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AMERICAN LOGIC IN THE 1920s

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In 1934 Alonzo Church, Kurt Gödel, S. C. Kleene, and J. B. Rosser were all to be found in Princeton, New Jersey. In 1936 Church founded THE JOURNAL OF SYMBOLIC LOGIC. Shortly thereafter Alan Turing arrived for a two year visit. The United States had become a world center for cutting-edge research in mathematical logic. In this brief survey[1] we shall examine some of the writings of American logicians during the 1920s, a period of important beginnings and remarkable insights as well as of confused gropings.

The publication of Whitehead and Russell's monumental *Principia Mathematica* [18] during the years 1910–1913 provided the basis for much of the research that was to follow. It also provided the basis for confusion that remained a factor during the period we are discussing. In 1908, Henri Poincaré, a famous skeptic where mathematical logic was concerned, wrote pointedly ([13]):

> It is difficult to admit that the word *if* acquires, when written $\supset$, a virtue it did not possess when written *if*.

Principia provided no very convincing answer to Poincaré. Indeed the fact that the authors of *Principia* saw fit to place their first two "primitive propositions"

***1.1:** Anything implied by a true proposition is true.
***1.2:** $\vdash p \vee p \supset p$

under one and the same heading suggest that they had thought of what they were doing as just such a translation as Poincaré had derided. A radically different view was implied by the American logician Clarence Lewis in his explanation of ***1.1** ([12]):

> The main thing to be noted about this operation of inference is that it is not so much a piece of reasoning as a mechanical, or strictly mathematical, operation for which a rule has been given. No "mental" operation is involved except that required to recognize a

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[1] This paper was written for a Festschrift volume honoring Alonzo Church and is based on a talk given by the author at a symposium honoring Alonzo Church in 1990 at the State University of New York in Buffalo.

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> previous proposition followed by the main implication sign, and to set off what follows that sign as a new assertion.

In the same treatise Lewis put his finger on the source of the confusion:

> . . . the system takes the laws of the logical relations of propositions for granted in order to prove them.

The decade of the 1920s began with Emil L. Post's seminal doctoral dissertation[2]. Post made Lewis's influence on his thought quite explicit with numerous references to [12]. In his paper, Post studied the propositional calculus[3] of *Principia Mathematica* as a purely "formal" system making a clear distinction between the theorems of the system and theorems about the system.

> We here wish to emphasize that the *theorems* of this paper are *about* the logic of propositions but are *not included* therein. [italics original]

Post introduced the use of "truth tables" as a tool for proving theorems "*about* the logic of propositions." He used the symbols $+, -$ for the truth values and proved that the formulas provable in the *Principia Mathematica* axiomatization are precisely those whose truth tables consist of a column of $+$'s. Post noted that his result showed that the *Principia* propositional calculus is complete and consistent and that it provided an algorithmic procedure for determining whether a given formula was derivable in that system.[4] In Post's proof, he made systematic use of what has come to be called *structural induction*; it is important to realize that since Whitehead and Russell thought of their development as providing a *foundation* for arithmetic, and hence for mathematical induction, using induction to develop their system would have seemed hopelessly circular. In proving his main result, Post observed that the *Principia* rules enabled one to prove for each formula α a formula of the form $\alpha \equiv \beta$ where β is in what is now called *disjunctive normal form* and is obtainable from α by straightforward algebraic operations. Next, for each such β containing at least two propositional variables, one of its variables p could be "factored" yielding a provably equivalent formula of the form:

$$(\delta_1 \wedge p \wedge \neg p) \vee (\delta_2 \wedge p) \vee (\delta_3 \wedge \neg p)$$

where $\delta_1, \delta_2, \delta_3$ do not contain the variable p. The result then follows easily by induction.

Post not only gave a complete analysis of this part of the Whitehead-Russell system, but also explored a number of important generalizations: The two-valued *truth tables* that Post introduced were generalized to tables

[2] [14].

[3] Post did not use this term.

[4] In [1], Paul Bernays stated that he had obtained results similar to Post in his Göttingen *Habilitationsschrift* of 1918.

involving an arbitrary finite number of variables. The rules of inference of the propositional calculus were generalized to rules determined by *productions* of a form so general that (viewed from a later perspective, of course) all recursively enumerable sets of strings were encompassed among the sets of "theorems" of the "logics" that could be defined using such rules. Finally, the $\sim, \vee$ system of *Principia* was generalized to arbitrary systems of truth functions that are complete in the sense that every truth function can be obtained by their suitable iteration; in this last case, only the main result is given.[5]

In Alonzo Church's dissertation [2] of 1926, Church proposed to do for set theory (including Cantor's theory of transfinite numbers) what the creators of non-Euclidean geometry had done for geometry. Lobachevsky had explicitly considered the parallel postulate of Euclid as a mere assumption the consequences of whose denial could be studied, rather than as an immutable a priori property of space. So Church proposed to consider Zermelo's axiom of choice as likewise a mere assumption the consequences of whose denial could be studied. Indeed, Church proposed a number of different propositions each of which contradicted the axiom of choice as possible "alternatives" to Zermelo's assumption. Although Church's dissertation was written in the style of unformalized axiomatics, he was clearly acutely aware of set theory together with logic as a foundation for mathematics. Indeed, he presciently commented:

> It is not improbable that the set of postulates at the basis of our logic is not complete, even if the axiom of choice is included in the set, . . .

Church made his line of thought clear by explaining that he believed this to be true

> . . . because if it were complete it ought to be possible, in [each] case to construct a particular function F of the kind whose existence is required by the axiom of choice.

Noteworthy contributions to logic and foundations of mathematics were few and far between during the twenties. Among these, we may turn to C. H. Langford. In his [8], [9] Langford developed what has come to be known as *the method of elimination of quantifiers* to solve the decision problem for the first order theory of dense linear orders. However, despite this very important technical contribution, Langford remained badly confused.

For the authors of *Principia,* the logic they were developing and studying was the one true logic. Despite Post's emphasis on using mathematical methods to investigate systems of logics as formal combinatorial objects and Church's willingness to explore variant systems of logic, this attitude

[5]The proof was published only two decades later [15]. For further discussion of Post's dissertation and related matters see [6].

persisted and is perhaps seen most sharply in Langford's writings. In 1928 Church wrote:[6]

> The purpose of this paper is to discuss the possibility of a system of logic in which the law of the excluded middle is not assumed . . .

In this article, Church argued strongly for an attitude towards the principles of logic that is open to the possibility of studying variant systems. To this, Langford ([10]) hotly retorted:

> This means that we cannot have alternative logics; for logic is the system of all propositions expressing analytic facts of a certain kind, namely formal analytic facts, and there cannot in the nature of the case be more than one such system, actual or conceivable.

In Europe the year 1928 saw the publication of a little textbook of logic [7] written from the perspective of Hilbert's school that proved to be extremely influential. In particular the propositional calculus and first order logic were developed as formal combinatorial systems with properties that could be investigated by normal mathematical methods. The problem of the completeness of first order logic was presented and provided the young Kurt Gödel with his dissertation topic. The *Entscheidungsproblem,* the decision problem for first order logic, was characterized as *the* fundamental problem of mathematical logic; the fact that this problem is unsolvable was to be proved by Church and by Turing (independently of one another) eight years later. C. H. Langford [11] reviewed this book for the *Bulletin of the American Mathematical Society.* His review not only reflected his commitment to the one true logic, but indeed showed that he quite refused to permit himself to begin to understand what the authors were about.

Hilbert and Ackermann began with a formal treatment of the propositional calculus very much after the manner of Post's dissertation. They used four of the five formulas which the *Principia* development used as "primitive propositions."[7] These were

$$X \vee X \to X$$
$$X \to X \vee Y$$
$$X \vee Y \to Y \vee X$$
$$(X \to Y) \to [Z \vee X \to Z \vee Y]$$

Langford translated these into *Principia* notation, displaying them as

(a): $(p) : p \vee p \cdot \supset \cdot p,$
(b): $(p, q) : p \cdot \supset \cdot p \vee q,$
(c): $(p, q) : p \vee q \cdot \supset \cdot q \vee p,$
(d): $(p, q, r) : \cdot p \supset q \cdot \supset : \ r \vee p \cdot \supset \cdot r \vee q,$

[6] in [3]

[7] Paul Bernays had shown that the fifth was superfluous, being derivable from the remaining four.

and observed that they were presented "together with a rule of substitution and a rule of inference.[8]" The parenthetical expressions are (in *Principia* notation), universal quantifiers; Langford uses them to emphasize his conception of these formulas as representing "analytic formal" truths holding with arbitrary propositions substituted for the variables. Langford is dreadfully puzzled over the authors' proofs that these formulas regarded as axioms are "consistent, independent, and complete."

> All we have to do ... in order to see that propositions **(a)** – **(d)**, together with the two rules are consistent, is simply to observe that they are all plainly true.

Langford's comments on the proofs of independence are even worse:

> There is no occasion for examining this argument directly; it is quite certainly mistaken, for the sufficient reason that ... **(c)** follows from **(a)**, **(b)**, and **(d)**.

For, as Langford notes, the authors have shown that $p \supset p$ does follow from **(a)**, **(b)**, and **(d)**; insisting that there is no distinction to be made between $p \vee q$ and $q \vee p$ "since only one meaning occurs, namely, 'At least one of the two propositions is true'," Langford asserts that **(c)** is then just an instance of $p \supset p$. Completeness also falls by the wayside since "such true propositions as $\sim (p, q) \cdot p \vee q$ ... appear to be not derivable by the rules given."

It is a great relief to turn from Langford's nonsense to Haskell Curry's work. Curry, studying with Hilbert, turned to a fundamental analysis of the operation of substitution ([4]) previously taken to be primitive. This was the beginning of Curry's work in combinatory logic. Working from an entirely different direction, Church was led in the early 1930s to the λ-calculus, and eventually to the famous "thesis" that bears his name.[9] It was Church's students Kleene and Rosser who were to show that the λ-calculus and combinatory logic were fully equivalent.

Studies like this one can have the effect of bolstering a sense of our superiority to our logical forbears. But this would be a serious mistake. The lessons to be learned are rather that the development of the outlook on our subject that today we take for granted was attained only with great difficulty.

REFERENCES

[1] Paul Bernays, *Axiomatische Untersuchung des Aussagen-Kalküls der 'Principia Mathematica'*, ***Mathematische Zeitschrift***, vol. 25 (1926), pp. 305–320.

[8] Of course this last is just modus ponens.

[9] The path from the first systems Church introduced containing the λ operator (systems that proved to be inconsistent) to the notion of λ-definability and thus to the beginning of what we know today as computability or recursion theory is another story and a very interesting one, [5].

[2] Alonzo Church, *Alternatives to Zermelo's assumption*, ***Transactions of the American Mathematical Society***, vol. 29 (1926), pp. 178–208.

[3] ———, *On the law of the excluded middle*, ***Bulletin of the American Mathematical Society***, vol. 34 (1928), pp. 75–78.

[4] Haskell Curry, *An analysis of logical substitution*, ***American Journal of Mathematics***, vol. 51 (1929), pp. 363–384.

[5] Martin Davis, *Why Gödel didn't have Church's Thesis*, ***Information and Control***, vol. 54 (1982), pp. 3–24.

[6] ———, *Emil L. Post: His life and work*, In *Solvability, Provability, Definability: The Collected Works of Emil L. Post* [16], edited by Martin Davis, pp. xi–xviii.

[7] D. Hilbert and W. Ackermann, ***Grundzüge der theoretischen Logik***, Julius Springer, Berlin, 1928.

[8] C. H. Langford, *Some theorems on deducibility*, ***Annals of Mathematics, second series***, vol. 28 (1927), pp. 16–40.

[9] ———, *Theorems on deducibility (second paper)*, ***Annals of Mathematics, second series***, vol. 28 (1927), pp. 459–471.

[10] ———, *Concerning logical principles*, ***Bulletin of the American Mathematical Society***, vol. 34 (1928), pp. 573–582.

[11] ———, *Review of* [7], ***Bulletin of the American Mathematical Society***, vol. 36 (1930).

[12] C. I. Lewis, ***A survey of symbolic logic***, University of California Press, Berkeley, 1918.

[13] Henri Poincaré, ***Science and method***, Dover, New York, 1952, translated by F. Maitland from the original French edition of 1908.

[14] Emil L. Post, *Introduction to a general theory of propositional functions*, ***American Journal of Mathematics***, vol. 43 (1921), pp. 163–185, reprinted in [17], pages 264–283 and in [16], pages 21–43.

[15] ———, ***The two-valued iterative systems of mathematical logic***, Annals of Mathematics Studies, no. 5, Princeton University Press, 1941, reprinted in [16], pages 249–374.

[16] ———, ***Solvability, provability, definability: The collected works of Emil L. Post***, Birkhäuser, Boston, 1994, edited by Martin Davis.

[17] Jean van Heijenoort (editor), ***From Frege to Gödel***, Harvard University Press, 1967.

[18] Alfred Whitehead and Bertrand Russell, ***Principia mathematica***, vol. 1 (1910), 2 (1912), 3 (1913), Cambridge University Press.

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