MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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METR 2015–05 February 2015

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February 2015

Abstract

Discrete versions of basic inequalities in functional analysis such as the Sobolev inequality play key role in theoretical analysis of finite difference schemes. They have been shown for some simple difference operators, but are still left open for general operators, even including the standard central difference operators. In this paper, we propose a systematic approach for deriving such inequalities for a certain class of central-difference type operators. We illustrate the results by giving a generic a priori estimate for certain conservative schemes for the nonlinear Schrödinger equation.

1 Introduction

In this paper, we are concerned with numerical computation of evolutionary partial differential equations (PDEs), with our special interest on its theoretical analysis. Such numerical computation is an indispensable tool in modern science and engineering, and thus there is a long history with vast amount of studies based on wide range of methods—finite difference, finite element, discontinuous Galerkin, mesh free, and so on—both from practical and theoretical aspects. For theoretical aspects, we may say that the finite element methods are most developed, since they are supported directly by functional analysis theory, and thus now have very sophisticated theoretical backgrounds (see, for example, [3, 4] for basic results for elliptic problems, and [22] for parabolic problems). The backgrounds of other newer methods, for example the discontinuous Galerkin methods, are still rather weaker, but recently they have been extensively studied to rapidly catch up the finite element methods by reconstructing the functional analysis theory so that it allows discontinuous functions.

In contrast to these maturity or rapid developments, the theoretical aspects of classical finite difference methods for evolutionary PDEs seem to remain at a primitive level, and recently relatively few efforts have been newly devoted in this direction. The main reason for this might be that it is generally quite difficult to import the tools from functional analysis to the world of discrete grid points, and thus it seems most people believe that finite difference methods are not suitable for hard theoretical analysis except for very simple cases. Let us below illustrate this taking the history of the numerical schemes of the nonlinear Schrödinger (NLS) equation as an example. In the rest of this section, we use some notation without declaring; they will be given in Section 2.

Let us consider the cubic NLS on the circle $\mathbb{T}$:

$$\mathrm{i}u_t = -u_{xx} - \gamma |u|^2 u.$$

This has been challenged numerically in various ways (see, for example, a classical review by Taha–Ablowitz [21] and a modern review by Faou [7]). A milestone of such studies might be the celebrated finite difference scheme by Delfour–Fortin–Payre [6] in 1981, which had discrete counterparts of the invariants of NLS on the real line (i.e., when we consider the infinite number of grid points), and thus exhibited excellent long time numerical behavior. No theoretical error estimate for the DFP scheme had been given at this point. Then Akrivis–Dougalis–Karakashian [1] (1991) considered a Galerkin version of the DFP scheme, and with the aid of functional analysis theory, gave a complete error estimate. The crucial step of this estimate was to draw an a priori estimate $\|u\|_\infty < \infty$ out of the invariants

$$\|u\|_2 = \text{const.} \quad \text{and} \quad \frac{\gamma}{2}\|u\|_4^4 - \|u_x\|_2^2 = \text{const.}$$

The key tools there were the Sobolev inequality:

$$\|u\|_\infty \leq C\|u\|_{W^{1,2}(\mathbb{T})}, \tag{1}$$

and the Gagliardo–Nirenberg (GN) inequality (with the index of NLS type):

$$\|u\|_4^4 \leq C\|u\|_{W^{1,2}(\mathbb{T})}\|u\|_2^3. \tag{2}$$

In fact, combining the invariants and the GN we get the bound $\|u\|_{W^{1,2}(\mathbb{T})} < \infty$, which then implies by the Sobolev inequality the desired estimate (see Section 4). Note that in this process we can directly use the inequalities in continuous functional analysis theory, which clearly shows a strong advantage of finite element (or Galerkin) methods over finite difference methods. Once we have such a boundedness estimate, we can easily bound the nonlinear term, and accordingly gain an error estimate by standard argument. Afterwards, in 1998, Matsuo–Sugihara–Mori [20] (see also [10]) reconsidered the DFP scheme again in the context of structure-preserving methods and showed that the discrete "invariants" corresponding to the two continuous invariants mentioned above are actually strictly kept in finite domain with realistic boundary conditions. Then they considered discrete versions of the Sobolev and Gagliardo–Nirenberg inequalities to follow the argument of Akrivis *et al.* [1]. There, the main difficulty was the establishment of the discrete Gagliardo–Nirenberg inequality—let us below briefly look at the heart of this discussion.

In [20], "discrete" functional spaces L^p_d and $W^{1,p}_\mathrm{d}(\delta^+)$ and their associated norms:

$$\|\boldsymbol{U}\|_{L^p_{\rm d}} := \left(\sum_{k=0}^{N-1} |U_k|^p \Delta x\right)^{1/p}, \tag{3}$$

$$\|\boldsymbol{U}\|_{W^{1,p}_{\rm d}(\delta^+)} := \left(\sum_{k=0}^{N-1} \left(|U_k|^p + \left|\frac{U_{k+1}-U_k}{\Delta x}\right|^p\right)\Delta x\right)^{1/p} \tag{4}$$

were considered. Note that the latter depends explicitly on how we approximate the derivative u_x; above is the simplest version which employs the forward difference operator δ^+. Associated to these norms, the following discrete inequalities were shown in [20]: the discrete Sobolev inequality

$$\|\boldsymbol{U}\|_\infty \le C\|\boldsymbol{U}\|_{W^{1,2}(\delta^+)}, \tag{5}$$

and the discrete GN

$$\|\boldsymbol{U}\|^4_{L^4_{\rm d}} \le C\|\boldsymbol{U}\|_{W^{1,2}(\delta^+)}\|\boldsymbol{U}\|^3_{L^2_{\rm d}}. \tag{6}$$

The Sobolev inequality had been known in some works (for example, John [16]), but the proof of the discrete GN was more complicated, and was first proved in this work. Then it was shown that the periodic DFP scheme has the discrete invariants

$$\|\boldsymbol{U}^{(m)}\|_{L^2_{\rm d}}\text{=const.} \quad \text{and} \quad \frac{\gamma}{2}\|\boldsymbol{U}^{(m)}\|^4_{L^4_{\rm d}} - \|D_{\delta^+}\boldsymbol{U}^{(m)}\|^2_{L^2_{\rm d}}\text{=const.}, \tag{7}$$

where the symbol D_{δ^+} is the matrix representation of the forward difference operator δ^+. The key here is the fact that *in all of the derivative terms above, the forward difference operator δ^+ is used throughout*, which makes us possible to precisely follow the discussion in the continuous case to deduce an a priori estimate $\|\boldsymbol{U}^{(m)}\|_\infty < \infty$.

Let us get back to the history of NLS computations. Although the DFP scheme was fine in terms of its long time behavior, it was sometimes not accurate enough, since it employed the simplest low-order difference operators. Among numerous studies to overcome this drawback, here we note two examples by Matsuo *et al.* [19] (2002) and Kanazawa–Matsuo–Yaguchi [17] (2014), where high-order schemes keeping the discrete invariants were proposed by utilizing high-order central difference operators and high-order compact difference operators, respectively. The discrete invariants read

$$\|\boldsymbol{U}^{(m)}\|_{L^2_{\rm d}}\text{=const.} \quad \text{and} \quad \frac{\gamma}{2}\|\boldsymbol{U}^{(m)}\|^4_{L^4_{\rm d}} - \|D_{\delta^{\langle 1\rangle},2s}\boldsymbol{U}^{(m)}\|^2_{L^2_{\rm d}}\text{=const.} \tag{8}$$

in Matsuo *et al.*[19], where $D_{\delta^{\langle 1\rangle},2s}$ denotes the matrix representation of certain high-order central operators, and

$$\|\boldsymbol{U}^{(m)}\|_{L^2_{\rm d}}\text{=const.} \quad \text{and} \quad \frac{\gamma}{2}\|\boldsymbol{U}^{(m)}\|^4_{L^4_{\rm d}} - \|D_{\delta_{\rm c}^{\langle 1\rangle}}\boldsymbol{U}^{(m)}\|^2_{L^2_{\rm d}}\text{=const.} \tag{9}$$

in Kanazawa *et al.* [17], where $D_{\delta_{\rm c}^{\langle 1\rangle}}$ corresponds to some compact difference operators. The discrete invariants not only gave good qualitative nature to the schemes but also raised the expectation to the rigorous convergence analyses as before. *But this expectation has not become a reality so far.* We need the discrete inequalities (1) and (2) for the corresponding high-order difference operators. They, however, had not been known in the literature, and after some struggle it turned out that they were much more difficult to establish than we simply expected; even whether they actually hold or not were unclear.

The difficulty can be understood from the following two viewpoints. First observation is that the high-order central difference operators and the compact difference operators can have falsely larger kernel space. The kernel space of the differential operator ∂_x (under the periodic boundary condition) is $\mathrm{span}\{1\}$ (the constants), and the same for the forward difference operator δ^+: $\mathrm{span}\{(1,1,\ldots)^\top\}$. On the contrary, for example, the standard 2nd-order central difference operator $\delta^{\langle 1\rangle,2}$ ($\delta_k^{\langle 1\rangle,2}U_k := (U_{k+1}-U_{k-1})/2\Delta x$) has the kernel of dimension two, spanned by $(1,1,\ldots,1)^\top$ and $(1,-1,1,-1,\ldots)^\top$, when N is even. An implication of this is the relation

$$\sum_{k=0}^{N-1}\left|\frac{U_{k+1}-U_{k-1}}{2\Delta x}\right|^2\Delta x \le \sum_{k=0}^{N-1}\left|\frac{U_{k+1}-U_k}{\Delta x}\right|^2\Delta x, \tag{10}$$

which in turn means that the discrete inequalities with respect to $\delta^{\langle 1\rangle,2}$ are purely stronger than those of δ^+, if they hold.

The second observation is simply that as the desired order is increased, the expression of the high-order difference operators become much more complicated involving wider stencils. In [20], the discrete inequalities were proved by following the elementary proofs of the continuous ones. This strategy, however, becomes soon infeasible for general operators. For example, the discrete Sobolev inequality for the fourth order central difference operator should read

$$\|\boldsymbol{U}\|_{L^\infty_{\mathrm{d}}} \le c\left(\sum_{k=0}^{N-1}|U_k|^2\Delta x + \sum_{k=0}^{N-1}\left|\frac{-U_{k+2}+8U_{k+1}-8U_{k-1}+U_{k-2}}{12\Delta x}\right|^2\Delta x\right), \tag{11}$$

which is difficult to prove by a direct calculation. The situation would get worse as the order is increased. Even worse, the compact difference operators are only determined implicitly, and such a direct calculation cannot work.

From the reasons above, discrete inequalities for general difference operators seem to have remained open, to the best of the present authors' knowledge, and accordingly convergence analyses for the finite difference schemes utilizing such operators were few. Although we have restricted our attention to NLS up here, the situation is true also for other PDEs.

Based on the background above, the aim of the present paper is to prove the discrete inequalities for some central-difference type operators. There are two keys in this challenge. The first key is a new result on the standard 2nd-order central difference operator $\delta^{\langle 1\rangle,2}$ (Lemma 7); this associates the discrete world to the continuous one, so that we can import the known results in continuous world, avoiding cumbersome direct discrete calculations. The same idea has been already employed for forward difference operators (see, for example, Holden–Raynaud [14]), but it seems new for central-difference operator. The second key is the idea of "equivalent operators." We do not hope to establish the association above for every complicated operators—instead, we propose to collect operators that are in some sense equivalent to $\delta^{\langle 1\rangle,2}$, and establish the desired inequalities by reducing them to $\delta^{\langle 1\rangle,2}$. This at the same time gives rise to a framework of *abstract finite difference schemes*—we consider a generic scheme with a generic difference operator in a certain class, and discuss its property using the inequalities commonly shared by the operators in that class. Although similar translations can be found in the literature, they were rather for specific purposes, and it seems there has been no systematic study ever. Here let us mention a series of studies on compact difference operators: [15, 23, 24]. There some compact difference approximations of ∂_{xx} were considered. It was pointed out that in some cases such approximations can be expressed as $(FD_{\delta^+})^\top(FD_{\delta^+})$, where F is some constant matrix and D_{δ^+} denotes the difference matrix for δ^+. Utilizing this expression, they reduced the convergence analysis of certain compact difference operators to that of the simplest forward difference case. Although they considered such translations for specific operators, the idea is similar to the one in the present paper. Note that, however, the reason of their success was the fact that on the circle $\mathbb{T}$, $\dim(\ker(\partial_{xx})) = 1$, and all the operators mentioned above shares this property. Thus they could reduce their discussions to the known results on the forward difference operator.

The present paper is organized as follows. In Section 2, the notation and necessary definitions are surveyed. We will also introduce the concept of class of difference operators. Section 3 shows the main results. In Section 4 we show a simple a priori estimate example for NLS. Section 5 is for concluding remarks.

2 Basic definitions and results

In this section we introduce some basic definitions and resuls to be used throughout this paper.

2.1 Some notation

We consider numerical methods for partial differential equations (PDEs) on $[0, T] \times [0, L]$ under the periodic boundary condition. We also often regard this as PDEs on the torus $\mathbb{T}$ of length L. We denote the numerical solution as $U_k^{(m)} \simeq u(k\Delta x, m\Delta t)$, where $\Delta x = L/N$ and Δt are the mesh sizes in x, t, respectively. Also we denote the solution vector as $\boldsymbol{U}^{(m)} := [U_0^{(m)}, U_1^{(m)}, \ldots, U_{N-1}^{(m)}]^\top$. The time step (m) is omitted unless indispensable. Corresponding to the periodic boundary condition, we demand the approximate solution satisfies $U_0 = U_N$, and accordingly we consider a space of such vectors (which is essentially finite-dimensional):

$$\mathbb{S}_N := \{(U_k \in \mathbb{C})_{k\in\mathbb{Z}} \quad | \quad U_k = U_{k \bmod N}\}. \tag{12}$$

We write the N-dimensional vectors $\mathbf{1}_N = (1, 1, \ldots)^\top$ and $\hat{\mathbf{1}}_N = (1, -1, \ldots)^\top$. We also denote the $N \times N$ identity matrix by I_N. We drop N in the above expressions when no confusion occurs. We will also use $N \times N$ matrices L_i and

R_i $(i = 1, 2, \ldots)$ defined by

$$L_1 := \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \quad R_1 := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}, \tag{13}$$

$$L_2 := {L_1}^2, \ L_3 = {L_1}^3, \ldots, \qquad R_2 = {R_1}^2, \ R_3 = {R_1}^3, \ldots. \tag{14}$$

To simplify the notation, we often use the two variable function $F_k(x, y)$ $(k \geq 1)$ satisfying $R_k - L_k = F_k(R_1, L_1)(R_1 - L_1)$. It is easy to see $F_k(R_1, L_1) = {R_1}^{k-1} + {R_1}^{k-2} L_1 + \cdots + R_1 {L_1}^{k-2} + {L_1}^{k-1}$. Note that L_1 and R_1 are circulant matrices and thus commutes. Also note $L_1 R_1 = R_1 L_1 = I$, and thus the above expansion is equivalent to $F_k(R_1, L_1) = {R_1}^{k-1} + {R_1}^{k-3} + \cdots + {L_1}^{k-3} + {L_1}^{k-1}$.

2.2 Finite difference operators

Next we introduce finite-difference operators approximating the differential operator ∂_x.

We denote the standard forward difference operator by δ^+, whose concrete form is

$$\delta^+ U_k = \frac{U_{k+1} - U_k}{\Delta x}.$$

We denote the matrix expression of δ^+ by D_{δ^+}. For this simplest operator, the discrete Sobolev inequality (5) and Gagliardo–Nirenberg inequality (6) hold [10]. Unless stated otherwise, the constant "C" appearing in such inequalities means a generic constant. As mentioned above, the kernel space of this operator is one-dimensional: $\ker(\delta^+) = \mathrm{span}\{\mathbf{1}\}$. In this sense, δ^+ is quite a natural operator inheriting the correct kernel space from ∂_x.

Next, we introduce central-difference type operators. In what follows, when we discuss common properties of such operators, we will simply use the expression δ, which means a generic operator. Its matrix expression is D_δ. Since the matrix actually depends on N, and actually its characteristic can vary on N, we should write $D_{\delta,N}$; however, since basically no confusion occurs, we prefer to drop N. We denote $(D_\delta \boldsymbol{U})_{k+j}$ by δU_{k+j}.

Definition 1 (Central-difference type operators)**.** *For nonnegative integers A, B and real coefficients α_j, β_j, which are independent of N, we say δ is a central-difference type operator if it is in the form*

$$\left(\delta U_k + \sum_{j=1}^{A} \alpha_j (\delta U_{k+j} + \delta U_{k-j}) \right) = \sum_{j=1}^{B} \beta_j \frac{U_{k+j} - U_{k-j}}{2j\Delta x}, \tag{15}$$

and δU_k becomes an approximation of ∂_x of at least $O(\Delta x)$. We denote by $\Delta^{\langle 1 \rangle}$ the set of all such δ's.

Remark 1. Throughout this paper, we assume $N \geq \max(2A + 1, 2B + 1)$ for simplicity (i.e., difference operators are considered for such N's). But in most of the discussions below, this can be relaxed by appropriately considering the periodicity.

Note that unless $A = 0$, it is a non-local operator, in the sense that (15) forms a linear system with respect to δU_k $(k = 1, \ldots, N)$, which should be solved in order to obtain each value (see the compact difference operators below). The matrix representation of (15) reads

$$\left(I + \sum_{j=1}^{A} \alpha_j (R_j + L_j) \right) D_\delta = \sum_{j=1}^{B} \beta_j \frac{R_j - L_j}{2j\Delta x}. \tag{16}$$

Also note that here we are implicitly assuming that the coefficient matrix in the left hand side is invertible (otherwise the operator is not well-defined).

The standard $2s$-order central difference operators, which we denote by $\delta^{\langle 1 \rangle, 2s}$, belong to this class with $A = 0$, $B = s$, and the appropriate coefficients β_j's for achieving $O(\Delta x^{2s})$ (see e.g. [8]). An interesting observation is that the pseudospectral difference operator, $\delta^{\langle 1 \rangle, \infty}$, can be obtained from such difference operators taking the limit $s \to \infty$, whose concrete form (for even N) reads

$$D_{\delta^{\langle 1 \rangle, \infty}} := D_{\mathcal{F}}^{-1} \mathrm{diag}\,(0, \mathrm{i}\pi, \ldots, \mathrm{i}(N/2-1)\pi, 0, -\mathrm{i}(N/2-1)\pi, \ldots, -\mathrm{i}\pi)\, D_{\mathcal{F}}, \tag{17}$$

where $D_{\mathcal{F}}$ is the discrete Fourier transform matrix. Strictly speaking, this operator does not belong to $\Delta^{\langle 1 \rangle}$, since it cannot be expressed in the form (15). Nevertheless, it shares some properties with $\delta^{\langle 1 \rangle, 2s}$, and thus in what follows we will sometimes mention it. Note that, to keep the difference matrix real-valued, pseudospectral difference operator is considered only for even N (see Fornberg [8]).

Another typical class is the compact difference operators, $\delta_{\mathrm{c}}^{\langle 1 \rangle}$, originally introduced in Lele [18] for the use in the field of computational fluid dynamics. The parameters A, B are chosen in $A \geq 1$, $B \geq 1$, and the coefficients α_j, β_j's are chosen to achieve a desired accuracy. The number $2A + 1$ is called the left stencil width, and $2B + 1$ the right stencil width. In this paper, we consider the compact difference operator of $A \leq 2$, $B \leq 3$, which is satisfied in practice. Some typical compact difference operators are described in Table 1 [18] (the blank cells are zero).

Table 1: Constants and Accuracies of First Derivatives.

compact difference operator	α_1	α_2	β_1	β_2	β_3	accuracy
2nd-order central: C2			1			$O(\Delta x^2)$
4th-order central: C4			$\frac{4}{3}$	$-\frac{1}{3}$		$O(\Delta x^4)$
6th-order central: C6			$\frac{3}{2}$	$-\frac{3}{5}$	$\frac{1}{10}$	$O(\Delta x^6)$
4th-order tridiagonal: T4	$\frac{1}{3}$		$\frac{3}{2}$			$O(\Delta x^4)$
6th-order tridiagonal: T6	$\frac{1}{3}$		$\frac{14}{9}$	$\frac{1}{9}$		$O(\Delta x^6)$
8th-order tridiagonal: T8	$\frac{3}{8}$		$\frac{25}{16}$	$\frac{1}{5}$	$-\frac{1}{80}$	$O(\Delta x^8)$
6th-order pentadiagonal: P6	$\frac{17}{57}$	$-\frac{1}{114}$	$\frac{90}{57}$			$O(\Delta x^6)$
8th-order pentadiagonal: P8	$\frac{4}{9}$	$\frac{1}{36}$	$\frac{40}{27}$	$\frac{25}{54}$		$O(\Delta x^8)$
10th-order pentadiagonal: P10	$\frac{1}{2}$	$\frac{1}{20}$	$\frac{17}{12}$	$\frac{101}{150}$	$\frac{1}{100}$	$O(\Delta x^{10})$

When $\alpha = \beta = 0$, the operators reduce to the standard central difference operators; i.e., C2=$\delta^{\langle 1\rangle,2}$, C4=$\delta^{\langle 1\rangle,4}$, and C6=$\delta^{\langle 1\rangle,6}$. The rest are the compact difference operators. Observe that they achieve higher-order accuracy with narrower stencils; for example, compare C6 and P10, both of which refer to 7 points. The name "compact" comes from this feature.

The operators $\delta^{\langle 1\rangle,2s}$'s and $\delta_{\mathrm{c}}^{\langle 1\rangle}$'s have a key property in common: they share the same kernel space, which is quite crucial in the subsequent analyses. In order to state this more precisely, let us introduce an important subclass of $\Delta^{\langle 1\rangle}$ (Def. 2 below). Let us rewrite (16) with the matrix $T_N \in \mathbb{R}^{N\times N}$ by

$$T_N := I + \sum_{j=1}^{A} \alpha_j (R_j + L_j),$$

and such a matrix $S_N \in \mathbb{R}^{N\times N}$ satisfying

$$\sum_{j=1}^{B} \beta_j \left(\frac{R_j - L_j}{2j\Delta x} \right) = S_N D_{\delta^{\langle 1\rangle,2}}. \tag{18}$$

Such a matrix S_N always exists.

Lemma 1. *For every operator $\delta \in \Delta^{\langle 1\rangle}$, there exists a matrix S_N satisfying* (18)*. It is not unique, but can be chosen to a banded matrix with the constant band width $2B-1$ (except for the top right and bottom left elements due to periodicity).*

proof. Let us write the left hand side of (18) as D. First, it is easy to see the existence of the banded version. Recalling $R_j - L_j = F_j(R_1, L_1)(R_1 - L_1)$, we see

$$D_{\delta^{\langle 1\rangle,2s}} = \sum_{j=1}^{B} \frac{\beta_j}{2j\Delta x} (R_j - L_j) = \left(\sum_{j=1}^{B} \frac{\beta_j}{j} F_j(R_1, L_1) \right) D_{\delta^{\langle 1\rangle,2}}. \tag{19}$$

This shows we can set $S_N = \left(\sum_{j=1}^{B} (\beta_j/j) F_j(R_1, L_1) \right)$. Since $F_j(R_1, L_1) = {R_1}^{j-1} + {R_1}^{j-3} + \cdots + {L_1}^{j-3} + {L_1}^{j-1}$ as mentioned before, we see we can find a matrix S_N of the form

$$S_N = s_0 I + \sum_{i=1}^{B} s_i (R_i + L_i). \tag{20}$$

The above argument only depends on the band width B, and thus the constants s_i $(i = 0, \dots, B)$ are independent of N.

The non-uniqueness arises from the kernel space of D and $D_{\delta^{\langle 1\rangle,2}}$. Recall that any circulant matrix can be diagonalized by the DFT matrix $D_{\mathcal{F}}$ with the column vectors of $D_{\mathcal{F}}$ being the eigenvectors. In particular, the eigenvalues of $R_j - L_j$ are $2\mathrm{i}\sin(2jk\pi/N)$ for $k = 0, \dots, N-1$. Thus we see that D and $D_{\delta^{\langle 1\rangle,2}}$ always share the same zero eigenvalue for $k = 0$ (with the eigenvector $\mathbf{1}$). Let $\Lambda_D, \Lambda_2, \Lambda_S$ be the diagonalizations of $D, D_{\delta^{\langle 1\rangle,2}}$ and S_N, respectively, supposing a circulant S_N. Then (18) implies $\Lambda_D = \Lambda_S \Lambda_2$. Since Λ_D and Λ_2 shares the same zero eigenvalue (for $\mathbf{1}$), the corresponding element of Λ_S can be arbitrary. This generates infinite number of (circulant) S_N's. $\square$

Let us look into the kernel issue a bit deeper. As mentioned in Introduction, $D_{\delta^{\langle 1 \rangle,2}}$ can have a spurious zero eigenvector. This depends on the parity of N. When N is odd, $R_j - L_j$ has only one zero eigenvalue for $k = 0$ (with the notation above). Thus we conclude that $\ker(\delta) = \mathrm{span}\{\mathbf{1}\}$ for any $\delta \in \Delta^{\langle 1 \rangle}$. In this case, the operator is just normal (in terms of its kernel). When N is even, however, the situation gets quite complicated. Our first observation is that the matrices $R_j - L_j$ ($j \geq 1$) have at least two zero eigenvalues for $k = 0, N/2$, with the eigenvectors $\mathbf{1}, \hat{\mathbf{1}}$. Thus all $\delta \in \Delta^{\langle 1 \rangle}$ has a spurious zero eigenvector $\hat{\mathbf{1}}$, and $\dim(\ker(\delta)) \geq 2$. Next, we notice this inequality can be sometimes strict. Let us for example consider $R_2 - L_2$ for $N = 4m$ with some integer m; then it has additional zero eigenvalue for $k = N/4$. Thus, depending on the values of β_j's, there remains the possibility that $\dim(\ker(\delta)) > 2$. These observations tell us that the central-difference type operator of the form (15) can behave quite strangely, which makes the discrete functional analytic approach quite difficult. This does not happen for δ^+.

The primal goal of this paper is to control such strange behaviors. Motivated by the above observations, let us group difference operators in view of their kernel space. More specifically, if the matrices T_N, S_N are invertible, then such an operator δ can be translated from/to $\delta^{\langle 1 \rangle,2}$, sharing the common kernel space. It naturally leads us to the following definition. The subscript "2" comes from $\delta^{\langle 1 \rangle,2}$. (It can also mean the false kernel dimension 2.)

Definition 2 (Set $\Delta_2^{\langle 1 \rangle}$)**.** *Let $\Delta_2^{\langle 1 \rangle}$ denote the set of all $\delta \in \Delta^{\langle 1 \rangle}$ such that S_N is invertible for all N for which the operator is defined.*

This demands S_N is invertible for every fixed N, but there remains the possibility that it tends to be singular when $N \to \infty$. The next concept, *equivalent operators*, is to exclude this possibility.

Definition 3 (p-reducibility and p-equivalence)**.** *We say $\delta \in \Delta_2^{\langle 1 \rangle}$* is p-reducible *to $\delta^{\langle 1 \rangle,2}$ if there exists a constant C, independent of N, such that $\|{S_N}^{-1} T_N\|_p < C$ holds. We say $\delta \in \Delta_2^{\langle 1 \rangle}$* is p-equivalent *to $\delta^{\langle 1 \rangle,2}$, if it additionally satisfies $\|{T_N}^{-1} S_N\|_p < C$.*

The p-equivalent operators can be safely translated from/to the representative element $\delta^{\langle 1 \rangle,2}$ in terms of p-norm. The p-reducibility demands a weaker property; it does not care if the translation from $\delta^{\langle 1 \rangle,2}$ to δ is safe or not.

Since S_N is circulant, 2-equivalence can be, in principle, discussed in terms of the eigenvalues. When we need general p-equivalence for $p \neq 2$, the next lemma is useful.

Lemma 2. *If $\delta \in \Delta_2^{\langle 1 \rangle}$ is 1-equivalent (or -reducible, respectively) to $\delta^{\langle 1 \rangle,2}$, then it is p-equivalent (-reducible) to $\delta^{\langle 1 \rangle,2}$ for all $p \geq 1$.*

proof. Recall the Riesz–Thorin theorem (for example, [13]): for every $A \in \mathbb{R}^{N \times N}$ and $p \geq 1$, $\|A\|_p \leq \|A\|_1^{1/p} \|A\|_\infty^{1-1/p}$. In particular, when A is symmetric, $\|A\|_p \leq \|A\|_1$. The matrices T_N and S_N in (20) are symmetric. □

The next lemma gives a simple sufficient condition for 1-equivalence.

Lemma 3. *If T_N and S_N (in the form (20)) are diagonally dominant, then the corresponding operator δ is 1-equivalent to $\delta^{\langle 1 \rangle,2}$.*

proof. Let us write $M = \sum_{i=1}^{B} s_i(R_i + L_i)$. Note that $S_N = s_0 I + M$ is diagonally dominant if and only if $\|M\|_1/s_0 < 1$. Thus by the Neumann expansion we have

$$\|S_N{}^{-1}\|_1 \leq \frac{1}{s_0}\sum_{j=0}^{\infty}\left(\frac{1}{s_0}\|M\|_1\right)^j \leq \frac{1}{s_0 - \|M\|_1}. \tag{21}$$

This bound does not depend on N. The bound for $\|S_N\|_1$ is obvious, and the same argument applies to T_N. This implies the claim. (Note that the weaker claim on 2-equivalence is immediate from the Gershgorin circle theorem: every eigenvalue λ of S_N lies on the disk $\{x \in \mathbb{R} \mid |x - s_0| \leq \|M\|_1\}$.) □

Some of the central difference operators $\delta^{\langle 1\rangle,2s}$ and the compact difference operators $\delta_{\mathrm{c}}^{\langle 1\rangle}$ satisfies the sufficient condition, and thus 1-equivalent to $\delta^{\langle 1\rangle,2}$.

Lemma 4. *$\delta^{\langle 1\rangle,2s}$ $(s \leq 7)$ and $\delta_{\mathrm{c}}^{\langle 1\rangle}$ in Table 1 are 1-equivalent to $\delta^{\langle 1\rangle,2}$.*

proof. We show $S_N = \left(\sum_{j=1}^{B}(\beta_j/j)F_j(R_1, L_1)\right)$ obtained in Lemma 1 is diagonally dominant for $s \leq 7$. For $s = 2$, for instance, from Table 1, $\beta_1 = 4/3$, $\beta_2 = -1/3$. Thus

$$S_N = \frac{4}{3}F_1(R_1, L_1) - \frac{1}{6}F_2(R_1, L_1) = \frac{4}{3}I - \frac{1}{6}(R_1 + L_1).$$

That is, now $s_0 = 4/3$ and $s_1 = -1/6$, which defines a diagonally dominant matrix. Similarly,

$$s = 3\colon \quad s_0 = \frac{23}{15}, \quad s_1 = -\frac{3}{10}, \quad s_2 = \frac{1}{30},$$
$$s = 4\colon \quad s_0 = \frac{176}{105}, \quad s_1 = -\frac{57}{140}, \quad s_2 = \frac{8}{105}, \quad s_3 = -\frac{1}{140},$$

and so on. Since the coefficients in rational number expression would soon get incredibly cumbersome, we omit them for $s \geq 5$, and instead show the values of $s_0 - 2\sum_{i=1}^{B}|s_i|$ for $2 \leq s \leq 8$.

$$\begin{aligned}
s = 2 &: & 1&,\\
s = 3 &: & \tfrac{13}{15} &\simeq 0.8667,\\
s = 4 &: & \tfrac{73}{105} &\simeq 0.6952,\\
s = 5 &: & \tfrac{23}{45} &\simeq 0.5111,\\
s = 6 &: & \tfrac{1121}{3465} &\simeq 0.3235,\\
s = 7 &: & \tfrac{6143}{45045} &\simeq 0.1364,\\
s = 8 &: & -\tfrac{243}{5005} &\simeq -0.0486.
\end{aligned}$$

This shows that for $s \leq 7$, the matrices S_N's are diagonally dominant, and thus the above discussion applies.

For the compact difference operator in Table 1, T_N's are all diagonally dominant. Hence, again, checking the diagonal dominance of S_N suffices. For the operators listed in Table 1, $B \leq 3$, and thus we need to check the matrix

$$\sum_{j=1}^{3}\frac{\beta_j}{j}F_j(R_1, L_1) = \beta_1 I + \frac{\beta_2}{2}(R_1 + L_1) + \frac{\beta_3}{3}(R_1{}^2 + R_1 L_1 + L_1{}^2).$$

It is diagonally dominant if $\beta_1 + \beta_3/3 > |\beta_2| + 2|\beta_3|/3$. All the operators in Table 1 satisfy this. $\square$

For the pseudospectral difference operator $\delta^{\langle 1\rangle,\infty}$, S_N is generally dense, and thus Lemma 3 cannot be utilized. For this operator, a weaker result holds, which corresponds to the 2-reducibility. Let us first construct a S_N. Note that $D_{\delta^{\langle 1\rangle,2}}$ can be diagonalized as

$$
\begin{aligned}
D_{\mathcal{F}} D_{\delta^{\langle 1\rangle,2}} D_{\mathcal{F}}^{-1} &= (0, \lambda_1, \lambda_2, \ldots, \lambda_{\frac{N}{2}-1}, 0, -\lambda_{\frac{N}{2}-1}, \ldots, -\lambda_1), &(22)\\
\lambda_j &= \mathrm{i}\frac{N}{2}\sin\left(\frac{2j\pi}{N}\right). &(23)
\end{aligned}
$$

This, together with (17), we obtain a matrix S_N:

$$
\begin{aligned}
S_N &= D_{\mathcal{F}}^{-1}\mathrm{diag}\left(1, \frac{\eta_1}{\lambda_1}, \ldots, \frac{\eta_{N/2-1}}{\lambda_{N/2-1}}, 1, \frac{\eta_{N/2-1}}{\lambda_{N/2-1}}, \ldots, \frac{\eta_1}{\lambda_1}\right) D_{\mathcal{F}}, &(24)\\
\eta_j &= \mathrm{i}j\pi. &(25)
\end{aligned}
$$

Lemma 5. *Let S_N be the matrix defined in* (24)*. Then there exists a constant C independent of N such that $\|S_N{}^{-1}\|_2 < C$ holds.*

proof. It suffices to show that η_j/λ_j $(j = 1, \ldots, N/2-1)$ is bounded from below independent of N. This is obvious since they must lie on the curve f_∞:

$$
f_\infty(\theta) = \begin{cases} \dfrac{\theta}{\sin(\theta)} & \theta \in (0, \pi), \\ 1 & \theta = 0. \end{cases} \tag{26}
$$

Since $f_\infty(\theta) \geq 1$ for $\theta \in [0, \pi)$, we have the claim. $\square$

Since $f_\infty(\theta)$ is not bounded from above, $\delta^{\langle 1\rangle,\infty}$ does not have a property corresponding to the 2-equivalence. This is a crucial difference between $\delta^{\langle 1\rangle,\infty}$ and $\delta^{\langle 1\rangle,2s}$ for $s < \infty$.

2.3 Discrete Norms

We use the following discrete analogues of the Lebesgue space $L^p(\mathbb{T})$ and the Sobolev space $W^{1,p}(\mathbb{T})$.

$$
\begin{aligned}
L^p_{\mathrm{d}} &:= \{\boldsymbol{U} \in \mathbb{S}_N \mid \|\boldsymbol{U}\|_{L^p_{\mathrm{d}}} < \infty\}, &(27)\\
W^{1,p}_{\mathrm{d}}(\delta) &:= \{\boldsymbol{U} \in \mathbb{S}_N \mid \|\boldsymbol{U}\|_{W^{1,p}_{\mathrm{d}}(\delta)} < \infty\}. &(28)
\end{aligned}
$$

They are essentially the same space as $\mathbb{S}_N$ for each fixed N, since the discrete norm is always bounded. Nevertheless we use this definition for convergence analysis where the limit $N \to \infty$ is taken into account.

The discrete norms are defined below.

Definition 4 (Discrete Norms)**.** For every $\boldsymbol{U} \in \mathbb{S}_N$ and $\delta \in \Delta^{\langle 1\rangle}$,

$$
\|\boldsymbol{U}\|_{L^p_{\mathrm{d}}} := \left(\sum_{k=0}^{N-1} |U_k|^p \Delta x\right)^{1/p}, \tag{29}
$$

$$
\|\boldsymbol{U}\|_{W^{1,p}_{\mathrm{d}}(\delta)} := \left(\sum_{k=0}^{N-1} (|U_k|^p + |\delta U_k|^p)\Delta x\right)^{1/p}. \tag{30}
$$

Observe that $\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,p}(\delta)}$ explicitly depends on the difference operator δ, and accordingly the space $W_{\mathrm{d}}^{1,p}(\delta)$ differs for each δ. In order to clarify this point, we include δ in the expression of the discrete Sobolev norm.

For the discussions below, it is convenient to note some properties of the norms of the linear operators on L_{d}^p. Let $\mathcal{L}(L_{\mathrm{d}}^p, L_{\mathrm{d}}^p)$ be the space of linear operators $L_{\mathrm{d}}^p \to L_{\mathrm{d}}^p$. From the definition of L_{d}^p, we see the following result about the equivalence between the operator norms of $\mathcal{L}(L_{\mathrm{d}}^p, L_{\mathrm{d}}^p)$ and the corresponding matrix norms. From this reason, below we do not explicitly distinguish these two concepts.

Lemma 6. Let M be an operator in $\mathcal{L}(L_{\mathrm{d}}^p, L_{\mathrm{d}}^p)$, whose representation matrix is denoted again by $M \in \mathbb{R}^{N\times N}$. Then

$$\|M\|_{\mathcal{L}(L_{\mathrm{d}}^p, L_{\mathrm{d}}^p)} = \|M\|_p, \tag{31}$$

where $\|\cdot\|_p$ is the standard matrix norm.

proof. From the definition of $\|\cdot\|_{L_{\mathrm{d}}^p}$, it immediately follows fact that $\|\boldsymbol{U}\|_{L_{\mathrm{d}}^p} = \|\boldsymbol{U}\|_p(\Delta x)^{1/p}$ where $\|\cdot\|_p$ is the ordinary vector norm. Using this relation, we get

$$\|M\|_{\mathcal{L}(L_{\mathrm{d}}^p, L_{\mathrm{d}}^p)} = \sup_{\boldsymbol{U}\in L_{\mathrm{d}}^p} \frac{\|M\boldsymbol{U}\|_{L_{\mathrm{d}}^p}}{\|\boldsymbol{U}\|_{L_{\mathrm{d}}^p}} = \sup_{\boldsymbol{U}\in L_{\mathrm{d}}^p} \frac{\|M\boldsymbol{U}\|_p}{\|\boldsymbol{U}\|_p} = \|M\|_p. \tag{32}$$

Note that, for each fixed N, $\boldsymbol{U} \in L_{\mathrm{d}}^p$ if and only if $\boldsymbol{U} \in \mathbb{S}_N$. □

3 Main Results

In this section we establish the following theorems. The key there is the 2- or p-reducibility of operators.

Theorem 1 (Discrete Sobolev inequality)**.** *Let $\delta \in \Delta_2^{\langle 1\rangle}$ be an operator that is 2-reducible to $\delta^{\langle 1\rangle,2}$. Then for every $\boldsymbol{U} \in \mathbb{S}_N$ the following inequality holds.*

$$\|\boldsymbol{U}\|_\infty \le C\|\boldsymbol{U}\|_{W^{1,2}(\delta)}, \tag{33}$$

where C is a constant which depends on δ but not on $\boldsymbol{U}$ and N.

Theorem 2 (Discrete Gagliardo–Nirenberg inequality (NLS type))**.** *Let $\delta \in \Delta_2^{\langle 1\rangle}$ be an operator that is 2-reducible to $\delta^{\langle 1\rangle,2}$. Then for every $\boldsymbol{U} \in \mathbb{S}_N$ the following inequality holds.*

$$\|\boldsymbol{U}\|_{L_{\mathrm{d}}^4}^4 \le C\|\boldsymbol{U}\|_{W^{1,2}(\delta)}\|\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^3, \tag{34}$$

where C is a constant which depends on δ but not on $\boldsymbol{U}$ and N.

The discrete Gagliardo–Nirenberg inequality actually holds in more general index, although the authors do not know if it has ever been explicitly pointed out in the literature. We show the following result, which demands a stronger assumption. Theorem 2 is a special case of it with $p = 4, q = r = 2$ and $\sigma = 1/4$.

Theorem 3 (Discrete Gagliardo–Nirenberg inequality (general case))**.** *Let $\delta \in \Delta_2^{\langle 1 \rangle}$ be an operator that is 1-reducible to $\delta^{\langle 1 \rangle, 2}$. Let also $1 \le p, q, r \le \infty$ and $0 \le \sigma \le 1$ such that*

$$\frac{1}{p} = \sigma \left(\frac{1}{r} - 1 \right) + (1 - \sigma) \frac{1}{q}. \tag{35}$$

Then for every $\boldsymbol{U} \in \mathbb{S}_N$ the following inequality holds.

$$\| \boldsymbol{U} \|_{L_{\mathrm{d}}^p} \le C \| \boldsymbol{U} \|_{W^{1,r}(\delta)}^{\sigma} \| \boldsymbol{U} \|_{L_{\mathrm{d}}^q}^{1-\sigma}, \tag{36}$$

where C is a constant which depends on δ, p, q, r, σ but not on $\boldsymbol{U}$ and N.

Remark 2. By slightly modifying the discussion, we can also prove the general version of the GN for δ^+. This seems new as well.

We start by proving the key lemma, which relates the "discrete" (finite-dimensional) to the "continuous" (infinite-dimensional) function space. The association is essentially done for the simplest central-difference operator $\delta^{\langle 1 \rangle, 2}$, and then extended to generic δ's by translation.

Lemma 7. *For every $\boldsymbol{U} \in \mathbb{S}_N$, we associate the piecewise linear function $\tilde{U} \in C(\mathbb{T})$ defined by*

$$\tilde{U}(k\Delta x) = \frac{|U_k| + |U_{k-1}|}{2} =: V_k \quad (k = 0, 1, \dots N-1), \tag{37}$$

$$\tilde{U}(x) = \frac{V_{k+1} - V_k}{\Delta x}(x - k\Delta x) + V_k, \quad x \in (k\Delta x, (k+1)\Delta x). \tag{38}$$

Then the following holds true with some constants $C_{1,p}, C_{2,p}, C_{3,\delta}, C_{4,\delta,p}$ which can depend on the specified elements but not on N.
(i) *For every $p \in \{1, \dots, \infty\}$, it holds*

$$C_{1,p} \| \boldsymbol{U} \|_{L_{\mathrm{d}}^p} \le \| \tilde{U} \|_{L^p} \le C_{2,p} \| \boldsymbol{U} \|_{L_{\mathrm{d}}^p}. \tag{39}$$

(ii) *For every $\delta \in \Delta_2^{\langle 1 \rangle}$ that is 2-reducible to $\delta^{\langle 1 \rangle, 2}$, it holds*

$$\| \tilde{U} \|_{W^{1,2}} \le C_{3,\delta} \| \boldsymbol{U} \|_{W_{\mathrm{d}}^{1,2}(\delta)}. \tag{40}$$

(iii) *For every $\delta \in \Delta_2^{\langle 1 \rangle}$ that is 1-reducible to $\delta^{\langle 1 \rangle, 2}$, it holds for every $p \in \{1, \dots, \infty\}$*

$$\| \tilde{U} \|_{W^{1,p}} \le C_{4,\delta,p} \| \boldsymbol{U} \|_{W_{\mathrm{d}}^{1,p}(\delta)}. \tag{41}$$

proof. (i) By easy calculation, we get the following identity for $\| \tilde{U} \|_{L^p}$:

$$\| \tilde{U} \|_{L^p}^p = \sum_{k=0}^{N-1} \frac{\Delta x}{p+1} \left({V_{k+1}}^p + {V_{k+1}}^{p-1} V_k + \cdots + V_{k+1} {V_k}^{p-1} + {V_k}^p \right). \tag{42}$$

First we obtain the left hand side of (39) by

$$\begin{aligned} \| \tilde{U} \|_{L^p}^p &\ge \sum_{k=0}^{N-1} \frac{\Delta x}{p+1} \left({V_{k+1}}^p + {V_k}^p \right) \\ &\ge \sum_{k=0}^{N-1} \frac{\Delta x}{p+1} \left(|U_{k+1}|^p + 2|U_k|^p + |U_{k-1}|^p \right) \\ &= \frac{4}{p+1} \| \boldsymbol{U} \|_{L_{\mathrm{d}}^p}^p. \end{aligned}$$

For the right hand side, we see

$$
\begin{aligned}
\|\tilde{U}\|_{L^p}^p &\le \sum_{k=0}^{N-1} \Delta x \ \max\left(V_{k+1}{}^p, V_k{}^p\right) \\
&\le \sum_{k=0}^{N-1} \Delta x \left(V_{k+1}{}^p + V_k{}^p\right) \\
&\le \sum_{k=0}^{N-1} \Delta x \left(2^p(|U_{k+1}|^p + |U_k|^p) + 2^p(|U_k|^p + |U_{k-1}|^p)\right) \\
&\le 2^{p+2}\|\boldsymbol{U}\|_{L_{\mathrm{d}}^p}^p,
\end{aligned}
$$

which proves (39).

(ii) To prove (40), we calculate the weak derivative of $\tilde{U}$ and use the triangle inequality to find for $x \in [k\Delta x, (k+1)\Delta x]$

$$
|\tilde{U}_x(x)| = \left|\frac{|U_{k+1}| - |U_{k-1}|}{2\Delta x}\right| \le \left|\frac{U_{k+1} - U_{k-1}}{2\Delta x}\right| \le |\delta_k^{\langle 1\rangle,2} U_k|. \tag{43}
$$

This estimate together with (39) reveal

$$
\begin{aligned}
\|\tilde{U}\|_{W_{\mathrm{d}}^{1,p}}^p &= \left(\|\tilde{U}\|_{L^p}^p + \|\tilde{U}_x\|_{L^p}^p\right) \\
&\le \max(C_{2,p}{}^2, 1)\left(\|\boldsymbol{U}\|_{L_{\mathrm{d}}^p}^p + \|\delta^{\langle 1\rangle,2}\boldsymbol{U}\|_{|L_{\mathrm{d}}^p}^p\right).
\end{aligned} \tag{44}
$$

This leads to the assertion for $\delta^{\langle 1\rangle,2}$. For other $\delta \in \Delta_2^{\langle 1\rangle}$ that is 2-reducible, from its definition there exist T_N and S_N such that $\|S_N{}^{-1}T_N\|_2$ has an upper bound which does not depend on N. Thus we have

$$
\begin{aligned}
\|\tilde{U}\|_{W^{1,2}}^2 &\le C_3\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta^{\langle 1\rangle,2})}^2 \\
&= C_3\left(\|\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^2 + \|D_{\delta^{\langle 1\rangle,2}}\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^2\right) \\
&= C_3\left(\|\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^2 + \|(T_N{}^{-1}S_N)^{-1}T_N{}^{-1}S_N D_{\delta^{\langle 1\rangle,2}}\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^2\right) \\
&\le C_3\left(\|\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^2 + \|S_N{}^{-1}T_N\|_2\|D_\delta\boldsymbol{U}\|_{L_{\mathrm{d}}^2}^2\right) \\
&\le C\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta)}^2.
\end{aligned} \tag{45}
$$

(iii) For $\delta \in \Delta_2^{\langle 1\rangle}$ that is 1-reducible, we see similarly to above that

$$
\begin{aligned}
\|\tilde{U}\|_{W^{1,p}}^p &\le C_3\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,p}(\delta^{\langle 1\rangle,2})}^p \\
&\le C\left(\|\boldsymbol{U}\|_{L_{\mathrm{d}}^p}^p + \|S_N{}^{-1}T_N\|_p\|D_\delta\boldsymbol{U}\|_{L_{\mathrm{d}}^p}^p\right) \\
&\le C\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,p}(\delta)}^p.
\end{aligned} \tag{46}
$$

This completes the proof. □

proof of Theorem 1 and Theorem 2. From Lemma 7 (i), (ii) and the continuous version of the Sobolev inequality (see, for example, [5]), we see

$$
\|\boldsymbol{U}\|_\infty = \|\tilde{U}\|_\infty \le C\|\tilde{U}\|_{W^{1,2}} \le C\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta)}. \tag{47}
$$

Next, for Theorem 2, we use Lemma 7 (i), (ii) and the continuous version of the Gagliardo–Nirenberg inequality (see, for example, [5]) to obtain

$$\|\boldsymbol{U}\|_{L^4_{\mathrm{d}}}^4 \leq C\|\tilde{U}\|_{L^4}^4 \leq C\|\tilde{U}\|_{W^{1,2}}\|\tilde{U}\|_{L^2}^3 \leq C\|\boldsymbol{U}\|_{W^{1,2}_{\mathrm{d}}(\delta)}\|\boldsymbol{U}\|_{L^2_{\mathrm{d}}}^3, \tag{48}$$

□

proof of Theorem 3. We use Lemma 7 (i), (iii) and the continuous version of the Gagliardo–Nirenberg inequality (see, for example, [5]) to obtain

$$\|\boldsymbol{U}\|_{L^p_{\mathrm{d}}}^p \leq C\|\tilde{U}\|_{L^p}^p \leq C\|\tilde{U}\|_{W^{1,r}}^{\sigma}\|\tilde{U}\|_{L^q}^{1-\sigma} \leq C\|\boldsymbol{U}\|_{W^{1,r}_{\mathrm{d}}(\delta)}^{\sigma}\|\boldsymbol{U}\|_{L^q_{\mathrm{d}}}^{1-\sigma}. \tag{49}$$

□

The pseudospectral difference operator $\delta^{\langle 1 \rangle,\infty}$ has the property similar to 2-reducibility (Lemma 5). Thus Lemma 7 also holds for this operator. We omit the proof.

Theorem 4 (Discrete inequalities for $\delta^{\langle 1 \rangle,\infty}$)**.** *For every $\boldsymbol{U} \in \mathbb{S}_N$ the following inequalities hold:*

$$\begin{aligned} \|\boldsymbol{U}\|_\infty &\leq C\|\boldsymbol{U}\|_{W^{1,2}(\delta^{\langle 1 \rangle,\infty})}, \\ \|\boldsymbol{U}\|_{L^4_{\mathrm{d}}}^4 &\leq C\|\boldsymbol{U}\|_{W^{1,2}(\delta^{\langle 1 \rangle,\infty})}\|\boldsymbol{U}\|_{L^2_{\mathrm{d}}}^3, \end{aligned}$$

where C is a constant which depends on $\delta^{\langle 1 \rangle,\infty}$ but not on $\boldsymbol{U}$ and N.

4 Application example

In this section we illustrate how the main results are useful, taking the nonlinear Schrödinger equation (NLS) as our working example.

We consider the cubic NLS on the circle $\mathbb{T}$:

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} - \gamma |u|^2 u, \quad x \in \mathbb{T},\ t > 0,\ \gamma > 0. \tag{50}$$

This equation has a conservative property in the sense that the solution u satisfies the following property:

$$\int_{\mathbb{T}} \left(-|u_x|^2 + \frac{\gamma}{2}|u|^4 \right) \mathrm{d}x = c_1, \tag{51}$$

$$\int_{\mathbb{T}} u^2 \mathrm{d}x = c_2. \tag{52}$$

Note that the estimate $\|u\|_\infty < c < +\infty$ follows from these invariants. From (51), we see

$$\|u\|_{W^{1,2}}^2 - c_1 - c_2 = \frac{\gamma}{2}\|u\|_{L^4}^4 \leq \frac{C\gamma}{2}\|u\|_{W^{1,2}}\|u\|_{L^2}^3 = \frac{{c_2}^{3/2}C\gamma}{2}\|u\|_{W^{1,2}},$$

where C is the constant in the GN inequality. This quadratic inequality implies that $\|u\|_{W^{1,2}}$ is bounded. This, together with the Sobolev inequality, shows the desired estimate.

Let $\delta \in \Delta_2^{\langle 1 \rangle}$, and consider a *generic* scheme:

$$\mathrm{i}\left(\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}\right) = -\delta^2 \left(\frac{U_k^{(m+1)} + U_k^{(m)}}{2}\right) - \gamma \left(\frac{|U_k^{(m+1)}|^2 + |U_k^{(m)}|^2}{2}\right)\left(\frac{U_k^{(m+1)} + U_k^{(m)}}{2}\right). \quad (53)$$

It has two conservation laws corresponding to (51)–(52) (we denote them again by c_1, c_2):

$$\sum_{k=0}^{N-1} \left(-|\delta U_k^{(m)}|^2 + \frac{\gamma}{2}|U_k^{(m)}|^4\right)\Delta x = c_1, \quad (54)$$

$$\sum_{k=0}^{N-1} |U_k^{(m)}|^2 \Delta x = c_2. \quad (55)$$

Note that the same generic difference operator appears in the scheme (53) and the discrete energy function (54). This is crucial in the analysis below. The schemes with $\delta = \delta^{\langle 1 \rangle, 2s}$ were discussed in [19], and those with $\delta = \delta_{\mathrm{c}}^{\langle 1 \rangle}$ in [17]. For both cases, no theoretical analysis was given, due to the lack of the corresponding discrete inequalities. Now we have them, and can give such analyses.

Let us confirm that we can universally deduce the essential estimate $\|\boldsymbol{U}^{(m)}\|_\infty < C_\delta < +\infty$.

Theorem 5 (A priori estimate on the approximate solution)**.** *Let $\delta \in \Delta_2^{\langle 1 \rangle}$ be an operator that is 2-reducible to $\delta^{\langle 1 \rangle, 2}$. Then for $m = 0, 1, 2, \ldots$,*

$$\|\boldsymbol{U}^{(m)}\|_\infty \leq C_{\delta, c_1, c_2, \gamma} \quad (56)$$

holds, where $C_{\delta, c_1, c_2, \gamma}$ is a constant that can depend on the specified elements but not on N and m.

proof. The argument goes exactly the same as in the continuous case, thanks to the new discrete inequalities.

From (54) and (55), and from the discrete Gagliardo–Nirenberg inequality for δ (Theorem 2), we see

$$\|\boldsymbol{U}^{(m)}\|^2_{W_{\mathrm{d}}^{1,2}(\delta)} - c_1 - c_2 = \frac{\gamma}{2}\|\boldsymbol{U}^{(m)}\|_4^4 \leq \frac{{c_2}^{3/2} C\gamma}{2}\|\boldsymbol{U}^{(m)}\|_{W_{\mathrm{d}}^{1,2}(\delta)}, \quad (57)$$

where this time C denotes the constant in Theorem 2. This implies the boundedness of $\|\boldsymbol{U}^{(m)}\|_{W_{\mathrm{d}}^{1,2}(\delta)}$. Then from the discrete Sobolev inequality for δ (Theorem 1), we have the claim. □

Once the estimate is at hand, we can obtain a full convergence result, by following the discussion in [10]. Let us here just present the conclusion. Let us denote the error in the numerical solution by $e_k^{(m)} := u_k^{(m)} - U_k^{(m)}$, where $u_k^{(m)} := u(k\Delta x, m\Delta t)$. We evaluate the error at a goal time $T = M\Delta t$: $\|\boldsymbol{e}^{(M)}\|_2$.

Theorem 6. *Let $\delta \in \Delta_2^{\langle 1 \rangle}$ be the generic difference operator of order s, which is 2-reducible to $\delta^{\langle 1 \rangle,2}$, and $u \in C^2[[0,T], C^{s+2}]$ be the true solution to NLS. Assume that the constants c_1 and c_2 in Theorem 5 can be bounded from above for all N. Then there exists a constant c, C which depends on δ, γ, c_1, c_2 and the true solution u, such that the following estimate holds.*

$$\|\boldsymbol{e}^{(M)}\|_2^2 \leq \frac{CT}{1 - c\Delta t}(\Delta t^4 + \Delta x^{2s}) e^{\frac{cT}{1-c\Delta t}} \tag{58}$$

5 Conclusion

In this paper, we have provided a unified analysis of the discrete inequalities regarding the central-difference type operators, which include the $2s$-order central difference schemes $\delta^{\langle 1 \rangle,2s}$ $(s \geq 1)$ and the compact difference operators $\delta_{\mathrm{c}}^{\langle 1 \rangle}$. There the key was to first prove the fundamental result for the representative difference operator $\delta^{\langle 1 \rangle,2}$, and then reduce other cases to it by the idea of equivalent (or reducible) operators. This sort of unified approach for difference operators seems not so common in the literature, if not completely new. We demonstrated the results taking NLS as an example. For NLS, the schemes themselves had been known [19, 17], but the convergence analysis has been left open, due to the lack of the required discrete inequalities. In the present paper, we have filled this gap.

The results in the present paper can be applied to wide range of schemes utilizing central-difference type operators. For example, there many dissipative schemes for the Cahn–Hilliard equation [9, 11], the phase-field crystal equation [25, 26], among others. It is possible to consider their higher-order versions by employing central-difference type operators, and for such schemes the results in the present paper are expected to be useful.

Some possible future works are commented below. First, although in this paper we proved the p-reducibility/equivalence for only limited member of $\Delta_2^{\langle 1 \rangle}$, the present authors conjecture that they hold in wider subset of, or even all of $\Delta_2^{\langle 1 \rangle}$. Preliminary numerical tests by the present authors support this view. In order to theoretically establish this, however, we have to discard the argument based on the diagonal dominance of the translation matrices, and find some new mathematical tools. Second, in this paper we have established the discrete Gagliardo–Nirenberg inequality for general index. It is an interesting and important topic to seek for PDE schemes where such an inequality is hoped. Last but not least, we hope to extend the idea of equivalent operators to construct a consistent big framework of discrete functional analysis, so that finite difference methods become really competitive to finite element methods. This is quite a big challenge, and should be beyond the ability of the present authors alone. We hope this view is shared by many researchers in related fields.

6 References

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