(i)$ in (1) is the $N_R \times N_T$ matrix of complex fading coefficients which are assumed to be stationary and ergodic. The (n_R, n_T)th element of the matrix $\boldsymbol{H}(i)$ represents the fading coefficient value at time i between the n_Rth receiver antenna and the n_Tth transmitter antenna. These fading coefficients are assumed to be slowly varying over the duration of a codeword. We assume that elements of the matrix $\boldsymbol{H}(i)$ are independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and $1/2$-variance per dimension. Of course, this gives rise to the so-called Rayleigh-fading channel model, which has often been used to model land-mobile wireless communication channels without a direct line-of-sight path [15].

We assume that the instantaneous value of the fading coefficient matrix $\boldsymbol{H}(i)$ is known to both the transmitter and the receiver. This assumption can be satisfied, for example, by employing a channel estimation scheme such as pilot symbol insertion or training bits. The transmitter may be assumed to be informed of those receiver estimated CSI via a delay- and error-free feedback path. This is a reasonable assumption when the channel varies at a much slower rate compared to the data rate of the system. In a time-duplexed system, the transmitter may also estimate its own CSI values using the reverse link received signals.

As we will see shortly, the capacity will be dependent on the number of transmitter and receiver antennas only through the relative parameters defined as $n = \max\{N_R, N_T\}$ and $m = \min\{N_R, N_T\}$.

III. Single Receiver Antenna Systems

We start by considering the capacity of a multiple transmit and single receiver antenna system with adaptive transmission techniques; i.e., $N_T = n$ and $N_R = m = 1$. In this case, the received signal $\boldsymbol{y}$ is a scalar which we denote as y. Note that, for convenience, we will drop the time index i whenever this causes no confusion.

In general, we may decompose the fading coefficient matrix $\boldsymbol{H}$ using the singular value decomposition [16], [17]

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^H \tag{2}$$

where $\boldsymbol{U}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{V}$ are matrices of dimension $N_R \times N_R$, $N_R \times N_T$, and $N_T \times N_T$, respectively. The matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices satisfying $\boldsymbol{U}\boldsymbol{U}^H = \boldsymbol{U}^H\boldsymbol{U} = \boldsymbol{I}_{N_R}$ and $\boldsymbol{V}\boldsymbol{V}^H = \boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}_{N_T}$. The matrix $\boldsymbol{\Lambda} = [\lambda_{i,j}]$ is a diagonal matrix with diagonal entries being equal to the nonnegative square roots of the eigenvalues of either $\boldsymbol{H}\boldsymbol{H}^H$ or $\boldsymbol{H}^H\boldsymbol{H}$, and, thus, are uniquely determined. For later use, we may also define the following $m \times m$ matrix:

$$\boldsymbol{W} = \begin{cases} \boldsymbol{H}\boldsymbol{H}^H, & \text{if } N_R \le N_T \\ \boldsymbol{H}^H\boldsymbol{H}, & \text{if } N_R > N_T. \end{cases} \tag{3}$$

Note that $\boldsymbol{W}$ can have at most m nonzero eigenvalues and thus correspondingly at most m diagonal entries of the matrix $\boldsymbol{\Lambda}$ are nonzero. It is also worth mentioning that the distribution of the matrix $\boldsymbol{W}$ is given by the well-known Wishart distribution [18]. In the present case of $N_T = n$ and $N_R = m = 1$, it is easily seen that a singular value decomposition of $\boldsymbol{H}$ is defined by

$$\begin{aligned} \boldsymbol{U} &= 1 \\ \boldsymbol{\Lambda} &= \left[\sqrt{\lambda_1}, 0, \ldots, 0\right] \\ \boldsymbol{V} &= [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \cdots\ \boldsymbol{v}_n] \end{aligned} \tag{4}$$

where

$$\lambda_1 = \sum_{i=1}^{n} |h_{i,j}|^2 \quad \text{and} \quad \boldsymbol{v}_1 = \frac{\boldsymbol{H}^H}{\sqrt{\lambda_1}}.$$

Defining the transformations $\tilde{\boldsymbol{y}} = \boldsymbol{U}^H\boldsymbol{y}$, $\tilde{\boldsymbol{x}} = \boldsymbol{V}^H\boldsymbol{x}$, and $\tilde{\boldsymbol{n}} = \boldsymbol{U}^H\boldsymbol{n}$, we see that the channel in (1) is equivalent to

$$\tilde{\boldsymbol{y}} = \boldsymbol{\Lambda}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{n}}. \tag{5}$$

If the average transmit power is constrained as

$$E\{\boldsymbol{x}^H\boldsymbol{x}\} = \operatorname{tr}[E\{\boldsymbol{x}\boldsymbol{x}^H\}] = P$$

then we also have that

$$E\{\tilde{\boldsymbol{x}}^H\tilde{\boldsymbol{x}}\} = \operatorname{tr}\left[E\{\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^H\}\right] = P. \tag{6}$$

From (4) and (5) we see that, for the present case of $N_R = 1$, the channel is equivalent to the following scalar channel:

$$\tilde{y} = \sqrt{\lambda_1}\tilde{x}_1 + \tilde{n} \tag{7}$$

where $\tilde{x}_1$ is the first component of the vector $\tilde{\boldsymbol{x}}$.

Hence, we see that only that energy contained in the component $\tilde{x}_1$ is useful in detecting the signal, and thus we may as well set $\tilde{x}_2 = \tilde{x}_3 = \cdots = \tilde{x}_{N_T} = 0$. Then, from (6) we have that

$$E\{\tilde{x}_1^2\} = P. \tag{8}$$

A. Optimal Adaptive Transmission

Let us define the received signal-to-noise ratio (SNR) $\gamma(i)$ for a given value of the channel coefficient matrix as

$$\gamma(i) = \lambda_1(i)\,\frac{P}{N_0}. \tag{9}$$

We let the transmitter adapt its instantaneous transmit power $P(\gamma(i))$ according to the channel variations, subject to the average power constraint

$$\int_\gamma P(\gamma) f_\gamma(\gamma)\,d\gamma \le P \tag{10}$$

where $P(\gamma)$ denotes the time-varying instantaneous adaptive power and $f_\gamma(\gamma)$ is the probability distribution function of $\gamma(i)$.

Note that the channel (5) with this adaptive transmission is mathematically equivalent to the scalar channel treated in [3] with the same average received SNR. Thus, observing that the instantaneous received SNR is given by $P(\gamma(i))\frac{\gamma(i)}{P}$, the average capacity of the channel in (7), and also in (1), can be defined similarly to [3] as

$$C = \max_{P(\gamma):\ \int_\gamma P(\gamma) f_\gamma(\gamma)\,d\gamma = P} \int_\gamma \log\left(1 + \frac{P(\gamma)\gamma}{P}\right) f_\gamma(\gamma) d\gamma. \tag{11}$$

Thus, the coding theorem and converse proven in [3] apply directly to the equivalent channel model in (5), and the maximizing power adaptation rule is thereby easily shown to be the water-filling algorithm [3], [19] given as

$$\frac{P(\gamma)}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}, & \text{if } \gamma \ge \gamma_0 \\ 0, & \text{if } \gamma \le \gamma_0 \end{cases} \tag{12}$$

where the cutoff value γ_0 is chosen to satisfy the power constraint (10) as

$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) f_\gamma(\gamma)\,d\gamma = 1. \tag{13}$$

From (11), the capacity of the multiple transmit antenna system under this optimal power and rate allocation scheme is then given by

$$C = \int_{\gamma_0}^{\infty} \log\left(\frac{\gamma}{\gamma_0}\right) f_\gamma(\gamma)\, d\gamma. \tag{14}$$

In order to evaluate the above capacity, we recall from above that λ_1 is the sum of n squared complex Gaussian random variables. Hence, from (9), γ is an n-Erlang random variable having the distribution function [20]

$$f_\gamma(\gamma) = \frac{1}{(n-1)!\overline{\gamma}}\left(\frac{\gamma}{\overline{\gamma}}\right)^{n-1} \exp\left(-\frac{\gamma}{\overline{\gamma}}\right), \qquad \gamma > 0, \tag{15}$$

where we have defined the average SNR as $\overline{\gamma} = \frac{P}{N_0}$.

From (15), we see that this channel is, in fact, equivalent to a single transmitter antenna system with receiver antenna diversity and maximal ratio combining. However, in this case, we are transmitting from multiple antennas. For example, given $\boldsymbol{H}$, the actual transmission scheme is such that if the instantaneous value of $\gamma(i)$ defined in (9) is greater than γ_0 satisfying (13), then the transmitted signals on the N_T antennas are given by $\boldsymbol{x} = \frac{\boldsymbol{H}^H}{\|\boldsymbol{H}\|}\tilde{x}_1$, where $\tilde{x}_1$ is the capacity-achieving signal for the system in (7) with $\tilde{x}_1^2 = P(\frac{1}{\gamma_0} - \frac{1}{\gamma})$. On the other hand, if the instantaneous value of $\gamma(i)$ defined in (9) is less than γ_0, then transmission from all antennas are cut off; i.e., no signal is transmitted from any antenna.

From the equivalence of the scalar system in (7) to the maximal ratio combining receiver diversity system, it then also follows that the properties of the cutoff value γ_0 and capacity expression given in [14] for the receiver diversity system holds for this transmitter diversity system verbatim. In fact, substituting (15) into (13) we get the equation that the cutoff value γ_o must satisfy to be

$$\frac{\Gamma(n, \mu)}{\mu} - \Gamma(n-1, \mu) = \overline{\gamma}(n-1)! \tag{16}$$

where $\Gamma(n, \mu)$ denotes the complementary incomplete gamma function $\Gamma(a, x) = \int_x^\infty e^{-t} x^{a-1}\, dt$ and

$$\mu = \frac{\gamma_0}{\overline{\gamma}}. \tag{17}$$

It was shown in [14] that there exists a unique μ, and thus a unique γ_0, that satisfies (16), and that this γ_0 always satisfies $0 \le \gamma_0 \le 1$. Specifically, $\lim_{\overline{\gamma}\longrightarrow 0} \gamma_0 = 0$ and $\lim_{\overline{\gamma}\longrightarrow\infty} \gamma_0 = 1$.

In general, solution of (16) requires numerical root finding techniques. However, it can be shown that for $\overline{\gamma}$ large (i.e., for large SNR), a reasonable approximation for γ_0 is given by

$$\gamma_0 \approx \frac{n-1}{\frac{1}{\overline{\gamma}} + n - 1}, \qquad \text{for } \overline{\gamma} \gg 1. \tag{18}$$

Substituting (15) into (14) and following the same steps as in [14] we can also obtain the equivalent capacity formula for the multiple transmit antenna system with optimal power adaptation to be

$$C = \log_2(e)\left(\sum_{k=1}^{n-1} \frac{\mathcal{P}_k(\mu)}{k} + E_1(\mu)\right) \text{ bits/channel use} \tag{19}$$

where μ is given in (17), $E_1(\mu)$ is the exponential integral function [21], [22] defined as

$$E_1(\mu) = \int_\mu^\infty \frac{e^{-t}}{t}\, dt$$

and the Poisson sum $\mathcal{P}_k(\mu)$ is

$$\mathcal{P}_k(\mu) = e^{-\mu} \sum_{j=0}^{k-1} \frac{\mu^j}{j!}.$$

In Fig. 1, we have shown the exact cutoff value and its approximation given by (18). From Fig. 1 it is clear that the above approximation is indeed good for large $\overline{\gamma}$ and the approximation becomes tighter as $N_T = n$ increases. In fact, if $N_T = n > 3$, the approximation (18) becomes tight for all SNR > 0 dB, as one can see from Fig. 1.

Fig. 2 plots the exact capacity of the optimal adaptive transmission scheme for the multiple transmit antenna system with a single receiver antenna. Also shown in Fig. 2 is the capacity of a similar system with only receiver channel state information as derived in [12]. Note that the asymptotic capacity of the receiver CSI only system tends to $\log(1 + \frac{P}{N_0})$ for large N_T as shown in [12]. We observe that when the CSI is available at both ends of the communication system, large capacity gains are possible compared to a system with only the receiver CSI. Of course, the price one has to pay for these large capacity gains is the outage probability determined by the cutoff value. Still, when the delay caused by the outage is within acceptable limits it is possible to gain large capacity improvements in multiple transmit antenna systems with the optimal adaptive transmission scheme proposed above.

B. Maximal Gain Transmission

Suppose now that instead of the above scheme we employ the simpler technique of choosing the transmitter antenna corresponding to the largest channel gain coefficient H_{1,n_T} and then transmit only on that particular antenna. We call this strategy the maximal gain transmission technique. In this subsection, we derive the capacity of this scheme, the adaptive power allocation rule that achieves it, and show that, in fact, this scheme is equivalent to a receiver diversity system with selection combining [14], [15]. We also provide simple approximations to the capacity and the cutoff value in this case.

With this new transmission scheme received signal can be written as

$$y(i) = h(i)x(i) + n(i) \tag{20}$$

where $x(i)$ is the transmitted signal at time i (which can be on any antenna) and $h(i)$ is the corresponding fading coefficient.

Analogously to the previous case, we may define

$$\gamma(i) = \frac{P|h(i)|^2}{N_0}. \tag{21}$$

Since $h(i) = \max\{H_{1,1}(i), H_{1,2}(i), \ldots, H_{1,n}(i)\}$, it is easily shown that the probability density function (pdf) of γ is given by

$$f_\gamma(\gamma) = \frac{n}{\overline{\gamma}} \exp\left(-\frac{\gamma}{\overline{\gamma}}\right)\left(1 - \exp\left(-\frac{\gamma}{\overline{\gamma}}\right)\right)^{n-1}. \tag{22}$$

Comparing the pdf in (22) with the pdf of the received SNR of a receiver diversity system with selection combining, given in [14], we observe that, in fact, they are identical. Thus, substituting (22) into (13), we obtain an equation that must be satisfied by the cutoff value of the adaptive transmission rule that achieves the capacity in the maximal gain transmission scheme to be

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\frac{\exp(-(1+k)\mu)}{(1+k)\mu} - E_1((1+k)\mu)\right] = \frac{\overline{\gamma}}{n} \tag{23}$$

where μ is given by (17) and $E_1(\mu)$ is the exponential integral defined earlier.

Again, (23) is identical to the equation that determines the cutoff for the selection combining receiver diversity system obtained in [14]. As a result, properties of the cutoff value given in [14] directly applies to this system as well. Specifically, $0 \le \gamma_0 \le 1$.

Substituting the series representation [21], [22]

$$E_1(x) = -E - \log(x) - \sum_{k=1}^{\infty} \frac{(-x)^k}{k.k!}$$

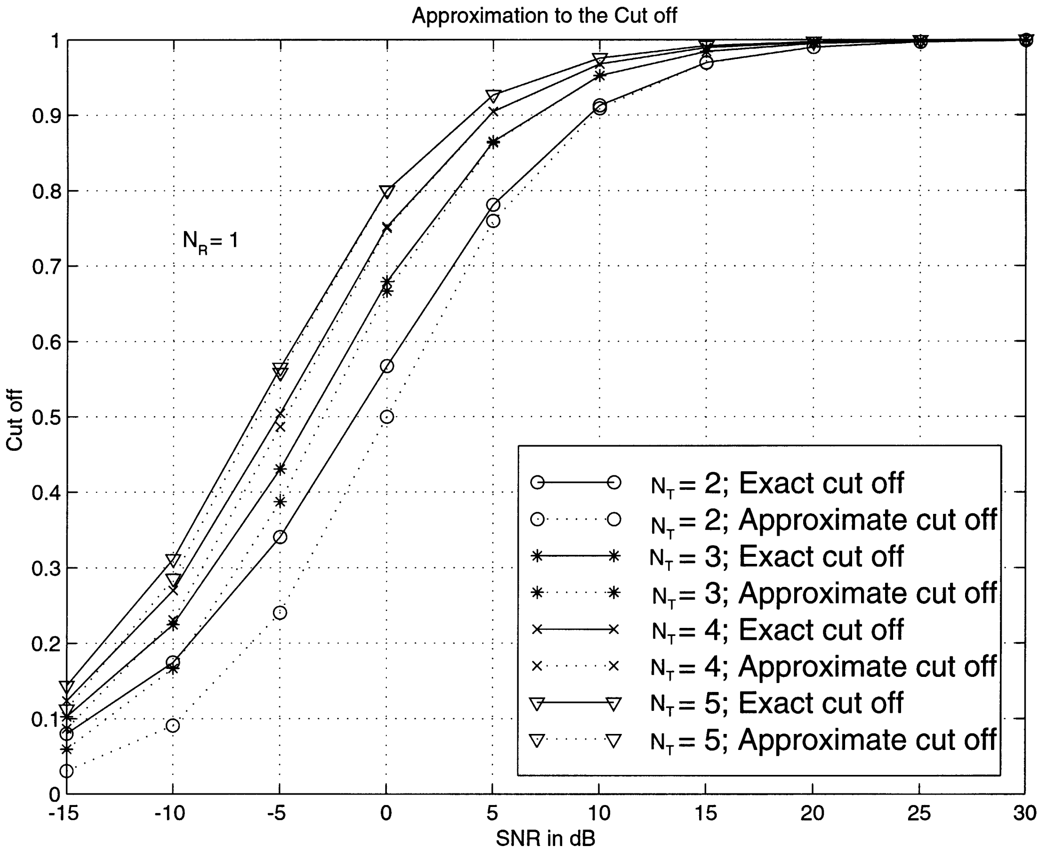

Fig. 1. Approximation to the optimal cut-off value versus SNR (in decibels). $N_R = 1$.

of the exponential integral function, where E is the Euler's constant ($E = 0.5772156649015325\cdots$) [22], and after some manipulations, we may obtain the following nth-order approximation to (23):

$$\frac{\gamma_0}{n} \approx \frac{1}{n} + \mu\left(\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\log(1+k)\right) + \frac{\mu^n}{n(n-1)}.$$

Hence, for $\overline{\gamma}$ large, an $(n-1)$th-order approximation to the cutoff value γ_0 can be given as

$$\gamma_0 \approx \frac{1}{\frac{M_n}{\overline{\gamma}}+1}, \qquad \text{for } \overline{\gamma} \gg 1 \tag{24}$$

where we have defined the constant M_n as

$$M_n = n\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\log(1+k).$$

Thus, the capacity of a maximal gain transmission system with $N_T = n$ antennas and a single receiver antenna is equal to that of a selection combining receiver diversity system with $N_R = n$ receiver antennas. However, it should be noted that in this case the codewords are transmitted from different transmitter antennas at each time instant depending on which antenna corresponds to the largest fading gain. This is somewhat similar to a particular implementation of Bell Labs layered space–time (BLAST) architecture [10] where one periodically rotates the transmit antennas. However, BLAST does not assume knowledge of fading coefficients at the transmitter and thus there is no associated cutoff value, and the order of antenna rotation is predetermined.

Using the results derived in [14] for the selection combining receiver diversity scheme, we have the capacity of maximal gain transmission scheme

$$C = n\mu\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\mathcal{J}_1((1+k)\mu), \tag{25}$$

where μ is given in (17) and the integral $\mathcal{J}_p(\mu)$ is defined as in [14] to be

$$\mathcal{J}_p(\mu) = \int_1^\infty t^{p-1}\log(t)e^{-\mu t}\,dt, \qquad \text{for } p = 1, 2, \ldots. \tag{26}$$

Using the fact that $\mathcal{J}_1(\mu) = \frac{E_1(\mu)}{\mu}$, we may write the capacity in (25) as

$$C = n\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\frac{E_1((1+k)\mu)}{1+k}. \tag{27}$$

It can also be shown that for $n \gg 1$ and $\overline{\gamma} \gg 1$, the above capacity is well approximated by

$$C \approx \log(\overline{\gamma}) - \left[E + \log(\gamma_0) + n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{\log(1+k)}{1+k}\right], \quad \text{for } n \gg 1, \text{ and } \overline{\gamma} \gg 1. \tag{28}$$

In particular, for $\overline{\gamma} > 1$, the capacity asymptote for large N_T is given by

$$\lim_{n\longrightarrow\infty} C = \log(\overline{\gamma}) - [E + \log(\gamma_0)] \approx \log(\overline{\gamma}), \qquad \text{for } \overline{\gamma} \gg 1. \tag{29}$$

Fig. 3 plots the exact capacity expression in (27), evaluated with both exact and approximate cutoff value γ_0. The figure shows that unless

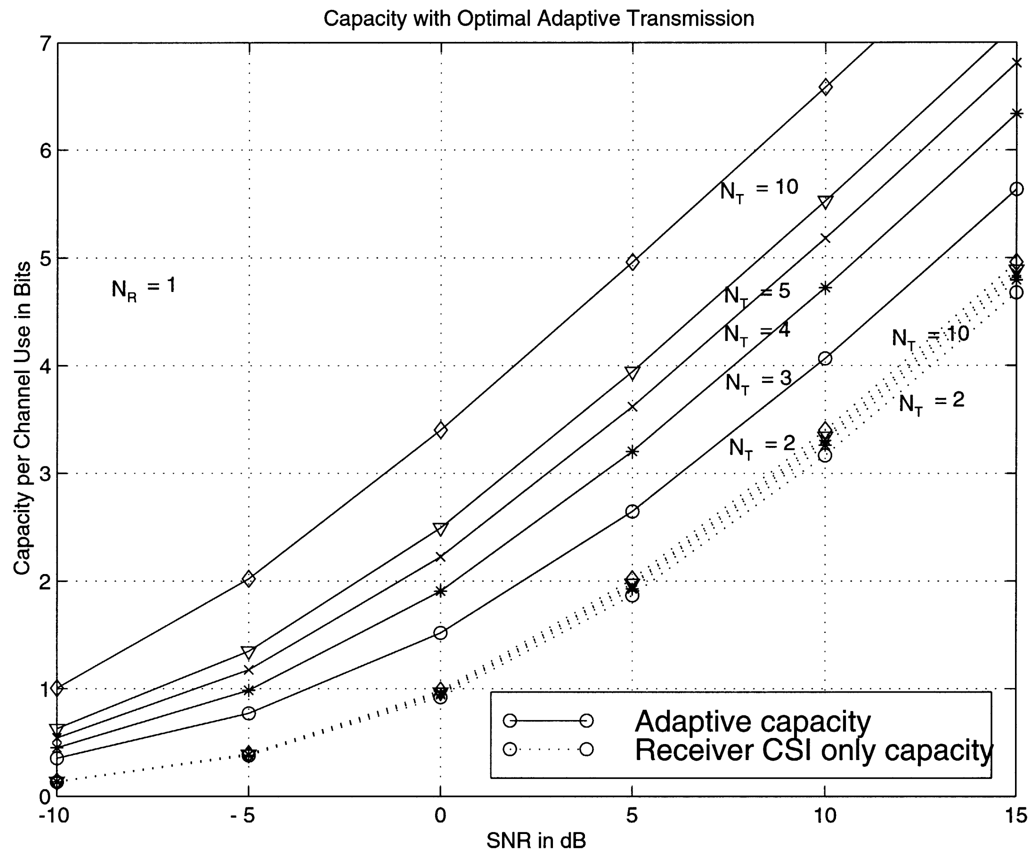

Fig. 2. Exact and approximate capacities for the optimal adaptive transmission versus SNR (in decibels). $N_R = 1$.

both SNR and $N_T = n$ are small, the capacity deviation due to the use of the approximate cutoff value given by (24) is not significant. In fact, from these plots it is evident that for N_T as low as 4, the error due to the approximation is negligible. Also, comparison of Fig. 3 with Fig. 2 illustrates the capacity loss due to the suboptimal transmission scheme.

IV. Capacity With Multiple Antennas at Both Transmitter and Receiver

We now turn to the situation in which there are multiple antennas at both transmitter and receiver ends. In this case, applying singular value decomposition of the matrix $\boldsymbol{H}$ in (2), we still have the equivalent channel model given in (5). In analogy with (9) we may define

$$\boldsymbol{\Lambda}'(i) = \sqrt{\frac{P}{mN_0}}\,\boldsymbol{\Lambda}(i) \tag{30}$$

and, as before, let the transmit power vary with the observed channel state information subject to the average power constraint P. If we define $\tilde{\boldsymbol{Q}} = \tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^H$, then the instantaneous transmit power can be written as $\tilde{\boldsymbol{x}}^H\tilde{\boldsymbol{x}} = \mathrm{tr}[\tilde{\boldsymbol{Q}}]$, and the average power constraint becomes $E\{\mathrm{tr}[\tilde{\boldsymbol{Q}}]\} \le P$. Hence, in this case the adaptive transmission strategy based on the observed channel state information can be achieved by letting $\tilde{\boldsymbol{Q}}$ be a function of $\boldsymbol{\Lambda}'(i)$. Thus, we denote the instantaneous value of $\tilde{\boldsymbol{Q}}(i)$ as $\tilde{\boldsymbol{Q}}(\boldsymbol{\Lambda}'(i))$. Then, we may define the average capacity of the vector, time-varying channel with adaptive transmission scheme to be

$$C = \max_{\tilde{\boldsymbol{Q}}(\boldsymbol{\Lambda}')>0,\, \mathrm{tr}(E\{\tilde{\boldsymbol{Q}}(\boldsymbol{\Lambda}')\})=P} E_{\boldsymbol{\Lambda}'}\left\{\log\det\left(\boldsymbol{I} + \boldsymbol{\Lambda}'\,\frac{\tilde{\boldsymbol{Q}}(\boldsymbol{\Lambda}')}{(P/m)}\,\boldsymbol{\Lambda}'\right)\right\}. \tag{31}$$

It can be shown that the above maximization is achieved by a diagonal $\tilde{\boldsymbol{Q}}(\boldsymbol{\Lambda}')$ and that the diagonal entries are given by a matrix water-filling formula to be, for $i = 1, \ldots, m$

$$\frac{\tilde{\boldsymbol{Q}}_{i,i}}{(P/m)} = \begin{cases} \frac{1}{\gamma_{i,0}} - \frac{1}{\gamma_i}, & \text{if } \gamma_i \ge \gamma_{i,0} \\ 0, & \text{if } \gamma_i \le \gamma_{i,0} \end{cases} \tag{32}$$

where γ_i for $i = 1, \ldots, m$ are defined as

$$\gamma_i = \overline{\gamma}\lambda_i \tag{33}$$

and λ_i are the eigenvalues of the Wishart distributed matrix $\boldsymbol{W}$ defined in (3). We have also redefined $\overline{\gamma}$ as

$$\overline{\gamma} = \frac{P}{mN_0}. \tag{34}$$

The cutoff values $\gamma_{i,0}$ in (32) are chosen to satisfy the power constraint

$$\begin{aligned} P &= \mathrm{tr}(E\{\tilde{\boldsymbol{Q}}(\boldsymbol{\Lambda}')\}) \\ &= \frac{P}{m}\sum_{i=1}^{m}\int_{\gamma_{i,0}}^{\infty}\left(\frac{1}{\gamma_{i,0}} - \frac{1}{\gamma_i}\right) f_{\gamma_i}(\gamma_i)\, d\gamma_i \end{aligned} \tag{35}$$

where $f_{\gamma_i}(\gamma_i)$ denotes the pdf of the ith nonzero eigenvalue of the Wishart matrix $\boldsymbol{W}$. If we let $f_\gamma(\gamma)$ denote the pdf of any unordered γ_i, for $i = 1, \ldots, m$, then (35) leads to

$$\int_{\gamma_0}^{\infty}\left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) f_\gamma(\gamma)\, d\gamma = 1 \tag{36}$$

where γ_0 is the cutoff transmission value corresponding to any eigenvalue.

The probability distribution function $p_\lambda(\lambda)$ of an unordered eigenvalue of a Wishart distributed matrix was given in [12], and can be

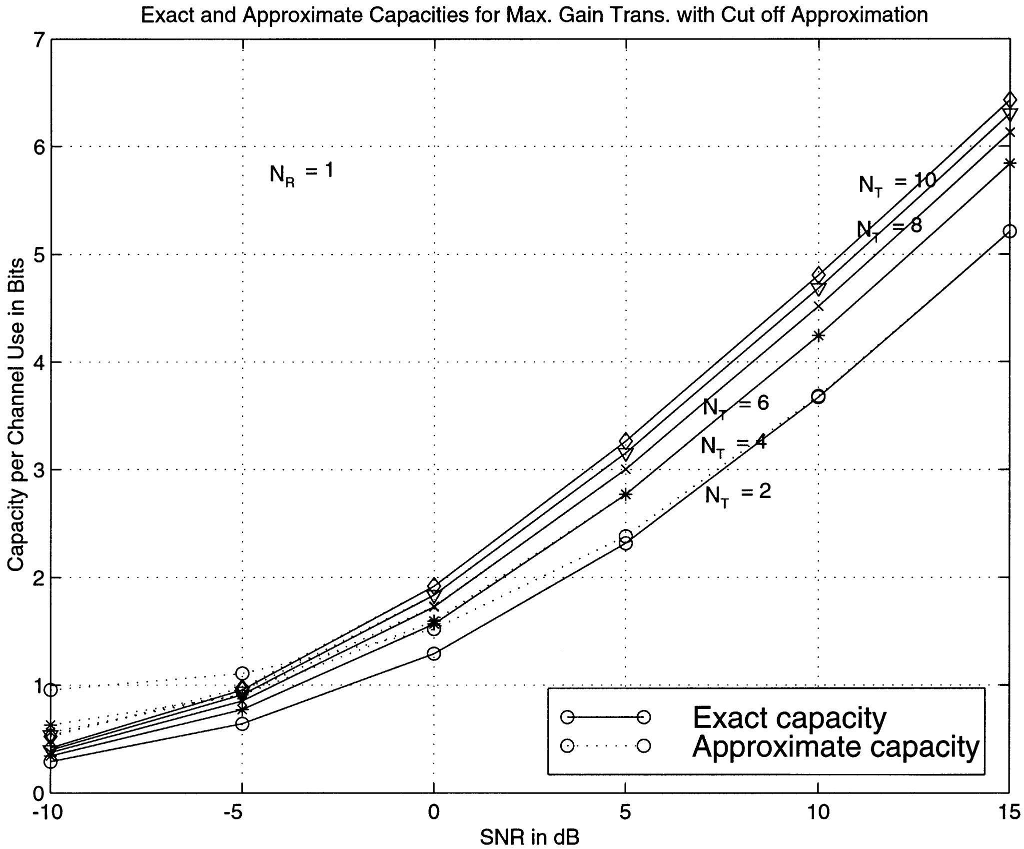

Fig. 3. Capacity of the maximal gain transmission scheme evaluated with both exact and approximate cutoff values versus SNR (in decibels). $N_R = 1$.

written as

$$p_\lambda(\lambda) = \frac{e^{-\lambda}\lambda^{n-m}}{m} \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \left[L_{k-1}^{n-m}(\lambda)\right]^2 \quad (37)$$

where the associated Laguerre polynomial of order k, $L_k^{n-m}(\lambda)$, for $k \geq 0$, is defined by [21], [22]

$$L_k^a(\lambda) = \frac{1}{k!} e^\lambda \lambda^{-(a)} \frac{d^k}{d\lambda^k} \left[e^\lambda \lambda^{a+k}\right]. \quad (38)$$

Then from the definition in (33) we have that

$$f_\gamma(\gamma) = \frac{1}{\overline{\gamma}} p_\lambda \left(\frac{\gamma}{\overline{\gamma}}\right). \quad (39)$$

Substituting (39) in (36) and introducing a change of variable we see that the cutoff value must satisfy

$$\sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \int_\mu^\infty \left(\frac{1}{\mu} - \frac{1}{\gamma}\right) e^{-\gamma}\gamma^{n-m} \left[L_{k-1}^{n-m}(\gamma)\right]^2 d\gamma = m\overline{\gamma} \quad (40)$$

where $\mu$is as defined in (17).

In the following subsection, we show that for $\overline{\gamma} > 0$, (40) has a unique solution μ.

A. Uniqueness of the Cutoff Value

Intuitively one would expect (40) to have a unique solution μ. In fact, by studying the properties of (40) we may show that this indeed holds true.

For convenience, let us define the integrand in (40) to be

$$f_{n-m,k}(\gamma, z) = \left(\frac{1}{z} - \frac{1}{\gamma}\right) e^{-\gamma}\gamma^{n-m} \left[L_{k-1}^{n-m}(\gamma)\right]^2.$$

Next, define the function $F(z)$ as

$$F(z) = \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \int_z^\infty f_{n-m,k}(\gamma, z)\, d\gamma - m\overline{\gamma}. \quad (41)$$

Note that (40) is then equivalent to the case of $F(z) = 0$. Differentiating (41) with respect to z gives

$$F'(z) = -\sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \frac{1}{z^2} \int_z^\infty \times e^{-\gamma}\gamma^{n-m} \left[L_{k-1}^{n-m}(\gamma)\right]^2 d\gamma \quad (42)$$

and we immediately notice that, since the integrand in (42) is positive

$$F'(z) < 0, \qquad \text{for } z > 0. \quad (43)$$

Similarly, one can also show that $F''(z) > 0$ for $z > 0$.

Next, either relying on the normalization property of a pdf or by explicitly recalling the integral equation [22, eq. 7.414.9] we have that

$$\lim_{z \to 0^+} \int_z^\infty e^{-\gamma}\gamma^{n-m} \left[L_{k-1}^{n-m}(\gamma)\right]^2 d\gamma = \frac{(n-m+k-1)!}{(k-1)!}, \qquad \text{for } n-m \geq 0. \quad (44)$$

Using [22, eq. 7.414.12], for $n - m > 0$, we also have that

$$\lim_{z \to 0^+} \int_z^\infty e^{-\gamma}\gamma^{n-m-1} \left[L_{k-1}^{n-m}(\gamma)\right]^2 d\gamma = \frac{\Gamma(n-m)\Gamma(n-m+k)}{\Gamma(n-m+1)[(k-1)!]^2} \times \frac{d^{k-1}}{dh^{k-1}} \left[\frac{F\left(\frac{n-m}{2}, \frac{n-m}{2} + \frac{1}{2}; n-m+1; \frac{4h}{(1+h)^2}\right)}{(1-h)(1+h)^{n-m}}\right]_{h=0}, \quad \text{for } n-m > 0 \quad (45)$$

where $F(a,\, b;\, c;\, x)$ is the hypergeometric function defined as [21], [22]

$$F(a,\, b;\, c;\, x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \tag{46}$$

where the hypergeometric coefficient $(a)_k$ is defined as the product

$$(a)_k = a(a+1)\cdots(a+k-1)$$

with $(a)_0 = 1$.

Applying a transformation formula for a hypergeometric function [22, eq. 9.134.2] to (45), we have for $n-m>0$

$$\lim_{z\longrightarrow 0^+} \int_z^{\infty} e^{-\gamma}\gamma^{n-m-1}\left[L_{k-1}^{n-m}(\gamma)\right]^2 d\gamma = \frac{(n-m+k-1)!}{(n-m)(k-1)!}, \qquad \text{for } n-m>0. \tag{47}$$

Substitution of (44) and (47) in (41) shows that, for $n-m>0$

$$\lim_{z\longrightarrow 0^+} F(z) = +\infty, \qquad \text{for } n-m>0. \tag{48}$$

Similarly, for $n-m=0$

$$\lim_{z\longrightarrow 0^+} F(z) = \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \lim_{z\longrightarrow 0^+}\left[\frac{1}{z} - E_1(z)\right] = +\infty, \qquad \text{for } n-m=0 \tag{49}$$

where E is Euler's constant and in the last step we have used the fact that

$$\lim_{z\longrightarrow 0} z\log(z) = 0.$$

Also, from (41), it is easily seen that

$$\lim_{z\longrightarrow +\infty} F(z) = -m\overline{\gamma}, \qquad \text{for } n-m\geq 0. \tag{50}$$

Thus, from (43) and (48)–(50) it follows that for $z>0$, the function $F(z)$ has a unique zero for all $n-m\geq 0$. From (17), then we see that for any $\overline{\gamma}>0$ there exists a unique cutoff value γ_0 for any $n-m\geq 0$ which satisfies (40), as we expected.

B. Evaluation of the Cutoff Value for Multiple-Antenna Systems

Substituting the polynomial representation

$$L_k^a(\lambda) = \sum_{p=0}^{k} (-1)^p \binom{k+a}{k-p} \frac{\lambda^p}{p!} \tag{51}$$

into (40) we obtain

$$\sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \binom{n-m+k-1}{k-1-p} \times \binom{n-m+k-1}{k-1-q} G_{p,\,q}(\mu) = m\overline{\gamma} \tag{52}$$

where we have defined the integral $G_{p,\,q}(\mu)$ to be

$$G_{p,\,q}(\mu) = \int_{\mu}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\gamma}\right) e^{-\gamma}\gamma^{n-m+p+q}\, d\gamma, \qquad \text{for } p+q = 0,\,1,\,\ldots,\,2(m-1). \tag{53}$$

Next, we consider the two cases of $n-m>0$ and $n-m=0$ separately in order to obtain an explicit solution to (52).

1) Case 1: $n-m>0$: Note that, when $n-m>0$, for $p+q = 0,\,\ldots,\,2(m-1)$, we have that $n-m+p+q-1\geq 0$ and $n-m+p+q\geq 1>0$. Then, we easily have that

$$G_{p,\,q}(z) = \frac{\Gamma(n-m+p+q+1,\mu)}{\mu} - \Gamma(n-m+p+q,\mu), \qquad \text{for } p+q = 0,1,\ldots,2(m-1) \text{ and } n-m>0 \tag{54}$$

where $\Gamma(a,\, x)$ is the complementary incomplete Gamma function and we have also made use of the integral identity

$$\int_{\mu}^{\infty} e^{-\gamma}\gamma^n\, d\gamma = n!e^{-\mu}\sum_{j=0}^{n}\frac{\mu^j}{j!}, \qquad \text{for } n\geq 0$$

which can be verified straightforwardly via repeated application of integration by parts.

Substituting (54) into (52) we obtain a closed-form equation that can be solved for a unique z (which is known to exist by the previous section), in general, via numerical root finding.

However, as we did in the single receiver antenna case, we may also obtain an approximate solution for the cutoff γ_0 by investigating small μ behavior of (52). In fact, following a similar procedure as in the case of a single receiver antenna, we may show that

$$\gamma_0 \approx \frac{B_1}{m + \frac{B_2}{\overline{\gamma}}}, \qquad \text{for } \overline{\gamma} \gg 1 \tag{55}$$

where we have defined the constants B_1 and B_2 to be the sums

$$B_1 = \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \times \binom{n-m+k-1}{k-1-p}\binom{n-m+k-1}{k-1-q} \times (n-m+p+q)!$$

and

$$B_2 = \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \times \binom{n-m+k-1}{k-1-p}\binom{n-m+k-1}{k-1-q} \times (n-m+p+q-1)!.$$

Note that for $n>1$ and $m=1$, $B_1=1$ and $B_2 = \frac{1}{n-1}$, and thus, (55) reduces to (18), as one would expect.

2) Case 2: $n-m=0$: When $n-m=0$, for $p+q = 0,\,\ldots,\,2(m-1)$, we still have that $n-m+p+q\geq 0$. However, in this case $n-m+p+q-1\geq -1$. For $n-m=0$, (52) reduces to

$$\sum_{k=1}^{m}\sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!}\binom{k-1}{k-1-p}\binom{k-1}{k-1-q} G_{p,q}(\mu) = m\overline{\gamma}$$

and, similarly, the integral $G_{p,\,q}(\mu)$ in (53) becomes

$$G_{p,\,q}(\mu) = \int_{\mu}^{\infty}\left(\frac{1}{\mu} - \frac{1}{\gamma}\right) e^{-\gamma}\gamma^{p+q}\, d\gamma, \qquad \text{for } p+q = 0,\,1,\,\ldots,\,2(m-1).$$

Then, we can easily show that

$$G_{p,\,q}(\mu) = \begin{cases} \frac{\Gamma(1,\,\mu)}{\mu} - E_1(\mu), & \text{if } p+q=0 \\ \frac{\Gamma(p+q+1,\,\mu)}{\mu} - \Gamma(p+q,\,\mu), & \text{if } p+q>0. \end{cases} \tag{56}$$

On substituting (56) into (52), again we may obtain a closed-form equation in μ that can be solved for a unique solution. It is also easily verified that this general equation reduces to the corresponding equation given in [14] for the case of $N_R = N_T = 1$.

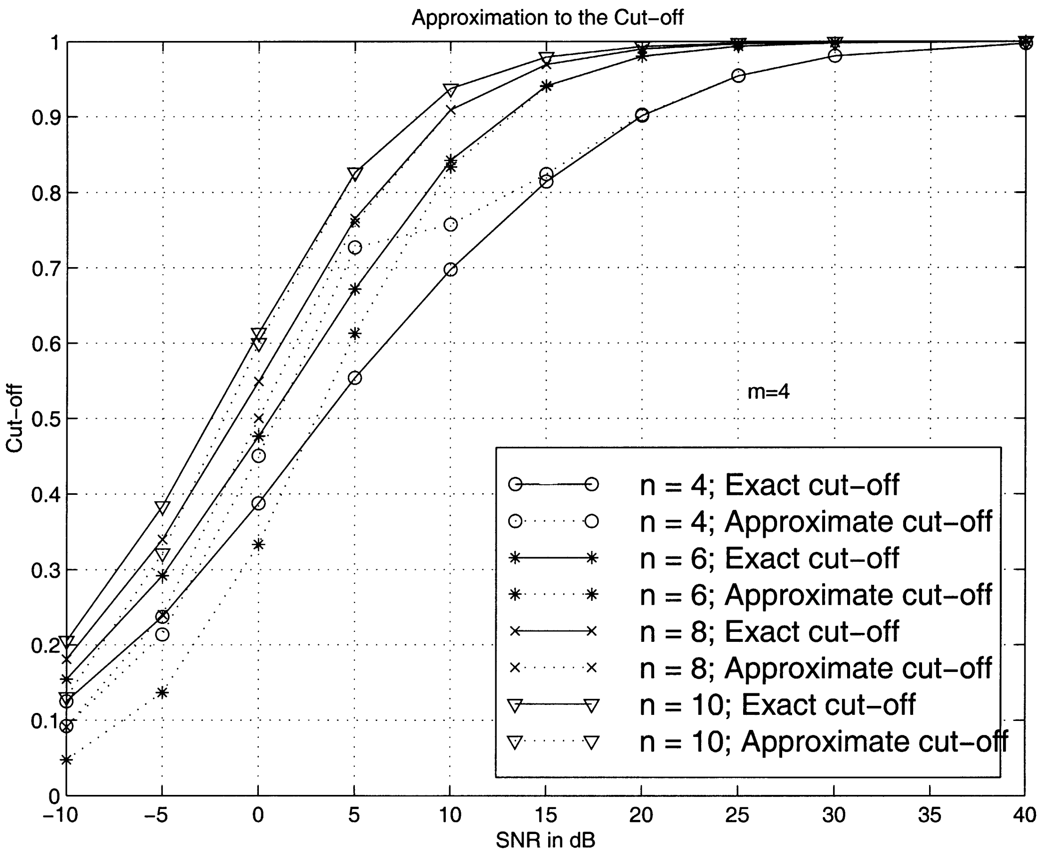

Fig. 4. Approximation to the optimal cutoff value versus SNR (in decibels). $m = 4$.

As we did earlier for the case of $n - m > 0$, we may obtain an approximate solution to the cutoff γ_0 that satisfies (52), which will eliminate the need to perform numerical root finding. However, due to the singularity of the exponential integral function $E_1(\mu)$ near zero, this becomes more involved than the previous case. Still, after some manipulations, we may show that a reasonable approximation for large $\overline{\gamma}$ is

$$\gamma_0 \approx \frac{ma_2 + \sqrt{m^2 a_2^2 - 4\overline{\gamma} a_1 a_3}}{2a_1} \tag{57}$$

where we have defined the constants a_1, a_2, and a_3 as

$$\begin{aligned} a_1 &= m^2 \left[1 - \frac{1 + D_3}{\overline{\gamma}}\right] \\ a_2 &= D_2 + m\left[\overline{\gamma} + \log(\overline{\gamma}) + 2 - E\right] \\ a_3 &= m + D_1 \end{aligned}$$

and where E is the Euler's constant and D_1, D_2, D_3 are the following sums:

$$\begin{aligned} D_1 = &\sum_{k=1}^{m} \sum_{p=0}^{k-1} \sum_{\substack{q=0 \\ p+q\neq 0}}^{k-1} \frac{(-1)^{p+q}}{p!q!} \binom{k-1}{k-1-p} \binom{k-1}{k-1-q} \\ &\times (p+q)!, \\ D_2 = &\sum_{k=1}^{m} \sum_{p=0}^{k-1} \sum_{\substack{q=0 \\ p+q\neq 0}}^{k-1} \frac{(-1)^{p+q}}{p!q!} \binom{k-1}{k-1-p} \binom{k-1}{k-1-q} \\ &\times (p+q-1)!, \\ D_3 = &\sum_{k=1}^{m} \sum_{p=0}^{k-1} \sum_{\substack{q=0 \\ p+q\neq 0}}^{k-1} \frac{(-1)^{p+q}}{p!q!} \binom{k-1}{k-1-p} \binom{k-1}{k-1-q} \\ &\times \frac{p+q}{2}(p+q-1)!. \end{aligned}$$

Fig. 4 shows the typical behavior of the cutoff value for a multiple-antenna system with $m = 4$, along with the cutoff approximations derived above. Note that, in the case of $n = m$, the cutoff approximation deviates considerably from the true cutoff for small values of $\overline{\gamma}$. However, as we will see in the next section, even these cutoff values will be effective in approximating the true capacity. It is clear that for all the other cases, derived approximations to the cutoff value do closely estimate the true cutoff value for reasonably high SNRs and large n. From Fig. 4, we may also observe that still γ_0 lies in the range $0 \leq \gamma_0 \leq 1$, and specifically $\gamma_0 \longrightarrow 1$ as $\overline{\gamma} \longrightarrow \infty$.

C. Evaluation of Capacity

Substituting (32) into (31), we obtain the capacity of the multiple-antenna system

$$C = m \int_{\gamma_0}^{\infty} \log\left(\frac{\gamma}{\gamma_0}\right) f_\gamma(\gamma)\, d\gamma, \tag{58}$$

where γ_0 is the cutoff transmission value corresponding to any unordered eigenvalue derived in the previous section, and $f_\gamma(\gamma)$ is the pdf of any scaled, unordered eigenvalue given in (39).

Using the explicit form of the pdf (39) and the representation of associated Laguerre polynomial given in (51), we can write (58) as

$$\begin{aligned} C = &\sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \\ &\times \binom{n-m+k-1}{k-1-p}\binom{n-m+k-1}{k-1-q} \mathcal{J}_{n-m+p+q+1}(\mu) \end{aligned} \tag{59}$$

where $\mathcal{J}_p(\mu)$, for $p = 1, 2, \ldots$, is the integral defined in (60). The integral $\mathcal{J}_p(\mu)$ can be evaluated in closed form and was given in [14] as

$$\mathcal{J}_p(\mu) = (p-1)!\left[E_1(\mu) + \sum_{j=1}^{p-1} \frac{\mathcal{P}_j(\mu)}{j}\right]. \tag{60}$$

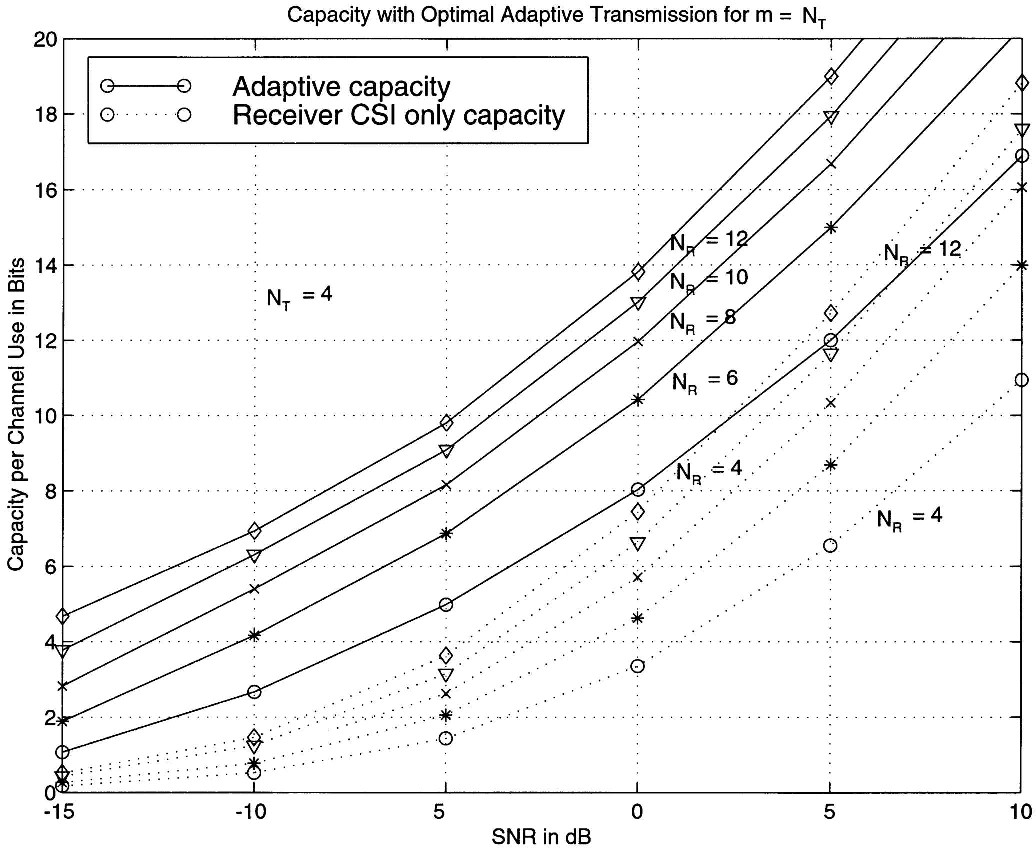

Fig. 5. Capacity of the multiple-antenna system with optimal adaptive transmission versus SNR (in decibels). $m = 4$ ($N_T = m$ in the receiver CSI only system).

Substituting (60) into (59) we obtain the capacity of multiple-antenna system

$$C = \log_2(e) \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1} \sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \times \binom{n-m+k-1}{k-1-p}\binom{n-m+k-1}{k-1-q}(n-m+p+q)! \times \left[E_1(z) + \sum_{j=1}^{n-m+p+q} \frac{\mathcal{P}_j(z)}{j}\right] \text{ bits/channel use.} \quad (61)$$

It is easy to verify that for $m = 1$, (61) reduces to (19) obtained previously for a system with multiple transmit antennas and a single receiver antenna, as required.

Fig. 5 plots the capacity of a multiple-antenna system for $m = 4$ with different values of n versus the SNR. Shown on the same figure is the capacity of the corresponding multiple-antenna system with only receiver CSI obtained in [12]. While the capacity of a multiple-antenna system with CSI at both transmitter and receiver is invariant to which end of the link has the larger number of antennas, this is not the case with only receiver CSI. Thus, Fig. 5 specifically corresponds to the case when the receiver CSI system has $N_T = m$ and $N_R = n$. Again, it is clear from Fig. 5 that large capacity improvements can be achieved with adaptive power and rate allocation when CSI is available at both ends of the system as compared to the case when only receiver CSI is available.

In Fig. 6, we have shown the capacity of the same system as that considered in Fig. 5, but this time comparing it with a receiver-CSI-only system with $N_T = n$ and $N_R = m$. In this case, the receiver-CSI-only system has a lower capacity than in the previous case thereby resulting in a larger capacity gap compared to the adaptive transmission system. However, the capacity of the adaptive transmission scheme is invariant under the swapping of the transmitter and receiver antennas and also is larger than either of the cases with only receiver CSI. Further, comparing these results with the capacity plots for $N_R = 1$ given earlier, we see that large capacity gains are available when multiple antennas are used at both ends of the communications link.

In Fig. 7, we have shown the capacity evaluated with both the exact cutoff value and the approximate cutoff value given by either (55) or (57) for a system with $m = 4$. This figure shows that the derived approximate cutoff values are indeed reasonable when the SNR is sufficiently large. Moreover, they confirm the earlier remark that although the cutoff estimate given in (57) deviates from the true cutoff more than that of (55), the approximation (57) nevertheless results in a reasonable capacity estimate. Fig. 7 shows that the capacity computed with approximate cutoff values tend to get closer and closer to the exact capacity either as SNR becomes large or the maximum of the number of antennas n grows.

Finally, Fig. 8 plots the capacity versus the minimum number of antennas m at one of the ends of the system against a fixed but large maximum number of antennas n at the other end. Fig. 8 corresponds to $n = 18$. As observed in the case when CSI is available only at the receiver, studied in [10] and [12], from Fig. 8 we see that again the capacity is almost linear in the minimum number of antennas m. In Fig. 8, we have also included the capacity approximations computed with the estimated cutoff values. Note that these capacity approximations are in good agreement with the exact capacities for the values of SNR and number of antennas considered.

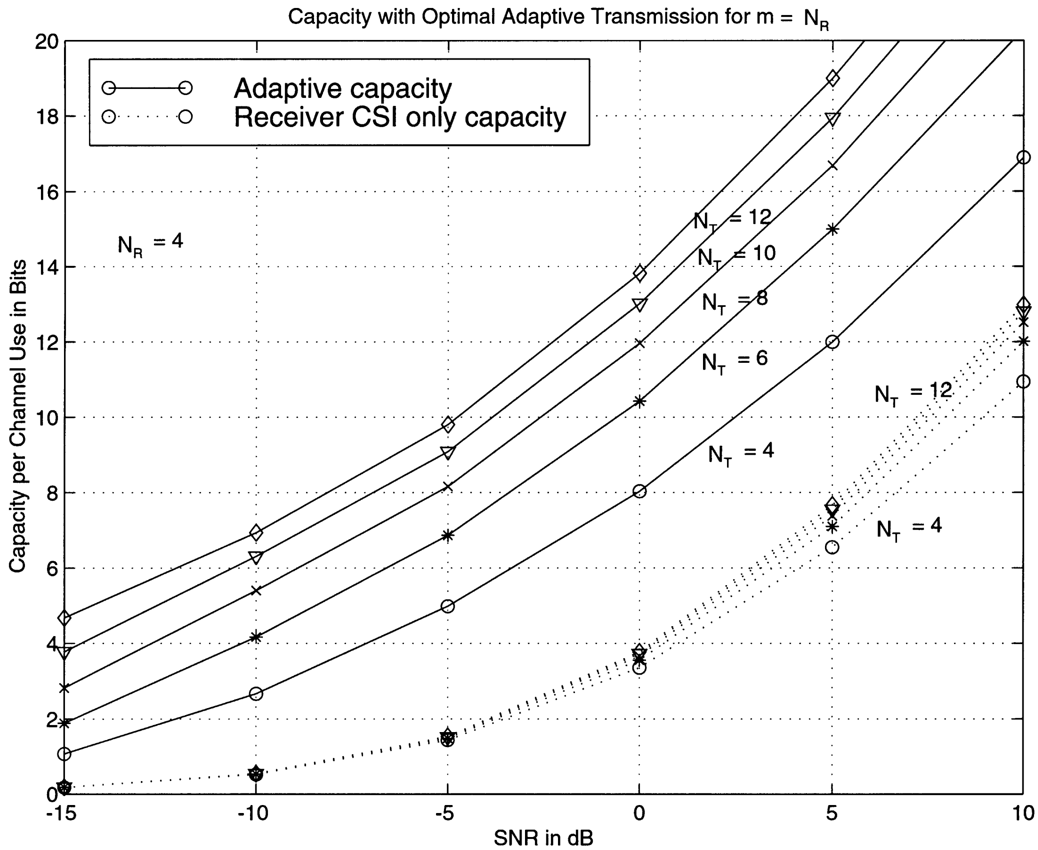

Fig. 6. Capacity of the multiple-antenna system with optimal adaptive transmission versus SNR (in decibels). $m = 4$ ($N_R = m$ in the receiver CSI only system).

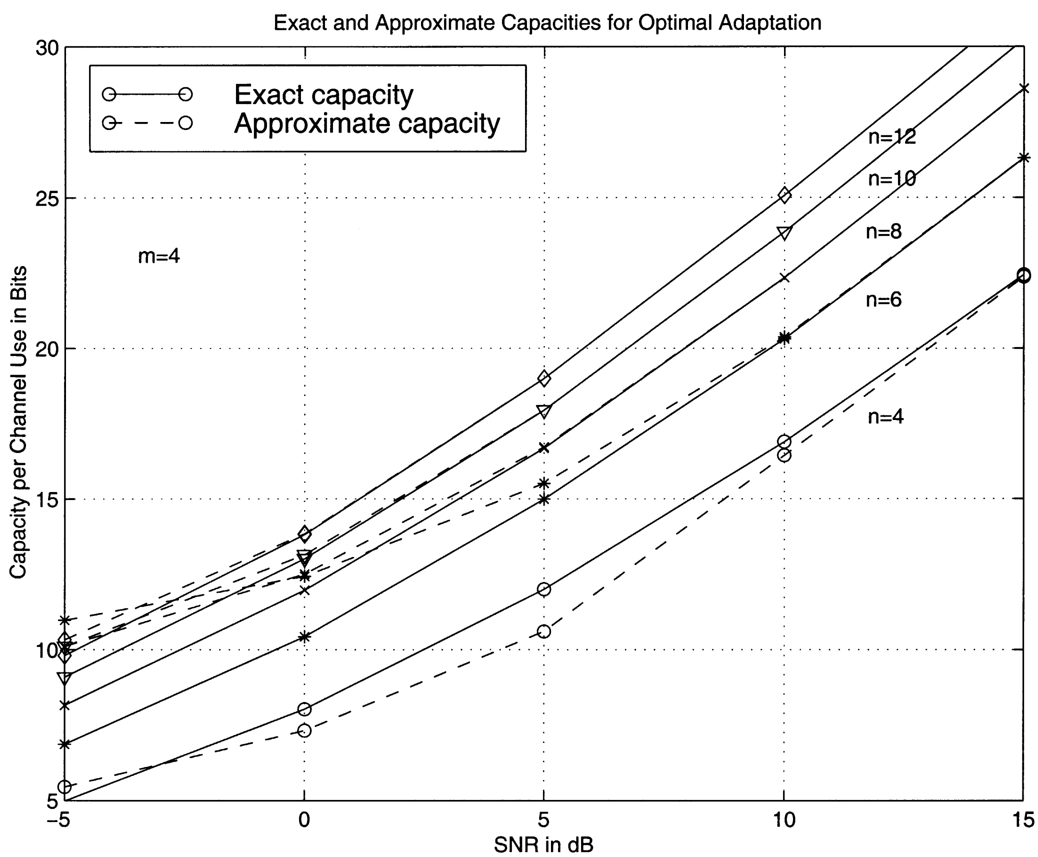

Fig. 7. Multiple-antenna system capacity evaluated with both exact and approximate cutoff values versus SNR (in decibels). $m = 4$.

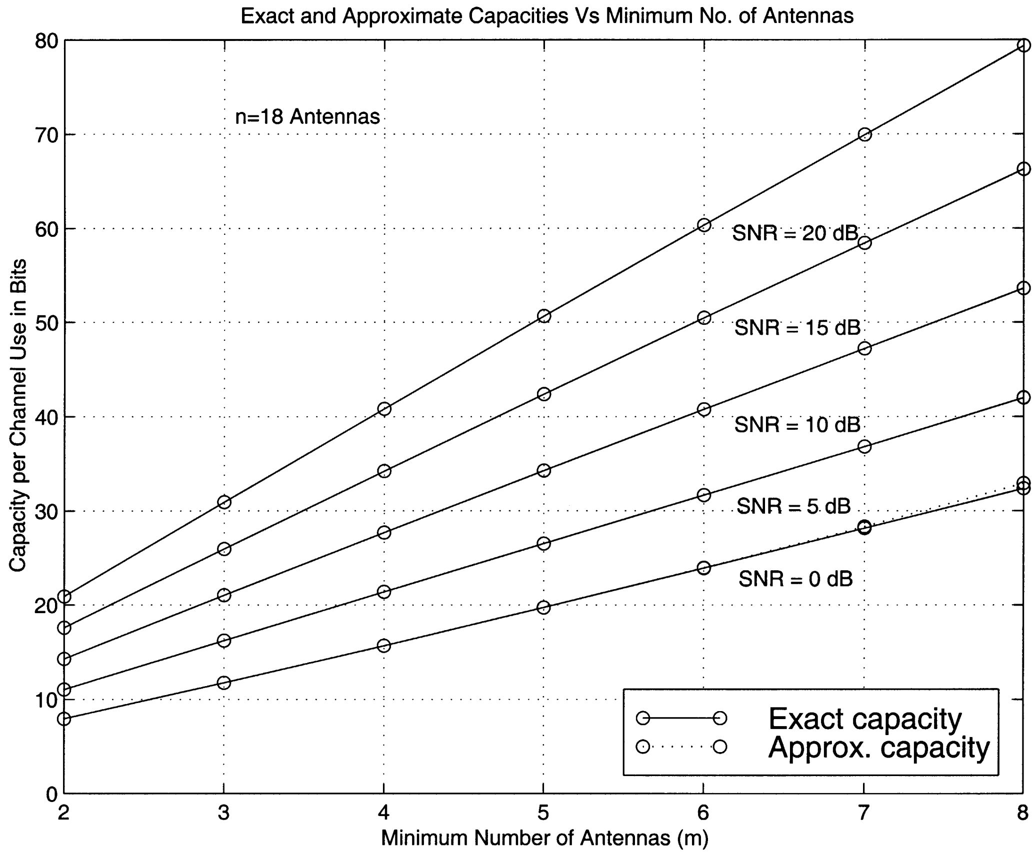

Fig. 8. Multiple-antenna system capacity evaluated with both exact and approximate cutoff values versus minimum number of antennas (m). $n = 18$.

D. Outage Probability

As remarked earlier, the large capacity gains possible with the adaptive transmission scheme derived above, compared to the capacity of a system with CSI available only at the receiver, come at the price of channel outage. This is so because the optimal adaptive power and rate allocation scheme would not be transmitting at all if all the observed γ_i's were less than the cutoff value γ_0, thus resulting in channel outage. Hence, in order to put the extraordinary capacity gains offered by the power and rate adaptation schemes in perspective, it is necessary to take into account the associated outage probability values. In what follows, we provide a simple upper bound for this outage.

We denote the largest eigenvalue of the Wishart distributed matrix $\boldsymbol{W}$ as $\lambda_{\max}$ and the outage probability of a multiple antenna system by $P_{\text{out}}^{n,m}$. Then, it is easily seen that

$$P_{\text{out}}^{n,m} = \int_0^{\mu} f_{\lambda_{\max}}(\lambda)\, d\lambda \tag{62}$$

where $f_{\lambda_{\max}}(\lambda)$ is the pdf of the largest eigenvalue of the Wishart matrix $\boldsymbol{W}$. An upper bound for this pdf in the case of real Gaussian random variables was derived in [23] for the case of $m = n$. Following [24], we may generalize this upper bound for any m and n and show that in the case of complex Wishart matrices

$$f_{\lambda_{\max}}(\lambda) \le \frac{1}{\Gamma(n)\Gamma(m)} \lambda^{n+m-2} e^{-\lambda}. \tag{63}$$

From (63) and (62), we have the following upper bound for the outage probability of the multiple-antenna system:

$$\begin{aligned} P_{\text{out}}^{n,m} &\le \frac{1}{\Gamma(n)\Gamma(m)} \left[\Gamma(n+m-1) - \Gamma(n+m-1,\, \mu)\right] \\ &\equiv p_1. \end{aligned} \tag{64}$$

Note that, for $m = 1$, this upper bound for the outage probability in fact gives the exact value of the outage. This is clear by observing that for $m = 1$ the right-hand side of (63) reduces to the exact pdf for this case, given by (15).

Fig. 9 plots this upper bound for the outage probability as a function of SNR for $m = 2$. As one would expect, the outage probability decreases with increasing SNR values. Also, it is clear from this plot that the outage probability bound decreases rapidly when the maximum number of antennas n increases for a fixed m.

Unfortunately, though, the above bound becomes very loose when the SNR is low and the maximum number of antennas are large. Especially, in some of these cases the right-hand side of (63) may become larger than unity rendering it completely useless. In order to circumvent this shortcoming, we may derive another bound which is always less than or equal to unity. Note that this bound is valid only for the case of $m = n$.

In order to derive this bound, we denote the smallest eigenvalue of the Wishart matrix $\boldsymbol{W}$ by $\lambda_{\min}$. It is shown in [24] that when $m = n$ the pdf $f_{\lambda_{\min}}(\lambda)$ of $\lambda_{\min}$ is given by $f_{\lambda_{\min}}(\lambda) = me^{-m\lambda}$. Since, $P_{\text{out}}^{m,m} \le \mathrm{P}(\lambda_{\min} < \mu)$, we have that

$$\begin{aligned} P_{\text{out}}^{m,m} &\le 1 - e^{-m\mu} \\ &\equiv p_2. \end{aligned} \tag{65}$$

Combining (64) and (65) we have that for $m = n$

$$P_{\text{out}}^{m,m} \le \min\{p_1,\, p_2\}. \tag{66}$$

Fig. 10 shows this upper bound for $m = n$ multiple-antenna system outage probability versus the SNR for different values of m. The conclusion one can draw by observing these plots is that employing multiple antennas at both ends of the communication link and adapting power and rate not only provides large capacity gains but also helps in decreasing the outage probability considerably.

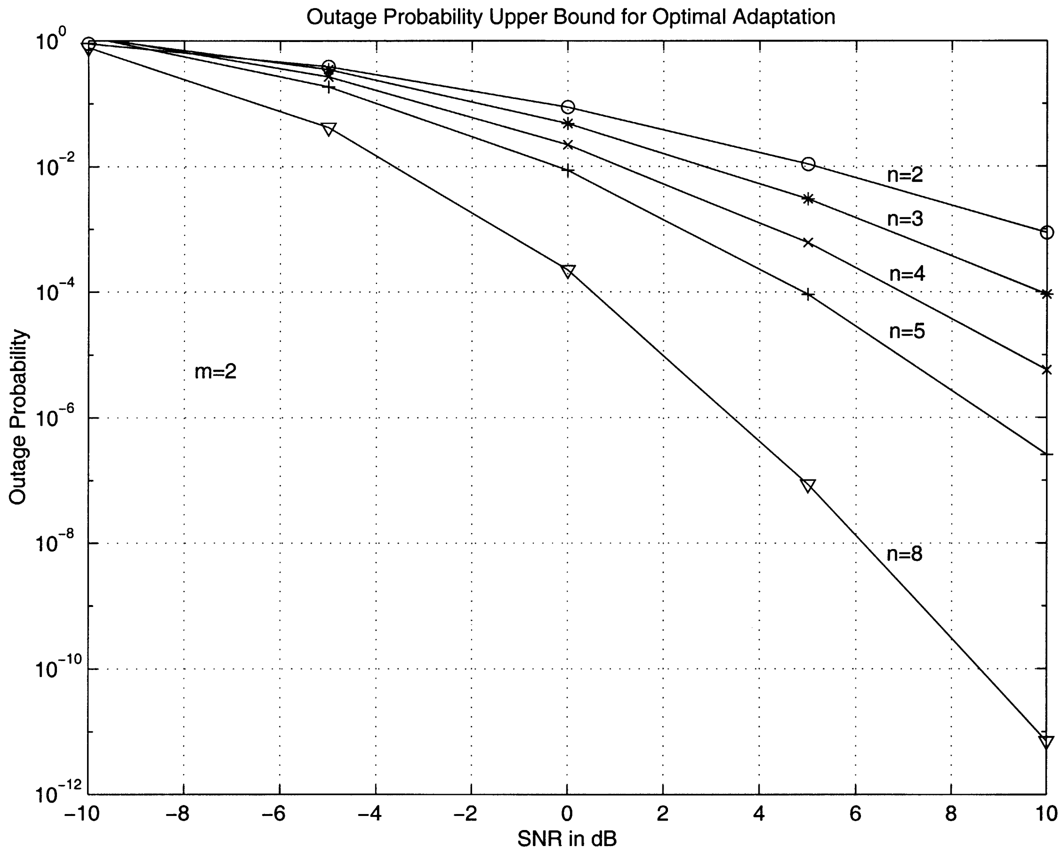

Fig. 9. Upper bound for outage probability of a multiple-antenna system versus SNR. $m = 2$.

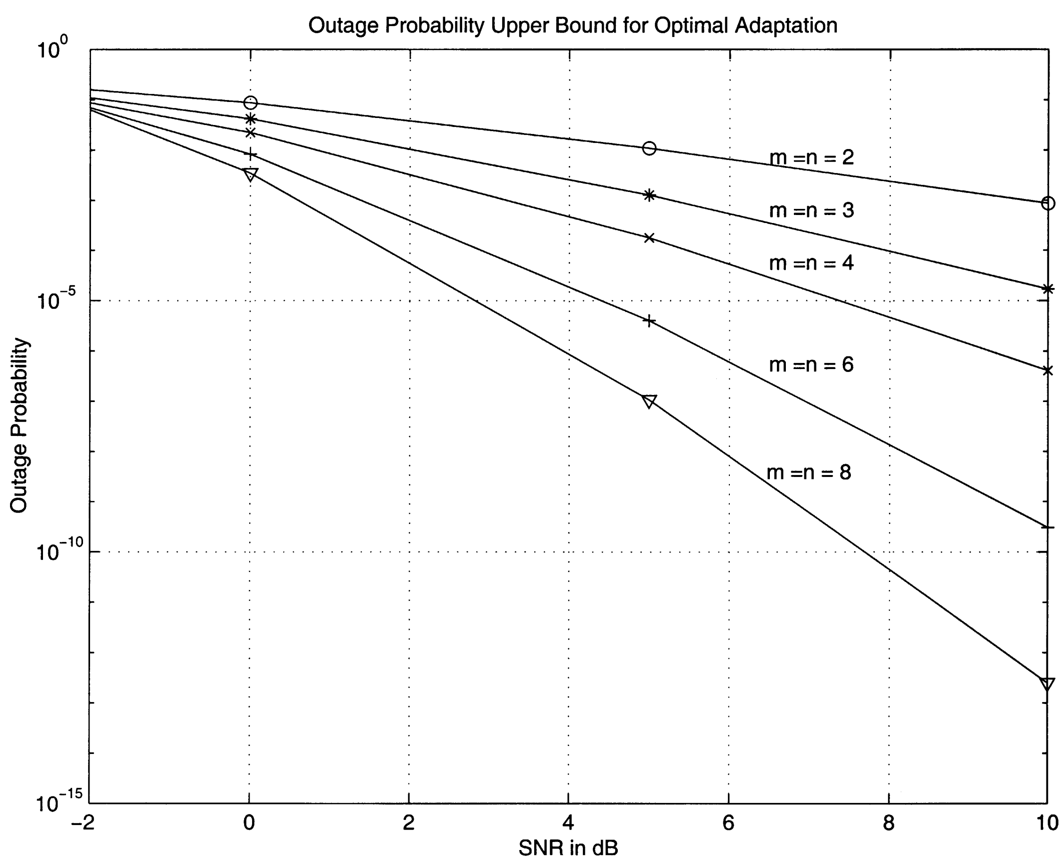

Fig. 10. Upper bound for outage probability of a multiple-antenna system versus SNR. $m = n$.

V. Conclusion

We have considered the capacity of multiple-antenna systems in Rayleigh flat fading under the assumption that CSI is available at both ends of the system. First, we derived the capacity of such systems in the case when only the transmitter is equipped with multiple antennas. We showed that the capacity of this system is, in fact, the same as a receiver-only diversity system with maximal ratio combining. We also proposed a transmission diversity scheme (maximal gain transmission) that is mathematically equivalent to a receiver-only diversity system with selection combining and evaluated its capacity.

Next, we derived capacity expressions for a general system with multiple antennas at both transmitter and receiver. We showed that the optimal power allocation is given by a matrix water-filling algorithm. We obtained an equation that determines the cutoff value for such systems, which can be evaluated via numerical root-finding, and a corresponding closed-form expression for the capacity with optimal power and rate adaptation. We evaluated this capacity for some representative situations and demonstrated similarities with the capacity of such systems when CSI is available only at the receiver end.

In all these cases, the only step that required numerical techniques in determining the capacity is the evaluation of the cutoff value γ_0. In order to circumvent this problem, we also derived approximations to the cutoff value for all cases considered. Numerical results show that these approximations yield good capacity estimates when the SNR or the number of antennas is sufficiently large.

From these capacity computations for multiple-antenna systems with adaptive transmission techniques we observe that large capacity gains are possible compared to the receiver-CSI-only systems. The tradeoff for these increased capacity values is the outage probability incurred by the adaptive power and rate allocation schemes. We derived simple upper bounds for this outage probability and showed that the channel outage probability may also be decreased by increasing the number of antennas.

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On the Separability of Demodulation and Decoding for Communications Over Multiple-Antenna Block-Fading Channels

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***Abstract*—We study the separability of demodulation and decoding for communications over multiple-antenna block-fading channels when bit-linear linear dispersion (BL-LD) codes are used. We assume the channel is known to the receiver only, and find necessary and sufficient conditions on the dispersion matrices for the separation of demodulation and decoding at the receiver without loss of optimality.**

***Index Terms*—Multiple-antenna systems, space–time codes.**

I. Introduction

We consider a multiple-input–multiple-output (MIMO) communication setup with t transmit antennas and r receive antennas. Information-theoretic results by Foschini and Gans [1] and Telatar [2] have sparked tremendous interest and effort in the design of practical channel codes for communications over multiple-antenna channels (i.e., MIMO chan-

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