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Nonconvex Embeddings of the Exceptional Simplicial 3-Spheres with 8 Vertices

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As shown by D. Barnette (1973, *J. Combin. Theory Ser. A* **14**, 37–53) there are precisely 39 simplicial 3-spheres with 8 vertices. Thirty-seven of these are boundary complexes for convex 4-polytopes. In this paper we supply nonconvex embeddings in Euclidean 4-space for the remaining two 3-spheres. We discuss the properties of the embeddings as well as the techniques used to demonstrate their validity.

1. INTRODUCTION

The question of which d-dimensional simplicial complexes are topological spheres as well as the determination of possible geometric embeddings of those spheres in $(d+1)$-dimensional Euclidean space has a long history. The particular problem we are interested in here has its origin in the work of Brückner in 1909 [9], which purported to list all simple 4-polytopes with 8 facets (equivalently, all simplicial 4-polytopes with 8 vertices). Strictly speaking, what Brückner actually did was produce potential Schlegel diagrams for these 4-polytopes. There were errors in Brückner's enumeration that were corrected when Grünbaum and Sreedharan [19] gave a complete enumeration of the simplicial 4-polytopes with 8 vertices. This was done by analyzing the examples of Brückner and comparing them with the simplicial spheres obtained by adding a vertex to 1 of the 5 simplicial polytopes with 7 vertices. It is shown in [19] that there are 37 examples that are *convexly embeddable* (that is, they can be realized as boundary complexes of convex 4-polytopes), while one of Brückner's examples cannot be represented in this way. We will refer to this exceptional 3-sphere as $\mathscr{M}$. (This complex is the so-called "Brückner Sphere".)

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Later, in [6], Barnette discovered a second non-polytopal 3-sphere with 8 vertices, which we will refer to as $\mathscr{M}'$. In a later paper [7], Barnette proves that the addition of $\mathscr{M}'$ completes the list of 3-spheres with 8 vertices.

We can summarize the current work with the following theorem.

THEOREM 1.1. *$\mathscr{M}$ and $\mathscr{M}'$ are embeddable as geometric simplicial complexes in Euclidean 4-space, but neither is convexly embeddable there.*

The organization of this paper is as follows. In Section 2 we review the structure of the complex $\mathscr{M}$ and introduce an embedding. Section 3 discusses the various techniques used in verifying the embedding. Section 4 discusses some of the geometric properties of the embedding, introduces a second embedding of $\mathscr{M}$ and describes an embedding for $\mathscr{M}'$. Finally, in Section 5, we mention some natural questions concerning embeddings of combinatorially described objects.

2. A DESCRIPTION OF $\mathscr{M}$

We define an *embedding* of an abstract simplicial cell complex to be a mapping of its vertices and cells into $\mathbb{R}^d$ for which (i) no two vertices are mapped into the same point (ii) the image of every cell is the convex hull of the images of its vertices and (iii) the images of two cells intersect in $\mathbb{R}^d$ if and only if the cells intersect in the combinatorial description. We will call the image of a cell a *face* of the embedding, while *facet* will be used to denote either the maximal cells themselves or their images under the embedding. (We hope this will not cause any confusion.) The three conditions above and the definition of an abstract simplicial cell complex imply that a non-empty intersection of two faces must in turn be a face. Note also that the above definitions *do not* permit subdivision of simplices in the construction of a geometric realization.

In [19], $\mathscr{M}$ is presented via the combinatorial relationships indicated in Table I. $\mathscr{M}$ is a simplical 3-sphere with eight vertices (labelled 1, 2, ..., 8) and twenty facets (labeled with capital letters).

TABLE I

The Simplices of $\mathscr{M}$

A: 1234	B: 1237	C: 1267	F: 1347
H: 1567	J: 2345	L: 2367	M: 3467
N: 3456	O: 4567	P: 2358	Q: 2368
R: 3568	S: 1268	T: 1568	U: 1248
V: 2458	W: 1478	X: 1578	Y: 4578

TABLE II

The Vertices of the Embedding $\mathbb{M}_1$ of $\mathscr{M}$

vertex 1	$= \{6, 12, -18, -6\}$
vertex 2	$= \{0, 0, 0, 0\}$
vertex 3	$= \{12, 0, 0, 0\}$
vertex 4	$= \{0, 12, 0, 0\}$
vertex 5	$= \{12, 18, -6, -6\}$
vertex 6	$= \{8, 8, -8, -7\}$
vertex 7	$= \{24, 24, -24, -12\}$
vertex 8	$= \{-12, -36, -60, -240\}$

Consider the mapping of vertices given in Table II. This mapping constitutes an embedding of $\mathscr{M}$; the techniques used to verify that it is an embedding are shown in Section 3. By [19], this complex cannot be the boundary complex of a convex polytope.

In the remainder of the paper, $\mathbb{M}_1$ will refer not only to the embedding described above but also to the image of $\mathscr{M}$ under the embedding. This image is also called the *geometric realization* or the *realization* of $\mathscr{M}$.

3. VERIFYING THE EMBEDDING

To verify that the choices for the locations of the vertices given in Table II define an embedding we simply need to check that its facets intersect only at common boundaries.

Below, we will describe an algorithm for determining when a k-simplex intersects an l-simplex in $\mathbb{R}^d$, where $k+l=d$. First, we prove a lemma to show that checking such intersections is sufficient to determine whether the simplices of $\mathbb{M}_1$ intersect only where required to by the combinatorial description of $\mathscr{M}$.

LEMMA 3.1. *If P and Q are polytopes in $\mathbb{R}^d$, then P and Q intersect if and only if a k-face of P intersects an l-face of Q for some $k+l \leqslant d$.*

Proof. The "if" direction is trivial. To prove "only if," first note that if S and T are both full dimensional, i.e., d-polytopes in $\mathbb{R}^d$, then the two intersect if and only if $\partial S \cap T$ or $S \cap \partial T$ is not empty. (This is easy to see if $d=1$. For larger d, the result reduces to the one dimensional case by restricting attention to an arbitrary line through a point in the intersection.)

Let P be a p-polytope and Q be a q-polytope in $\mathbb{R}^d$ where $p+q>d$. Let ∂P and ∂Q denote the relative boundaries of P and Q, respectively.

We now claim that P and Q intersect only if $\partial Q \cap P$ or $Q \cap \partial P$ is not empty. To see this, note that if the two polytopes intersect, then the affine hull of P must intersect the affine hull of Q. The intersection is necessarily a j-flat with $j \geqslant p+q-d > 0$. Let P' be the intersection of this j-flat with P and Q' be the intersection of this j-flat with Q. Clearly, $P \cap Q \neq \emptyset$ if and only if $P' \cap Q' \neq \emptyset$. If P' and Q' are both j-dimensional, then the result follows from the above note. Otherwise, $P' \subset \partial P$ or $Q' \subset \partial Q$ and the claim is evident.

To prove the lemma, note that if $\dim(P) + \dim(Q) \leqslant d$, then the result is vacuously true. Otherwise, the above claim allows us to replace P or Q with a proper face that still intersects the other polytope. We may iteratively replace the polytopes with proper subfaces until the sum of the dimensions is $\leqslant d$. ▮

The next lemma gives an algorithm for determining when two simplices of the appropriate dimensions intersect.

Let A and B be two simplices in $\mathbb{R}^d$ determined by vertices $\{a_1, a_2, ..., a_k\}$ and $\{b_1, b_2, ..., b_l\}$, respectively. Assume that the sum of the simplices' dimensions is d (equivalently, that the total number of vertices is $d+2$). Let V denote the union of these vertex sets and assume that the vertices of V are in general position. Note that every subset consisting of d vertices determines a hyperplane (i.e., a $d-1$-flat). For each pair of vertices $x, y \in V$, their complement in V determines a hyperplane, denoted $H_{x,y}$.

LEMMA 3.2. *Let A and B be two simplices as above. Choose some $x \in A$. The simplices A and B intersect in their relative interiors if and only if for every vertex $y \in A$ with $y \neq x$, x and y lie on opposite sides of $H_{x,y}$ and for every vertex $w \in B$, x and w lie on the same side of $H_{x,w}$.*

Proof. Suppose first there is a non-empty intersection of interiors of the two simplices. Let z be an arbitrary point in that intersection. The point z may be expressed as a sum in two ways,

$$z = \sum_{i=1}^{k} \lambda_{a_i} a_i = \sum_{j=1}^{l} \sigma_{b_j} b_j,$$

where $\sum \lambda_{a_i} = \sum \sigma_{b_j} = 1$ and all the λ_{a_i} and σ_{b_j} are positive. Thus

$$0 = \sum_{i=1}^{k} \lambda_{a_i} a_i - \sum_{j=1}^{l} \sigma_{b_j} b_j.$$

Let y be any vertex in A. Choose a point in $H_{x,y}$ to act as the origin, and let $n_{x,y}$ be the normal vector to $H_{x,y}$. For each vertex v in $V \setminus \{x, y\}$ (now considered as a vector relative to the origin) we see that $n_{x,y} \cdot v = 0$ since

$v \in H_{x,y}$. If we form the dot product of $n_{x,y}$ with the identity $\sum_{i=1}^{k} \lambda_{a_i} a_i - \sum_{j=1}^{l} \sigma_{b_j} b_j = 0$ we are left with

$$n_{x,y} \cdot \{\lambda_x x + \lambda_y y\} = 0.$$

Since λ_x and λ_y are positive, x and y must lie on opposite sides of $H_{x,y}$.

If w is any vertex in B, we construct $n_{x,w}$ in a similar fashion. After taking the dot product we are left with

$$n_{x,w} \cdot \{\lambda_x x - \sigma_w w\} = 0.$$

Again, the coefficients are positive, so x and w must lie on the same side of $H_{x,w}$.

The proof of the converse is essentially the reverse of the argument above. However, the accounting is simplified by embedding the vertices of V in $\mathbb{R}^{d+1}$. We embed A into $\mathbb{R}^{d+1}$ via the map sending vertex $a_i \mapsto \binom{1}{a_i} = \tilde{a}_i$ and embed B via the map sending vertex $b_j \mapsto \binom{-1}{-b_j} = \tilde{b}_j$. We will refer to the union of these two sets as $\tilde{V}$. We now seek non-negative λ_{a_i} and σ_{b_j} such that:

$$0 = \sum_{i=1}^{k} \lambda_{a_i} \tilde{a}_i + \sum_{j=1}^{l} \sigma_{b_j} \tilde{b}_j.$$

(Restricting attention to the first coordinate, this equation implies that $\sum_{i=1}^{k} \lambda_{a_i} = \sum_{j=1}^{l} \sigma_{b_j}$, so this linear dependence would imply that a convex combination of the a_i equals a convex combination of the b_j.)

Let $\tilde{H}_{x,y}$ be the hyperplane in $\mathbb{R}^{d+1}$ that passes through the origin and every point in $\tilde{V}$ except for $\tilde{x}$ and $\tilde{y}$ (the images of x and y). Let $\operatorname{dist}(\tilde{x}, \tilde{H}_{x,y})$ be the distance (in $\mathbb{R}^{d+1}$) between $\tilde{x}$ and the hyperplane $\tilde{H}_{x,y}$. Set $\lambda_x = 1$. For every other a_i and every b_j, let

$$\lambda_{a_i} = \operatorname{dist}(\tilde{a}_i, \tilde{H}_{a_i,x}) / \operatorname{dist}(\tilde{x}, \tilde{H}_{a_i,x})$$

$$\sigma_{b_j} = \operatorname{dist}(\tilde{b}_j, \tilde{H}_{b_j,x}) / \operatorname{dist}(\tilde{x}, \tilde{H}_{b_j,x}).$$

Note that all λ_{a_i} and σ_{b_j} are positive.

We claim that $\sum_{i=1}^{k} \lambda_{a_i} \tilde{a}_i + \sum_{j=1}^{l} \sigma_{b_j} \tilde{b}_j = 0$ as required. To see this, consider the set of vectors, $\{\tilde{n}_{x,v}\}$, defined as the normals to the $\tilde{H}_{x,v}$ where x is fixed and v ranges over the remaining vertices in V. This set must span $\mathbb{R}^{d+1}$ since the points of V are in general position in $\mathbb{R}^d$.

If $y \in A$, then

$$\tilde{n}_{x,y} \cdot \left\{ \sum_{i=1}^{k} \lambda_{a_i} \tilde{a}_i + \sum_{j=1}^{l} \sigma_{b_j} \tilde{b}_j \right\} = \lambda_x (\tilde{n}_{x,y} \cdot \tilde{x}) + \lambda_y (\tilde{n}_{x,y} \cdot \tilde{y}).$$

By construction, the absolute values of the two terms are equal (λ_y was defined as the relevant ratio). By hypothesis, x and y are on opposite sides of $H_{x,y}$, so $\tilde{x}$ and $\tilde{y}$ are on opposite sides of $\tilde{H}_{x,y}$ and the two terms cancel.

If $w \in B$,

$$\tilde{n}_{x,w} \cdot \left\{ \sum_{i=1}^{k} \lambda_{a_i} \tilde{a}_i + \sum_{j=1}^{l} \sigma_{b_j} \tilde{b}_j \right\} = \lambda_x(\tilde{n}_{x,w} \cdot \tilde{x}) + \sigma_w(\tilde{n}_{x,w} \cdot \tilde{w}).$$

By hypothesis, x and w are on the same side of $H_{x,w}$, so the embedding places $\tilde{x}$ and $\tilde{w}$ on opposite sides of $\tilde{H}_{x,w}$ and again the two terms cancel.

Since the sum has zero dot product with every member of a spanning set of vectors, it must identically be zero, so the intersection of the relative interiors of A and B is nonempty. ∎

4. A DISCUSSION OF THE EMBEDDING

Ultimately, we would like to understand how the combinatorics of $\mathscr{M}$ dictate the geometric properties of its various realizations.

Our first question is whether $\mathbb{M}_1$ is star-shaped. (By this we mean not that the complex itself is star-shaped, but rather that the set of all points "inside," i.e., the bounded component of the complement of the complex in $\mathbb{R}^4$, is star-shaped.) Below we show $\mathbb{M}_1$ is not star-shaped, but we will later introduce a second realization of $\mathscr{M}$ that is star-shaped.

The key step is reducing the question to a linear programming problem. A d-dimensional geometric simplicial sphere $\mathscr{S}$ in $\mathbb{R}^{d+1}$ is star-shaped from some point z if and only if for each d-simplex in $\mathscr{S}$ with vertices $\{v_1, v_2, ..., v_{d+1}\}$, the $(d+1)$-simplex determined by $\{z, v_1, v_2, ..., v_{d+1}\}$ is contained in the "inside" of $\mathscr{S}$. In such a case, an orientation on the sphere (corresponding to an outward normal for the realization) will determine an ordering of the vertices $\{v_1, v_2, ..., v_{d+1}\}$ that ensures that the determinant of $\{z, v_1, v_2, ..., v_{d+1}\}$ maintains a consistent sign as we vary over all the d-simplices of $\mathscr{S}$. By allowing z to vary, we obtain a system of linear inequalities in the coordinates of $z = \{z_1, z_2, z_3, z_4\}$. Thus a given simplicial d-sphere $\mathscr{S}$ is star-shaped if and only if there exists a solution to this set of linear inequalities. (Note that this method actually finds a "star center" whenever one exists.)

In the case of $\mathbb{M}_1$, we used an orientation on $\mathscr{M}$, took the corresponding determinants and after some simplification obtained the system of inequalities in Table III. As the origin lies in the convex hull of $\mathbb{M}_1$, the system required shifting by a fixed vector (or slack variable) to eliminate the possibility of missing a solution with negative coordinates. A calculation

TABLE III

Inequalities Used to Determine If $\mathbb{M}_1$ Is Star-Shaped or Not

$-(z_3-3z_4)\leqslant 0$
$-z_2-2z_4\leqslant 0$
$z_1-2z_2-z_3\leqslant 0$
$-12+z_1+z_2-2z_3+7z_4\leqslant 0$
$-(216+23z_1-13z_2-5z_3+48z_4)\leqslant 0$
$z_3-z_4\leqslant 0$
$-z_2-z_3\leqslant 0$
$-72+6z_1+6z_2+17z_3-16z_4\leqslant 0$
$-12+z_1+z_2-17z_3+20z_4\leqslant 0$
$72+11z_1-6z_2+16z_4\leqslant 0$
$5z_2+21z_3-6z_4\leqslant 0$
$-(125z_2+181z_3-64z_4)\leqslant 0$
$-1548+129z_1-31z_2-113z_3+20z_4\leqslant 0$
$229z_1-328z_2-169z_3+80z_4\leqslant 0$
$-5244+693z_1-235z_2-229z_3+36z_4\leqslant 0$
$-(55z_1+21z_3-8z_4)\leqslant 0$
$15z_1+41z_3-11z_4\leqslant 0$
$-(912+59z_1-76z_2+17z_3+8z_4)\leqslant 0$
$84+6z_1-8z_2+z_3+z_4\leqslant 0$
$-(492+24z_1-41z_2+7z_4)\leqslant 0$

in *Mathematica* then determined that this system cannot be satisfied and so $\mathbb{M}_1$ is not star-shaped.

However, a star-shaped embedding of $\mathscr{M}$ does exist. In order to contrast the two embeddings, we must first present a descriptive language to characterize the various nonconvex features of an embedding. No such

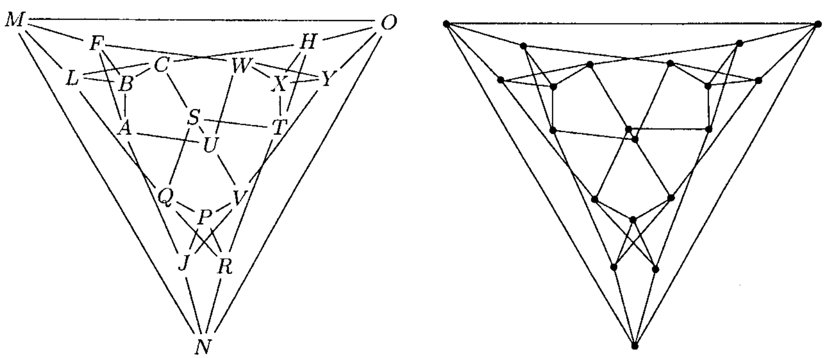

FIG. 1. Each node in the graph corresponds to a facet of $\mathscr{M}$. The graph's edges join neighboring facets (those that intersect in a 2-cell). For simplicity, the remaining figures will omit the facet labels.

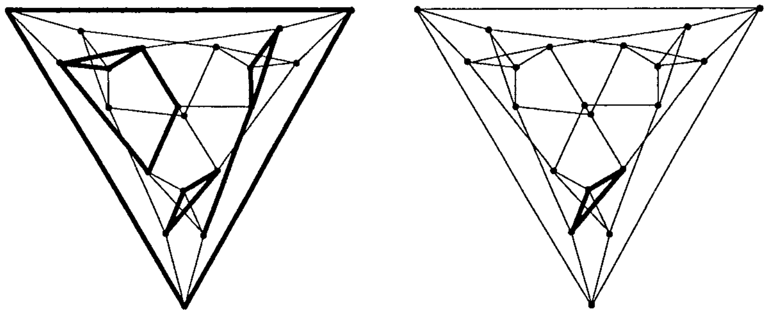

FIG. 2. The graph from Fig. 1 has been modified by identifying ridges versus creases. Creases are marked with dark lines. The facet graph for $\mathbb{M}_1$ is on the left. The facet graph for $\mathbb{M}_2$ is on the right.

description seems to exist for dimensions $d \geqslant 4$. We will describe two different perspectives—"locally defined" convex features versus "globally defined" convex features.

The local description begins with the graph shown in Fig. 1. In this graph, each node is a facet of $\mathscr{M}$ and edges connect those pairs that intersect in a 2-cell. (If $\mathscr{M}$ were a simplicial polytope, this graph simply would be the ridge graph of $\mathscr{M}$, or equivalently, the edge graph of its dual.) The choice of embedding will modify this graph by assigning a sign to each edge.

Since $\mathscr{M}$ is a sphere, we may define an outward pointing normal for each facet in an embedding. This normal allows us to define a closed half-space that includes the facet and at least part of the "inside" of the complex. (In the convex case, these closed half-spaces are supporting hyperplanes and contain the entire polytope.) Suppose M and N are neighboring facets. This implies they share exactly three vertices. The fourth vertex of M may lie inside or outside the closed half-space defined above. In the former case, the graph's edge will be assigned $+$ and called a *ridge*, in the latter case, the graph's edge will be assigned $-$ and called a *crease*.

Intuitively, the two facets may bend inward, so that any segment connecting points from the interior of each facet lies inside the complex, as in convex polytopes, or the two facets may bend outward, which ensures that the complex is not the boundary of its convex hull.

None of the facets in our embeddings are coplanar, so a sign may be assigned to each edge. A natural convention would be to assign "0" to neighboring facets that are coplanar.

This augmented graph is called the *facet graph* for the embedding. For convex polytopes, the facet graph contains only ridges. In this context, the result in [19] may be rephrased as "there exists no embedding of $\mathscr{M}$ into $\mathbb{R}^4$ whose facet graph has no creases." Figure 2 shows the graphs that correspond to the embeddings $\mathbb{M}_1$ and $\mathbb{M}_2$. ($\mathbb{M}_2$ will be described later.) Here the graph of Fig. 1 is supplemented with dark lines to indicate which edges are creases.

Each set of creases that is path-connected in the facet graph forms a *locally defined nonconvex feature* of the embedding. Such features can take a variety of forms and may possess a non-trivial topology. We begin the classification with the simplest cases. If the set of all the 2-faces that contain a vertex v of $\mathscr{M}$ consists entirely of creases then this set is called a *rank* 0 *dent* (or simply a 0-*dent*). Similarly, if the set of all the 2-faces that contain a particular edge of $\mathscr{M}$ consists entirely of creases *and* the set is not part of a 0-dent then this set is called a 1-*dent*. In higher dimensions, we could extend this in the obvious way. Since any crease that is not part of a 0-dent or 1-dent can be considered to be a 2-dent (trivially), any locally defined nonconvex feature may be considered to be a union of dents of various ranks.

$\mathbb{M}_1$ possesses four separate locally defined nonconvex features. The first feature is simply the 1-dent specified by $\overline{25}$. The second is the 1-dent specified by $\overline{46}$. The third consists of two 1-dents—those specified by $\overline{26}$ and $\overline{27}$. The final feature consists of the 1-dent specified by $\overline{15}$ and the 2-dent $\{568\}$.

We now describe a globally defined notion of convexity and nonconvexity. We will define (extending Hajós and Heppes in [20]) the *support set* of an embedding $\mathbb{P}$ of a simplicial complex $\mathscr{P}$ as the set of points of $\mathbb{P}$ that are contained in a supporting hyperplane of $\mathbb{P}$. Any face of $\mathbb{P}$ in the support set of $\mathbb{P}$ we will call a *supporting* face of $\mathbb{P}$. Similarly, if a face of $\mathbb{P}$ is *not* in the support set it will be called a *non-supporting* face of $\mathbb{P}$. Note then that any supporting face of $\mathbb{P}$ must necessarily be on the boundary of the convex hull of $\mathbb{P}$. It is straightforward to check that any vertex corresponding to a 0-dent cannot be a supporting face and any edge corresponding to a 1-dent cannot be a supporting face. However, not every non-supporting vertex or edge will correspond to a dent. Note that any crease cannot be a supporting face and any facet containing a crease cannot be a supporting face.

If $\mathscr{E}$ is an embedding of a d-sphere such that for every $k < d-1$, every non-supporting k-face corresponds to a j-dent where $j \leqslant k$, then $\mathscr{E}$ is called *expanded*. We note that $\mathbb{M}_1$ is not expanded. While the non-supporting edges $\overline{15}$, $\overline{25}$ and $\overline{27}$ correspond to 1-dents, vertex 6 is non-supporting but there is no 0-dent in the facet graph. A natural question is whether a given facet graph may be realized as an expanded embedding. To see that this question is non-trivial, we note that there are facet graphs of simplicial 2-spheres that are realizable in $\mathbb{R}^3$ but not realizable via an expanded embedding. A separate question is whether every simplicial sphere possesses an expanded embedding.

TABLE IV

These Are Simplices of $\mathscr{M}'$, in the Notation of [6], with $q = p'$

A: $abde$	B: $cefb$	C: $afcb$	D: $abdp$
E: $bdep$	F: $bcep$	G: $cefp$	H: $acfp$
J: $adfp$	K: $abcp$	L: $defp$	M: $abcq$
N: $abeq$	P: $acdq$	Q: $adeq$	R: $bcfq$
S: $befq$	T: $cdfq$	V: $defq$	

$\mathbb{M}_1$ was constructed by starting with coordinates for the 8 vertices with only one point lying inside the convex hull and then systematically eliminating excess intersections. In particular, we used the observation that if an edge intersects a simplex, then one can eliminate the intersection by determining where the edge intersects the simplex and then moving the edge along the line that extends it past the boundary of the simplex. Trial and error played a role.

If A and B are any sets, the *symmetric difference* $A \Delta B$ is defined to be the set $(A \cup B) \setminus (A \cap B)$. If A and B are simplicial complexes, then their symmetric difference is also a simplicial complex. If A and B are subsets of $\mathbb{R}^d$ then their symmetric difference is a subset of $\mathbb{R}^d$.

A second embedding of $\mathscr{M}$ was discovered by employing the close connection between $\mathscr{M}$ and the polytope P^8_{32} (in the notation of [19]). If we relabel the vertices $\{1, 2, 3, 4, 5, 6, 7, 8\}$ of P^8_{32} as $\{8, 6, 7, 1, 2, 3, 4, 5\}$, we see that P^8_{32} includes all but three of $\mathscr{M}$'s facets. Moreover, if $\mathscr{K}$ denotes the abstract simplex $\{23458\}$, $\mathscr{M} = P^8_{32} \Delta \mathscr{K}$. Thus, removing the 4-simplex $\{23458\}$ from any realization of the (relabelled) convex polytope P^8_{32} gives a realization, $\mathbb{M}_2$, of $\mathscr{M}$. Its facet graph is shown on the right in Fig. 2. The only locally nonconvex feature is the 1-dent corresponding to $\overline{25}$. Because of its construction, this realization is necessarily star-shaped.

Finally, we note that Barnette's sphere ($\mathscr{M}'$, as described in [6] see Table IV) is closely related both to $\mathscr{M}$ and to the polytope P^8_{27} (in the notation of [19]). Let $\mathbb{K}_1$ denote the 4-simplex $\{34567\}$; then $\mathbb{M}_2 \Delta \mathbb{K}_1$ yields an embedding of $\mathscr{M}'$ whose nonconvex features are just two 1-dents. (Here, the vertices $\{a, b, c, d, e, f, p, q = p'\}$ of Barnette's description are associated with vertices $\{1, 2, 8, 7, 3, 5, 6, 4\}$ of $\mathscr{M}$.) Similarly, if $\mathbb{K}_2$ denotes the 4-simplex $\{13578\}$, then $P^8_{27} \Delta \mathbb{K}_2$ will produce an embedding of $\mathscr{M}'$ with just a single 1-dent. (Here, the vertices $\{a, b, c, d, e, f, p, q = p'\}$ of Barnette's description are associated with vertices $\{1, 4, 5, 7, 2, 3, 6, 8\}$ of P^8_{27}.) Note that by removing a 4-simplex $\mathbb{K}_3 = \{defpq\}$ from this second embedding of $\mathscr{M}'$, we could arrive at a third embedding for $\mathscr{M}$. Its facet graph would have two 1-dents as its non-convex features.

5. CONCLUSION

It is our hope that others may find the techniques outlined in this paper helpful in other embedding problems. While it traditionally has been more convenient (and perhaps more interesting) to study the question of embeddability of cell complexes from either a purely topological viewpoint, or as convex polytopes, or by allowing subdivision (i.e., piece-wise linear embeddings), we feel there are many interesting geometric questions concerning the embeddability of cell complexes as defined in Section 3.

In the study of other simplicial spheres there are many questions that may prove to be of interest. What is the minimum number of creases possible in a realization of a given simplicial sphere? How are the nonconvex features of simplicial spheres related in various realizations? (For example, are certain cells always creases? Always ridges? What is the effect of forcing a cell to be a ridge or forcing it to be a crease?) For which spheres is it possible to find an embedding such that the image is star-shaped? Beyond the issue of being star-shaped, one might consider other measures of how "nice" a realization might be (e.g. the minimum number of convex sets the realization may be split into).

What is the connection between a nonconvex embedding of a simplicial sphere and an embedding of its (combinatorial) dual? Barnette and Wegner showed in [8] that the 2-skeleton of the dual of $\mathscr{M}$ is not geometrically realizable with convex plane polygons as the 2-faces, but it does not appear to be known what happens if the condition that the polygons of the 2-skeleton be convex is replaced with non-self-intersecting.

The complex $\mathscr{M}$ is shellable. Is there a connection between shellability and geometric embeddings? Is a complex with a star-shaped realization necessarily shellable? Bruggesser and Mani [10] showed that every decomposition of a d-sphere (here defined as any subdivision of a d sphere with the usual polytopal face intersection properties that possesses a subdivision isomorphic with a triangulation of the boundary of a $(d+1)$ simplex) has a subdivision that is shellable.

Conjecture 5.1. Every shellable combinatorial simplicial d-sphere is embeddable in $\mathbb{R}^{d+1}$.

If this turns out to be true, every combinatorial d-sphere would possess a subdivision that is geometrically realizable in $\mathbb{R}^{d+1}$.

The general question of nonconvex embeddings, especially in higher dimensions, is largely unexplored. There has been some interesting work on nonconvex embeddings in 3-dimensional Euclidean space (see [11, 12, 14–18, 20]), but considerably less done in higher dimensions. Altshuler and Steinberg in [2, 3], and with Bokowski in [1] determined that there are

1296 (combinatorial) simplicial 3-spheres with nine vertices, and of those 1142 may be realized as the boundaries of convex 4-polytopes. The remaining 154 are shown not to be realizable as polytopes, but one wonders in what fashion these may be realized nonconvexly. Let $s(d, n)$ be the number of triangulations with n labeled vertices of a $(d-1)$ sphere and $c(d, n)$ be the number of combinatorial types of simplicial d polytopes with n labeled vertices. Kalai has shown [21] that for $d \geqslant 5$

$$\lim_{n \to \infty} [c(d, n)/s(d, n)] = 0$$

and that for every $b \geqslant 4$

$$\lim_{d \to \infty} [c(d, d+b)/s(d, d+b)] = 0.$$

These results indicate that in some sense, the number of simplicial spheres dominates the number of simplicial polytopes. Such a large set deserves closer inspection. While it is well known that every simplicial d-sphere is realizable as a simplicial complex in $\mathbb{R}^{2d+1}$ (this is true of any simplicial d-complex, see [13, Exercise 25, p. 67], we may conjecture the following.

Conjecture 5.2. Every combinatorial simplicial d-sphere is embeddable in $\mathbb{R}^{d+1}$

Note. The calculations used to verify the embedding of $\mathscr{M}$ and to check whether $\mathscr{M}$ is star-shaped are available as a *Mathematica* notebook from `http://math.washington.edu/~gwilliam`.

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