Equiintegrability and Controlled Convergence for the Henstock-Kurzweil Integral

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1. INTRODUCTION AND PRELIMINARIES

J.Kurzweil and J.Jarnek have shown a characterization of equi-integrability for the Henstock-Kurzweil integral on m -dimensional compact subintervals of $\mathbb{R}^m$, Main Theorem 1.1, [5]. There are also other characterizations of equi-integrability for the Henstock-Kurzweil integral in [8] and [6]. Our characterization of equi-integrability sharpens the result in [6] (see Theorem 5 and Corollary 9, [6]). The original proofs in [6] does not consider pointwise bounded Henstock-Kurzweil-equintegrable sequences.

We firstly get a characterization of equiintegrability for Henstock-Kurzweil integral, Theorem 2.3. This theorem yields a controlled convergence theorem for the Henstock-Kurzweil integral of real-valued functions, Theorem 2.4. There is a considerable work dealing with controlled convergence theorems for Henstock-Kurzweil integral (see e.g. [5], [3], [9], [10] etc.). The real valued version of Theorem 3.3 in [4] and Theorem 6 in [1] are other controlled convergence theorems for Henstock-Kurzweil integral.

We denote by $\mathcal{I}$ the family of all non-degenerate closed subintervals of $[0, 1]$ and by λ the Lebesgue measure on $[0, 1]$. We will identify an interval function $F : \mathcal{I} \rightarrow \mathbb{R}$ with the point function $F(t) = F([0, t])$, $t \in [0, 1]$; and conversely, we will identify a point function $F : [0, 1] \rightarrow \mathbb{R}$ with the interval function

$F([u, v]) = F(v) - F(u)$, $[u, v] \in \mathcal{I}$. An interval function $F : \mathcal{I} \rightarrow \mathbb{R}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each nonoverlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$. The intervals $I, J \in \mathcal{I}$ are said to be nonoverlapping if $\text{int}(I) \cap \text{int}(J) = \emptyset$, where $\text{int}(I)$ denotes the interior of I .

A pair (I, t) of an interval $I \in \mathcal{I}$ and a point $t \in I$ is called tagged interval, t is the tag of I . A $\mathcal{HK}$ -partition π in $[0, 1]$ is a finite collection of tagged intervals (I, t) with nonoverlapping intervals I . A function $\delta : A \rightarrow (0, +\infty)$ is said to be a gauge on $A \subset [0, 1]$. An $\mathcal{HK}$ -partition π in $[0, 1]$ is

- an $\mathcal{HK}$ -partition of $[0, 1]$ if $\cup_{(I,t) \in \pi} I = [0, 1]$,
- A -tagged if for all $(I, t) \in \pi$ we have $t \in A$,
- δ -fine, if for every tagged interval $(I, t) \in \pi$ we have $I \subset (t - \delta(t), t + \delta(t))$.

Let us now recall a few necessary definitions.

Definition 1.1. A function $f : [0, 1] \rightarrow \mathbb{R}$ is called Henstock-Kurzweil integrable ($\mathcal{HK}$ -integrable) on $[0, 1]$ if there exists $w_f \in \mathbb{R}$ satisfying the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0, 1]$ such that for every δ -fine $\mathcal{HK}$ -partition π of $[0, 1]$, we have

$$\left| \sum_{(I,t) \in \pi} f(t)\lambda(I) - w_f \right| < \epsilon.$$

We write $(HK) \int_S f = w_f$ and call w_f $\mathcal{HK}$ -integral of f over $[0, 1]$.

The function f is said to be $\mathcal{HK}$ -integrable on a set $A \subset [0, 1]$ if the function $f \cdot \chi_A : [0, 1] \rightarrow \mathbb{R}$ is $\mathcal{HK}$ -integrable on $[0, 1]$, where χ_A denotes the characteristic function of A . We write $(HK) \int_{[0,1]} f \cdot \chi_A = (HK) \int_A f$ for $\mathcal{HK}$ -integral of f on A .

If f is $\mathcal{HK}$ -integrable on $[0, 1]$, then we define the function $F : [0, 1] \rightarrow \mathbb{R}$ by $F(I) = (HK) \int_I f$, for all $I \in \mathcal{I}$. This function is said to be the primitive of f on $[0, 1]$. By the real valued version of Theorem 3.3.5, [7], F is additive and by Theorem 7.4.1, [7], F is continuous on $[0, 1]$.

Definition 1.2. A family $\mathcal{M}$ of functions $f : [0, 1] \rightarrow \mathbb{R}$ is said to be $\mathcal{HK}$ -equiintegrable if each $f \in \mathcal{M}$ is $\mathcal{HK}$ -integrable and for every $\epsilon > 0$ there exists a gauge δ on $[0, 1]$, such that for every δ -fine $\mathcal{HK}$ -partition π of $[0, 1]$, we have

$$\left| \sum_{(I,t) \in \pi} f(t)\lambda(I) - (HK) \int_{[0,1]} f \right| < \epsilon \quad \text{for all } f \in \mathcal{M}.$$

A function $F : [0, 1] \rightarrow \mathbb{R}$ is said to be ACG^* on $A \subset [0, 1]$ if A is the union of a sequence of subsets A_i such that F is $AC^*(A_i)$ for each i , i.e., for every $\epsilon > 0$ there is $\eta > 0$ such that for any finite or infinite sequence $([a_k, b_k])$ of pairwise nonoverlapping subintervals in $\mathcal{I}$ with at least one a_k, b_k belonging to A_i satisfying

$$(1.1) \quad \sum_k |F(b_k) - F(a_k)| < \epsilon,$$

whenever $\sum_k (b_k - a_k) < \eta$. Replacing (1.1) by

$$\sum_k \omega(F, [a_k, b_k]) < \epsilon,$$

where $\omega(F, [a_k, b_k]) = \sup\{|F(b'_k) - F(a'_k)| : [a'_k, b'_k] \subset [a_k, b_k]\}$, we obtain the definition of a function F that is ACG_* on $A \subset [0, 1]$.

A sequence of functions (F_n) is said to be $UACG^*$ on $A \subset [0, 1]$ if A is the union of a sequence of subsets (A_i) such that (F_n) is $UAC^*(A_i)$ for each i , that is, η in the definition of $AC^*(A_i)$ with F replaced by F_n is independent of n . The notion of a sequence (F_n) that is $UACG_*(A)$ is similarly defined.

By Lemma 5.3.2 and 5.3.3, [10], we obtain immediately the following.

Lemma 1.3. *Let $F : [0, 1] \rightarrow \mathbb{R}$ be a continuous additive interval function and let A be a subset of $[0, 1]$. Then F is $AC^*(A)$ if and only if F is $AC_*(A)$.*

The sequence (F_n) is said to satisfy the uniformly strong Lusin condition, or briefly USL , if for every $\epsilon > 0$ and every subset $Z \subset S$ with $\lambda(Z) = 0$ there exists a gauge δ on Z such that for each δ -fine Z -tagged $\mathcal{HK}$ -partition π in $[0, 1]$, we have $\sum_{(I,t) \in \pi} |F_n(I)| < \epsilon$ for all $n \in \mathbb{N}$.

2. THE MAIN RESULTS

The main theorems are Theorem 2.3 and Theorem 2.4. The following lemmas make it possible to present clearly Theorem 2.3. The notion of $\mathcal{HK}$ -equiintegrability does not allow one to ignore sets of measure zero (see [2]). Nevertheless, Lemma 2.1 allows us to ignore sets of measure zero in the notion of $\mathcal{HK}$ -equiintegrability. The "only if part" of the lemma is proven in [2], Exercise 13.9. The "if part" is straightforward.

Lemma 2.1. *Let (f_n) be a pointwise bounded sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ and let E be subset of $[0, 1]$ such that $\lambda(S \setminus E) = 0$. Then, the sequence (f_n) is $\mathcal{HK}$ -equiintegrable if and only if the sequence $(f_n \cdot \chi_E)$ is $\mathcal{HK}$ -equiintegrable.*

The next lemma shows that the condition " (f_n) is $\mathcal{HK}$ -equiintegrable" implies that pointwise bounded and USL notions are equivalent.

Lemma 2.2. *Let (f_n) be a sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ and let $f : [0, 1] \rightarrow \mathbb{R}$ be a function. If we have*

- $f_n(s) \rightarrow f(s)$ almost everywhere in $[0, 1]$,
- (f_n) is $\mathcal{HK}$ -equiintegrable,

then the following statements are equivalent

- (i) (f_n) is pointwise bounded,
- (ii) (F_n) is USL .

Proof. By hypothesis, there exists $Z \subset [0, 1]$ with $\lambda(Z) = 0$ such that (f_n) converges pointwise in $E = [0, 1] \setminus Z$ to f .

Assume that (f_n) is pointwise bounded and denote $f^{(E)} = f \cdot \chi_E$, $f_n^{(E)} = f_n \cdot \chi_E$, $n \in \mathbb{N}$. Then, the sequence $(f_n^{(E)})$ converges pointwise in $[0, 1]$ to $f^{(E)}$ and by Lemma 2.1 the sequence $(f_n^{(E)})$ is $\mathcal{HK}$ -equiintegrable. By Theorem 3.3.7, [7], we get also that each F_n is the primitive of $\mathcal{HK}$ -integrable function $f_n^{(E)}$. Therefore, we obtain by Theorem 8, [6] that (F_n) is USL .

Conversely, we assume that (F_n) is USL . Let z be an arbitrary element of Z . Since (f_n) is $\mathcal{HK}$ -equiintegrable, by Lemma 3.5.6, [7], for the given $\epsilon > 0$ there exists a gauge δ_1 on Z such that for every δ_1 -fine Z -tagged $\mathcal{HK}$ -partition π in $[0, 1]$, we have

$$\left| \sum_{(I,s) \in \pi} (f_n(s)\lambda(I) - F_n(I)) \right| \leq \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Since (F_n) is USL , there exists a gauge δ_2 on Z such that for every δ_2 -fine Z -tagged $\mathcal{HK}$ -partition π in $[0, 1]$, we have $\sum_{(I,s) \in \pi} |F_n(I)| < \epsilon$, for all $n \in \mathbb{N}$.

Now, if we denote $\delta = \min\{\delta_1, \delta_2\}$ and consider any δ -fine $\{z\}$ -tagged $\mathcal{HK}$ -partition $\pi = \{(I, z)\}$ in $[0, 1]$, then we get

$$|f_n(z)\lambda(I)| \leq |f_n(z)\lambda(I) - F_n(I)| + |F_n(I)| \leq 2\epsilon,$$

for all $n \in \mathbb{N}$. Thus, we have $|f_n(z)| \leq \frac{2\epsilon}{\lambda(I)} = M > 0$, for all $n \in \mathbb{N}$. Since z was arbitrary, the last result yields that (f_n) is pointwise bounded. $\square$

Theorem 2.3. *Let (f_n) be a sequence of $\mathcal{HK}$ -integrable functions $f_n : [0, 1] \rightarrow \mathbb{R}$ and let $f : [0, 1] \rightarrow \mathbb{R}$ be a function. If we have*

- $f_n(s) \rightarrow f(s)$ almost everywhere in $[0, 1]$,
- (F_n) is USL , where F_n 's are the primitives of f_n 's,

then the following statements are equivalent

- (A) *every subsequence of (f_n) has a subsequence $\mathcal{HK}$ -equiintegrable,*
- (B) *every subsequence of (F_n) has a subsequence $UACG^*(W)$, where $W \subset [0, 1]$ such that $\lambda([0, 1] \setminus W) = 0$.*

Proof. (A) $\Rightarrow$ (B) Let us consider any subsequence $(G_n) \subset (F_n)$. We denote by (g_n) the subsequence of (f_n) which is the sequence corresponding to (G_n) . For the subsequence $(g_n) \subset (f_n)$ there exists a subsequence $(h_n) \subset (g_n)$ which is $\mathcal{HK}$ -equiintegrable. Let (H_n) be the subsequence of (G_n) which is the sequence corresponding to (h_n) . Therefore by Corollary 9, [6], the subsequence $(H_n) \subset (G_n)$ is $UACG^*(W)$, where $W \subset [0, 1]$ such that $\lambda([0, 1] \setminus W) = 0$. Thus, (H_n) is the required subsequence of (G_n) .

(B) $\Rightarrow$ (A) Let (g_n) be any subsequence of (f_n) . We denote by (G_n) the subsequence of (F_n) which is the sequence corresponding to (g_n) . For the subsequence $(G_n) \subset (F_n)$ there exists a subsequence $(H_n) \subset (G_n)$ which is $UACG^*(W)$, where $W \subset [0, 1]$ such that $\lambda([0, 1] \setminus W) = 0$. Let (h_n) be the subsequence of (g_n) which is the sequence corresponding to (H_n) . Hence, by Theorem 5 in [6], for the sequence (h_n) there exists a sequence $(h_n^{(*)})$ such that

- $h_n^{(*)}(s) = h_n(s)$ almost everywhere in $[0, 1]$ for all $n \in \mathbb{N}$,
- there is a $\mathcal{HK}$ -equiintegrable subsequence $(h_{n_k}^{(*)}) \subset (h_n^{(*)})$.

Then, there exists $Z \subset [0, 1]$ with $\lambda(Z) = 0$ such that $h_{n_i}^{(*)}(s) = h_{n_i}(s)$ for every $s \in E = [0, 1] \setminus Z$ and $i \in \mathbb{N}$, and consequently the sequence $(h_{n_k}^{(*)}(s))$ converges to $f(s)$ for each $s \in E$. Hence, by Theorem 3.3.4 and Theorem 3.3.7 in [7], for each $I \in \mathcal{I}$, we have

$$(HK) \int_I h_{n_k}^{(*)} = (HK) \int_I h_{n_k} = H_{n_k}(I),$$

and since (H_{n_k}) is USL and $(h_{n_k}^{(*)})$ converges pointwise in $E = [0, 1] \setminus Z$ to f , we get by Lemma 2.2 that $(h_{n_k}^{(*)})$ is pointwise bounded. Consequently, we obtain by Lemma 2.1 that (h_{n_k}) is $\mathcal{HK}$ -equiintegrable. Thus, (h_{n_k}) is required subsequence of (g_n) and the proof is finished. $\square$

Theorem 2.4. *If a sequence (f_n) of $\mathcal{HK}$ -integrable functions $f_n : S \rightarrow \mathbb{R}$ and a function $f : S \rightarrow \mathbb{R}$ satisfy the following conditions*

- (i) $f_n(s) \rightarrow f(s)$ almost everywhere in $[0, 1]$,
- (ii) (F_n) is USL , where F_n 's are the primitive of f_n 's,
- (iii) (F_n) satisfies the condition (B) from Theorem 2.3,

then the function f is $\mathcal{HK}$ -integrable on $[0, 1]$ and for every $I \in \mathcal{I}$ we have $\lim_{n \rightarrow \infty} F_n(I) = F(I)$, where F is the primitive of f .

Proof. We firstly show the theorem for the case when $I = [0, 1]$. Let (g_n) be any subsequence of (f_n) . Then, we obtain by Theorem 2.3 that there exists a subsequence $(h_n) \subset (g_n)$ such that (h_n) is $\mathcal{HK}$ -equiintegrable.

By hypothesis there exists $Z \subset [0, 1]$ with $\lambda(Z) = 0$ such that (h_n) converges pointwise in $E = S \setminus Z$ to f . Denote $f^{(E)} = f \cdot \chi_E$ and $h_n^{(E)} = h_n \cdot \chi_E$, $n \in \mathbb{N}$. Then, the sequence $(h_n^{(E)})$ converges pointwise in $[0, 1]$ to $f^{(E)}$. By Lemma 2.2 and Lemma 2.1 we have also that $(h_n^{(E)})$ is $\mathcal{HK}$ -equiintegrable. Therefore, by Theorem 3.5.2 in [7], the function $f^{(E)}$ is $\mathcal{HK}$ -integrable on $[0, 1]$ and

$$(2.1) \quad \lim_{n \rightarrow \infty} (HK) \int_{[0,1]} h_n^{(E)} = (HK) \int_{[0,1]} f^{(E)}.$$

Hence, we obtain by Theorem 3.3.7 in [7] that the function f is $\mathcal{HK}$ -integrable on $[0, 1]$ and $\lim_{n \rightarrow \infty} (HK) \int_{[0,1]} h_n = (HK) \int_{[0,1]} f$. Since every subsequence $(g_n) \subset (f_n)$ has a subsequence $(h_n) \subset (g_n)$ satisfying the above property, the sequence itself has the property. Thus, we have $\lim_{n \rightarrow \infty} F_n([0, 1]) = F([0, 1])$, where F is the primitive of f on $[0, 1]$.

Now, we show the theorem for an arbitrary $I \in \mathcal{I}$. Denote $f^{(I)} = f \cdot \chi_I$ and $f_n^{(I)} = f_n \cdot \chi_I$, $n \in \mathbb{N}$. Then, by Theorem 3.3.4 in [7], $f^{(I)}$ and each $f_n^{(I)}$ is $\mathcal{HK}$ -integrable on $[0, 1]$. Note that

$$F_n^{(I)}(J) = \begin{cases} F_n(I \cap J) & I \cap J \neq \emptyset \\ 0 & I \cap J = \emptyset \end{cases}$$

for every $J \in \mathcal{I}$, where $F_n^{(I)}$'s are the primitives of $f_n^{(I)}$'s.

Evidently $(f_n^{(I)})$ satisfies (i). Let us show that $(F_n^{(I)})$ satisfies (ii) and (iii). Firstly, we show that $(F_n^{(I)})$ is *USL*. By $\partial(I)$ the set of endpoints of I is denoted. Let us consider any $\epsilon > 0$ and an arbitrary $Z \subset [0, 1]$ with $\lambda(Z) = 0$. We can define a gauge δ_0 on Z such that for every δ_0 -fine Z -tagged $\mathcal{HK}$ -partition π in $[0, 1]$, we have

$$(J, z) \in \pi \text{ and } z \notin \partial I \Rightarrow J \subset I \text{ or } J \subset S \setminus I$$

Since (F_n) is *USL*, there exists a gauge δ_1 on Z such that for each δ_1 -fine Z -tagged $\mathcal{HK}$ -partition π in $[0, 1]$, we have $\sum_{(J,z) \in \pi} |F_n(J)| < \frac{\epsilon}{3}$, for all $n \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} \sum_{(J,z) \in \pi} |F_n^{(I)}(J)| &= \sum_{z \in \partial(I): (J,z) \in \pi} |F_n^{(I)}(J)| + \sum_{z \notin \partial(I): (J,z) \in \pi} |F_n^{(I)}(J)| = \\ &= \sum_{z \in \partial(I): (J,z) \in \pi} |F_n^{(I)}(J)| + \sum_{z \in \text{int}(I): (J,z) \in \pi} |F_n(J)| < 2 \cdot \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for each δ -fine Z -tagged $\mathcal{HK}$ -partition $\pi = \{(J, z)\}$ in $[0, 1]$, where $\delta = \min\{\delta_0, \delta_1\}$.

This means that $(F_n^{(I)})$ is *USL*.

Secondly, since each $F_n^{(I)}$ is continuous on $[0, 1]$ and for every $J \in \mathcal{I}$ we have $\omega(F_n^{(I)}, J) \leq \omega(F_n, J)$, we obtain by Lemma 1.3 that $(F_n^{(I)})$ satisfies (iii).

Thus, by applying the same arguments as in the case when $I = [0, 1]$ we infer $\lim_{n \rightarrow \infty} F_n^I(I) = F(I)$. Finally, since I was arbitrary we get $\lim_{n \rightarrow \infty} F_n(I) = F(I)$, for every $I \in \mathcal{I}$. $\square$

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