How do read-once formulae shrink?

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Abstract

Let f be a de Morgan read-once function of n variables. Let f_ε be the random restriction obtained by independently assigning to each variable of f, the value 0 with probability $(1-\varepsilon)/2$, the value 1 with the same probability, and leaving it unassigned with probability ε. We show that f_ε depends, on the average, on only $O(\varepsilon^\alpha n + \varepsilon n^{1/\alpha})$ variables, where $\alpha = \log_{\sqrt{5}-1} 2 \simeq 3.27$. This result is asymptotically the tightest possible. It improves a similar result obtained recently by Håstad, Razborov and Yao.

1 Introduction

Obtaining non-trivial lower bounds on the complexity of Boolean functions is currently a very difficult task. Only a handful of methods yielding such lower bounds are currently known and even they work only in suitably restricted models. The current state of affairs in this respect is summerized in the books of Dunne [5] and Wegener [15] and the survey paper of Boppana and Sipser [2].

Many of the currently known methods for obtaining complexity lower bounds use the tool of *random restrictions*. A random restriction f_ε of a Boolean function f is obtained by randomly assigning Boolean values to some of f variables. The parameter $0 \le \varepsilon \le 1$ denotes the probability in which each variable remains unassigned. With the complementary probability $1-\varepsilon$ each variables is assigned, with equal probabilities, the values 0 or 1. The choices made for each variable are assumed to be independent of the choices made for all the other variables.

Arguments using random restrictions are used in the chain of works by Furst, Saxe and Sipser [6], Yao [16] and Håstad [7] that culminate in the proof that bounded depth (and unbounded fan-in) circuits for the parity function have exponential size. Other arguments using random restrictions were used by Karchmer and Wigderson [10] in their proof that monotone circuits for *st*-connectivity require super-logarithmic depth (and hence super-polynomial monotone formula size).

Random restrictions were first used, however, by Subbotovskaya [18]. She showed that the (de Morgan) formula size of any function is expected to shrink by a factor of at least $\varepsilon^{1.5}$ when hit by a random restriction leaving only fraction ε of the variables unassigned. This immediately implied an $\Omega(n^{1.5})$ lower bound on the (de Morgan) formula size of the parity function of n variables. The bound for the parity function was later improved to a tight $\Omega(n^2)$ lower bound by Khrapchenko [19],[20], using a different method.

Andreev [17] renewed the interest in the shrinkage of formulae under random restrictions when he showed how to obtain, using Subbotovskaya's arguments coupled with some new ideas, an $\Omega(n^{2.5-o(1)})$ lower bound on the formula size of a natural function of n variables whose formula size is known to

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be at most $O(n^3)$. It was later observed by Nisan and Impagliazzo [12] that Andreev's arguments actually give a lower bound of $\Omega(n^{\Gamma+1-o(1)})$ for Andreev's function, where $\Gamma \geq 1.5$ is the largest exponent that can replace the 1.5 in Subbotovskaya's result. This exponent Γ is called the *shrinkage exponent* of de Morgan formulae. Nisan and Impagliazzo [12] were also able, using some nice ideas, to show that $\Gamma \geq 1.55$. Paterson and Zwick [13] were then able to improve this to $\Gamma \geq \frac{5-\sqrt{3}}{2} \simeq 1.63$. The parity function shows, as noticed by many researchers, that $\Gamma \leq 2$. Håstad [8] announced, very recently, that $\Gamma = 2$.

Paterson and Zwick [13] also considered the problem of finding the shrinkage exponents of monotone and of read-once formulae, denoted respectively by Γ_m and Γ^*. They showed that $\Gamma \leq \Gamma_m \leq \Gamma^* \leq \alpha$ where $\alpha = \log_{\sqrt{5}-1} 2 \simeq 3.27$ and conjectured that $\Gamma_m = \Gamma^* = \alpha$. The conjecture $\Gamma_m = \alpha$ is interesting as it would imply an $\Omega(n^{\alpha-o(1)})$ lower bound on the monotone formula size of the majority function. A proof that $\Gamma^* = \alpha$ can be seen as a first step towards proving that $\Gamma_m = \alpha$.

Håstad, Razborov and Yao [9] proved recently that $\Gamma^* = \alpha$. It is the purpose of this paper to give a tighter analysis of the shrinkage of read-once formulae, leading in particular to an alternative proof of the result $\Gamma^* = \alpha$. Both the work of Håstad, Razborov and Yao [9] and the current work use results of Valiant [14] and Boppana [1] concerning the amplification properties of read-once formulae. In this work we also use some extensions of Boppana's results obtained by Dubiner and Zwick [3],[4].

2 Preliminaries

Definition 2.1 (Formulae) A (de Morgan) *formula* in the n variables $x_1, \ldots, x_n$ is defined recursively as follows: (i) for $1 \leq i \leq n$, the variables x_i and their negations $\overline{x_i}$ are formulae; (ii) if f and g are formulae then so are $(f \wedge g)$, $(f \vee g)$. A formula is *monotone* if no negated variables appear in it. A formula is *read-once* if every variable appears in it at most once. Formulae define Boolean functions in the obvious way. When no confusion arises, we use the same symbol (e.g., f) to denote, both a formula and the function it defines. The *size* of a formula f is the number of occurrences of variables (and negated variables) in it. The formula complexity of a function f, denoted by $L(f)$, is the minimal size of a formula that defines f.

In the sequel it will be convenient to treat read-once formulae as monotone formulae. This is easily done by replacing negated variables appearing in a read-once formula by the un-negated variables. As each of these variables does not appear anywhere else in the formula, this will not affect the shrinkage properties of this read-once formula.

Definition 2.2 (Amplification) Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, we define its *amplification function* $f : [0,1] \to [0,1]$ as follows: $f(p) = \Pr[f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = 1]$, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are independent random variables each assuming the value 1 with probability p and the value 0 with probability $1-p$. Again, we usually use the same symbol (e.g., f) for both a Boolean function and its amplification function. A Boolean function $f : \{0,1\}^n \to \{0,1\}$ *amplifies* (p,q) to (p',q') if $f(p) = p'$ and $f(q) = q'$.

Definition 2.3 (Random restrictions) Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and a number $0 \leq \varepsilon \leq 1$, the random restriction f_ε is the random function $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are independent random variables and

$$\mathbf{x}_i = \begin{cases} 0 & \text{with probability } (1-\varepsilon)/2; \\ 1 & \text{with probability } (1-\varepsilon)/2; \\ x_i & \text{with probability } \varepsilon. \end{cases}$$

We denote by $E_\varepsilon(f) = E[L(f_\varepsilon)]$ the expected formula size of f_ε. If f is a read-once formula, then $E_\varepsilon(f)$ is just the expected number of variables on which f_ε depends.

The following results are easily verified.

Lemma 2.4 ([1]) *Let f be a read-once formula.*

1. *If $f = f_1 \wedge f_2$ then $f(x) = f_1(x) f_2(x)$.*
2. *If $f = f_1 \vee f_2$ then $f(x) = 1 - (1 - f_1(x))(1 - f_2(x))$.*

Lemma 2.5 ([9]) *Let f be a read-once formula.*

1. *If $f = f_1 \wedge f_2$ then $E_\varepsilon(f) = f_1(\frac{1+\varepsilon}{2}) E_\varepsilon(f_2) + f_2(\frac{1+\varepsilon}{2}) E_\varepsilon(f_1)$.*
2. *If $f = f_1 \vee f_2$ then $E_\varepsilon(f) = (1 - f_1(\frac{1-\varepsilon}{2})) E_\varepsilon(f_2) + (1 - f_2(\frac{1-\varepsilon}{2})) E_\varepsilon(f_1)$.*

Lemma 2.5 was first obtained by Håstad, Razborov and Yao [9]. To verify the first part of it, for example, note that $f_1(\frac{1+\varepsilon}{2})$ and $f_2(\frac{1+\varepsilon}{2})$ are the probabilities that $(f_1)_\varepsilon$ and $(f_2)_\varepsilon$, respectively, are not identically zero.

Definition 2.6 (Shrinkage exponents) The *shrinkage exponent* Γ_C of a class C of Boolean functions is defined as the least upper bound of all $\gamma \geq 1$ for which $E_\varepsilon(f) = O(\varepsilon^\gamma L(f) + 1)$ for every $f \in C$ and $0 \leq \varepsilon \leq 1$ (the constant implied by this big-O notation should depend only on C, and not on f and ε). We denote by Γ, Γ_m and Γ^* the shrinkage exponents, respectively, of all the Boolean functions, of the monotone Boolean functions and the read-once Boolean functions.

The current knowledge on the shrink exponents is summarized in the following Theorem

Theorem 2.7 ([8],[9]) *1.* $\Gamma = 2$.

2. $\Gamma^* = \alpha$ *where* $\alpha = \log_{\sqrt{5}-1} 2 \simeq 3.27$.
3. $\Gamma \leq \Gamma_m \leq \Gamma^*$.

The result $\Gamma = 2$ was obtained by Håstad [8]. The result $\Gamma^* = \alpha$ was obtained by Håstad, Razborov and Yao [9]. The other relation is obvious. To obtain the $\Gamma^* = \alpha$ result, Håstad, Razborov and Yao [9] proved the following:

Theorem 2.8 ([9]) *If f is a read-once function of n variables then*

$$E_\varepsilon(f) = O(\varepsilon^\alpha (\ln \frac{1}{\varepsilon})^{\alpha-1} n + (\ln n)^{-1}).$$

Our main result is the following improvement of Håstad, Razborov and Yao's result:

Theorem 2.9 *If f is a read-once function of n variables then*

$$E_\varepsilon(f) = O(\varepsilon^\alpha n + \varepsilon n^{1/\alpha}) \quad .$$

Examples given by Paterson and Zwick [13] and Håstad, Razborov and Yao [9] show that this result is asymptotically the tightest possible.

Theorem 2.9 will follow from the results of Section 4. We first show there (Theorem 4.1) that an upper bound on $E_\varepsilon(f)$ may be obtained using any real function $M(s, x, y)$ that satisfies certain inequalities. (This is reminiscent of the methods used in [4]). We then construct (in Theorem 4.2) a funtion $M(s, x, y)$ using which we get the correct asymptotical behavior of $E_\varepsilon(f)$ (for the most shrink resistent f). The proof of Theorem 4.1 is simple. The proof of Theorem 4.2, on the other hand, is a bit tedious. It is given in Section 5. In the next section we summarize some of the amplification results of [1],[4] that are needed here.

3 Amplification bounds

Let

$$a = \tfrac{\sqrt{5}-1}{2} \simeq 0.62$$

$$\alpha = \log_{\sqrt{5}-1} 2 \simeq 3.27 \quad , \quad \beta = \log_{\frac{\sqrt{5}+1}{2}} 2 \simeq 1.44 \quad , \quad \gamma = \exp(\tfrac{2-a-\ln 4}{\beta}) \simeq 0.997$$

$$\delta = \exp(\tfrac{1-a-\ln 4}{\beta}) \simeq 0.498 \quad , \quad c_0 = \delta \, (\ln \tfrac{\gamma}{\delta})^{1/\beta} \simeq 0.387$$

The following function plays a central role in the study of the amplification properties of read-once functions.

$$G(x) = \begin{cases} x \, (\ln \frac{\gamma}{x})^{1/\beta} / c_0 & \text{if } 0 \le x \le \delta \\ 1 & \text{if } \delta \le x \le 1-\delta \\ (1-x) \left(\ln \frac{\gamma}{1-x}\right)^{1/\beta} / c_0 & \text{if } 1-\delta \le x \le 1 \end{cases} .$$

Clearly $G(x) = G(1-x)$ for every $0 \le x \le 1$. The function $G(x)$ defined here is a slight variation of a similar function defined in [4]. The function $G(x)$ satisfies the following inequality:

Lemma 3.1 (Boppana's inequality) *For every $0 < x, y \le 1$ we have*

$$\left(\frac{G(x)}{x}\right)^{\beta} + \left(\frac{G(y)}{y}\right)^{\beta} \le \left(\frac{G(xy)}{xy}\right)^{\beta} .$$

A proof of this Lemma will be given in the appendix. A similar inequality was proved in [4] for the function $\tilde{G}(x) = x \, (\ln \frac{\gamma}{x})^{1/\beta}$ if $0 \le x \le \frac{1}{2}$ and $\tilde{G}(x) = (1-x) \left(\ln \frac{\gamma}{1-x}\right)^{1/\beta}$ if $\frac{1}{2} \le x \le 1$. Graphs of the functions $G(x)$ and $\tilde{G}(x)$ are given in the appendix. The function $\tilde{G}(x)$ is not unimodal, it has two maxima at $x \simeq 0.498$ and $x \simeq 0.502$. While this caused no problems in [4], it does cause problems here. To avoid them we slightly altered the definition of $\tilde{G}(x)$ to get the unimodal function $G(x)$.

An inequality of the form used in Lemma 3.1 was first obtained by Boppana [1]. He showed that the entropy function $H(x) = x \ln \frac{1}{x} + (1-x) \ln \frac{1}{1-x}$ satisfies such an inequality. To get good amplification bounds, one wants a symmetric function $F(x)$ (i.e., $F(x) = F(1-x)$ for $0 \le x \le 1$) that satisfies Lemma 3.1 and which has a small asymptotical behavior near $x = 0$ and $x = 1$. The function $G(x)$ (and also $\tilde{G}(x)$) are asymptotically optimal in this respect. For each function $F(x)$ that satisfies the required conditions we have $F(x) = \Omega(G(x))$.

Strengthening arguments of Boppana [1], it was shown by Dubiner and Zwick [4] that

Theorem 3.2 *If $G(x)$ is symmetric (i.e., $G(x) = G(1-x)$ for $0 \le x \le 1$) and satisfies Boppana's inequality (Lemma 3.1) and if f is a read-once function of n variables that amplifies (x_0, y_0) to (x_1, y_1) then*

$$n^{1/\alpha} \ge \int_{x_1}^{y_1} \frac{dz}{G(z)} \Big/ \int_{x_0}^{y_0} \frac{dz}{G(z)} .$$

From this we immediately get

Corollary 3.3 *If f is a read-once function of n variables, $\varepsilon \le 0.02$ and $x = f(\frac{1-\varepsilon}{2})$, $y = f(\frac{1+\varepsilon}{2})$ then*

$$\varepsilon n^{1/\alpha} \ge \int_{x}^{y} \frac{dz}{G(z)} .$$

In the next sections we rely both on Corollary 3.3, and on the methods used to obtain it, to get a proof of Theorem 2.9.

4 Shrinkage bounds

Theorem 2.9 will follow easily from the following two theorems.

Theorem 4.1 *If $M : [0,\infty) \times [0,1]^2 \to [0,\infty)$ satisfies the following two conditions:*

$$\begin{aligned} M(s_1+s_2, x_1x_2, y_1y_2) &\geq y_2M(s_1,x_1,y_1) + y_1M(s_2,x_2,y_2) \quad (1) \\ M(s,x,y) &= M(s,1-y,1-x) \quad (2) \end{aligned}$$

for every $0 \leq s$, $0 < x \leq y < 1$ and $0 \leq s_1, s_2$, $0 < x_1 \leq y_1 < 1$, $0 < x_2 \leq y_2 < 1$ such that

$$\int_{x_1}^{y_1} \frac{dz}{G(z)} \leq {s_1}^{1/\alpha} \quad , \quad \int_{x_2}^{y_2} \frac{dz}{G(z)} \leq {s_2}^{1/\alpha} \quad (3)$$

then for every read-once function f of n variables we have

$$E_\varepsilon(f) \leq \varepsilon \cdot \frac{M(\varepsilon^\alpha n, f(\frac{1-\varepsilon}{2}), f(\frac{1+\varepsilon}{2}))}{M(\varepsilon^\alpha, \frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2})} \quad .$$

Proof : By induction on n, the size of f. If $n = 1$ then $E_\varepsilon(f) = \varepsilon$ and the inequality holds (with equality). Suppose the inequality holds for all read-once functions of less than n variables. Let f be a read-once function of n variables. We show that the inequality also holds for f. Either $f = f_1 \wedge f_2$ or $f = f_1 \vee f_2$ where f_1 and f_2 depend respectively on n_1 and n_2 variables, were $n_1 + n_2 = n$ and $n_1, n_2 < n$. By the induction hypothesis we can assume therefore that the inequality holds for both f_1 and f_2. Assume at first that $f = f_1 \wedge f_2$. Let

$$\begin{array}{ccccc} s_0 = \varepsilon^\alpha & , & x_0 = \frac{1-\varepsilon}{2} & , & y_0 = \frac{1+\varepsilon}{2} \\ s_1 = \varepsilon^\alpha n_1 & , & x_1 = f_1(x_0) & , & y_1 = f_1(y_0) \\ s_2 = \varepsilon^\alpha n_2 & , & x_2 = f_2(x_0) & , & y_2 = f_2(y_0) \\ s = s_1 + s_2 = \varepsilon^\alpha n & , & x = x_1x_2 = f(x_0) & , & y = y_1y_2 = f(y_0) \end{array}$$

By Lemma 2.5, the induction hypothesis and the conditions on $M(s,x,y)$ we have

$$E_\varepsilon(f_1 \wedge f_2) \;=\; y_1E_\varepsilon(f_2) + y_2E_\varepsilon(f_1) \;\leq\; \varepsilon \cdot \frac{y_2M(s_1,x_1,y_1) + y_1M(s_2,x_2,y_2)}{M(s_0,x_0,y_0)} \;\leq\; \varepsilon \cdot \frac{M(s,x,y)}{M(s_0,x_0,y_0)} \quad .$$

We know, by Corollary 3.3, that $\int_{x_1}^{y_1} \frac{dz}{G(z)} \leq {s_1}^{1/\alpha}$ and $\int_{x_2}^{y_2} \frac{dz}{G(z)} \leq {s_2}^{1/\alpha}$ so the inequality $y_2M(s_1,x_1,y_1) + y_1M(s_2,x_2,y_2) \leq M(s,x,y)$ is guarenteed to hold. The proof for the case $f = f_1 \vee f_2$ is dual to the proof given here (it uses condition (2)). The induction step is thus established and this completes the proof of the Theorem. □

Theorem 4.2 *For small enough $c_1 > 0$ and large enough $c_2, c_3 > 0$ the function*

$$M(s,x,y) = \begin{cases} s^{1/\alpha}(1 + c_2 s^{1/\alpha}) \max\{G(x), G(y)\} & \text{if } s \leq c_1; \\ c_3 s & \text{otherwise,} \end{cases}$$

satisfies the conditions of Theorem 4.1. A valid choice of c_1, c_2, c_3 is $c_1 = 10^{-18}$, $c_2 = 10^2$ and $c_3 = 10^{13}$.

A proof of Theorem 4.2 will be given in the next section.

Theorem 2.9 follows easily from Theorems 4.1 and 4.2. For small enough ε we have $M(\varepsilon^\alpha, \frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}) = \varepsilon(1+c_2\varepsilon) > \varepsilon$ and therefore $E_\varepsilon(f) \leq M(\varepsilon^\alpha n, f(\frac{1-\varepsilon}{2}), f(\frac{1+\varepsilon}{2}))$. Now $M(\varepsilon^\alpha n, f(\frac{1-\varepsilon}{2}), f(\frac{1+\varepsilon}{2}))$ is $O(\varepsilon^\alpha n)$ if $\varepsilon^\alpha n \geq c_1$ and is $O(\varepsilon n^{1/\alpha})$ if $\varepsilon^\alpha n \leq c_1$.

5 Inequalities

In the proof of Theorem 4.2, we use the following special cases of Hölder's inequality.

Lemma 5.1 (Hölder's inequality) *If $t_1 + t_2 = 1$ where $0 \le t_1, t_2$ then, as $\alpha = log_{\sqrt{5}-1} 2 \simeq 3.27$ and $\beta = \log_{\frac{\sqrt{5}+1}{2}} 2 \simeq 1.44$ satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have*

1. $(x^\beta + y^\beta)^{1/\beta} \ge t_1^{1/\alpha} x + t_2^{1/\alpha} y$ *for every* $0 \le x, y$.
2. $(x^\beta + y^\beta)^{1/\beta} \ge t_1^{1/\alpha} x + t_2^{1/\alpha} y + 0.001$ *for every* $1 \le x, y \le 3$ *and* $t_1 \le \frac{1}{10}, t_2 \ge \frac{9}{10}$.
3. $x + y \ge (\alpha x)^{1/\alpha} (\beta y)^{1/\beta}$ *for every* $0 \le x, y$.

By combining Boppana's and Hölder's inequalities we get

Lemma 5.2 *If $0 < y_1, y_2 \le 1$ and $t_1 + t_2 = 1$, $0 \le t_1, t_2$ then*

$$\frac{G(y_1 y_2)}{y_1 y_2} \ge \left[\left(\frac{G(y_1)}{y_1} \right)^\beta + \left(\frac{G(y_2)}{y_2} \right)^\beta \right]^{1/\beta} \ge t_1^{1/\alpha} \frac{G(y_1)}{y_1} + t_2^{1/\alpha} \frac{G(y_2)}{y_2} .$$

The following extension of Lemma 5.2 plays a central role in the proof of Theorem 4.2.

Lemma 5.3 *If $\int_{x_1}^{y_1} \frac{dz}{G(z)}, \int_{x_2}^{y_2} \frac{dz}{G(z)} \le 10^{-5}$ where $0 < x_1 \le y_1 < 1$, $0 < x_2 \le y_2 < 1$ then*

1. *if $(y_1 - a)^2 + (y_2 - a)^2 \ge 0.01$ then*

$$\left(\frac{\max\{G(x_1 x_2), G(y_1 y_2)\}}{y_1 y_2} \right)^\beta \ge \left(\frac{\max\{G(x_1), G(y_1)\}}{y_1} \right)^\beta + \left(\frac{\max\{G(x_2), G(y_2)\}}{y_2} \right)^\beta .$$

2. *if $t_1 + t_2 = 1$ where $0 \le t_1, t_2$ and either $(y_1 - a)^2 + (y_2 - a)^2 \ge 0.01$ or $t_1 \le \frac{1}{10}, t_2 \ge \frac{9}{10}$ then*

$$\frac{\max\{G(x_1 x_2), G(y_1 y_2)\}}{y_1 y_2} \ge t_1^{1/\alpha} \frac{\max\{G(x_1), G(y_1)\}}{y_1} + t_2^{1/\alpha} \frac{\max\{G(x_2), G(y_2)\}}{y_2} .$$

The proof of this Lemma is lengthy and it is presented in the appendix. We just offer here some intuitive arguments for its validity. If $\int_{x_1}^{y_1} \frac{dz}{G(z)}, \int_{x_2}^{y_2} \frac{dz}{G(z)} \le 10^{-5}$ then $G(x_i)$ and $G(y_i)$, for $i = 1, 2$, and $G(x_1 x_2)$ and $G(y_1 y_2)$ are extremely close (see next Lemma). The required inequalities will hold therefore unless both inequalities (i.e., Boppana's and Hölder's inequalities) are satisfied with near equality. Equality in Boppana's inequality is obtained only when $y_1 = y_2 = a$ or when $y_1 = 1$ or $y_2 = 1$. Equality in Hölder's inequality, when $y_1 = y_2 = a$ holds only if $t_1 = t_2 = \frac{1}{2}$. The first inequality is not claimed to hold (and in fact, does not hold) when $y_1 \approx y_2 \approx a$. The first inequality does hold however when $y_1 \approx y_2 \approx a$ but $t_1 \le \frac{1}{10}, t_2 \ge \frac{9}{10}$, as the required slackness is obtained in Hölder's inequality. A more careful analysis is required to show that both inequalities do hold when $y_1 \approx 1$ or $y_2 \approx 1$.

A final ingredient needed in the proof of Theorem 4.2 is the following:

Lemma 5.4

1. *If $\int_x^y \frac{dz}{G(z)} \le r$ then $y - x \le r$.*
2. *If $\int_x^y \frac{dz}{G(z)} \le r \le 0.01$ and $0 < x \le y \le 0.1$ then $y^{1+4r} \le x$.*
3. *If $\int_x^y \frac{dz}{G(z)} \le r \le 0.01$ and $0.9 < x \le y < 1$ then $(1 - x)^{1+4r} \le (1 - y)$.*

Proof : Part 1 follows easily as $0 \le G(x) \le 1$ for every $0 \le x \le 1$ and therefore $y - x \le \int_x^y \frac{dz}{G(z)} \le r$. To prove part 2, note that if $x \le y^{1+4r}$ and $0 < x \le y \le 0.1$ then

$$\begin{aligned} \int_x^y \frac{dz}{G(z)} &> c_0 \int_x^y \frac{dz}{z(\ln \frac{1}{z})^{1/\beta}} = c_0\alpha\left[\left(\ln\frac{1}{x}\right)^{1/\alpha} - \left(\ln\frac{1}{y}\right)^{1/\alpha}\right] \\ &\ge c_0\alpha\left(\ln\frac{1}{y}\right)^{1/\alpha}[(1+4r)^{1/\alpha} - 1] > r \quad . \end{aligned}$$

The last inequality here holds as for $y \le 0.1$ we have $c_0\alpha\left(\ln\frac{1}{y}\right)^{1/\alpha} > 1$, and for $r \le 0.01$ we have $[(1+4r)^{1/\alpha} - 1] > r$. This is, of course, a contradiction and part 2 is proved. Part 3 follows from part 2 by symmetry. □

We may now present a proof of Theorem 4.2.

Proof (of Theorem 4.2) : Let $0 < x \le y < 1$ and $0 \le s$. Condition (2) follows easily as $G(x) = G(1-x)$ and $G(y) = G(1-y)$. Assume now that $0 \le s_1, s_2$ and that $0 < x_1 \le y_1 < 1$, $0 < x_2 \le y_2 < 1$ satisfy condition (3). We break the proof of condition (1) into three cases:

Case 1 $s_1, s_2 \ge c_1$.

This is the easiest case. We certainly have $s_1 + s_2 \ge c_1$. Condition (1) is easily satisfied as

$$\begin{aligned} M(s_1+s_2, x_1x_2, y_1y_2) &= c_3(s_1+s_2) = c_3s_1 + c_3s_2 = \\ M(s_1,x_1,y_1) + M(s_2,x_2,y_2) &\ge y_2M(s_1,x_1,y_1) + y_1M(s_2,x_2,y_2) \quad . \end{aligned}$$

Case 2 $s_1 \ge c_1$ but $s_2 \le c_1$ (or vice versa).

We have to show that

$$\begin{aligned} M(s_1+s_2, x_1x_2, y_1y_2) &= c_3(s_1+s_2) \ge \\ y_2M(s_1,x_1,y_1) + y_1M(s_2,x_2,y_2) &= y_2(c_3s_1) + y_1(s_2^{1/\alpha}(1+c_2s_2^{1/\alpha})\max\{G(x_2),G(y_2)\}) \end{aligned}$$

or equivalently that

$$c_3[(1-y_2)s_1 + s_2] \ge y_1 \cdot s_2^{1/\alpha}(1+c_2s_2^{1/\alpha})\max\{G(x_2),G(y_2)\} \quad .$$

As $s_2 \le c_1 \le s_1$ and $0 < y_1 < 1$ it is enough to show that

$$c_3[c_1(1-y_2) + s_2] \ge (1 + c_1^{1/\alpha}c_2)s_2^{1/\alpha}\max\{G(x_2),G(y_2)\} \quad .$$

By Hölder's inequality (third part of Lemma 5.1), we have

$$[c_1(1-y_2) + s_2] \ge (\beta c_1(1-y_2))^{1/\beta}(\alpha s_2)^{1/\alpha}$$

and as a consequence it is enough to show that

$$c_4(1-y_2)^{1/\beta} \ge \max\{G(x_2),G(y_2)\}$$

where $c_4 = \alpha^{1/\alpha}(\beta c_1)^{1/\beta}c_3/(1+c_1^{1/\alpha}c_2)$ ($c_4 \simeq 5.9$ for the specific values given). Now if $y_2 \le 0.52$ and $c_4(0.48)^{1/\beta} \ge 1$ (as is the case for the specific values given) then the inequality is easily satisfied. Assume therefore that $y_2 \ge 0.52$. By Lemma 5.4(1) we know that $x_2 \ge 0.51$ and therefore $G(x_2) \ge G(y_2)$. It is enough, therefore, to show that

$$c_4(1-y_2)^{1/\beta} \ge (1-x_2)\left(\ln\frac{1}{1-x_2}\right)^{1/\beta}/c_0 \ge G(x_2) \quad .$$

By Lemma 5.4(3) and the fact that $s_2 \le c_1$ we get that

$$(1-x_2)^{1.01} \le (1-x_2)^{1+4c_1^{1/\alpha}} \le 1-y_2 \quad .$$

The required inequality follows then from the fact that

$$(c_0c_4)^\beta \left(\frac{1}{1-x_2}\right)^{\beta-1.01} \ge \ln\frac{1}{1-x_2}$$

for every $x_2 \ge 0.52$, provided that $c_0c_4 \ge 2$ (which is again satisfied).

Case 3 $s_1, s_2 \le c_1$.

We have to show that

$$\frac{M(s_1+s_2, x_1x_2, y_1y_2)}{y_1y_2} \ge \frac{M(s_1,x_1,y_1)}{y_1} + \frac{M(s_2,x_2,y_2)}{y_2} \quad .$$

We do not know here whether s_1+s_2 is larger or smaller than c_1. If c_3 is large enough, however, (and $c_3 = 10^{13}$ is large enough) then in either case we have

$$M(s_1+s_2, x_1x_2, y_1y_2) \ge (s_1+s_2)^{1/\alpha}(1+c_2(s_1+s_2)^{1/\alpha})\max\{G(x_1x_2), G(y_1y_2)\} \quad .$$

It is therefore enough to prove the following inequality:

$$\begin{aligned}\frac{\max\{G(x_1x_2),G(y_1y_2)\}}{y_1y_2} \quad &\ge \quad \frac{1+c_2s_1^{1/\alpha}}{1+c_2(s_1+s_2)^{1/\alpha}}\cdot\left(\frac{s_1}{s_1+s_2}\right)^{1/\alpha}\cdot\frac{\max\{G(x_1),G(y_1)\}}{y_1} \\ &+ \quad \frac{1+c_2s_2^{1/\alpha}}{1+c_2(s_1+s_2)^{1/\alpha}}\cdot\left(\frac{s_2}{s_1+s_2}\right)^{1/\alpha}\cdot\frac{\max\{G(x_2),G(y_2)\}}{y_2} \quad .\end{aligned}$$

Note that $\frac{1+c_2s_i^{1/\alpha}}{1+c_2(s_1+s_2)^{1/\alpha}} \le 1$ for $i=1,2$. If either $(y_1-a)^2+(y_2-a)^2 \ge 0.01$ or $t_1 = s_1/(s_1+s_2) \le 0.1$, $t_2 = s_2/(s_1+s_2) \ge 0.9$ (or vice versa) the required inequality follows from Lemma 5.3. We may assume, therefore, that $(y_1-a)^2+(y_2-a)^2 \le 0.01$ and that $0.1 \le t_1, t_2 \le 0.9$. If $0.5 \le x \le 0.75$ then $0.8 \le G(x) \le 1$ and $-2 \le G'(x) \le 0$ and therefore $G(x_i)/G(y_i) \le 1+3(y_i-x_i) \le 1+3s_i^{1/\alpha}$ and

$$\frac{1+c_2s_i^{1/\alpha}}{1+c_2(s_1+s_2)^{1/\alpha}}\cdot\frac{G(x_i)}{G(y_i)} \le \frac{(1+c_2s_i^{1/\alpha})(1+3s_i^{1/\alpha})}{1+c_2(s_1+s_2)^{1/\alpha}} \le 1$$

provided that $c_2 \ge 3/((\frac{10}{9})^{(1/\alpha)}-1-3c_1^{1/\alpha})$. This clearly holds when $c_2 = 10^2$. The last inequality, in conjunction with Lemma 5.2, imply condition (1). □

6 Concluding remarks

We have obtained a tight analysis of the shrinkage of read-once formulae. The more interesting problem, that of the shrinkage of monotone formulae remains open. We hope that some of the ideas used here may prove useful when attacking that problem.

Acknowlegement

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A Proof of Lemma 3.1

Let

$$\tilde{G}(x) = \begin{cases} x\left(\ln \frac{\gamma}{x}\right)^{1/\beta} & \text{if } 0 \le x \le 0.5 \\ (1-x)\left(\ln \frac{\gamma}{1-x}\right)^{1/\beta} & \text{if } 0.5 \le x \le 1 \end{cases} .$$

A graph of the function $\tilde{G}(x)$ is given in Figure 1. As can be seen from the enlarged graph given on the right of the Figure, the function $\tilde{G}(x)$ has a small cusp at $x = \frac{1}{2}$. It can be easily checked that the function $\tilde{G}(x)$ attains its two maxima at $x = \delta = \exp(\frac{1-a-\ln 4}{\beta}) \simeq 0.497954$ and at $x = 1 - \delta \simeq 0.502046$. The function $G(x)$ defined in Section 3 is obtained from $\tilde{G}(x)$ by replacing the cusp with a short line segement connecting the two maxima points, as shown by the dashed line in the right half of Figure 1. The function $G(x)$ is also scaled so that $G(x) = 1$ for every $\delta \le x \le 1 - \delta$. It is proved in [4] that the function $\tilde{G}(x)$ satisfies Boppana's inequality:

Lemma A.1 *For every $0 < x, y \le 1$ we have*

$$\left(\frac{\tilde{G}(x)}{x}\right)^{\beta} + \left(\frac{\tilde{G}(y)}{y}\right)^{\beta} \le \left(\frac{\tilde{G}(xy)}{xy}\right)^{\beta} .$$

□

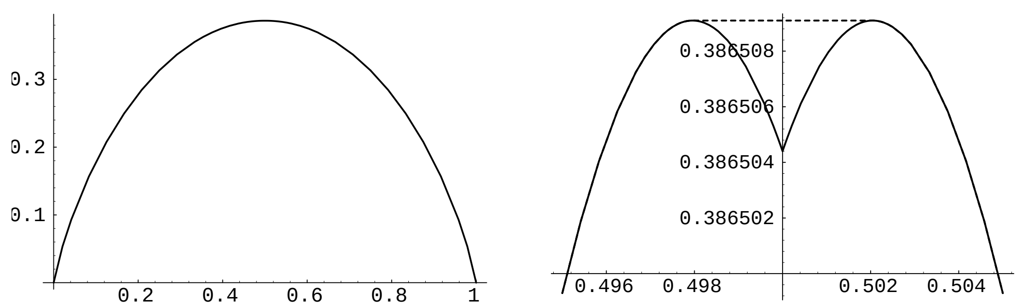

Figure 1: The function $\tilde{G}(x)$.

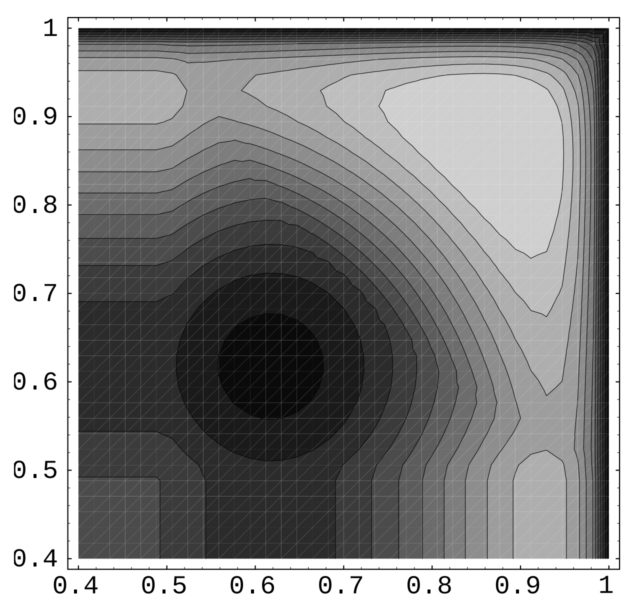

Figure 2: A contour plot of the function $\tilde{F}(x, y)$.

A contour plot of the funtion

$$\tilde{F}(x,y) = \left(\frac{\tilde{G}(xy)}{xy}\right)^\beta - \left(\frac{\tilde{G}(x)}{x}\right)^\beta - \left(\frac{\tilde{G}(y)}{y}\right)^\beta$$

for $0.4 \le x, y \le 1$ is given in Figure 2. Nothing interesting happens to $\tilde{F}(x,y)$ when $x \le 0.4$ or $y \le 0.4$. The function $\tilde{F}(x,y)$ has a global minimum point at $(x,y) = (a,a)$, where $a = \frac{\sqrt{5}-1}{2} \simeq 0.62$, whose value is 0. The value 0 is also attained along the lines $x = 1$ and $y = 1$. In all other points the strict inequality $\tilde{F}(x,y) > 0$ is satisfied. The function $\tilde{F}(x,y)$ attains its global maximum at $(x,y) = (b,b)$ where $b \simeq 0.88$. The value of the global maximum is about 0.011.

Lemma 3.1 claims that the inequality in Lemma A.1 continues to hold when $\tilde{G}(x)$ is replaced by $G(x)$. This is hardly surprising as the definition of $G(x)$ differs from that of $\tilde{G}(x)$ only at points in which the inequality in Lemma A.1 is strict. To the request of the referee, we include the technical proof of Lemma 3.1.

Proof of Lemma 3.1 : Note that $G(x) = \tilde{G}(x)/c_0$ for $x \le \delta$ and for $x \ge 1-\delta$ and that $G(x) \ge \tilde{G}(x)/c_0$ for $\delta \le x \le 1-\delta$. The inequlity we have to prove is therefore automatically satisfied when $x \notin [\delta, 1-\delta]$ and $y \notin [\delta, 1-\delta]$. Assume therefore that $\delta \le x \le 1-\delta$. The case $\delta \le y \le 1-\delta$ is symmetric. We consider four different cases.

Case 1 $0 < y \le \delta$.

We have to verify that $(\frac{c_0}{x})^\beta + \ln\frac{\gamma}{y} \le \ln\frac{\gamma}{xy}$ or equivalently that $\ln\frac{1}{x} \ge (\frac{c_0}{x})^\beta$ for every $\delta \le x \le 1-\delta$. The even stronger one-variable inequality, $\ln\frac{\gamma^{1/2}}{x} \ge (\frac{c_0}{x})^\beta$, for every $\delta \le x \le 1-\delta$, is easily verified.

Case 2 $\delta \le y \le 1-\delta$.

We have to verify that $(\frac{c_0}{x})^\beta + \left(\frac{c_0}{y}\right)^\beta \le \ln\frac{\gamma}{xy}$ for every $\delta \le x \le 1-\delta$. This follows from the fact that $\ln\frac{\gamma^{1/2}}{x} \ge (\frac{c_0}{x})^\beta$ and that $\ln\frac{\gamma^{1/2}}{y} \ge \left(\frac{c_0}{y}\right)^\beta$ for every $\delta \le x, y \le 1-\delta$.

Case 3 $1-\delta \le y \le \frac{\delta}{x}$.

We have to verify that

$$\ln\frac{\gamma}{xy} - (\tfrac{c_0}{x})^\beta - \left(\frac{1}{y} - 1\right)^\beta \ln\frac{\gamma}{1-y} \ge 0$$

for every $\delta \le x \le 1-\delta$. The partial derivative of the above expression, with respect to x, is

$$-\frac{1}{x} + \frac{\beta c_0}{x^2}\left(\frac{c_0}{x}\right)^{\beta-1} = -\frac{1}{x}\left(1 - \beta\left(\frac{c_0}{x}\right)^\beta\right) .$$

which is anihilated only when $x = \beta^{1/\beta} \cdot c_0 = \delta$. It is therefore enough to check that the desired inequality holds on the boundary of the region considered in this case, i.e., along the line segments $\{(x, 1-\delta) : \delta \le x \le 1-\delta\}$, $\{(\delta, y) : 1-\delta \le y \le 1\}$ and $\{(1-\delta, y) : 1-\delta \le y \le \frac{\delta}{1-\delta}\}$, and along the hyperbola segment $\{(x, \frac{\delta}{x}) : \delta \le x \le 1-\delta\}$. That the inequality holds on the first line segment follows from the previous case. The fact that it holds on the other two line segments follows from Lemma A.1. The hyperbola segment is dealt with in the next case.

Case 4 $\frac{\delta}{x} \le y \le 1$.

We have to verify that

$$\left(\frac{c_0}{xy}\right)^\beta - (\tfrac{c_0}{x})^\beta - \left(\frac{1}{y} - 1\right)^\beta \ln\frac{\gamma}{1-y} \ge 0$$

for every $\delta \le x \le 1-\delta$. The partial derivative of the above expression, with respect to x, is

$$\frac{\beta c_0^\beta}{x^{\beta+1}}(1 - \left(\frac{1}{y}\right)^\beta)$$

which is anihilated only when $y = 1$. It is again enough to verify the inequality on the boundary of the region considered. The inequality is satisfied with equality along the line segment $\{(x, 1) : 1-\delta \le$

$x \le \delta\}$. It holds on the line segment $\{(1-\delta, y) : \frac{\delta}{1-\delta} \le y \le 1\}$ because of Lemma A.1. It remains again to verify the inequality along the hyperbola segment $\{(x, \frac{\delta}{x}) : \delta \le x \le 1-\delta\}$. Specifically, we have to verify that

$$(\tfrac{c_0}{\delta})^\beta - (\tfrac{c_0}{x})^\beta - (\tfrac{x}{\delta} - 1)^\beta \ln \tfrac{\gamma}{1-\delta/x} \ge 0$$

for every $\delta \le x \le 1-\delta$. This one-variable inequality is again easily verified. It is satisfied with equality when $x = \delta$ and is strict otherwise. □

In the proof of Lemma 5.3, we rely on some additional properties of the function $G(x)$ brought in the following Lemma. These properties say, in essence, that the inequality of Lemma 3.1 is very loose in most cases.

Lemma A.2

1. $\left(\frac{G(x)}{x}\right)^\beta + \left(\frac{G(y)}{y}\right)^\beta + 10^{-4} \le \left(\frac{G(xy)}{xy}\right)^\beta$ *if* $0 < x, y \le 0.9997$ *and* $(x-a)^2 + (y-a)^2 \ge 0.01$.
2. $\left(\frac{G(x)}{x}\right)^\beta + (1-y) \le \left(\frac{G(xy)}{xy}\right)^\beta$ *for every* $0 < x \le 0.5$ *and* $0.999 \le y \le 1$.
3. $G^\beta(x) + G^\beta(y) \le G^\beta(xy)$ *for every* $0.7 \le x \le 1$, $0.9995 \le y \le 1$.

Proof : 1. The point $(x, y) = (a, a)$ is the only local minimum of the function $F(x, y) = \left(\frac{G(xy)}{xy}\right)^\beta - \left(\frac{G(x)}{x}\right)^\beta - \left(\frac{G(y)}{y}\right)^\beta$ inside the open unit square $(0, 1)^2$. It is therefore enough to verify the required inequality on the line segments $\{(x, 0.9997) : 0 < x \le 1\}$, $\{(0.9997, y) : 0 < y \le 1\}$ and on the circle $\{(x, y) : (x-a)^2 + (y-a)^2 = 0.01\}$. These one-variable inequalities are again easily verfied. The minimum along the line segments is attained at $(x, y) = (0.9997, 0.9997)$ and its value is about $1.3 \cdot 10^{-4}$. The minimum along the circle is attained at $(x, y) = (a-0.1, a)$ or $(x, y) = (a, a-0.1)$ and its value is about 0.0043.

2. We consider three different cases. If $0 < x \le \delta$, we have to verify that $\ln \frac{\gamma}{x} + c_0^\beta(1-y) \le \ln \frac{\gamma}{xy}$, or equivalently that $c_0^\beta(1-y) \le \ln \frac{1}{y}$ for every $0.999 \le y \le 1$. This one-variable inequality is easily verified. If $\delta \le x \le \frac{\delta}{y}$, we have to verify that $(1/x)^\beta + (1-y) \le c_0^{-\beta} \cdot \ln \frac{\gamma}{xy}$, or equivalently that $(1/x)^\beta - c_0^{-\beta} \ln \frac{\gamma}{x} \le c_0^{-\beta} \ln \frac{1}{y} - (1-y)$, for every $0.999 \le y \le 1$. The left-hand side is increasing in x and it is therefore enough to verify the inequality when x is maximal, i.e., when $x = \frac{\delta}{y}$. The inequality then is $(y/\delta)^\beta + (1-y) \le c_0^{-\beta} \ln \frac{\gamma}{\delta}$, for every $0.999 \le y \le 1$, and this one-variable inequality is again easily verified. Finally, if $\frac{\delta}{y} \le x \le \frac{1}{2}$, we have to verify that $(1/x)^\beta + (1-y) \le (1/xy)^\beta$, or equivalently that $(1-y)/((1/y)^\beta - 1) \le (1/x)^\beta$, for every $0.999 \le y \le 1$. This is again easily verified as $(1-y)/((1/y)^\beta - 1) \le \beta^{-1} < 0.7$, for every $0.999 \le y \le 1$, and $(1/x)^\beta \ge 2^\beta > 2.7$ for every $x \le \frac{1}{2}$.

3. It can be verified using differentiation that the function $G^\beta(xy) - G^\beta(x) - G^\beta(y)$ has no critical points within the rectangle $(0.7, 1) \times (0.9995, 1)$. It is therefore enough to verify the inequality on the boundary of this rectangle. These one-variable inequalities are easily verified. □

B Proof of Lemma 5.3

Proof of Lemma 5.3(1) : Assume at first that $y_1, y_2 \le 0.9997$ and that $(y_1-a)^2 + (y_2-a)^2 \ge 0.01$. In view of Lemma A.2(1) it is enough to prove, in this case, that

$$\left(\frac{G(x_i)}{y_i}\right)^\beta - \left(\frac{G(y_i)}{y_i}\right)^\beta \le 5 \cdot 10^{-5}$$

for $i = 1, 2$. If $y_i \le 0.5$ then $G(x_i) \le G(y_i)$ and there is nothing to prove. Assume therefore that $y_i \ge 0.5$. By Lemma 5.4(1) we know that $y_i - x_i \le 10^{-5}$ and it can be easily verified that the required condition holds.

Assume therefore that $y_2 \geq 0.9997$ (the case $y_1 \geq 0.9997$ is handled similarly). We know in this case that $G(x_2) \geq G(y_2)$. If $x_1 \leq y_1 \leq 0.5$ then $G(x_1) \leq G(y_1)$ and it is enough to show, in this case, that

$$\left(\frac{G(y_1y_2)}{y_1y_2}\right)^\beta - \left(\frac{G(y_1)}{y_1}\right)^\beta \;\geq\; 1 - y_2 \geq \left(\frac{G(x_2)}{y_2}\right)^\beta \quad .$$

The first inequality here follows, as $y_1 \leq 0.5$ and $y_2 \geq 0.999$, from Lemma A.2(2). The second inequality follows from the following chain of inequalities

$$1 - y_2 \geq (1-x_2)^{1.005} \geq \left(\frac{1-x_2}{0.9997c_0}\right)^\beta \ln\frac{\gamma}{1-x_2} \geq \left(\frac{G(x_2)}{y_2}\right)^\beta \quad .$$

The first inequality here follows from Lemma 5.4(3), the third inequality follows from the definition of $G(x)$ and the second inequality can be easily verified to hold for $x_2 \geq 0.99965$.

If, on the other hand, $y_1 \geq 0.5$ (and $y_2 \geq 0.9997$) then we know that $G(x_1) \geq G(y_1)$ (as well as $G(x_2) \geq G(y_2)$) and it is enough to show that

$$G^\beta(x_1x_2) \geq y_2^\beta G^\beta(x_1) + y_1^\beta G^\beta(x_2) \quad .$$

If $x_1 \geq 0.7$, we know by Lemma A.2(3) that $G^\beta(x_1x_2) \geq G^\beta(x_1) + G^\beta(x_2)$, which is more than required. We may assume therefore that $0.499 \leq x_1 \leq 0.7$ and $0.999 \leq x_2$ (Lemma 5.4(1) is used here) and therefore $x_1x_2 \geq \delta$ and $G(x_1x_2) \geq G(x_1)$. It is therefore enough to show that

$$G^\beta(x_1)(1-y_2^\beta) \geq 0.8(1-y_2^\beta) \geq \left(\frac{1-x_2}{c_0}\right)^\beta \ln\frac{1}{1-x_2} \geq G^\beta(x_2) \quad .$$

The first inequality here follows from the fact that for $0.499 \leq x_1 \leq 0.7$ we have $G^\beta(x_1) \geq 0.8$. The third inequality follows again from the definition of $G(x)$. By Lemma 5.4(3) we get that $(1-x_2) < (1-y_2)^{1/1.01}$. The second inequality follows then from the fact that for $y_2 \geq 0.9997$ we have

$$0.8c_0^\beta(1-y_2^\beta) \geq (1-y_2)^{\beta/1.01}\ln\frac{1}{1-y_2} \quad .$$

This completes the proof of the first part.

Proof of Lemma 5.3(2) : If $(y_1-a)^2 + (y_2-a)^2 \geq 0.01$, the claim follows immediately from the first part and from Hölder's inequality (Lemma 5.1(1)), as in Lemma 5.2. Assume therefore that $(y_1-a)^2 + (y_2-a)^2 \leq 0.01$ and that $t_1 \leq 0.1, t_2 \geq 0.9$. It is easy to check that in this case $1 \leq \left(\frac{G(y_1)}{y_1}\right)^\beta, \left(\frac{G(y_2)}{y_2}\right)^\beta \leq 3$. By Boppana's inequaliy (Lemma 3.1) and Lemma 5.1(2) we get that

$$\frac{G(y_1y_2)}{y_1y_2} \geq t_1^{1/\alpha}\frac{G(y_1)}{y_1} + t_2^{1/\alpha}\frac{G(y_2)}{y_2} + 0.001 \quad .$$

As in the proof of the previous part, it is enough, therefore, to show that $\frac{G(x_i)}{y_i} - \frac{G(y_i)}{y_i} \leq 0.0005$ and this can again be easily verified. This completes the proof of the Lemma. □

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