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Research Article

Toeplitz Operators on Abstract Hardy Spaces Built upon Banach Function Spaces

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Let X be a Banach function space over the unit circle $\mathbb{T}$ and let $H[X]$ be the abstract Hardy space built upon X. If the Riesz projection P is bounded on X and $a \in L^\infty$, then the Toeplitz operator $T_a f = P(af)$ is bounded on $H[X]$. We extend well-known results by Brown and Halmos for $X = L^2$ and show that, under certain assumptions on the space X, the Toeplitz operator T_a is bounded (resp., compact) if and only if $a \in L^\infty$ (resp., $a = 0$). Moreover, $\|a\|_{L^\infty} \le \|T_a\|_{\mathscr{B}(H[X])} \le \|P\|_{\mathscr{B}(X)}\|a\|_{L^\infty}$. These results are specified to the cases of abstract Hardy spaces built upon Lebesgue spaces with Muckenhoupt weights and Nakano spaces with radial oscillating weights.

1. Introduction

The Banach algebra of all bounded linear operators on a Banach space E will be denoted by $\mathscr{B}(E)$. Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. For $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, a function of the form $p(t) = \sum_{k=-n}^{n} \alpha_k t^k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{-n, \dots, n\}$ and $t \in \mathbb{T}$, is called a trigonometric polynomial of order n. The set of all trigonometric polynomials is denoted by $\mathscr{P}$. The Riesz projection is the operator P which is defined on $\mathscr{P}$ by

$$P\colon \sum_{k=-n}^{n} \alpha_k t^k \longmapsto \sum_{k=0}^{n} \alpha_k t^k. \tag{1}$$

For $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T}$ in the complex plane. For $f \in L^1$, let

$$\widehat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\varphi}\right) e^{-in\varphi} d\varphi, \quad n \in \mathbb{Z}, \tag{2}$$

be the sequence of the Fourier coefficients of f. The classical Hardy spaces H^p are given by

$$H^p := \left\{ f \in L^p \colon \widehat{f}(n) = 0 \ \forall n < 0 \right\}. \tag{3}$$

It is well known that the Riesz projection extends to a bounded linear operator on L^p if and only if $1 < p < \infty$. For $a \in L^\infty$, the Toeplitz operator T_a with symbol a on H^p, $1 < p < \infty$, is given by

$$T_a f = P(af). \tag{4}$$

Toeplitz operators have attracted the mathematical community for the many decades since the classical paper by Toeplitz [1]. Brown and Halmos [2, Theorem 4] proved that a necessary and sufficient condition that an operator on H^2 is a Toeplitz operator is that its matrix with respect to the standard basis of H^2 is a Toeplitz matrix, that is, the matrix of the form $(a_{k-j})_{j,k\in\mathbb{Z}_+}$. The norm of T_a on the Hardy space H^2 coincides with the norm of its symbol in L^∞ (actually, this result was already in a footnote of [1]). Brown and Halmos also observed, as a corollary, that the only compact Toeplitz operator on H^2 is the zero operator. We here mention [3, Part B, Theorem 4.1.4] and [4, Theorem 1.8] for the proof of the Brown-Halmos theorem. An analogue of this result is true for Toeplitz operators acting on H^p, $1 < p < \infty$ [5, Theorem 2.7].

In this paper, we will consider the so-called Banach function spaces X in place of L^p. As usual, we equip the unit circle $\mathbb{T}$ with the normalized Lebesgue measure $dm(\tau) =$

$|d\tau|/(2\pi)$. Denote by L^0 the set of all measurable complex-valued functions on $\mathbb{T}$, and let L^0_+ be the subset of functions in L^0 whose values lie in $[0,\infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by $\mathbb{I}_E$. A mapping $\rho : L^0_+ \to [0,\infty]$ is called a function norm if, for all functions f, g, f_n ($n \in \mathbb{N}$) in L^0_+, for all constants $c \geq 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

(a) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(cf) = c\rho(f)$, $\rho(f+g) \leq \rho(f) + \rho(g)$,

(b) $0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),

(c) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(d) $\rho(\mathbb{I}_E) < \infty$, $\int_E f(\tau)dm(\tau) \leq C_E \rho(f)$,

with $C_E \in (0,\infty)$ depending on E and ρ but independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is a Banach space under the norm $\|f\|_X := \rho(|f|)$. Such a space X is called a Banach function space. If ρ is a function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(\tau)\, g(\tau)\, dm(\tau) : f \in L^0_+,\ \rho(f) \leq 1\right\}, \quad g \in L^0_+. \tag{5}$$

The Banach function space X' determined by the function norm ρ' is called the associate space (or Köthe dual space) of X. The associate space X' is a subspace of the dual space X^*. The simplest examples of Banach function spaces are the Lebesgue spaces L^p, $1 \leq p \leq \infty$. The class of all Banach function spaces includes all Orlicz spaces, as well as all rearrangement-invariant Banach function spaces (see, e.g., [6, Chap. 3]). We are mainly interested in non-rearrangement-invariant Banach function spaces. Two typical examples of non-rearrangement-invariant Banach function spaces are weighted Lebesgue space and weighted Nakano spaces (weighted variable Lebesgue spaces) considered in the last section of the paper.

Following [7, p. 877], we will consider abstract Hardy spaces $H[X]$ built upon a Banach function space X over the unit circle $\mathbb{T}$ as follows:

$$H[X] := \left\{f \in X : \hat{f}(n) = 0\ \forall n < 0\right\}. \tag{6}$$

This definition makes sense because X is continuously embedded in L^1 in view of axiom (d). It can be shown that $H[X]$ is a closed subspace of X. It is clear that if $1 \leq p \leq \infty$, then $H[L^p]$ is the classical Hardy space H^p.

It follows from axiom (d) that $\mathscr{P} \subset L^\infty \subset X$. We will restrict ourselves to Banach function spaces X such that the Riesz projection defined initially on $\mathscr{P}$ by formula (1) extends to a bounded linear operator on the whole space X. The extension will again be denoted by P. If $a \in L^\infty$ and $P \in \mathscr{B}(X)$, then the Toeplitz operator defined by formula (4) is bounded on $H[X]$ and

$$\|T_a\|_{\mathscr{B}(H[X])} \leq \|P\|_{\mathscr{B}(X)} \|a\|_{L^\infty}. \tag{7}$$

The Brown-Halmos theorem [2, Theorem 4] was extended by the author [8, Theorem 4.5] to the case of reflexive rearrangement-invariant Banach function spaces with nontrivial Boyd indices. Note that the nontriviality of the Boyd indices implies the boundedness of the Riesz projection.

The first aim of this paper is to show that the Brown-Halmos theorem remains true for abstract Hardy spaces $H[X]$ built upon reflexive Banach function spaces X (not necessarily rearrangement-invariant) if $P \in \mathscr{B}(X)$. Further, we show that, under mild assumptions on a Banach function space X, a Toeplitz operator T_a is compact on the abstract Hardy space $H[X]$ built upon X if and only if $a = 0$. These results are specified to the case of Hardy spaces built upon Lebesgue spaces with Muckenhoupt weights and upon Nakano spaces with certain radial oscillating weights. Both classes of spaces in our examples are not rearrangement-invariant.

For $f \in X$ and $g \in X'$, we will use the following pairing:

$$\langle f, g\rangle := \int_{\mathbb{T}} f(\tau)\, \overline{g(\tau)}\, dm(\tau). \tag{8}$$

For $n \in \mathbb{Z}$ and $\tau \in \mathbb{T}$, put $\chi_n(\tau) = \tau^n$. Then the Fourier coefficients of a function $f \in L^1$ can be expressed by $\hat{f}(n) = \langle f, \chi_n\rangle$ for $n \in \mathbb{Z}$.

Theorem 1 (main result 1). *Let X be a reflexive Banach function space over the unit circle $\mathbb{T}$ such that the Riesz projection P is bounded on X. Suppose $A \in \mathscr{B}(H[X])$ and there is a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that*

$$\left\langle A\chi_j, \chi_k\right\rangle = a_{k-j} \quad \forall j,k \in \mathbb{Z}_+. \tag{9}$$

Then there is a function $a \in L^\infty$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover

$$\|a\|_{L^\infty} \leq \|T_a\|_{\mathscr{B}(H[X])} \leq \|P\|_{\mathscr{B}(X)} \|a\|_{L^\infty}. \tag{10}$$

We need the notion of a function with absolutely continuous norm to formulate the result on the noncompactness of nontrivial Toeplitz operators. Following [6, Chap. 1, Definition 3.1], a function f in a Banach function space X is said to have absolutely continuous norm in X if $\|f\mathbb{I}_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets satisfying $\mathbb{I}_{E_n} \to \emptyset$ almost everywhere as $n \to \infty$. The set of all functions in X of absolutely continuous norm is denoted by X_a. It is known that a Banach function space X is reflexive if and only if X and X' have absolutely continuous norm (see [6, Chap. 1, Corollary 4.4]).

Theorem 2 (main result 2). *Let X be a Banach function space over the unit circle $\mathbb{T}$ such that $\mathbb{I}_E \in X_a$ for every measurable subset $E \subset \mathbb{T}$. If the Riesz projection P is bounded on X and $a \in L^\infty$, then the Toeplitz operator $T_a \in \mathscr{B}(H[X])$ is compact if and only if $a = 0$.*

The paper is organized as follows. Section 2 contains results on the density of the set of all trigonometric polynomials $\mathscr{P}$ (resp., the set of all analytic polynomials $\mathscr{P}_A$)

in a Banach function space X (resp., in the abstract Hardy space $H[X]$ built upon X). We also show that the norm of a function f in X can be calculated in terms of $\langle f, p\rangle$, where $p \in \mathscr{P}$, under the assumption that X' is separable. Further, we prove that every bounded linear operator on a separable Banach function space, whose matrix is of the form $(a_{k-j})_{j,k\in\mathbb{Z}}$, is an operator of multiplication by a function $a \in L^\infty$ and the sequence of its Fourier coefficients is exactly $\{a_k\}_{k\in\mathbb{Z}}$. Finally, we prove that if the characteristic functions of all measurable sets $E \subset \mathbb{T}$ have absolutely continuous norms in X, then the sequence $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on the abstract Hardy space $H[X]$. In Section 3, we provide proofs of our main results, using auxiliary results from the previous section. In Section 4, we specify our main results to the case of Hardy spaces built upon weighted Lebesgue spaces $L^p(w)$ with Muckenhoupt weights w and to the case of weighted Nakano spaces $L^{p(\cdot)}(w)$ with certain radial oscillating weights. In both cases, it is known that the Riesz projection is bounded.

2. Auxiliary Results

2.1. Density of Continuous Function and Trigonometric Polynomials in Banach Function Spaces. The following statement can be proved by analogy with [9, Lemma 1.3].

Lemma 3. *Let X be a Banach function space over the unit circle $\mathbb{T}$. The following statements are equivalent:*

(a) *the set $\mathscr{P}$ of all trigonometric polynomials is dense in the space X;*

(b) *the space C of all continuous functions on $\mathbb{T}$ is dense in the space X;*

(c) *the Banach function space X is separable.*

2.2. Density of Analytic Polynomials in Abstract Hardy Spaces. Let $m \in \mathbb{Z}_+$. A function of the form $q(t) = \sum_{k=0}^{m} \alpha_k t^k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{0, \ldots, m\}$ and $t \in \mathbb{T}$, is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials is denoted by $\mathscr{P}_A$.

Lemma 4. *Let X be a separable Banach functions space over the unit circle $\mathbb{T}$. If the Riesz projection P is bounded on X, then the set $\mathscr{P}_A$ is dense in $H[X]$.*

Proof. If $f \in H[X] \subset X$, then by Lemma 3, there exists a sequence $p_n \in \mathscr{P}$ such that $\|f - p_n\|_X \to 0$ as $n \to \infty$. It is clear that $f = Pf$ and $Pp_n \in \mathscr{P}_A$. Since $P \in \mathscr{B}(X)$, we finally have

$$\begin{aligned} \|f - Pp_n\|_X &= \|Pf - Pp_n\|_X \\ &\le \|P\|_{\mathscr{B}(X)} \|f - p_n\|_X \longrightarrow 0 \end{aligned} \tag{11}$$

as $n \to \infty$. Thus $\mathscr{P}_A$ is dense in X. □

2.3. A Formula for the Norm in a Banach Function Space

Lemma 5. *Let X be a Banach function space over the unit circle $\mathbb{T}$. If the associate space X' is separable, then for every $f \in X$*

$$\|f\|_X = \sup\{|\langle f, p\rangle| : p \in \mathscr{P}, \|p\|_{X'} \le 1\}. \tag{12}$$

Proof. By [6, Theorem 2.7 and Lemma 2.8], for every $f \in X$

$$\|f\|_X = \sup\{|\langle f, g\rangle| : g \in X', \|g\|_{X'} \le 1\}. \tag{13}$$

By the lattice property of the associate space X', we have $\mathscr{P} \subset X'$. Hence, equality (13) implies that for $f \in X$

$$\|f\|_X \ge \sup\{|\langle f, p\rangle| : p \in \mathscr{P}, \|p\|_{X'} \le 1\}. \tag{14}$$

Fix $g \in X'$ such that $0 < \|g\|_{X'} \le 1$. Since X' is separable, it follows from Lemma 3 that there exists a sequence $q_n \in \mathscr{P} \setminus \{0\}$ such that $\|q_n - g\|_{X'} \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, put $p_n := (\|g\|_{X'}/\|q_n\|_{X'}) q_n \in \mathscr{P}$. Then for every $n \in \mathbb{N}$

$$\|p_n\|_{X'} = \|g\|_{X'} \le 1, \tag{15}$$

$$\|g - p_n\|_{X'} \le \|g - q_n\|_{X'} + \|q_n\|_{X'} \left(1 - \frac{\|g\|_{X'}}{\|q_n\|_{X'}}\right). \tag{16}$$

Hence

$$\lim_{n\to\infty} \|g - p_n\|_{X'} = 0. \tag{17}$$

It follows from Hölder's inequality for Banach function spaces (see [6, Chap. 1, Theorem 2.4]) and (17) that

$$\lim_{n\to\infty} |\langle f, g\rangle - \langle f, p_n\rangle| \le \lim_{n\to\infty} \|f\|_X \|g - p_n\|_{X'} = 0. \tag{18}$$

Thus, taking into account (15) and (18), we deduce for every function $g \in X'$ satisfying $0 < \|g\|_{X'} \le 1$ that

$$\begin{aligned} |\langle f, g\rangle| &= \lim_{n\to\infty} |\langle f, p_n\rangle| \le \sup_{n\in\mathbb{N}} |\langle f, p_n\rangle| \\ &\le \sup\{|\langle f, p\rangle| : p \in \mathscr{P}, \|p\|_{X'} \le 1\}. \end{aligned} \tag{19}$$

This inequality and equality (13) imply that

$$\|f\|_X \le \sup\{|\langle f, p\rangle| : p \in \mathscr{P}, \|p\|_{X'} \le 1\}. \tag{20}$$

Combining inequalities (14) and (20), we arrive at equality (12). □

2.4. Multiplication Operators. We start this subsection with the following result by Maligranda and Persson on multiplication operators acting on Banach function spaces.

Lemma 6 (see [10, Theorem 1]). *Let X be a Banach function space over the unit circle $\mathbb{T}$. If $a \in L^0$, then the multiplication operator*

$$\begin{aligned} M_a \colon X &\longrightarrow X, \\ f &\longmapsto af, \end{aligned} \tag{21}$$

is bounded on X if and only if $a \in L^\infty$ and $\|M_a\|_{\mathscr{B}(X)} = \|a\|_{L^\infty}$.

It is easy see that

$$\langle M_a\chi_j,\chi_k\rangle = \langle a,\chi_{k-j}\rangle = \widehat{a}(k-j) \quad \forall j,k\in\mathbb{Z}. \tag{22}$$

The following lemma shows that every bounded operator with such a property is a multiplication operator.

Lemma 7. *Let X be a separable Banach functions space over the unit circle $\mathbb{T}$. Suppose $A\in\mathscr{B}(X)$ and there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers such that*

$$\langle A\chi_j,\chi_k\rangle = a_{k-j} \quad \forall j,k\in\mathbb{Z}. \tag{23}$$

Then there exists a function $a\in L^\infty$ such that $A=M_a$ and $\widehat{a}(n)=a_n$ for all $n\in\mathbb{Z}$.

Proof. Put $a := A\chi_0 \in X$. Since $X\subset L^1$, we infer from (23) that

$$\widehat{a}(n) = \langle a,\chi_n\rangle = \langle A\chi_0,\chi_n\rangle = a_n, \quad n\in\mathbb{Z}. \tag{24}$$

If $f=\sum_{k=-m}^{m}\widehat{f}(k)\chi_k\in\mathscr{P}$, then $af\in X\subset L^1$ and the jth Fourier coefficient of af is calculated by

$$(af)^\wedge(j) = \sum_{k\in\mathbb{Z}}\widehat{a}(j-k)\widehat{f}(k) = \sum_{k=-m}^{m} a_{j-k}\widehat{f}(k). \tag{25}$$

On the other hand, from (23), we get for $j\in\mathbb{Z}$

$$\begin{aligned}(Af)^\wedge(j) = \langle Af,\chi_j\rangle &= \sum_{k=-m}^{m}\widehat{f}(k)\langle A\chi_k,\chi_j\rangle \\ &= \sum_{k=-m}^{m} a_{j-k}\widehat{f}(k).\end{aligned} \tag{26}$$

By (25) and (26), $(af)^\wedge(j)=(Af)^\wedge(j)$ for all $j\in\mathbb{Z}$. Therefore, $Af=af$ for all $f\in\mathscr{P}$ in view of the uniqueness theorem for Fourier series (see, e.g., [11, Chap. I, Theorem 2.7]). Since the space X is separable, the set $\mathscr{P}$ is dense in X by Lemma 3. Therefore $Af=af$ for $f\in X$. This means that $A=M_a\in\mathscr{B}(X)$. It remains to apply Lemma 6. □

2.5. Weak Convergence of the Sequence $\{\chi_k\}_{k\in\mathbb{Z}_+}$ to Zero on the Abstract Hardy Space. Recall that the annihilator of a subspace S of a Banach space E is the set $S^\perp$ of all linear functionals $\Lambda\in E^*$ such that $\Lambda(x)=0$ for all $x\in S$ (see, e.g., [12, p. 110]).

Lemma 8. *If X is a Banach function space such that $\mathbb{I}_E\in X_a$ for every measurable subset $E\subset\mathbb{T}$, then $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on $H[X]$.*

Proof. By [12, Theorem 7.1], $(H[X])^*$ is isometrically isomorphic to $X^*/(H[X])^\perp$. Since $\chi_k\in H[X]$ for all $k\ge 0$, in view of the above fact, it is sufficient to prove that $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on the whole space X instead of the subspace $H[X]$.

By [6, Chap. 1, Corollary 3.14], if $\mathbb{I}_E\in X_a$ for every measurable subset $E\subset\mathbb{T}$, then $(X_a)^*$ is isometrically isomorphic to X'. In view of [6, Chap. 1, Theorem 2.2], X' is a Banach function space, which is continuously embedded into L^1 due to axiom (d) of the definition of a Banach function norm. Thus, for every $\Lambda\in X^*$, there exists a function $g\in X'\subset L^1$ such that $\Lambda(f)=\langle f,g\rangle$ for all $f\in X$. In particular, if $f=\chi_k$ with $k\ge 0$, then

$$\Lambda(\chi_k) = \langle\chi_k,g\rangle = \overline{\langle g,\chi_k\rangle} = \overline{\widehat{g}(k)}. \tag{27}$$

By the Riemann-Lebesgue lemma (see, e.g., [11, Chap. I, Theorem 2.8]) and (27), $\Lambda(\chi_k)\to 0$ as $k\to\infty$ for every $\Lambda\in X^*$; that is, $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on X, which completes the proof. □

3. Proof of the Main Results

3.1. Proof of Theorem 1. We follow the scheme of the proof of [8, Theorem 4.5] (see also [5, Theorem 2.7]). Without loss of generality, we may assume that the operator A is nonzero. For $n\in\mathbb{Z}_+$, put $b_n := \chi_{-n}A\chi_n$. Then taking into account Lemma 6 and that $A\in\mathscr{B}(H[X])$, we get

$$\begin{aligned}\|b_n\|_X &\le \|\chi_{-n}\|_{L^\infty}\|A\chi_n\|_X = \|A\chi_n\|_{H[X]} \\ &\le \|A\|_{\mathscr{B}(H[X])}\|\chi_n\|_{H[X]} = \|A\|_{\mathscr{B}(H[X])}\|1\|_X.\end{aligned} \tag{28}$$

Consider the following subset of the associate space:

$$V := \left\{ y\in X' : \|y\|_{X'} < \frac{1}{\|A\|_{\mathscr{B}(H[X])}\|1\|_X} \right\}. \tag{29}$$

It follows from Hölder's inequality for Banach function spaces (see [6, Chap. 1, Theorem 2.4]) and (28) and (29) that

$$|\langle b_n,y\rangle| \le \|b_n\|_X\|y\|_{X'} < 1 \quad \forall y\in V,\ n\in\mathbb{Z}_+. \tag{30}$$

Since X is reflexive, in view of [6, Chap. 1, Corollaries 4.3-4.4], we know that X' is canonically isometrically isomorphic to X^*. Applying the Banach-Alaoglu theorem (see, e.g., [13, Theorem 3.17]) to V, X', and $\{b_n\}_{n\in\mathbb{Z}_+}\subset X = X^{**} = (X')^*$, we deduce that there exists a $b\in X$ such that some subsequence $\{b_{n_k}\}_{k\in\mathbb{Z}_+}$ of $\{b_n\}_{n\in\mathbb{Z}_+}$ converges to b in the weak topology on X. In particular

$$\lim_{k\to+\infty}\langle b_{n_k},\chi_j\rangle = \langle b,\chi_j\rangle \quad \forall j\in\mathbb{Z}. \tag{31}$$

On the other hand, the definition of b_n and equality (9) imply that

$$\langle b_{n_k},\chi_j\rangle = \langle A\chi_{n_k},\chi_{n_k+j}\rangle = a_j \quad \text{whenever } n_k+j\in\mathbb{Z}_+. \tag{32}$$

It follows from (31) and (32) that

$$\langle b,\chi_j\rangle = a_j \quad \forall j\in\mathbb{Z}. \tag{33}$$

Now define the mapping B by

$$\begin{aligned} B:\ &\mathscr{P}\longrightarrow X, \\ &f\longmapsto bf. \end{aligned} \tag{34}$$

Assume that f and g are trigonometric polynomials of orders m and r, respectively. Then

$$f = \sum_{k=-m}^{m} \widehat{f}(k)\,\chi_k, \qquad g = \sum_{j=-r}^{r} \widehat{g}(j)\,\chi_j. \tag{35}$$

It follows from (9) and (33) that for $n \geq \max\{m, r\}$

$$\begin{aligned} \langle Bf, g\rangle &= \sum_{k=-m}^{m}\sum_{j=-r}^{r} \widehat{f}(k)\,\widehat{g}(j)\,\langle b\chi_k, \chi_j\rangle \\ &= \sum_{k=-m}^{m}\sum_{j=-r}^{r} \widehat{f}(k)\,\widehat{g}(j)\,a_{j-k} \\ &= \sum_{k=-m}^{m}\sum_{j=-r}^{r} \widehat{f}(k)\,\widehat{g}(j)\,\langle A\chi_{k+n}, \chi_{j+n}\rangle \\ &= \sum_{k=-m}^{m}\sum_{j=-r}^{r} \widehat{f}(k)\,\widehat{g}(j)\,\langle \chi_{-n}A(\chi_n\chi_k), \chi_j\rangle \\ &= \langle \chi_{-n}A(\chi_n f), g\rangle. \end{aligned} \tag{36}$$

It is clear that $\chi_n f \in H[X]$ for $n \geq \max\{m, r\}$. Therefore, taking into account Lemma 6, we see that for $n \geq \max\{m, r\}$

$$\begin{aligned} \left\|M_{\chi_{-n}} A M_{\chi_n} f\right\|_X &\leq \|\chi_{-n}\|_{L^\infty} \|A\chi_n f\|_{H[X]} \\ &\leq \|A\|_{\mathscr{B}(H[X])} \|\chi_n f\|_{H[X]} \\ &\leq \|A\|_{\mathscr{B}(H[X])} \|\chi_n\|_{L^\infty} \|f\|_X \\ &= \|A\|_{\mathscr{B}(H[X])} \|f\|_X. \end{aligned} \tag{37}$$

By Hölder's inequality for Banach function spaces (see [6, Chap. 1, Theorem 2.4]) from (36) and (37), we obtain

$$\begin{aligned} |\langle Bf, g\rangle| &\leq \limsup_{n\to\infty} \left|\left\langle M_{\chi_{-n}} A M_{\chi_n} f, g\right\rangle\right| \\ &\leq \limsup_{n\to\infty} \left\|M_{\chi_{-n}} A M_{\chi_n} f\right\|_X \|g\|_{X'} \\ &\leq \|A\|_{\mathscr{B}(H[X])} \|f\|_X \|g\|_{X'}. \end{aligned} \tag{38}$$

Since a Banach function space X is reflexive and the Lebesgue measure is separable, it follows from [6, Chap. 1, Corollaries 4.4 and 5.6] that the spaces X and X' are separable. Then Lemma 5 and inequality (38) yield

$$\begin{aligned} \|Bf\|_X &= \sup\left\{|\langle Bf, g\rangle| : g \in \mathscr{P},\ \|g\|_{X'} \leq 1\right\} \\ &\leq \|A\|_{\mathscr{B}(H[X])} \|f\|_X, \end{aligned} \tag{39}$$

for all $f \in \mathscr{P}$. In view of Lemma 3, $\mathscr{P}$ is dense in X. Then (39) implies that the linear mapping B defined in (34) extends to an operator $B \in \mathscr{B}(X)$ such that

$$\|B\|_{\mathscr{B}(X)} \leq \|A\|_{\mathscr{B}(H[X])}. \tag{40}$$

We deduce from (33) that

$$\left\langle B\chi_j, \chi_k\right\rangle = \left\langle b, \chi_{k-j}\right\rangle = a_{k-j} \quad \forall j, k \in \mathbb{Z}. \tag{41}$$

By Lemma 7, there exists a function $a \in L^\infty$ such that $B = M_a$ and $a_n = \widehat{a}(n)$ for all $n \in \mathbb{Z}$. Moreover

$$\|B\|_{\mathscr{B}(X)} = \|M_a\|_{\mathscr{B}(X)} = \|a\|_{L^\infty}. \tag{42}$$

It follows from the definition of the Toeplitz operator T_a that

$$\left\langle T_a\chi_j, \chi_k\right\rangle = \widehat{a}(k-j), \quad j, k \in \mathbb{Z}_+. \tag{43}$$

Combining this fact with equality (9), we arrive at

$$\left\langle T_a\chi_j, \chi_k\right\rangle = a_{k-j} = \left\langle A\chi_j, \chi_k\right\rangle, \quad j, k \in \mathbb{Z}_+. \tag{44}$$

Since $T_a\chi_j, A\chi_j \in H[X] \subset H^1$, by the uniqueness theorem for Fourier series (see, e.g., [11, Chap. I, Theorem 2.7]), it follows from (44) that $T_a\chi_j = A\chi_j$ for all $j \in \mathbb{Z}_+$. Therefore

$$T_a p = Ap \quad \forall p \in \mathscr{P}_A. \tag{45}$$

In view of Lemma 4, the set $\mathscr{P}_A$ is dense in $H[X]$. This fact and equality (45) imply that $T_a = A$ and

$$\|T_a\|_{\mathscr{B}(H[X])} = \|A\|_{\mathscr{B}(H[X])}. \tag{46}$$

Combining inequality (40) with equalities (42) and (46), we arrive at the first inequality in (10). The second inequality in (10) is obvious.

3.2. Proof of Theorem 2. It is clear that if $a = 0$, then T_a is the zero operator, which is compact. Now assume that T_a is compact. Then it maps weakly convergent sequences in $H[X]$ into strongly convergent sequences in $H[X]$ (see, e.g., [14, Section 7.5, Theorem 4]). Since $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges to zero weakly on $H[X]$ in view of Lemma 8, we have

$$\lim_{k\to\infty} \|T_a\chi_k\|_{H[X]} = 0. \tag{47}$$

By [6, Chap. 1, Theorem 2.7 and Lemma 2.8], for $k \in \mathbb{Z}_+$,

$$\begin{aligned} \|T_a\chi_k\|_{H[X]} &= \|T_a\chi_k\|_X \\ &= \sup\left\{|\langle T_a\chi_k, g\rangle| : g \in X',\ \|g\|_{X'} \leq 1\right\}. \end{aligned} \tag{48}$$

Since $L^\infty \subset X'$, there exists a constant $c \in (0, \infty)$ such that

$$c^{-1}\|\chi_m\|_{X'} \leq \|\chi_m\|_{L^\infty} = 1, \quad m \in \mathbb{Z}. \tag{49}$$

For all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}_+$ such that $k + n \in \mathbb{Z}_+$, we have

$$\widehat{a}(n) = \langle T_a\chi_k, \chi_{k+n}\rangle. \tag{50}$$

Then from (48)–(50) we obtain for all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}_+$ such that $k + n \in \mathbb{Z}_+$

$$\|T_a\chi_k\|_{H[X]} \geq \left|\left\langle T_a\chi_k, c^{-1}\chi_{k+n}\right\rangle\right| = c^{-1}|\widehat{a}(n)|. \tag{51}$$

Passing in this inequality to the limit as $k \to \infty$ and taking into account (47), we see that $\widehat{a}(n) = 0$ for all $n \in \mathbb{Z}$. By the uniqueness theorem for Fourier series (see, e.g., [11, Chap. I, Theorem 2.7]), this implies that $a = 0$ a.e. on $\mathbb{T}$.

4. Toeplitz Operators on Hardy Spaces Built upon Weighted Lebesgue and Nakano Spaces

4.1. The Case of Hardy Spaces Built upon Lebesgue Spaces with Muckenhoupt Weights. A measurable function $w : \mathbb{T} \to [0, \infty]$ is referred to as a weight if $0 < w(\tau) < \infty$ almost everywhere on $\mathbb{T}$. If X is a Banach function space over the unit circle and w is a weight, then

$$X(w) := \{f \in L^0 \colon fw \in X\} \tag{52}$$

is a normed space equipped with the norm $\|f\|_{X(w)} := \|fw\|_X$. Moreover, if $w \in X$ and $1/w \in X'$, then $X(w)$ is a Banach function space (see [15, Lemma 2.5]).

Let $1 < p < \infty$ and w be a weight. It is well known that the Riesz projection P is bounded on the weighted Lebesgue space $L^p(w)$ if and only if the weight w satisfies the Muckenhoupt A_p-condition; that is,

$$\begin{aligned} \sup_{I \subset \mathbb{T}} \left(\frac{1}{m(I)} \int_I w^p(\tau)\, dm(\tau) \right)^{1/p} \\ \cdot \left(\frac{1}{m(I)} \int_I w^{-p'}(\tau)\, dm(\tau) \right)^{1/p'} < \infty, \end{aligned} \tag{53}$$

where the supremum is taken over all subarcs I of the unit circle $\mathbb{T}$ and $1/p + 1/p' = 1$ (see [16] and also [5, Section 1.46], [3, Section 5.7.3(h)]). In the latter case, we will write $w \in A_p(\mathbb{T})$. It is clear that if $w \in A_p(\mathbb{T})$, then $w \in L^p$ and $1/w \in L^{p'}$. Hence $L^p(w)$ is a Banach function space whenever $w \in A_p(\mathbb{T})$. It is well known that if $1 < p < \infty$, then $L^p(w)$ is reflexive. We denote the corresponding Hardy space by $H^p(w) := H[L^p(w)]$.

Surprisingly enough, we were not able to find the following results explicitly stated in the literature.

Corollary 9. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{T})$. If $A \in \mathcal{B}(H^p(w))$ and there exists a sequence $\{a_n\}_{n \in \mathbb{Z}}$ of complex numbers satisfying (9), then there exists a function $a \in L^\infty$ such that $A = T_a$ and $\widehat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover*

$$\|a\|_{L^\infty} \le \|T_a\|_{\mathcal{B}(H^p(w))} \le \|P\|_{\mathcal{B}(L^p(w))} \|a\|_{L^\infty}. \tag{54}$$

This is an immediate consequence of Theorem 1. For the weight $w = 1$, it is proved in [5, Theorem 2.7].

Corollary 10. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{T})$. If $a \in L^\infty$, then the Toeplitz operator $T_a \in \mathcal{B}(H^p(w))$ is compact if and only if $a = 0$.*

This corollary follows from Theorem 2.

4.2. The Case of Hardy Spaces Built upon Nakano Spaces with Radial Oscillating Weights. We denote by $\mathcal{P}_C(\mathbb{T})$ the set of all continuous functions $p : \mathbb{T} \to (1, \infty)$. For $p \in \mathcal{P}_C(\mathbb{T})$, let $L^{p(\cdot)}$ be the set of all functions $f \in L^0$ such that

$$\int_{\mathbb{T}} \left| \frac{f(\tau)}{\lambda} \right|^{p(\tau)} dm(\tau) < \infty, \tag{55}$$

for some $\lambda = \lambda(f) > 0$. This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 \colon \int_{\mathbb{T}} \left| \frac{f(\tau)}{\lambda} \right|^{p(\tau)} dm(\tau) \le 1 \right\}, \tag{56}$$

(see, e.g., [17, p. 73] or [18, p. 77]). If p is constant, then $L^{p(\cdot)}$ is nothing but the Lebesgue space L^p. The spaces $L^{p(\cdot)}$ are referred to as Nakano spaces. We refer to Maligranda's paper [19] for the role of Hidegoro Nakano in the study of these spaces.

Since $\mathbb{T}$ is compact, we have

$$1 < \min_{t \in \mathbb{T}} p(t), \max_{t \in \mathbb{T}} p(t) < \infty. \tag{57}$$

In this case, the space $L^{p(\cdot)}$ is reflexive and its associate space is isomorphic to the space $L^{p'(\cdot)}$, where $1/p(\tau) + 1/p'(\tau) = 1$ for all $\tau \in \mathbb{T}$ (see, e.g., [17, Section 2.8] and [18, Section 3.2]).

Let Sf be the Cauchy singular integral of a function $f \in L^1(\mathbb{T})$ defined by

$$(Sf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\mathbb{T} \setminus \mathbb{T}(t, \varepsilon)} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{T}, \tag{58}$$

where $\mathbb{T}(t, \varepsilon) := \{\tau \in \mathbb{T} \colon |\tau - t| < \varepsilon\}$. For a weight $w : \mathbb{T} \to [0, \infty]$, consider the weighted Nakano space

$$L^{p(\cdot)}(w) := \{f \in L^0 \colon fw \in L^{p(\cdot)}\}. \tag{59}$$

It follows from [15, Theorem 6.1] that if the operator S is bounded on $L^{p(\cdot)}(w)$, then

$$\sup_{I \subset \mathbb{T}} \frac{1}{m(I)} \|w \chi_I\|_{L^{p(\cdot)}} \|w^{-1} \chi_I\|_{L^{p'(\cdot)}} < \infty, \tag{60}$$

where the supremum is taken over all subarcs $I \subset \mathbb{T}$. In particular, in this case, $w \in L^{p(\cdot)}$ and $1/w \in L^{p'(\cdot)}$, whence $L^{p(\cdot)}(w)$ is a Banach function space by [15, Lemma 2.5(b)].

We say that an exponent $p \in \mathcal{P}_C(\mathbb{T})$ is locally log-Hölder continuous (cf. [17, Definition 2.2]) if there exists a constant $C_{p(\cdot)} \in (0, \infty)$ such that

$$\begin{aligned} |p(t) - p(\tau)| \le \frac{C_{p(\cdot)}}{-\log |t - \tau|} \\ \forall t, \tau \in \mathbb{T} \text{ satisfying } |t - \tau| < \frac{1}{2}. \end{aligned} \tag{61}$$

The class of all locally log-Hölder continuous exponents will be denoted by $LH(\mathbb{T})$. Notice that some authors also denote this class by $\mathbb{P}^{\log}(\mathbb{T})$ (see, e.g., [20, Section 1.1.4]).

Following [21, Section 2.3], denote by W the class of all continuous functions $\varrho : [0, 2\pi] \to [0, \infty)$ such that $\varrho(0) = 0$, $\varrho(x) > 0$, if $0 < x \le 2\pi$, and ϱ is almost increasing; that is, there is a universal constant $C > 0$ such that $\varrho(x) \le C\varrho(y)$ whenever $x \le y$. Further, let $\mathbb{W}$ be the set of all functions $\varrho : [0, 2\pi] \to [0, \infty]$ such that $x^\alpha \varrho(x) \in W$ and $x^\beta / \varrho(x) \in W$

for some $\alpha, \beta \in \mathbb{R}$. Clearly, the functions $\varrho(x) = x^\gamma$ belong to $\mathbb{W}$ for all $\gamma \in \mathbb{R}$. For $\varrho \in \mathbb{W}$, put

$$\Phi_\varrho^0(x) := \limsup_{y\to 0} \frac{\varrho(xy)}{\varrho(y)}, \quad x \in (0,\infty). \tag{62}$$

Since $\varrho \in \mathbb{W}$, one can show that the limits

$$m(\varrho) := \lim_{x\to 0} \frac{\log \Phi_\varrho^0(x)}{\log x}, \qquad M(\varrho) := \lim_{x\to\infty} \frac{\log \Phi_\varrho^0(x)}{\log x} \tag{63}$$

exist and $-\infty < m(\varrho) \le M(\varrho) < +\infty$. These numbers were defined under some extra assumptions on ϱ by Matuszewska and Orlicz [22, 23] (see also [24] and [25, Chapter 11]). We refer to $m(\varrho)$ (resp., $M(\varrho)$) as the lower (resp., upper) Matuszewska-Orlicz index of ϱ. For $\varrho(x) = x^\gamma$, one has $m(\varrho) = M(\varrho) = \gamma$. Examples of functions $\varrho \in \mathbb{W}$ with $m(\varrho) < M(\varrho)$ can be found, for instance, in [25, p. 93]. Fix pairwise distinct points $t_1, \dots, t_n \in \Gamma$ and functions $w_1, \dots, w_n \in \mathbb{W}$. Consider the following weight:

$$w(t) := \prod_{k=1}^n w_k(|t - t_k|), \quad t \in \Gamma. \tag{64}$$

Each function $w_k(|t - t_k|)$ is a radial oscillating weight. This is a natural generalization of the so-called Khvedelidze weights $w(t) = \prod_{k=1}^n |t - t_k|^{\lambda_k}$, where $\lambda_k \in \mathbb{R}$ (see, e.g., [5, Section 5.8]).

Theorem 11. *Let $p \in LH(\mathbb{T})$. Suppose $w_1, \dots, w_n \in \mathbb{W}$ and the weight w is given by (64). The Cauchy singular integral operator S is bounded on $L^{p(\cdot)}(w)$ if and only if for all $k \in \{1, \dots, n\}$*

$$0 < \frac{1}{p(t_k)} + m(w_k), \frac{1}{p(t_k)} + M(w_k) < 1. \tag{65}$$

The sufficiency portion of Theorem 11 was obtained by Kokilashvili et al. [21, Theorem 4.3] (see also [20, Corollary 2.109]) for more general finite Carleson curves in place of $\mathbb{T}$. The necessity portion was proved by the author [26, Corollary 4.3] for Jordan Carleson curves.

Lemma 12. *Let $p \in LH(\mathbb{T})$. Suppose $w_1, \dots, w_n \in \mathbb{W}$ and the weight w is given by (64). Then the weighted Nakano space $L^{p(\cdot)}(w)$ is a reflexive Banach function space and the Riesz projection P is bounded on $L^{p(\cdot)}(w)$.*

Proof. In view of Theorem 11, the operator S is bounded on the space $L^{p(\cdot)}(w)$. As was observed above, the boundedness of the operator S on the space $L^{p(\cdot)}(w)$ implies that $w \in L^{p(\cdot)}$ and $1/w \in L^{p'(\cdot)}(w)$ by [15, Theorem 6.1]. Hence $L^{p(\cdot)}(w)$ is a reflexive Banach function space thanks to [15, Lemma 2.5 and Corollary 2.8]. By [15, Lemma 6.4], the operator $P = (I+S)/2$ is bounded on $L^{p(\cdot)}(w)$. □

Consider the Hardy space $H^{p(\cdot)}(w) := H[L^{p(\cdot)}(w)]$ built upon the weighted Nakano space $L^{p(\cdot)}(w)$, where $p \in LH(\mathbb{T})$ and w is a weight as in Theorem 11.

Theorem 1 and Lemma 12 yield the following.

Corollary 13. *Let $p \in LH(\mathbb{T})$. Suppose $w_1, \dots, w_n \in \mathbb{W}$ and the weight w is given by (64). If $A \in \mathscr{B}(H^{p(\cdot)}(w))$ and there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying (9), then there exists a function $a \in L^\infty$ such that $A = T_a$ and $\widehat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover*

$$\|a\|_{L^\infty} \le \|T_a\|_{\mathscr{B}(H^{p(\cdot)}(w))} \le \|P\|_{\mathscr{B}(L^{p(\cdot)}(w))} \|a\|_{L^\infty}. \tag{66}$$

Similarly, Theorem 2 and Lemma 12 imply the following.

Corollary 14. *Let $p \in LH(\mathbb{T})$. Suppose $w_1, \dots, w_n \in \mathbb{W}$ and the weight w is given by (64). If $a \in L^\infty$, then the Toeplitz operator $T_a \in \mathscr{B}(H^{p(\cdot)}(w))$ is compact if and only if $a = 0$.*

4.3. Concluding Remarks. After this paper was submitted for publication, Leśnik posted the paper [27] in arXiv, where among other results he proved analogues of Theorems 1 and 2 for Toeplitz operators acting between abstract Hardy spaces $H[X]$ and $H[Y]$ built upon distinct rearrangement-invariant Banach function spaces X and Y. The set of allowed symbols in [27] coincides with the set $M(X, Y)$ of pointwise multipliers from X to Y, which may contain unbounded functions. Thus, his results complement ours in a nontrivial way but are not more general than ours, because Leśnik restricts himself to rearrangement-invariant spaces X and Y only. On the other hand, the main aim of this paper is to consider the questions of the boundedness and compactness of Toeplitz operators on an abstract Hardy space $H[X]$ in the case when X is an arbitrary, not necessarily rearrangement-invariant, Banach function space.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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References

[1] O. Toeplitz, "Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen," *Mathematische Annalen*, vol. 70, no. 3, pp. 351–376, 1911.

[2] A. Brown and P. R. Halmos, "Algebraic properties of Toeplitz operators," *Journal Für Die Reine Und Angewandte Mathematik*, vol. 213, pp. 89–102, 1964.

[3] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading. Vol. 1*, vol. 92 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2002.

[4] V. V. Peller and S. V. Khrushchev, “Hankel operators, best approximations, and stationary Gaussian processes,” *Russian Mathematical Surveys*, vol. 37, no. 1, pp. 53–124, 1982.

[5] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer, Berlin, Germany, 2nd edition, 2006.

[6] C. Bennett and R. Sharpley, *Interpolation of Operators*, vol. 129 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1988.

[7] Q. H. Xu, “Notes on interpolation of Hardy spaces,” *Annales de l'Institut Fourier*, vol. 42, no. 4, pp. 875–889, 1992.

[8] A. Yu. Karlovich, “Norms of Toeplitz and Hankel operators on Hardy type subspaces of rearrangement-invariant spaces,” *Integral Equations and Operator Theory*, vol. 49, no. 1, pp. 43–64, 2004.

[9] A. Yu. Karlovich, “On the essential norm of the Cauchy singular integral operator in weighted rearrangement-invariant spaces,” *Integral Equations and Operator Theory*, vol. 38, no. 1, pp. 28–50, 2000.

[10] L. Maligranda and L. E. Persson, “Generalized duality of some Banach function spaces,” *Indagationes Mathematicae (Proceedings)*, vol. 92, no. 3, pp. 323–338, 1989.

[11] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover Publications, Inc., New York, corrected edition, 1976.

[12] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, NY, USA, 1970.

[13] W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.

[14] W. Cheney, *Analysis for Applied Mathematics*, vol. 208 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001.

[15] A. Yu. Karlovich, “Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces,” *Journal of Integral Equations and Applications*, vol. 15, no. 3, pp. 263–320, 2003.

[16] R. Hunt, B. Muckenhoupt, and R. Wheeden, “Weighted norm inequalities for the conjugate function and Hilbert transform,” *Transactions of the American Mathematical Society*, vol. 176, pp. 227–251, 1973.

[17] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Birkhäuser, Basel, Switzerland, 2013.

[18] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, Germany, 2011.

[19] L. Maligranda, “Hidegoro Nakano (1909–1974)—on the centenary of his birth,” in *Banach and function spaces III (ISBFS 2009)*, pp. 99–171, Yokohama Publ., Yokohama, 2011.

[20] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko, *Integral operators in non-standard function spaces. Vol. 1*, vol. 248 of *Operator Theory: Advances and Applications*, Birkhäuser/Springer, [Cham], 2016.

[21] V. Kokilashvili, N. Samko, and S. Samko, “Singular operators in variable spaces *Lp(·)(Ω,ρ)* with oscillating weights,” *Mathematische Nachrichten*, vol. 280, no. 9-10, pp. 1145–1156, 2007.

[22] W. Matuszewska and W. Orlicz, “On certain properties of φ-functions,” *Bulletin de l'Academie Polonaise des Sciences. Serie des Sciences Mathematiques, Astronomiques et Physiques*, vol. 8, pp. 439–443, 1960.

[23] W. Matuszewska and W. Orlicz, “On some classes of functions with regard to their orders of growth,” *Studia Mathematica*, vol. 26, pp. 11–24, 1988.

[24] L. Maligranda, “Indices and interpolation,” *Polska Akademia Nauk. Instytut Matematyczny. Dissertationes Mathematicae. Rozprawy Matematyczne*, vol. 234, 49 pages, 1985.

[25] L. Maligranda, *Orlicz Spaces and Interpolation, Seminars in Mathematics*, vol. 5, Dep. Mat., Universidade Estadual de Campinas, Campinas SP, Brazil, 1989.

[26] A. Yu. Karlovich, “Singular integral operators on variable Lebesgue spaces with radial oscillating weights,” in *Operator Algebras, Operator Theory and Applications. Proceedings of the 18th International Workshop on Operator Theory and Applications (IWOTA), Potchefstroom, South Africa, July 3–6, 2007*, vol. 195 of *Operator Theory: Advances and Applications*, pp. 185–212, 2010.

[27] K. Leśnik, “Toeplitz and Hankel operators between distinct Hardy spaces,” https://arxiv.org/abs/1708.00910.