Classification of hermitian forms with the neighbour method

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Abstract

The neighbour method of Kneser can be adapted to the hermitian case. Generalizing results of [Hof91], we show that it can be used to classify any genus in a hermitian space of dimension ≥ 2 by neighbour steps at suitable primes. The method was implemented for positive definite hermitian lattices (not necessarily free) over $\mathbb{Q}(\sqrt{d})$. A table of class numbers of unimodular genera and the largest minima attained in those genera is given. We also describe a generalization of the LLL-algorithm to lattices in positive hermitian spaces over number fields.

1 Introduction

The neighbour method was introduced by Kneser [Kne57] for integral $\mathbb{Z}$-lattices in a quadratic space. The geometrical idea is to give a "global" construction that changes the localizations of a lattice only at one prime. The invariant factors of a lattice in its neighbours are $(1/p, 1, \ldots, 1, p)$. At primes p not dividing the determinant of a lattice L, all neighbours can be determined in a convenient way. Let the neighbourhood of a lattice L at p denote the minimal set containing L and all integral p-neighbours of its elements. A simple geometrical argument shows, that for $p \nmid \det(L)$ the neighbourhood of L contains all lattices with the same determinant that are locally equal to L at all primes but p. The connection to the genus of L is established using "strong approximation". This situation carries over to hermitian spaces over a number field.

We fix some notation: Let $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of integers, non-negative integers, rational, real, and complex numbers respectively. Troughout this paper E/F denotes an extension of degree 2 of number fields and $x \to \overline{x}$ the nontrivial automorphism of E over F. The rings of integers are denoted by $\mathcal{O}_E$ and $\mathcal{O}_F$. Let V be an n-dimensional E-space with a regular hermitian form $h : V \times V \mapsto E$ (i.e. h is E-linear in the first variable, $\forall x, y\,:\; h(x,y) = \overline{h(y,x)}$ and $\phi : V \to V^* : (\phi(y))(x) = h(x,y)$ is bijective). Let Ω_F denote the set of prime spots of F and for an arbitrary spot $\mathfrak{p}$ let $F_\mathfrak{p}$ be the localization of F at $\mathfrak{p}$ and $(\mathcal{O}_F)_\mathfrak{p}$ the integers in $F_\mathfrak{p}$. Let $E_\mathfrak{p} = E \otimes_F F_\mathfrak{p}$, $(\mathcal{O}_E)_\mathfrak{p} = \mathcal{O}_E \otimes_{\mathcal{O}_F} (\mathcal{O}_F)_\mathfrak{p}$ and $V_\mathfrak{p} = V \otimes_F F_\mathfrak{p}$. There is a canonical injection $(\mathcal{O}_E)_\mathfrak{p} \subset E_\mathfrak{p}$. If $\mathfrak{p}$ is inert or ramified then $E_\mathfrak{p}$ is a field (the localization of E at the prime above $\mathfrak{p}$). If $\mathfrak{p}$ is split, we define a ring isomorphism $E_\mathfrak{p} \simeq F_\mathfrak{p} \times F_\mathfrak{p}$ by $e \otimes f \mapsto (ef\,, \overline{e}f)$ (choosing a prime $\mathfrak{P} \mid \mathfrak{p}$ and an isomorphism $E_\mathfrak{P} \simeq F_\mathfrak{p}$). We define an involution of $E_\mathfrak{p}$ over $F_\mathfrak{p}$ by $e \otimes f \mapsto \overline{e} \otimes f$. $V_\mathfrak{p}$ is a free $E_\mathfrak{p}$-module of rank n in an obvious way and we can extend h to a hermitian form

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$h : V_{\mathfrak{p}} \times V_{\mathfrak{p}} \mapsto E_{\mathfrak{p}}$ by $h(v_1 \otimes f_1 \, , v_2 \otimes f_2) := h(v_1, v_2) \otimes f_1 f_2$ for $f_1, f_2 \in F_{\mathfrak{p}}$ and $v_1, v_2 \in V$.

Definition 1.1 *We say that the hermitian space (V,h) is (positive) definite, if for all archimedean spots ν of F the completions are $F_\nu \simeq \mathbb{R}$ and $E_\nu \simeq \mathbb{C}$ and all the hermitian spaces (V_ν, h) are (positive) definite. Otherwise (V,h) is called indefinite. The space (V,h) is called isotropic at $\mathfrak{p}$, if $(V_{\mathfrak{p}}, h)$ has an isotropic vector $x \in V_{\mathfrak{p}} \setminus \{0\}$ with $h(x,x) = 0$.*

A hermitian space (W,h) over a ring R in our context can be the n-dimensional vector space (V,h) over E or the free $E_{\mathfrak{p}}$-module $(V_{\mathfrak{p}}, h)$, where h is the induced hermitian form. An isomorphism of hermitian spaces (W,h) and (W',h') over R is an isomorphism $\phi : W \to W'$ of R-modules, that respects the hermitian forms, i.e. with $h'\big(\phi(x), \phi(y)\big) = h(x,y)$ for $x, y \in W$. Isomorphisms in the category of hermitian spaces are called isometries.

Definition 1.2 *The unitary group of a hermitian space (W,h) over R is the group of automorphisms. We define the unitary and the special unitary group by*

$$\begin{aligned} \mathbf{U}(W,h) &:= \big\{\phi \in \mathbf{GL}(W,R) \mid \forall x,y \in W : h\big(\phi(x),\phi(y)\big) = h(x,y)\big\} \\ \mathbf{SU}(W,h) &:= \big\{\phi \in \mathbf{U}(W,h) \mid \det(\phi) = 1\big\} \, . \end{aligned}$$

According to [Lan36] two spaces (V,h) and (V',h') over E are isometric if and only if the localizations (V_ν, h) and (V'_ν, h) are isometric for all archimedean and discrete valuations ν of F.

Now let $\mathcal{O}$ be a Dedekind domain with quotient field k. An $\mathcal{O}$-lattice L in a space $W \simeq k^n$ is a finitely generated $\mathcal{O}$-module, that spans W as a vector space. It is not necessarily free. However, it admits a representation of the form

$$L = \sum_{i=1}^{n} \mathfrak{a}_i x_i \; , \qquad (1)$$

where $x_i \in k$ and $\mathfrak{a}_i$ are fractional ideals of $\mathcal{O}$. A k-base $x_1, \ldots, x_n$ of the vector space W for which we can write L in the form (1) is called a *pseudo-base* of L, and $\mathfrak{a}_i$ the *coefficient ideal* of x_i in L. The ideal class of $\prod_{i=1}^{n} \mathfrak{a}_i$ in the ideal class group C_E of E does not depend on the special pseudo-base, it is called the *Steinitz class* of L. For the basic facts about lattices over Dedekind domains we refer to [O'M63] Chap. 8. Now, with E/F and V as above we define the localization of an $\mathcal{O}_E$-lattice $L \subset V$ at a prime $\mathfrak{p} \in \Omega_F$ by $L_{\mathfrak{p}} = L \otimes_{\mathcal{O}_F} (\mathcal{O}_F)_{\mathfrak{p}} \subset E_{\mathfrak{p}}$. To include the localization at split primes we extend the notion of a lattice (following [Shi64]): An $\mathcal{O}$-lattice L' in a free R-module W will denote either the localization of an $\mathcal{O}_E$-lattice L at a split prime $\mathfrak{p} \in \Omega_F$ (where $\mathcal{O} = (\mathcal{O}_E)_{\mathfrak{p}} \simeq (\mathcal{O}_F)_{\mathfrak{p}} \times (\mathcal{O}_F)_{\mathfrak{p}}$, $R = E_{\mathfrak{p}}$, $W = V_{\mathfrak{p}}$) or a lattice over a Dedekind domain $\mathcal{O}$ with quotient field R.

Definition 1.3 *Let L and M be $\mathcal{O}$-lattices in a free R-module W. The relative discriminant of L in M is the $\mathcal{O}$-module in R defined by*

$$[L/M]_R := \big\langle \{\det \alpha \mid \alpha \in \mathbf{GL}(W,R) \text{ and } \alpha L \subseteq M\} \big\rangle_{\mathcal{O}} \, .$$

We will simply write $[L/M]$ instead of $[L/M]_R$ in most cases.

The relative discriminant has the following properties, where the case of $(\mathcal{O}_E)_{\mathfrak{p}}$-lattices at a split prime (with $(\mathcal{O}_E)_{\mathfrak{p}} = (\mathcal{O}_F)_{\mathfrak{p}} \times (\mathcal{O}_F)_{\mathfrak{p}}$) can be derived considering the projections $L \cdot (1,0)$ and $L \cdot (0,1)$ as $(\mathcal{O}_F)_{\mathfrak{p}}$-lattices (see [Shi64]).

Lemma 1.4 *Let L, M, N be $\mathcal{O}_E$ lattices in an E-space W. We have:*

1. $[L/M]\,[M/N] = [L/N]$.

2. *For* $\alpha \in \mathbf{GL}(W, E)$*:* $[L/\alpha L] = \det(\alpha)\mathcal{O}_E$.

3. *If* $L \supseteq M$ *and* $L/M \simeq \mathcal{O}_E/\mathfrak{a}$ *as* $\mathcal{O}_E$*-modules, then* $[L/M] = \mathfrak{a}$.

4. *For a prime* $\mathfrak{p} \in \Omega_F$ *:* $[L/M]_E\,(\mathcal{O}_E)_\mathfrak{p} = [L_\mathfrak{p}/M_\mathfrak{p}]_{E_\mathfrak{p}}$.

5. *Let* K *be a subfield of the number field* E*: Then* $[L/M]_K = \mathrm{N}_{E/K}\big([L/M]_E\big)$.

Given number fields E/K we denote by $\mathrm{N}_{E/K}(a)$, $\mathrm{N}_{E/K}(\mathfrak{a})$ the norm of elements in E as well as the norm of $\mathcal{O}_E$-ideals, and in the case $K = \mathbb{Q}$ the latter will also denote the positive generator of the $\mathbb{Z}$-ideal $\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{a})$.
The theory of lattices in hermitian spaces develops along the lines of the quadratic case. Isomorphisms in the category of hermitian spaces with lattices, are called isometries of (hermitian) lattices. Automorphisms of lattices are denoted as follows:

Definition 1.5 *Let* L *be an* $\mathcal{O}$ *lattice in a hermitian space* (W, h)*. Then*

$$\begin{aligned} \mathbf{U}(L,h) &:= \big\{\phi \in \mathbf{U}(W,h) \mid \phi(L) = L\big\} \\ \mathbf{SU}(L,h) &:= \big\{\phi \in \mathbf{U}(L,h) \mid \det(\phi) = 1\big\} \end{aligned}$$

Let us recall some definitions for hermitian lattices:

Definition 1.6 *Let* L *be an* $\mathcal{O}$*-lattice in a hermitian space* (W, h)*. The dual lattice* $L^\#$ *and the discriminant ideal* $\mathfrak{d}(L)$ *are defined by*

$$\begin{aligned} L^\# &:= \big\{x \in W \mid h(x, L) \subseteq \mathcal{O}\big\} \qquad (2) \\ \mathfrak{d}(L) &:= [L^\#/L]\,. \end{aligned}$$

If $L \subseteq L^\#$ *we say that* L *is integral, and* L *is called unimodular iff* $L = L^\#$*.*
The norm ideal $\mathfrak{n}(L)$ *of* L *is the fractional ideal generated by* $\{h(x,x) \mid x \in L\}$ *(the* $\mathcal{O}_F$ *or* $(\mathcal{O}_F)_\mathfrak{p}$ *ideal if* $\mathcal{O} = \mathcal{O}_E$ *or* $\mathcal{O} = (\mathcal{O}_E)_\mathfrak{p}$*, respectively). If* $F = \mathbb{Q}$ *and* $\mathfrak{n}(L) \subseteq 2\mathbb{Z}$ *then* L *is called even.*

The discriminant ideal of a lattice L subject to (1) can be expressed by the Gram matrix of h with respect to the pseudo-base $(x_1, \ldots, x_n)$ and the coefficient ideals $\mathfrak{a}_i$ as follows:

$$\mathfrak{d}(L) = \prod_{i=1}^{n} (\mathfrak{a}_i \overline{\mathfrak{a}_i})\ \det\Big(\big(h(x_i, x_j)\big)_{1 \le i,j \le n}\Big)\,. \qquad (3)$$

Let L be an $\mathcal{O}_E$-lattice in the n-dimensional hermitian E-space (V, h). We may regard L also as $\mathbb{Z}$-lattice in the quadratic space $(V, \mathrm{tr}_{E/\mathbb{Q}} \circ h)$ over $\mathbb{Q}$, where $\mathrm{tr}_{E/\mathbb{Q}}$ denotes the trace. Let $L^\#_\mathbb{Z}$ denote the dual of L as a $\mathbb{Z}$-lattice with respect to the $\mathbb{Q}$-bilinear form $T := \mathrm{tr}_{E/\mathbb{Q}} \circ h$ (defined analogous to (2)). From the definition of the different $\mathcal{D}_{E/\mathbb{Q}}$ of E over $\mathbb{Q}$ it is immediate that

$$L^\#_\mathbb{Z} = \mathcal{D}_{E/\mathbb{Q}}^{-1}\, L^\#\,. \qquad (4)$$

Let $\det_B(M)$ denote the determinant of a $\mathbb{Z}$-lattice M in an m-dimensional $\mathbb{Q}$-space (W, B) with bilinear form B, i.e. $\det_B(M) := \det\Big(\big(B(z_i, z_j)\big)_{1 \le i,j \le m}\Big)$ for any $\mathbb{Z}$-base $(z_1, \ldots, z_m)$ of M. Let $\mathrm{d}_{E/\mathbb{Q}}$ denote the discriminant of E. Then (4) and 5. of Lemma 1.4 imply

$$\big|\det_T\big(L^\#_\mathbb{Z}\big)\big| = \big|\mathrm{d}_{E/\mathbb{Q}}\big|^{-n}\, \mathrm{N}_{E/\mathbb{Q}}\big(\mathfrak{d}(L)\big)\,. \qquad (5)$$

Since we have strong approximation only with respect to $\mathbf{SU}(V,h)$ and not for the unitary group, we need to define the notion of the "special genus" of a lattice, analogous to the spinor genus of a lattice in a quadratic space.

Definition 1.7 *Let L be an $\mathcal{O}_E$-lattice in the hermitian space (V,h). The class, genus and the special genus $\mathrm{gen}^0(L)$ of L are defined by*

$$\begin{aligned} \mathrm{cl}(L) &= \{M \text{ an } \mathcal{O}_E\text{-lattice} \mid \exists\, \alpha \in \mathbf{U}(V,h) : L = \alpha M\} \\ \mathrm{gen}(L) &= \{M \text{ an } \mathcal{O}_E\text{-lattice} \mid \forall\, \mathfrak{p} \in \Omega_F\ \exists\, \alpha_\mathfrak{p} \in \mathbf{U}(V_\mathfrak{p},h) : L_\mathfrak{p} = \alpha_\mathfrak{p} M_\mathfrak{p}\} \\ \mathrm{gen}^0(L) &= \{M \text{ an } \mathcal{O}_E\text{-lattice} \mid \exists\, \beta \in \mathbf{U}(V,h)\ \forall\, \mathfrak{p} \in \Omega_F\ \exists\, \alpha_\mathfrak{p} \in \mathbf{SU}(V_\mathfrak{p},h) : \\ &\qquad L_\mathfrak{p} = \beta \circ \alpha_\mathfrak{p} M_\mathfrak{p}\} \end{aligned}$$

The local theory of hermitian forms is developed in [Jac62] for the ramified and inert case where $E_\mathfrak{p}$ is a field, and in [Shi64] for the case $E_\mathfrak{p} = F_\mathfrak{p} \times F_\mathfrak{p}$. We will state the results where they are needed.

In Section 2 we describe the properties of the neighbour construction and the application of the strong approximation theorem for $\mathbf{SU}(V,h)$. For hermitian spaces the neighbour method was introduced by Hoffmann [Hof91] under conditions that imply $\mathrm{gen}(L) = \mathrm{gen}^0(L)$. We generalize some of his results and prove that all classes in a genus $\mathrm{gen}(L)$ have representatives that can be constructed from L by successive neighbour steps at different primes. A suitable finite set of primes can be determined a priori.

Section 3 gives a detailed description of all the neighbours of one lattice. An explicit formula for the number of neighbours is established in Lemma 3.3.

In Section 4 some details of the practical neighbour algorithm are given. We define a reduction procedure for lattices in a totally positive hermitian space that is analogous to the LLL-algorithm and gives a similar bound for the minimum.

Results of computations are given in Section 5. We list the number of classes and the largest minimum attained for unimodular hermitian genera that were classified completely. More data for these genera including lists of classes and the masses can be obtained at `http://www.math.uni-sb.de/~schulzep/`.

2 The neighbour method

In this section we reformulate results of Hoffmann [Hof91] in a more general context. Hoffmann considered modular hermitian lattices of odd dimension and assumed that every square in the group of ideal classes of E was trivial. But only the properties of the localization $L_\mathfrak{p}$ with $\mathfrak{P} \mid \mathfrak{p}$ are relevant for the neighbour construction with respect to the prime $\mathfrak{P} \subset \mathcal{O}_E$. We assume that $\mathfrak{P}$ does not divide the discriminant of L. We could easily generalize to the case that L is "locally modular" i.e. $L_\mathfrak{p}$ becomes unimodular by suitable scaling of the hermitian form. The main facts about the set of classes generated by successive neighbour constructions are stated in Proposition 2.5, Corollary 2.7, Theorem 2.9. They show that $\mathrm{gen}(L)$ is contained in the closure of $\{L\}$ under neighbour constructions at a finite number of primes not dividing $\mathfrak{d}(L)$. Analyzing the special genera in $\mathrm{gen}(L)$ we can give a finite number of lattices $L_i \in \mathrm{gen}(L)$ (each constructed from L by neighbouring) such that $\mathrm{gen}(L) \subseteq \bigcup_i \mathfrak{N}(L_i, \mathfrak{P})$ for suitable $\mathfrak{P}$.

The following definition and lemma are similar to those given in [HS94] for $\mathbb{Z}$-lattices in a quadratic space.

Definition 2.1 *Let L be an integral $\mathcal{O}_E$-lattice in V, $\mathfrak{P}$ a prime ideal in $\mathcal{O}_E$ that does not divide $\mathfrak{d}(L)$.*

1. *An $\mathcal{O}_E$-lattice M is called a $\mathfrak{P}$-neighbour of L iff the following holds: M is integral and we have the $\mathcal{O}_E$-module isomorphisms*

$$M/(L \cap M) \simeq \mathcal{O}_E/\mathfrak{P} \;\; \text{and} \;\; L/(L \cap M) \simeq \mathcal{O}_E/\overline{\mathfrak{P}}\,.$$

2. *Let $x \in L \setminus \mathfrak{P}L$ with $h(x,x) \in \mathfrak{P}\overline{\mathfrak{P}}$. Such a vector is called admissible in this context. The $\mathfrak{P}$-neighbour of L at an admissible vector x is*

$$L(\mathfrak{P},x) := \mathfrak{P}^{-1}x + \underbrace{\{y \in L \mid h(x,y) \in \mathfrak{P}\}}_{:= L_x}$$

If there are only neighbours of one prime ideal $\mathfrak{P}$ in a specific context, we use the notation $L(x) = L(\mathfrak{P},x)$.

Using the invariant factor theorem it is easily seen that if L is a neighbour of M then $\mathfrak{d}(L) = \mathfrak{d}(M)$.

Lemma 2.2 *Let L be as above. An $\mathcal{O}_E$-lattice M is a $\mathfrak{P}$-neighbour of L iff there is an $x \in L \setminus \mathfrak{P}L$ with $h(x,x) \in \mathfrak{P}\overline{\mathfrak{P}}$ and $M = L(x)$.*

Proof: "$\Rightarrow$": Let $x \in \mathfrak{P}M \setminus \mathfrak{P}L \subset L$. From the integrality of M we have $h(x,x) \in \mathfrak{P}\overline{\mathfrak{P}}$ and $L_x \supseteq L \cap M$. Also $L \supsetneq L_x$ because L is integral, $x \notin \mathfrak{P}L$ and $\mathfrak{P} \not| \mathfrak{d}(L)$. So $L_x/(L \cap M)$ is a true submodule of the simple module $L/(L \cap M)$ and thus $L_x = L \cap M$. Similarly $M \supseteq L(x) \supsetneq L \cap M$ yields $M = L(x)$.
"$\Leftarrow$:" It is easy to see that $h(x,x) \in \mathfrak{P}\overline{\mathfrak{P}}$ implies $L(x)$ is integral and $L(x) \cap L = L_x$, and $x \in L \setminus \mathfrak{P}L$ implies $L \cap \mathfrak{P}^{-1}x = \mathcal{O}_E x$. Thus we have:

$$L(x)/(L(x) \cap L) \simeq (L + L(x))/L \simeq (\mathfrak{P}^{-1}x + L)/L \simeq \mathfrak{P}^{-1}x/(L \cap \mathfrak{P}^{-1}x) \simeq \mathcal{O}_E/\mathfrak{P}$$

To determine $L/(L \cap L(x))$ consider the module homomorphism $\phi: L \to \mathcal{O}_E/\overline{\mathfrak{P}}$: $\phi(y) = h(y,x) + \overline{\mathfrak{P}}$. It is surjective because $\mathfrak{P} \not| \mathfrak{d}(L)$ and the kernel of ϕ is clearly L_x. Thus $L/(L \cap L(x)) \simeq \mathcal{O}_E/\overline{\mathfrak{P}}$ as was claimed by the Lemma. ∎

What makes this notion so valuable is the following fact: By successive construction of $\mathfrak{P}$-neighbours starting with a lattice L we obtain all classes of lattices M which have a representative that is locally equal to L at all spots $\mathfrak{q} \in \Omega_F \setminus \{\mathfrak{p}\}$ and with $\mathfrak{d}(M) = \mathfrak{d}(L)$.

Definition 2.3 *Notation being as above let*

$$\mathfrak{N}(L,\mathfrak{P}) = \{M \text{ an } \mathcal{O}_E\text{-lattice} \mid \exists \beta \in \mathbf{U}(V,h)\, \exists L_0 = L, L_1, \ldots, L_k = \beta M : L_{i+1} \text{ is a } \mathfrak{P}\text{-neighbour of } L_i\}$$

The (oriented) neighbour graph $\mathrm{NG}(L,\mathfrak{P})$ of a lattice L at $\mathfrak{P}$ is the graph with vertices from $\{\mathrm{cl}(M) \mid M \in \mathfrak{N}(L,\mathfrak{P})\}$ and the edges $(\mathrm{cl}(M), cl(N))$ for $M, N \in \mathfrak{N}(L,\mathfrak{P})$ with N a $\mathfrak{P}$-neighbour of M.

Proposition 2.4 *Let L, M be integral $\mathcal{O}_E$-lattices with $\mathfrak{d}(L) = \mathfrak{d}(M)$ and let $\mathfrak{P}$ be a prime of $\mathcal{O}_E$ with $\mathfrak{P} \not| \mathfrak{d}(L)$ and set $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$. Assume that there is an $e \in \mathbb{N}_0$ such that $\mathfrak{P}^e M \subseteq L$. Then there are integral lattices $L_0 = L, L_1, \ldots, L_k = M$ such that L_{i+1} is a $\mathfrak{P}$-neighbour of L_i for all $i \in \{0, \ldots, k-1\}$.*
Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_F$ with $\mathcal{O}_F \supseteq \mathfrak{a} \supseteq \mathrm{tr}_{E/F}(\mathcal{O}_E)$. Assume that for the norm ideals of L and M holds $\mathfrak{n}(L), \mathfrak{n}(M) \subseteq \mathfrak{a}$. Then we may choose the lattices $L_1, \ldots, L_k$ as above with $\mathfrak{n}(L_i) \subseteq \mathfrak{a}$.

Proof: For the first statement we reformulate the proof of Th. 4.7 of [Hof91] in our context. Suppose $e \in \mathbb{N}_0$ is minimal with $\mathfrak{P}^e L \subseteq M$. If $e = 0$ we are done, since $\mathfrak{d}(L) = \mathfrak{d}(M)$ and $L \subseteq M$ implies $L = M$.
Let $e \geq 1$. Since $\mathfrak{P}^{e-1} M \not\subseteq L$, we may choose $x \in \mathfrak{P}^e M \setminus \mathfrak{P} L \subseteq L \setminus \mathfrak{P} L$. Clearly $h(x,x) \in \mathfrak{P}\overline{\mathfrak{P}}$ and $h(x, M) \subseteq \mathfrak{P}$ since M is integral. Let $L_1 = L(x)$. Then $\mathfrak{P}^e M \subseteq L \cap M \subseteq L_x \subset L_1$. The index $[M/(L_1 \cap M)]$ is smaller than $[M/(L \cap M)]$ because $\mathfrak{P}^{-1} x \subseteq M$ and $\mathfrak{P}^{-1} x \not\subseteq L$ implies $L \cap M \subsetneq L(x) \cap M$. We proceed by applying this construction to L_i until $M = L_i$ after finitely many steps, since the index $[M/(L_i \cap M)]$ is strictly decreasing. For the second statement assume that $\mathfrak{n}(L), \mathfrak{n}(M) \subseteq \mathfrak{a}$. Then for L_1 as above holds $\mathfrak{n}(L_1) \subseteq \mathfrak{a}$ and thus $\mathfrak{n}(L_2), \ldots, \mathfrak{n}(L_k) \subseteq \mathfrak{a}$. ∎

Proposition 2.5 *Let L, $\mathfrak{P}$ be as above and let $\mathfrak{P} \mid \mathfrak{p} \in \Omega_F$. Then*

1. $$\mathfrak{N}(L, \mathfrak{P}) = \Big\{ M \text{ an } \mathcal{O}_E\text{-lattice} \mid \ \mathfrak{d}(L) = \mathfrak{d}(M) \text{ and } \exists \beta \in \mathbf{U}(V,h) \ \forall \mathfrak{q} \in \Omega_F \setminus \{\mathfrak{p}\} : L_\mathfrak{q} = \beta M_\mathfrak{q} \Big\}$$

2. *If $\mathfrak{a}$ is an ideal in $\mathcal{O}_F$ with $\mathcal{O}_F \supseteq \mathfrak{a} \supseteq \mathrm{tr}_{E/F}(\mathcal{O}_E)$, then the set*
$$N(L, \mathfrak{P}, \mathfrak{a}) := \big\{ \mathrm{cl}(M) \in \mathrm{NG}(L, \mathfrak{P}) \mid \ \mathfrak{n}(M) \subseteq \mathfrak{a} \big\}$$
of vertices is connected in $\mathrm{NG}(L, \mathfrak{P})$.

Proof: 1. "$\subseteq$": Let x be an admissible vector for L at $\mathfrak{P}$. For $\mathfrak{q} \neq \mathfrak{p}$ we have $L(x)_\mathfrak{q} = L_\mathfrak{q}$.
"$\supseteq$": For any $\mathcal{O}_E$-lattices L, M with $\mathfrak{p}^e M \subseteq L$ for some $e \in \mathbb{N}_0$ there is an $e' \in \mathbb{N}_0$ and $a \in E$ with $N_{E/F}(a) = 1$ such that $\mathfrak{P}^{e'} a M \subseteq L$. This is trivially possible with $a = 1$ if $\mathfrak{P}$ is not split and in the split case we can choose $a = \overline{b} b^{-1}$ where $\overline{b}$ is a generator of a suitable power of $\mathfrak{P}$ as described in [Hof91], Prop.3.2. Then the inclusion follows from Prop.2.4.
The second assertion is a consequence of the second part of Prop.2.4. ∎

On the other hand, if $n \geq 2$ the strong approximation theorem for $\mathbf{SU}(V,h)$ implies that a neighbourhood $\mathfrak{N}(L, \mathfrak{P})$ (at suitable $\mathfrak{P}$) contains at least the special genus of L. Strong approximation on $\mathbf{SU}(V,h)$ for indefinite h was proved in [Shi64], for a survey of the general situation for classical groups refer to the article of Kneser in [AD66].

Theorem 2.6 *(Strong Approximation): With the notation as above assume that the dimension of V is ≥ 2. Let $S \subseteq \Omega_F$. Assume that V is indefinite or that $V_\mathfrak{q}$ is isotropic at a prime $\mathfrak{q} \in \Omega_F \setminus S$. Let T be a finite set of primes with $T \subset S$. Let L be an $\mathcal{O}_F$-lattice in V and for all $\mathfrak{q} \in T$ let $\sigma_\mathfrak{q} \in \mathbf{SU}(V_\mathfrak{q}, h)$. Then for every positive integer e there exists $\sigma \in \mathbf{SU}(V,h)$ subject to:*

$$\begin{aligned} \forall \mathfrak{q} \in T : \quad & (\sigma - \sigma_\mathfrak{q}) L_\mathfrak{q} \subset \mathfrak{q}^e L_\mathfrak{q} \\ \forall \mathfrak{q} \in S \setminus T : \quad & \sigma L_\mathfrak{q} = L_\mathfrak{q} \end{aligned}$$

Corollary 2.7 *Let $\dim(V) \geq 2$, $L \subset V$ be an $\mathcal{O}_E$-lattice and $\mathfrak{P}$ a prime in $\mathcal{O}_E$ with $\mathfrak{P} \nmid \mathfrak{d}(L)$. Let $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$ and assume that $V_\mathfrak{p}$ is isotropic or V is indefinite. Then*

$$\mathfrak{N}(L, \mathfrak{P}) \supseteq \mathrm{gen}^0(L)\,.$$

Proof: Let $M \in \mathrm{gen}^0(L)$ and for all $\mathfrak{q} \in \Omega_F$ let $\beta \sigma_\mathfrak{q} L_\mathfrak{q} = M_\mathfrak{q}$ with $\beta \in \mathbf{U}(V,h)$ and $\sigma_\mathfrak{q} \in \mathbf{SU}(V_\mathfrak{q}, h)$. Apply the above theorem with $S = \Omega_F \setminus \{\mathfrak{p}\}$, $T = \{\mathfrak{q} \mid \beta^{-1} M_\mathfrak{q} \neq$

$L_{\mathfrak{q}}\}$ and e such that $\mathfrak{q}^e L_{\mathfrak{q}} \subseteq \sigma_{\mathfrak{q}} L_{\mathfrak{q}}$ for $\mathfrak{q} \in T$. We can find $\sigma' \in \mathbf{SU}(V,h)$ with $\sigma' L_{\mathfrak{q}} = \sigma_{\mathfrak{q}} L_{\mathfrak{q}}$ for all $\mathfrak{q} \in S$. Then the assertion is a consequence of Prop.2.5. ∎

The hypothesis that $V_{\mathfrak{p}}$ is isotropic (in the definite case) comes true for all split $\mathfrak{P}$ as well as for $\dim_E(V) \geq 3$.

Let us briefly collect some facts about $(\mathcal{O}_E)_{\mathfrak{q}}$-lattices in the hermitian space $V_{\mathfrak{q}}$ (see [Jac62] for inert and ramified $\mathfrak{q}$ and [Shi64] Sect.3 for the split case):
If $\mathfrak{q} \in \Omega_F$ is split or inert, then every $(\mathcal{O}_E)_{\mathfrak{q}}$-lattice $L^{\mathfrak{q}}$ has an orthogonal $(\mathcal{O}_E)_{\mathfrak{q}}$-base.
If $\mathfrak{q} \in \Omega_F$ is ramified, then every lattice $L^{\mathfrak{q}}$ is an orthogonal sum of 1 and 2-dimensional sublattices. If $\mathfrak{q}$ is split or inert or ramified with $\mathfrak{q} \nmid 2$, then all unimodular lattices $L^{\mathfrak{q}} \subset V_{\mathfrak{q}}$ are isometric. In the remaining case of ramified $\mathfrak{q} \mid 2$, we need the norm ideal $\mathfrak{n}(L^{\mathfrak{q}})$ as additional invariant to identify $\mathrm{cl}(L^{\mathfrak{q}})$. This leads to

Lemma 2.8 *Let L, $\mathfrak{P}$ be as in Corollary 2.7. If $\mathfrak{P}$ is split or inert or $\mathfrak{P} \nmid 2$, then $\mathfrak{N}(L,\mathfrak{P}) \subseteq \mathrm{gen}(L)$. If $\mathfrak{P}$ is ramified and $\mathfrak{P} \mid 2$, then for $M \in \mathfrak{N}(L,\mathfrak{P})$ we have: $M \in \mathrm{gen}(L) \iff \mathfrak{n}(M) = \mathfrak{n}(L)$.*

Following [Shi64] we set

$$\begin{aligned} \mathcal{E}_{\mathfrak{q},0} &:= \{x \in (\mathcal{O}_E)^*_{\mathfrak{q}} \mid x\overline{x} = 1\} \\ \mathcal{E}(L_{\mathfrak{q}}) &:= \{\det(\alpha) \mid \alpha \in \mathbf{U}(L_{\mathfrak{q}},h)\} \quad . \end{aligned}$$

If $L_{\mathfrak{q}}$ splits a 1-dimensional sublattice it is immediate that $\mathcal{E}(L_{\mathfrak{q}}) = \mathcal{E}_{\mathfrak{q},0}$, and because it always splits at least a 2-dimensional sublattice we have $\mathcal{E}(L_{\mathfrak{q}}) \supseteq {\mathcal{E}_{\mathfrak{q},0}}^2$.
Using the neighbour construction at different primes (not dividing the discriminant of the genus) we reach all classes of lattices in $\mathrm{gen}(L)$. In view of Prop.2.5 this is a consequence of the following theorem setting $T = \{\mathfrak{q} \in \Omega_F \mid \mathfrak{q} \mid \mathfrak{d}(L) \cap \mathcal{O}_F\}$.

Theorem 2.9 *With the common notation let M and L be integral $\mathcal{O}_E$-lattices in a hermitian space V with $\dim_E(V) \geq 2$ and assume $M \in \mathrm{gen}(L)$. Let $T \subset \Omega_F$ be a finite set of primes. Then there is a lattice M' in the class of M such that $M'_{\mathfrak{q}} = L_{\mathfrak{q}}$ for all $\mathfrak{q} \in T$.*

Proof: Let $\alpha_{\mathfrak{q}} L_{\mathfrak{q}} = M_{\mathfrak{q}}$ with $\alpha_{\mathfrak{q}} \in \mathbf{U}(V_{\mathfrak{q}},h)\ \forall \mathfrak{q} \in \Omega_F$. Let us first assume, that there is a $\gamma \in \mathbf{U}(V,h)$ with $\det(\gamma)\det(\alpha_{\mathfrak{q}}) = \det(\beta_{\mathfrak{q}})$ where $\beta_{\mathfrak{q}} \in \mathbf{U}(L_{\mathfrak{q}},h)$ for all $\mathfrak{q} \in T$. Applying Theorem 2.6 to $\gamma\alpha_{\mathfrak{q}}{\beta_{\mathfrak{q}}}^{-1}$ we find $\delta \in \mathbf{SU}(V,q)$ with $\delta L_{\mathfrak{q}} = \gamma\alpha_{\mathfrak{q}}{\beta_{\mathfrak{q}}}^{-1} L_{\mathfrak{q}}$ for $\mathfrak{q} \in T$. Then $\delta^{-1}\gamma M_{\mathfrak{q}} = \delta^{-1}\gamma\alpha_{\mathfrak{q}}{\beta_{\mathfrak{q}}}^{-1} L_{\mathfrak{q}} = L_{\mathfrak{q}}$ for $\mathfrak{q}$ in T as was claimed by the theorem. To find a γ with the properties stated above it is sufficient to find a scalar $c \in E$ with $c\overline{c} = 1$ and $c\det(\alpha_{\mathfrak{q}}) \in \mathcal{E}(L_{\mathfrak{q}})$. For all split $\mathfrak{q} \in T$ fix a $\mathfrak{Q} \mid \mathfrak{q}$ and an isomorphism $E_{\mathfrak{Q}} \simeq F_{\mathfrak{q}}$. Then we have an isomorphism $E_{\mathfrak{q}} \simeq F_{\mathfrak{q}} \times F_{\mathfrak{q}}$ via $e \otimes s \mapsto (es, \overline{e}s)$. The map $E^*_{\mathfrak{q}} \to \{x \in E_{\mathfrak{q}} \mid x\overline{x} = 1\} : x \mapsto \overline{x}x^{-1}$ is surjective if $E_{\mathfrak{q}}$ is a field, and in the split case even the restriction to $F^*_{\mathfrak{q}} \times \{1\}$ is surjective. For $\mathfrak{q} \in T$ let $\det(\alpha_{\mathfrak{q}}) = \overline{r}_{\mathfrak{q}}{r_{\mathfrak{q}}}^{-1}$ with $r_{\mathfrak{q}} \in E_{\mathfrak{q}}$ and in the split case let $r_{\mathfrak{q}} = (r_{\mathfrak{q},1}, 1) \in F_{\mathfrak{q}} \times F_{\mathfrak{q}}$. Using strong approximation in E, we find $s \in E$ such that $sr_{\mathfrak{q},1}$ is a unit in the ring of integers of $E_{\mathfrak{Q}}$ and s is a unit at $\overline{\mathfrak{Q}}$ for all split $\mathfrak{q} \in T$ (where $\mathfrak{Q} \mid \mathfrak{q}$ is chosen as mentioned above) and such that $sr_{\mathfrak{q}} - 1 \in 4\mathfrak{Q}\,(\mathcal{O}_E)_{\mathfrak{q}}$ for ramified $\mathfrak{Q} \mid \mathfrak{q} \in T$. Let $c = \overline{s}s^{-1}$. For split primes $\mathfrak{q} \in T$ we have $c\det(\alpha_{\mathfrak{q}}) = \overline{s}s^{-1}(r_{\mathfrak{q},1}^{-1}, r_{\mathfrak{q},1}) = (\overline{s}(sr_{\mathfrak{q},1})^{-1}, \overline{s}^{-1}sr_{\mathfrak{q},1}) \in \mathcal{E}_{\mathfrak{q},0} = \mathcal{E}(L_{\mathfrak{q}})$. At ramified primes in T we have $sr_{\mathfrak{q}} = x^2_{\mathfrak{q}}$ by the local square theorem and thus $c\det(\alpha_{\mathfrak{q}}) = \overline{sr_{\mathfrak{q}}}(sr_{\mathfrak{q}})^{-1} \in \mathcal{E}^2_{\mathfrak{q},0} \subseteq \mathcal{E}(L_{\mathfrak{q}})$. For inert $\mathfrak{q}$ we always have $c\det(\alpha_{\mathfrak{q}}) \in \mathcal{E}_{\mathfrak{q},0} = \mathcal{E}(L_{\mathfrak{q}})$. ∎

There are finitely many special genera in one genus and the index is determined by the structure of the class group of E and "local factors". The relation between $\mathrm{gen}(L)$ and $\mathrm{gen}^0(L)$ is revealed in theorem 5.24 and 5.27 of [Shi64] (where the class is actually the special genus). We can summarize these results as follows:

Theorem 2.10 *Let C denote the group of ideal classes of E and let C_0 be the subgroup of those classes that have a representative $\mathfrak{a} = \overline{\mathfrak{a}}$. Define subgroups of the ideals of $\mathcal{O}_E$ by $J := \{\mathfrak{a} \in I_E \mid \mathrm{N}_{E/F}(\mathfrak{a}) = \mathcal{O}_F\}$ and $J_0 := \{a\mathcal{O}_E \mid a \in E \ , \ a\overline{a} = 1\}$. Let Ω_r be the set of those (ramified) primes $\mathfrak{q} \subset \mathcal{O}_F$ with $\mathcal{E}_{\mathfrak{q},0} \neq \mathcal{E}(L_\mathfrak{q})$. Set $\mathfrak{E}(L) = \prod_{\mathfrak{q}\in\Omega_r} \mathcal{E}_{\mathfrak{q},0}/\mathcal{E}(L_\mathfrak{q})$ and $R = \big\{\big(a\,\mathcal{E}(L_\mathfrak{q})\big)_{\mathfrak{q}\in\Omega_r} \in \mathfrak{E}(L) \mid \ a \in \mathcal{O}_E \text{ and } \overline{a}a = 1\big\}$. Let $H \subset J \times \mathfrak{E}(L)$ with $H := \big\{\big(a\mathcal{O}_E\,,\,(a\,\mathcal{E}(L_\mathfrak{q}))_{\mathfrak{q}\in\Omega_r}\big) \ \mid \ a \in E \ , \ a\overline{a} = 1\big\}$. Define a map*

$$\psi \ : \ \mathrm{gen}(L) \to J \times \mathfrak{E}(L) \ : \ M \mapsto \Big([L/M]\,,\, \big(\det(\alpha_\mathfrak{q})\,\mathcal{E}(L_\mathfrak{q})\big)_{\mathfrak{q}\in\Omega_r}\Big)\ ,$$

where we assume that $\alpha_\mathfrak{q} L_\mathfrak{q} = M_\mathfrak{q}$ with $\alpha_\mathfrak{q} \in \mathbf{U}(V_\mathfrak{q}, h)$ and where $[L/M]$ denotes the relative discriminant of L in M. Then we have:

1. *The map ψ induces a one-to-one correspondence of the special genera in $\mathrm{gen}(L)$ and the elements of $\big(J \times \mathfrak{E}(L)\big)\,/\,H$. The cosets of H are precisely the sets $(\mathfrak{a}_i, \theta_j)\,H$, where the $\mathfrak{a}_i$ run through a complete set of representatives of J/J_0 and θ_j through a complete set of representatives of $\mathfrak{E}(L)/R$.*

2. *The map $\mathfrak{a} \mapsto \mathfrak{a}^{-1}\overline{\mathfrak{a}}$ induces an isomorphism $C/C_0 \simeq J/J_0$, and the number of special genera in gen(L) is $|C/C_0| \cdot |\mathfrak{E}(L)/R|$.*

How does a neighbour step affect the special genus? Let $\mathfrak{P} \subset \mathcal{O}_E$ be a prime with $\mathfrak{P} \nmid \mathfrak{d}(L)$ and let x be an admissible vector for L at $\mathfrak{P}$. The invariant factors of L_x in L and $L(x)$ are $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ by Lemma 2.2, thus$[L/L(\mathfrak{P},x)] = \mathfrak{P}^{-1}\overline{\mathfrak{P}}$. If $n = \dim_E(V)$ is odd or if $\mathfrak{P}$ is not ramified or if $\mathfrak{P} \nmid 2$ then [Jac62] shows that $\mathcal{E}(L_\mathfrak{p}) = \mathcal{E}_{\mathfrak{p},0}$ and $\psi(L(\mathfrak{P},x)) = \big(\mathfrak{P}^{-1}\overline{\mathfrak{P}}\,,\,1\big) \in J \times \mathfrak{E}(L)$, which does not depend on L or x. Given an element $\big(\mathfrak{A}\,,\,1\big) \in J \times \mathfrak{E}(L)$ with $\mathfrak{A}$ coprime to $2\,\mathfrak{d}(L)$ we can induce the action of $\big(\mathfrak{A}\,,\,1\big)$ on $\big(J \times \mathfrak{E}(L)\big)\,/\,H$ by (multiple) neighbour steps on representatives of special genera in gen(L). The group $\big(J \times \mathfrak{E}(L)\big)\,/\,H$ is generated by representatives of $C/C_0 \simeq J/J_0$ and of $\mathfrak{E}(L)/R$. Every class in C/C_0 contains infinitely many primes so that it is represented by a $\mathfrak{P} \nmid 2\,\mathfrak{d}(L)$ and we can change between special genera belonging to the different relative discriminants in J/J_0 by single neighbour steps at split primes.
Let $u_i = \big(u_{i\mathfrak{q}}\,\mathcal{E}(L_\mathfrak{q})\big)_{\mathfrak{q}\in\Omega_r}$ be generators of $\mathfrak{E}(L)/R$. As in the proof of Theorem 2.9 we determine $c_i = r_i^{-1}\overline{r_i} \in E$ such that $c_i u_i = 1 \in \mathfrak{E}(L)$ and r_i is coprime to $2\,\mathfrak{d}(L)$. Then $(\mathcal{O}_E\,, u_i)\,H = (r_i^{-1}\overline{r_i}\mathcal{O}_E\,, 1)\,H$ and the action of u_i on the special genera is done by neighbour steps at $\mathfrak{P}_1, \ldots, \mathfrak{P}_s, \overline{\mathfrak{Q}}_1, \ldots, \overline{\mathfrak{Q}}_t$ if $r_i\mathcal{O}_E = \mathfrak{P}_1 \cdots \mathfrak{P}_s(\mathfrak{Q}_1 \cdots \mathfrak{Q}_t)^{-1}$.

The following corollary of Theorem 2.10 is useful in the special situation where Ω_r contains only one prime $\mathfrak{P} \mid \mathfrak{p}$ which can be used to construct the special genus.

Corollary 2.11 *Let $M \in \mathrm{gen}(L)$ with $[L/M] = a\,\mathcal{O}_E$ for $a \in E$ and $a\overline{a} = 1$. Let Ω_r be a set of ramified primes in $\mathcal{O}_F$, that contains all primes $\mathfrak{q}$ for which $\mathcal{E}_{\mathfrak{q},0}/\mathcal{E}(L_\mathfrak{q})$ is not trivial. Then there is a lattice $L' \in \mathrm{gen}^0(L)$ with ${L'}_\mathfrak{q} = M_\mathfrak{q}$ at all $\mathfrak{q} \in \Omega_F \setminus \Omega_r$.*

Proof: Assume $\alpha_\mathfrak{q} L_\mathfrak{q} = M_\mathfrak{q}$ with $\alpha_\mathfrak{q} \in \mathbf{U}(V_\mathfrak{q}, h)$ for all $\mathfrak{q} \in \Omega_F$. For $\mathfrak{q} \in \Omega_r$ choose $\beta_\mathfrak{q} \in \mathbf{U}(V_\mathfrak{q}, h)$ with $\det(\beta_\mathfrak{q}) = a\det(\alpha_\mathfrak{q})^{-1}$, which is a unit at $\mathfrak{q}$. Define the lattice L' by ${L'}_\mathfrak{q} = M_\mathfrak{q}$ for $\mathfrak{q} \in \Omega_F \setminus \Omega_r$ and ${L'}_\mathfrak{q} = \beta_\mathfrak{q} M_\mathfrak{q}$ for $\mathfrak{q}$ in the finite set Ω_r. ∎

3 Neighbours of one lattice

The number of neighbours of a given lattice is finite. The following Lemma is similar to the quadratic case (but somewhat more complicated, since the 2nd condition on the right hand side can be omitted for the quadratic $\mathfrak{P} \nmid 2$ case).

Lemma 3.1 *Let L be an integral $\mathcal{O}_E$-lattice with $\mathfrak{P} \not| \mathfrak{d}(L)$, $x, y \in L \setminus \mathfrak{P}L$ and $h(x,x), h(y,y) \in \mathfrak{P}\overline{\mathfrak{P}}$. Then*

$$L(x) = L(y) \iff \exists\, \lambda \in \mathcal{O}_E \setminus \mathfrak{P} : \; x - \lambda y \in \mathfrak{P}L \text{ and } h(x,y) \in \mathfrak{P}\overline{\mathfrak{P}}$$

Proof: "$\Rightarrow$": $x \in \mathfrak{P}L(x) = \mathfrak{P}L(y) \implies \exists \lambda_1 \in \mathcal{O}_E,\ z_1 \in \mathfrak{P}L_y : x = \lambda_1 y + z_1$. Similarly $y = \lambda_2 x + z_2$ with $\lambda_2 \in \mathcal{O}_E$ and $z_2 \in \mathfrak{P}L_x$. This implies $(1 - \lambda_1\lambda_2)x = \lambda_1 z_2 + z_1 \in \mathfrak{P}L$. Because $x \notin \mathfrak{P}L$ we have $1 - \lambda_1\lambda_2 \in \mathfrak{P}$ and thus $\lambda_1, \lambda_2 \notin \mathfrak{P}$. $h(x,y) \in \mathfrak{P}\overline{\mathfrak{P}}$ is a consequence of $h(\mathfrak{P}^{-1}x, \mathfrak{P}^{-1}y) \subseteq h(L(x), L(x)) \subseteq \mathcal{O}_E$.
"$\Leftarrow$:" Let $x = \lambda_1 y + z_1$ with $\lambda_1 \in \mathcal{O}_E \setminus \mathfrak{P}$, $z_1 \in \mathfrak{P}L$ and let $h(x,y) \in \mathfrak{P}\overline{\mathfrak{P}}$. Because λ_1 is a unit at $\mathfrak{P}$, we can also find $\lambda_2 \in \mathcal{O}_E \setminus \mathfrak{P}$ and $z_2 \in \mathfrak{P}L$ with $y = \lambda_2 x + z_2$. We prove $L(x) \subseteq L(y)$. Let $rx + z \in L(x)$ with $r \in \mathfrak{P}^{-1}$ and $z \in L_x$. Now $rx + z = r\lambda_1 y + rz_1 + z \in L(y) \iff rz_1 + z \in L_y$. We check $h(y, rz_1 + z) \in \mathfrak{P}$:

$$\begin{aligned} h(y, rz_1 + z) + \mathfrak{P} &= h(\lambda_2 x, rz_1 + z) + \mathfrak{P} = h(\lambda_2 x, rz_1) + \mathfrak{P} = \\ &= \lambda_2 \overline{r} h(x, x - \lambda_1 y) + \mathfrak{P} \subseteq \overline{\mathfrak{P}^{-1}}\mathfrak{P}\overline{\mathfrak{P}} = \mathfrak{P} \end{aligned}$$

Exchanging x and y proves the other inclusion. ∎

Given an integral lattice L and a prime $\mathfrak{P} \subset \mathcal{O}_E$ with $\mathfrak{P} \nmid \mathfrak{d}(L)$ we now determine a set $R(L, \mathfrak{P}) \subset L$ of representatives of the classes of admissible vectors with respect to the relation $x \sim x' \iff L(x) = L(x')$. For every neighbour M of L there will be exactly one $x \in R(L, \mathfrak{P})$ with $M = L(x)$. Let R_1 be a set of representatives of the projective $\mathcal{O}_E/\mathfrak{P}$-space $\mathbf{P}(L/\mathfrak{P}L)$ and for $x \in L \setminus \mathfrak{P}L$ let $[x]$ denote the class of x in $\mathbf{P}(L/\mathfrak{P}L)$.

Let $\mathfrak{p} \subset \mathcal{O}_F$ be the prime beneath $\mathfrak{P}$ and let $\pi \in \mathfrak{P} \setminus \mathfrak{P}\overline{\mathfrak{P}}$. We have to distinguish three cases depending on the decomposition behaviour of $\mathfrak{p}$ in $\mathcal{O}_E$.
1. $\mathfrak{p}$ split (i.e. $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}\overline{\mathfrak{P}}$ with $\mathfrak{P} \neq \overline{\mathfrak{P}}$): Every projective class $[x]$ with $x \in L \setminus \mathfrak{P}L$ contains an admissible vector and all admissible vectors in $[x]$ lead to the same neighbour. To prove this, we see that with π defined as above we have $\overline{\pi}x \in [x]$, and $\overline{\pi}x$ is admissible because $h(\overline{\pi}x, \overline{\pi}x) = \pi\overline{\pi}h(x,x) \in \mathfrak{P}\overline{\mathfrak{P}}$. On the other hand let $y_1, y_2 \in [x]$ and $h(y_1, y_1),\ h(y_2, y_2) \in \mathfrak{P}\overline{\mathfrak{P}}$. Then there are $\lambda \in \mathcal{O}_E \setminus \mathfrak{P}$, $p \in \mathfrak{P}$ and $z \in L$ such that $y_1 = \lambda y_2 + pz$. Now

$$\begin{aligned} h(y_1, y_1) = \lambda\overline{\lambda}h(y_2, y_2) + \lambda\overline{p}h(y_2, z) + \overline{\lambda}ph(z, y_2) + p\overline{p}h(z,z) \ \in \mathfrak{P}\overline{\mathfrak{P}} \\ \Rightarrow \overline{\lambda}ph(z, y_2) \in \overline{\mathfrak{P}} \ \Rightarrow\ ph(z, y_2) \in \overline{\mathfrak{P}} \cap \mathfrak{P} \end{aligned}$$

Then $h(y_1, y_2) + \mathfrak{P}\overline{\mathfrak{P}} = h(\lambda y_2 + pz, y_2) + \mathfrak{P}\overline{\mathfrak{P}} = ph(z, y_2) + \mathfrak{P}\overline{\mathfrak{P}} = \mathfrak{P}\overline{\mathfrak{P}}$ and that implies $L(y_1) = L(y_2)$. Thus, in the split case the set $R(L, \mathfrak{P}) := \{\overline{\pi}x \mid x \in R_1\}$ is a set of representatives of the admissible vectors.

Now assume $\mathfrak{P} = \overline{\mathfrak{P}}$. A necessary condition for a projective class $[x]$ to contain an admissible vector is $h(x,x) \in \mathfrak{P}$. Let $R_2 = \{x \in R_1 \mid h(x,x) \in \mathfrak{P}\}$. Thus, given a $\mathfrak{P}$ neighbour M of L we can find an $x \in R_2$ and an admissible $y \in [x]$ such that $M = L(y)$. Because for any $\lambda \in \mathcal{O}_E \setminus \mathfrak{P}$ the vector λy is also admissible and $L(y) = L(\lambda y)$, we can find an admissible $y' = x + z$ with $z \in \mathfrak{P}L$ and $M = L(y) = L(y')$. Choose a vector $y_x \in L \setminus \mathfrak{P}L$ such that $h(y_x, x) \notin \mathfrak{P}$. Such y_x exists, because $x \in L \setminus \mathfrak{P}L$ and $\mathfrak{P} \not| \mathfrak{d}(L)$. Consider the $\mathcal{O}_E/\mathfrak{P}$-linear mapping $\phi_x : \mathfrak{P}L/\mathfrak{P}^2L \to \mathfrak{P}/\mathfrak{P}^2 : y + \mathfrak{P}^2L \mapsto h(y,x) + \mathfrak{P}^2$. Because πy_x is not in the kernel of ϕ_x, the vector $z \in \mathfrak{P}L$ can be written in the form $z = \lambda\pi y_x + u$ with $\lambda \in \mathcal{O}_E$, $u \in \mathfrak{P}L$ and $h(u,x) \in \mathfrak{P}^2$. Let $y'' = y' - u = x + \lambda\pi y_x$. We have

$$\begin{aligned} h(y'', y'') + \mathfrak{P}^2 &= h(y', y') - h(u, y') - \overline{h(u, y')} + h(u,u) + \mathfrak{P}^2 = \\ &= -h(u,x) - h(u,z) - \overline{h(u,x)} - \overline{h(u,z)} + \mathfrak{P}^2 = \mathfrak{P}^2. \end{aligned}$$

Thus y'' is admissible and $L(y'') = L(y')$ since $h(y'', y') = h(y', y') - h(u, y') \in \mathfrak{P}^2$. Further, if $y_1 = x + \lambda_1 \pi y_x$ and $y_2 = x + \lambda_2 \pi y_x$ are admissible then

$$\begin{aligned} & L(y_1) = L(y_2) \iff h(y_1, y_2) \in \mathfrak{P}^2 \\ \iff \quad & h(y_1 - y_2, y_2) = \pi(\lambda_1 - \lambda_2) h(y_x, x + \pi\lambda_2 y_x) \in \mathfrak{P}^2 \iff \lambda_1 - \lambda_2 \in \mathfrak{P} \end{aligned}$$

We have proved the

Lemma 3.2 *Let $\mathfrak{P} \subset \mathcal{O}_E$ be a prime with $\mathfrak{P} = \overline{\mathfrak{P}}$. Let L be an $\mathcal{O}_E$-lattice in V with $\mathfrak{P} \not| \mathfrak{d}(L)$ and let $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$. Let R_2 be a set of representatives x of those projective classes $[x]$ in $\mathbf{P}(L/\mathfrak{P}L)$ with $h(x,x) \in \mathfrak{P}$. For every $x \in R_2$ choose a vector $y_x \in L$ with $h(y_x, x) \notin \mathfrak{P}$. Let R_E be a set of representatives of $\mathcal{O}_E/\mathfrak{P}$. Then for every $\mathfrak{P}$-neighbour M of L there is an $x \in R_2$ and $\lambda \in R_E$ uniquely determined such that $x + \pi\lambda y_x$ is admissible and $M = L(x + \pi\lambda y_x)$.*

Let us specialize further:
2. $\mathfrak{p}$ **ramified** (i.e. $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}^2$): With the notation from above all the vectors $x + \pi\lambda y_x$ with $x \in R_2$ and $\lambda \in R_E$ are admissible, because $h(x + \pi\lambda y_x, x + \pi\lambda y_x) \in \mathfrak{P} \cap \mathcal{O}_F \subset \mathfrak{P}^2$. In the ramified case the set $R(L, \mathfrak{P}) := \{x + \pi\lambda y_x \mid x \in R_2,\ \lambda \in R_E\}$ is a set of representatives of the admissible vectors leading to different neighbours. Note: Since $\mathcal{O}_E/\mathfrak{P} \simeq \mathcal{O}_F/\mathfrak{p}$ we can choose $R_E \subset \mathcal{O}_F$.
3. $\mathfrak{p}$ **inert** (i.e. $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}$): The trace form $E \times E \to F : (x, y) \mapsto \mathrm{tr}(xy)$ is non-degenerate and $\mathfrak{p}$ is unramified, i.e. $\mathfrak{p}$ does not divide the discriminant of the $\mathcal{O}_F$-lattice $\mathcal{O}_E$ with respect to this form. Thus the $\mathcal{O}_F/\mathfrak{p}$-bilinear form

$$\mathcal{O}_E/\mathfrak{P} \times \mathcal{O}_E/\mathfrak{P} \to \mathcal{O}_F/\mathfrak{p} : \ (\lambda, \mu) \mapsto \mathrm{tr}(\lambda\mu)$$

is non-degenerate. Let $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. For all $x \in R_2$ let $a_x \in \mathcal{O}_F$ with $h(x,x) \in \pi a_x + \mathfrak{p}^2$. Then

$$\begin{aligned} & x + \pi\lambda y_x \text{ is admissible} \\ \iff \quad & h(x + \pi\lambda y_x, x + \pi\lambda y_x) + \mathfrak{p}^2 = \pi a_x + \pi\,\mathrm{tr}(\lambda h(y_x, x)) + \mathfrak{p}^2 = \mathfrak{p}^2 \\ \iff \quad & \mathrm{tr}(\lambda h(y_x, x)) + a_x \in \mathfrak{p} \end{aligned}$$

We see that for all $x \in R_2$ the set of $\lambda + \mathfrak{P} \in \mathcal{O}_E/\mathfrak{P}$ leading to an admissible vector $x + \pi\lambda y_x$ is a 1-dimensional affine $\mathcal{O}_F/\mathfrak{p}$-space in $\mathcal{O}_E/\mathfrak{P}$. With the notation from above we set $R(L, \mathfrak{P}) := \{x + \pi\lambda y_x \mid x \in R_2,\ \lambda \in R_E \text{ and } \mathrm{tr}\big(\lambda\, h(y_x, x)\big) + a_x \in \mathfrak{p}\}$. Thus, in the inert case as well as in the ramified case there are exactly $\#(\mathcal{O}_F/\mathfrak{p})\,\#R_2$ different $\mathfrak{P}$-neighbours of the lattice L.

We want to determine $\#R_2$ and the number of neighbours of a lattice more explicitly. Let $\mathcal{O}_F/\mathfrak{p} = F_q$ be a field of characteristic p with $q = p^k$ elements. The number of isotropic vectors of a quadratic space $({F_q}^m, B)$ is determined in [Kne87] Sec. 12, depending only on the dimension and whether the space is hyperbolic or not.
First assume that $\mathfrak{p}$ is inert. Then the conjugation induces a non-trivial automorphism of $\mathcal{O}_E/\mathfrak{P}$ over $\mathcal{O}_F/\mathfrak{p}$ and the $2n$-dimensional $\mathcal{O}_F/\mathfrak{p}$-space $L/\mathfrak{P}L$ with the quadratic form $x \mapsto h(x,x) + \mathfrak{p}$ is regular, and is hyperbolic if and only if n is even. Note, that this holds also if $\mathfrak{p} \mid 2$ with the notion of a quadratic space given in [Kne87] Sec. 2. To see that the quadratic space is regular we take a diagonal base (see [Jac62]) of $(V_\mathfrak{p}, h)$ that gives us a diagonal $\mathcal{O}_E/\mathfrak{P}$-base of the hermitian form on $L/\mathfrak{P}L \simeq L_\mathfrak{p}/\mathfrak{P}L_\mathfrak{p}$. Then we use the fact that $\mathcal{O}_E/\mathfrak{P}$ is regular and anisotropic over $\mathcal{O}_F/\mathfrak{p}$ with respect to the norm form.
If $\mathfrak{p}$ is ramified and $\mathfrak{p} \nmid 2$, we consider the n-dimensional quadratic $\mathcal{O}_F/\mathfrak{p}$-space $L/\mathfrak{P}L$ with the form induced by h.

Now assume $\mathfrak{p} \mid 2$. Every element of F_q is a square. If $\mathfrak{p}$ is ramified and if we assume that $L_\mathfrak{p}$ is normal, i.e. $\mathfrak{n}(L_\mathfrak{p})\,(\mathcal{O}_E)_\mathfrak{p} = \{h(x,y) \mid x,y \in L_\mathfrak{p}\}$, then $L_\mathfrak{p}$ has an orthogonal base (see [Jac62]). As above this gives us a diagonal $\mathcal{O}_F/\mathfrak{p}$-base of $L/\mathfrak{P}L$. Using this base it is easy to see that there are exactly $q^{n-1}-1$ isotropic vectors.

Lemma 3.3 *Let L be an $\mathcal{O}_E$-lattice in an n-dimensional hermitian space and let $\mathfrak{P}$ be a prime in $\mathcal{O}_E$ with $\mathfrak{P} \nmid \mathfrak{d}(L)$. Let $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$ and $\mathcal{O}_F/\mathfrak{p} = F_q$. Assume further that $\mathfrak{n}(L)\,\mathcal{O}_E = \{h(x,y) \mid x,y \in L\}$ (this is redundant unless $\mathfrak{p}$ is ramified and $\mathfrak{p} \mid 2$). In the ramified case let d be the determinant of the n-dimensional $\mathcal{O}_F/\mathfrak{p}$-space $L/\mathfrak{P}L$ with the bilinear form $(x,y) \mapsto h(x,y) + \mathfrak{P}$. Then the number D of different $\mathfrak{P}$-neighbours of L is as follows:*

1. *If $\mathfrak{p}$ is split $D = (q^n-1)/(q-1)$.*
2. *If $\mathfrak{p}$ is inert*
   $D = q\left(q^{2n-1} + (-1)^n(q^n - q^{n-1}) - 1\right)/(q^2-1)$.
3. *If $\mathfrak{p}$ is ramified and (n is odd or $\mathfrak{p} \mid 2$)*
   $D = q\left(q^{n-1}-1\right)/(q-1)$.
4. *If $\mathfrak{p}$ is ramified and n is even and $\mathfrak{p} \nmid 2$*
   $D = q\left(q^{n-1} + \chi(q^{n/2} - q^{n/2-1}) - 1\right)/(q-1)$,
   where $\chi = 1$ if $(-1)^{n/2}d$ is a square in $\mathcal{O}_F/\mathfrak{p}$ and otherwise $\chi = -1$.

4 The Algorithm

The main algorithm to generate the neighbour graph $\mathrm{NG}(L,\mathfrak{P})$ is quite simple and we do not discuss the different strategies. Let S_e and S_u denote the sets that contain the already "expanded" and the "unexpanded" vertices of the part of $\mathrm{NG}(L,\mathfrak{P})$ that is already constructed. Then the algorithm can be outlined as follows:

START: Set $S_e \leftarrow \emptyset$ and $S_u \leftarrow \{\mathrm{cl}(L)\}$.

LOOP: Choose $M \in \mathrm{cl}(M) \in S_u$. Construct a set $N_1(M,\mathfrak{P})$ of classes, that contains the classes of all $\mathfrak{P}$-neighbours of M, which are finitely many as described in Sec. 3. Then determine $N' := N_1(M,\mathfrak{P}) \setminus (S_e \cup S_u)$ by testing the classes for isometry. Set $S_e \leftarrow S_e \cup \{\mathrm{cl}(M)\}$ and $S_u \leftarrow (S_u \cup N') \setminus \{\mathrm{cl}(M)\}$. Terminate if S_u is empty, otherwise goto LOOP.

It is trivial to verify that S_e contains all of $\mathrm{NG}(L,\mathfrak{P})$ when the algorithm terminates. If we can control whether $S_u \cup S_e = \mathrm{NG}(L,\mathfrak{P})$ (e.g. if we know the mass of $\mathrm{NG}(L,\mathfrak{P})$), then we might replace the condition "*if S_u is empty*" to terminate the algorithm by "*if* $\mathrm{NG}(L,\mathfrak{P}) = S_u \cup S_e$". Concerning the construction of $N_1(M,\mathfrak{P})$ we can take advantage of the action of the automorphisms $\mathbf{U}(M,h)$ on the set of neighbours of M. Let $\phi \in \mathbf{U}(M,h)$ and x be an admissible vector for M at $\mathfrak{P}$. Then $\phi(x)$ is also admissible and $M(\phi(x)) = \phi(M(x))$. It is not difficult (though rather technical in the case $\mathfrak{P} = \overline{\mathfrak{P}}$) to determine a set $R(M,\mathfrak{P})$ of representatives of neighbour vectors of M as described in Sec. 3. We can define this set in a way that makes it easy to find $y \in R(M,\mathfrak{P})$ with $M(y) = M(\phi(x))$, and thereby define an action of $\mathbf{U}(M,h)$ on $R(M,\mathfrak{P})$. To construct $N_1(M,\mathfrak{P})$ we take one representative x of each orbit of $R(M,\mathfrak{P})$ under the action of $\mathbf{U}(M,h)$ and collect the classes $\mathrm{cl}(M(x))$. To minimize the number of classes from other genera in the case $\mathfrak{P} \mid 2$ we can restrict the construction of the neighbour graph to the set $N\big(L,\mathfrak{P},\mathfrak{n}(L)+\mathrm{tr}_{E/F}(\mathcal{O}_E)\big)$ (as defined in Prop. 2.5) which is connected in $\mathrm{NG}(L,\mathfrak{P})$.

We describe now some important sub-algorithms that are needed to compute the neighbourhood of a lattice. Computations were done only for positive definite hermitian spaces over $E = \mathbb{Q}(\sqrt{D})$ with a squarefree $D < 0$. Then $\mathcal{O}_E = \mathbb{Z} + \omega\mathbb{Z}$, where $\omega = \sqrt{D}$ for $D \equiv 2, 3 \pmod 4$ and $\omega = (1+\sqrt{D})/2$ for $D \equiv 1 \pmod 4$. Calculations with ideals in $\mathcal{O}_E$ are done using Hermite normal forms as described in [Coh95] Chap. 5.2 to find $\mathbb{Z}$-bases of the sum and product of two ideals. A lattice $L = \sum_{i=1}^n \mathfrak{a}_i x_i$ is given by the ideals $\mathfrak{a}_i$ and the Gram matrix of h with respect to the pseudo-base $(x_1, \ldots, x_n)$. Shifting scalar factors between $\mathfrak{a}_i$ and x_i we obtain ideals that are reduced representatives in their ideal class (reduced as binary forms, see [Coh95] Chap. 5).

Neighbour step:

Lemma 81:2 of [O'M63] is crucial for computations with pseudo-bases of abstract lattices:

Lemma 4.1 *Let E be a number field, V an n-dimensional vector space over E and U an $n-1$-dimensional subspace. Let L be an $\mathcal{O}_E$-lattice in V. Then there is a vector $x \in V \setminus U$ with coefficient ideal $\mathfrak{a}$ such that $L = \mathfrak{a}x + (L \cap U)$.*

The proof of this lemma is constructive. A special case was realized as a subroutine that takes a lattice (i.e. ideals and a Gram matrix with respect to $x_1, \ldots, x_n$) and the coefficients of a vector $y = \lambda x_r + \mu x_s$ and gives the ideals and Gram matrix with respect to a base $\left(x'_i\right)_i$ with $x'_r = y$, $x'_s \in Ex_r + Ex_s$ and $x'_i = x_i$ for $i \neq r, s$. Iterating this, we can find a pseudo-base containing any given non-zero vector.

Now let $L = \sum_{i=1}^n \mathfrak{a}_i x_i$ be integral and assume that $\mathfrak{P}$ is a prime in $\mathcal{O}_E$ with $\mathfrak{P} \nmid \mathfrak{d}(L)$ and let $y \in L \setminus \mathfrak{P}L$ with $h(y,y) \in \mathfrak{P}\overline{\mathfrak{P}}$. As described above we find a pseudo-base $L = \mathfrak{b}y + \sum_{i=2}^n \mathfrak{b}_i y_i$ with fractional ideals $\mathfrak{b}, \mathfrak{b}_2, \ldots, \mathfrak{b}_n$. Differently from the case of lattices in quadratic spaces it is possible that $\mathfrak{b}y \not\subset L_y$ (with L_y as defined in 2.1) if $\mathfrak{P} \neq \overline{\mathfrak{P}}$. We distinguish two cases:

Assume $\mathfrak{b}y \subset L_y$: Since $L_y \neq L$ there is an index $j \in \{2, \ldots, n\}$ with $\mathfrak{b}_j y_j \not\subset L_y$ and we may assume that $j = n$. Then for $i = 2, \ldots, n-1$ we can find $\lambda_i \in \mathfrak{b}_n \mathfrak{b}_i^{-1}$ such that $\mathfrak{b}_i(y_i + \lambda_i y_n) \subset L_y$ in the following way: Let $r \in \mathfrak{b}_n$ with $h(y, ry_n) \in \mathcal{O}_E \setminus \mathfrak{P}$. Find $s \in \mathcal{O}_E$ with $sh(y, ry_n) - 1 \in \mathfrak{P}$. Then $\lambda_i := -r\overline{s}\overline{h(y, y_i)}$ will do. Clearly, with $\lambda_i \in \mathfrak{b}_n \mathfrak{b}_i^{-1}$ the pairs (y_i, y_n) and $(y'_i := y_i + \lambda_i y_n\ , y_n)$ span the same sublattices and since we know the invariant factors of L_y in L and $L(y)$ by Lemma 2.2 we conclude

$$\begin{aligned}
L &= \mathfrak{b}y + \sum_{i=2}^{n-1} \mathfrak{b}_i y'_i + \mathfrak{b}_n y_n \\
L_y &= \mathfrak{b}y + \sum_{i=2}^{n-1} \mathfrak{b}_i y'_i + \overline{\mathfrak{P}}\mathfrak{b}_n y_n \\
L(y) &= \mathfrak{P}^{-1}\mathfrak{b}y + \sum_{i=2}^{n-1} \mathfrak{b}_i y'_i + \overline{\mathfrak{P}}\mathfrak{b}_n y_n
\end{aligned}$$

Assume $\mathfrak{b}y \not\subset L_y$: Similar to the case above we find $\lambda_i \in \mathfrak{b}\mathfrak{b}_i^{-1}$ such that $\mathfrak{b}_i(y_i + \lambda_i y) \subset L_y$ for $i = 2, \ldots, n$. With $y'_i := y_i + \lambda_i y$ we have

$$\begin{aligned}
L &= \mathfrak{b}y + \sum_{i=2}^{n} \mathfrak{b}_i y'_i \\
L_y &= \overline{\mathfrak{P}}\mathfrak{b}y + \sum_{i=2}^{n} \mathfrak{b}_i y'_i \\
L(y) &= \overline{\mathfrak{P}}\mathfrak{P}^{-1}\mathfrak{b}y + \sum_{i=2}^{n} \mathfrak{b}_i y'_i
\end{aligned}$$

We now specialize to the situation where $\mathrm{tr}_{E/\mathbb{Q}} \circ h$ is a positive definite $\mathbb{Q}$-bilinear form. Equivalently, we may assume that F is totally real, E is totally complex, and for all embeddings $\sigma_i : F \to \mathbb{R}$ the form $x \mapsto \sigma_i(h(x,x))$ is positive definite. We say that h is "totally positive" in that case.

Test for isometry:
Here we assume $F = \mathbb{Q}$. Let $E = \mathbb{Q}(\sqrt{D})$ with a squarefree $D < 0$ and let h be a positive definite hermitian form on $V = E^n$. We use $\mathbb{Z}$-bases to perform tests for isometry: Given lattices L and M let $(z_i)_{1 \le i \le 2n}$ and $(z'_i)_{1 \le i \le 2n}$ be $\mathbb{Z}$-bases of L and M, respectively. Let $\nu \in E^*$ with $\overline{\nu} = -\nu$. Define two matrices $S_L, A_L \in \mathbf{M}(2n, \mathbb{Q})$ by $(S_L)_{ij} + \nu\,(A_L)_{ij} = h(z_i, z_j)$. Clearly S_L is symmetric and positive definite and A_L is antisymmetric. Let S_M, A_M be such matrices for the lattice M with respect to (z'_i).

Lemma 4.2 *The lattices L and M are in the same class if and only if*

$$\exists U \in \mathbf{GL}(2n, Z) : U^t S_M U = S_L \quad \text{and} \quad U^t A_M U = A_L \ . \tag{6}$$

Proof: Let $\psi_L, \psi_M : V \to \mathbb{Q}^{2n}$ denote the $\mathbb{Q}$-linear bijections given by the bases (z_i) and (z'_i). Clearly, the mapping $\phi_U : x \mapsto \psi_M^{-1}\big(U\psi_L(x)\big)$ is an isometry of L and M if and only if (6) holds and ϕ_U is E-linear. But the E-linearity of ϕ_U is a consequence of (6) together with the E-linearity (in the first variable) and regularity of the hermitian form. ∎

The search for a matrix U as above is done by choosing the column vectors u_i of U from the finite sets $D_i := \{x \in \mathbb{Z}^{2n} \mid x^t S_M x = (S_L)_{ii}\}$ and testing the other conditions as they arise in the usual backtrack search. This search is improved using the "fingerprint" and other invariants to detect wrong choices of u_i as described by Plesken and Souvignier in [PS97]. Generators of the automorphism group of a lattice are computed in a similar way. We use slightly modified versions of the sophisticated implementations [Sou95] as subroutines.

We can take advantage of the E-linearity of ϕ_U, if the matrices S_L, A_L are with respect to a base of L that reflects the structure of L as an $\mathcal{O}_E$-lattice: Assume $L = \sum_{i=1}^n \mathfrak{a}_i x_i$ and $\mathfrak{a}_i = \langle a_{i,1}, a_{i,2}\rangle_{\mathbb{Z}}$ $(i = 1, \ldots, n)$. Define the $\mathbb{Z}$-base of L by $z_i := a_{i,1}x_i$ and $z_{i+n} := a_{i,2}x_i$ for $1 \le i \le n$. Then (6) implies

$$u_{i+n} = \psi_M \phi_U(z_{i+n}) = \psi_M\big(a_{i,2}\, a_{i,1}^{-1}\, \psi_M^{-1}(u_i)\big) \ . \tag{7}$$

So we need not involve the sets $D_{n+1}, \ldots, D_{2n}$ in the search, but can just try the vectors determined by (7). Since $\mathfrak{a}_i$ is reduced, we have $\mathrm{N}(a_{i,2}\, a_{i,1}^{-1}) \ge 1$ and for free lattices this norm is about $1/4$ of the discriminant of E over $\mathbb{Q}$. Excluding the sets D_{n+i} from the search will therefore reduce the maximal norm of the vectors involved in the search. On the other hand, we can apply reduction of $\mathbb{Z}$-lattices and replace $(S_L\, ,\, A_L)$ by $(V^t S_L V\, ,\, V^t A_L V)$, where $V \in \mathbf{GL}(2n, Z)$ is a transformation to a reduced (e.g. LLL-reduced) $\mathbb{Z}$-base for S_L. Which of the strategies is better depends on the lattices L and M and, of course, on the pseudo-base $x_1, \ldots, x_n$ of L. To find pseudo-bases consisting of short vectors, we implemented an analogy of LLL-reduction described below, which is far better than the other methods we tried.

Reduction:
We use a notion of a reduced pseudo-base that arises as an analogy to the LLL-algorithm [LLL82], especially the block-LLL from [Sch94]. In [Fie97] Fieker suggests some LLL-versions for totally positive quadratic forms in the sense of [Hum40]. We are in a more special context, but our version gives an estimate for the quality of the result as formulated in Lemma 4.4.

For any discrete set $X \subset V$ in a rational or real quadratic space (V, B) with a positive definite bilinear form B let $\mu_B(X) := \min\big\{B(x,x) \mid x \in X \setminus \{0\}\big\}$.

Definition 4.3 *Let F be a totally real field and E an extension of degree 2. Let V be an n-dimensional E-space with a totally positive hermitian form h, i.e. the $\mathbb{Q}$-bilinear form $T := \mathrm{tr}_{E/\mathbb{Q}} \circ h$ is positive definite. Let $q, q_1 \in \mathbb{R}$ with $0 < q < 1$ and $0 < q_1 \leq 1$. Let $k \in \{2, \ldots, n\}$ and $C \in \mathbb{R}^+$. For a pseudo-base $(x_1, \ldots, x_n)$ of an $\mathcal{O}_E$-lattice $L = \sum_{i=1}^n \mathfrak{a}_i x_i$ let $p_i : V \to \left(\sum_{j=1}^{i-1} x_j E\right)^-$ denote the orthogonal projection and $x_i^* := p_i(x_i)$. For $1 \leq i \leq j \leq n$ let $L_{i,j} := \sum_{r=i}^{j} \mathfrak{a}_r x_r$. Then the pseudo-base is called (k, q, q_1)-reduced, iff*

1. $\forall\, i = 1, \ldots, n : \quad \mathfrak{a}_i \supseteq \mathcal{O}_E$
2. $\forall\, i = 1, \ldots, n$ *with* $b(i) := \min(n, i + k - 1)$ *:*
$$q\, T(x_i^*, x_i^*) \leq \mu_T\big(p_i(L_{i,b(i)})\big)$$
3. $\forall\, 1 \leq j < i \leq n$:
$$q_1\, T\big(p_j(x_i)\,,\, p_j(x_i)\big) \leq \mu_T\Big(p_j\big(\{x_i + \alpha x_j \mid \alpha \in \mathfrak{a}_i^{-1}\mathfrak{a}_j\}\big)\Big).$$

If in addition we have $\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{a}_i^{-1}) \leq C$ for $i = 1, \ldots, n$, we will use the term (k, q, q_1, C)-reduced pseudo-base.

Remarks: This can be achieved by reduction of the trace form restricting to E-linear transformations. The parameter q plays the role of the LLL-constant. We can achieve Condition 3 without affecting Condition 2 and 1. We will show that (k, q, q_1)-reduction implies (k, q, q_1, C)-reduction for suitable C.

An algorithm to find a (k, q, q_1)-reduced pseudo-base starting with $L = \sum_{i=1}^n \mathfrak{a}_i x_i$ may be formulated as follows:

START: For $j = 1, \ldots, n$ execute I-RED(j) below. Set $i \leftarrow 1$.

LOOP: If $i = n$ execute sub-algorithm S-RED(n) below and terminate.
Otherwise, let $g := \min(n, i + k - 1)$ and $L_{i,g} := \sum_{r=i}^{g} \mathfrak{a}_r x_r$. Decide, whether Condition 2 in Definition 4.3 above comes true (using LLL on $p_i(L_{i,g})$ as a $\mathbb{Z}$-lattice with appropriate constant or enumerating all vectors in $p_i(L_{i,g})$ with length $< q\, T(x_i^*, x_i^*)$).
If Condition 2 comes true, execute S-RED(i), set $i \leftarrow i + 1$ and goto LOOP.
If not, let $y \in L_{i,g}$ with
$$q\, T\big(p_i(y)\,,\, p_i(y)\big) \leq \mu_T\big(p_i(L_{i,g})\big) \quad \text{and} \quad T\big(p_i(y)\,,\, p_i(y)\big) \leq \sqrt{q}\, T(x_i^*, x_i^*)$$
Then (using Lemma 4.1) find another pseudo-base $L_{i,g} = \mathfrak{b} y + \sum_{r=i+1}^{g} \mathfrak{b}_r y_r$ and set $\mathfrak{a}_i \leftarrow \mathfrak{b}$, $x_i \leftarrow y$ and for $i + 1 \leq r \leq g$ set $\mathfrak{a}_r \leftarrow \mathfrak{b}_r$, $x_r \leftarrow y_r$. For $j = i + 1, \ldots, g$ execute I-RED(j) and for $j = i, \ldots, g$ execute S-RED(j). If $i = 1$, set $i \leftarrow 2$, else set $i \leftarrow \max(i - k + 1, 1)$. Goto LOOP.

Sub-algorithm S-RED(i): ("size reduction") For $j = i - 1, \ldots, 1$ do: If
$$q_1\, T(p_j(x_i + a x_j), p_j(x_i + a x_j)) \leq \mu_T\Big(p_j\big(\{x_i + \alpha x_j \mid \alpha \in \mathfrak{a}_i^{-1}\mathfrak{a}_j\}\big)\Big) \tag{8}$$
does not hold with $a = 0$, then determine $a \in \mathfrak{a}_i^{-1}\mathfrak{a}_j$ for which (8) holds and set $x_i \leftarrow x_i + a x_j$.

Sub-algorithm I-RED(i): (reduce ideal) If $\mathfrak{a}_i \not\supseteq \mathcal{O}_E$ or if
$$q\, T(a\, x_i^*\,,\, a\, x_i^*) \leq \mu_T(\mathfrak{a}_i x_i^*) \tag{9}$$
does not hold with $a = 1$, then determine $a \in \mathfrak{a}_i$ for which (9) holds. Set $x_i \leftarrow a x_i$ and $\mathfrak{a}_i \leftarrow a^{-1}\mathfrak{a}_i$.

Note that at the beginning of LOOP conditions 2 and 3 are valid up to $i-1$ and Condition 1 for all indices. Because condition 2 for $i=n$ is just the ideal-reduction and holds true after START and LOOP, the pseudo-base x_i will be (k,q,q_1)-reduced, if the algorithm ends.

The analysis of this algorithm is similar to the usual LLL. Let γ_r denote Hermite's constant, i.e. for any $\mathbb{Z}$-lattice M in an r-dimensional positive definite real or rational quadratic space (V,B) we have $\mu_B(M) \le \gamma_r\bigl(\det_B(M)\bigr)^{1/r}$. Assume that $L=\sum_{j=1}^n \mathfrak{a}_j x_j$ is (k,q,q_1)-reduced. We will first show that there is a constant C_1 such that

$$\forall\, i=1,\dots,n-1 \qquad T(x_i^*,x_i^*) \le C_1\, T(x_{i+1}^*,x_{i+1}^*) \tag{10}$$

We will use Condition 2 of Definition 4.3 only for $k=2$. With $m:=[E:\mathbb{Q}]$ and $L' := p_i(\mathfrak{a}_i x_i + \mathfrak{a}_{i+1}x_{i+1})$ we have (using (5) of Sec. 1)

$$\bigl(q\,T(x_i^*,x_i^*)\bigr)^2 \;\le\; \bigl(\mu_T(L')\bigr)^2 \;\le\; \gamma_{2m}^2\bigl(\det_T(L')\bigr)^{1/m} \tag{11}$$

$$\begin{aligned}\det_T(L') = |\,\mathrm{d}_E|^2\,\mathrm{N}_{E/\mathbb{Q}}\bigl(\mathfrak{d}(L')\bigr) &= |\,\mathrm{d}_E|^2\,\mathrm{N}_{E/\mathbb{Q}}\bigl(\mathfrak{d}(\mathfrak{a}_i x_i^*)\mathfrak{d}(\mathfrak{a}_{i+1}x_{i+1}^*)\bigr) \le \\ &\le |\,\mathrm{d}_E|^2\,\mathrm{N}_{E/\mathbb{Q}}\bigl(h(x_i^*,x_i^*)\bigr)\mathrm{N}_{E/\mathbb{Q}}\bigl(h(x_{i+1}^*,x_{i+1}^*)\bigr)\,,\end{aligned} \tag{12}$$

where we used $\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{a}_j) \le 1$ since $\mathfrak{a}_j \supseteq \mathcal{O}_E$. Since $h(x,x) \in F$ takes only totally positive values, the inequality of the arithmetic and geometric mean yields $\Bigl(\mathrm{N}_{E/\mathbb{Q}}\bigl(h(x,x)\bigr)\Bigr)^{1/m} \le m^{-1}\,T(x,x)$. Putting together these inequalities we get

$$\begin{aligned} & q^2\,T(x_i^*,x_i^*)^2 \le \gamma_{2m}^2|\,\mathrm{d}_E|^{2/m}m^{-2}\,T(x_i^*,x_i^*)\,T(x_{i+1}^*,x_{i+1}^*) \\ \Longrightarrow \qquad & T(x_i^*,x_i^*) \le \bigl(\tfrac{\gamma_{2m}}{qm}\bigr)^2\,|\,\mathrm{d}_E|^{2/m}\,T(x_{i+1}^*,x_{i+1}^*)\,. \end{aligned} \tag{13}$$

Thus (10) holds with $C_1 = \bigl(\frac{\gamma_{2m}}{qm}\bigr)^2\,|\,\mathrm{d}_E|^{2/m}$ which does not depend on L or T. Clearly (10) implies $C_1 \ge 1$ and

$$\mu_T(L) \le T(x_1,x_1) \le {C_1}^{i-1}\,T(x_i^*,x_i^*)\,,$$

i.e. $T(x_i^*,x_i^*)$ is bounded from below. Since in LOOP a possible base change fixes $T(x_j^*,x_j^*)$ for $j<i$ and multiplies $T(x_i^*,x_i^*)$ by a factor $\le \sqrt{q} < 1$, induction on i_0 shows that LOOP will be executed only finitely many times with $i \le i_0$. Now, let $0 \ne x = \sum_{r=1}^g a_r x_r \in L$ and we may assume $0 \ne a_g$. Then $x = \sum_{r=1}^{g-1} b_r x_r^* + a_g x_g^*$ with $b_r \in E$ and $a_g \in \mathfrak{a}_g$.

$$\begin{aligned} T(x,x) = \textstyle\sum_{i=1}^{g-1} T(b_i x_i^*, b_i x_i^*) + T(a_g x_g^*, a_g x_g^*) &\ge T(a_g x_g^*, a_g x_g^*) \ge \\ \ge q\,T(x_g^*,x_g^*) \ge q\,C_1^{1-g}T(x_1,x_1) &\ge q\,C_1^{1-n}T(x_1,x_1)\,. \end{aligned}$$

A similar argument as in (11) and (12) on the m-dimensional $\mathbb{Q}$-space $\bigl(p_i(\mathfrak{a}_i x_i)\,,\,T\bigr)$ $(i=1,\dots,n)$ yields:

$$\begin{aligned} & q\,T(x_i^*,x_i^*) \le \gamma_m|\,\mathrm{d}_E|^{1/m}m^{-1}\,\bigl(\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{a}_i\overline{\mathfrak{a}}_i)\bigr)^{1/m}\,T(x_i^*,x_i^*) \\ \Longrightarrow \qquad & \mathrm{N}_{E/\mathbb{Q}}(\mathfrak{a}_i^{-1}) \le \bigl(\tfrac{\gamma_m}{qm}\bigr)^{m/2}\,|\,\mathrm{d}_E|^{1/2}\,. \end{aligned} \tag{14}$$

We have proved the following:

Lemma 4.4 *With the notation from Definition 4.3, let $(x_1,\dots,x_n)$ be a (k,q,q_1)-reduced pseudo-base of the $\mathcal{O}_E$-lattice $L=\sum_{i=1}^n \mathfrak{a}_i x_i$.*

1. *Then $(x_1,\dots,x_n)$ is (k,q,q_1,C)-reduced with $C = \left(\dfrac{\gamma_m}{qm}\right)^{m/2}|\,\mathrm{d}_E|^{1/2}$.*

2. For $i = 1, \dots, n-1$ *holds*

$$T(x_i^*, x_i^*) \le C_1\, T(x_{i+1}^*, x_{i+1}^*) \quad \textit{with } C_1 = \left(\frac{\gamma_{2m}}{qm}\right)^2 |\,\mathrm{d}_E|^{2/m}.$$

3. We have $T(x_1, x_1) \le q^{-1}\, C_1^{n-1}\, \mu_T(L)$.

So far we did not make use of Condition 3 of Definition 4.3. This condition together with 1 and 2 of the lemma above implies $T(x_i, x_i) \le C_2(i)\, T(x_i^*, x_i^*)$ with suitable $C_2(i)$ not depending on L or T, but we do not carry this out in detail.

This algorithm was implemented (in a straightforward way without optimization) for lattices over imaginary quadratic fields using rational arithmetic, and it worked fine in these cases. If q is chosen very close to 1 (say $1 > q \ge 0.999$), in rare cases problems arise from a numerical explosion of the coefficients. Of course, a crucial point for the performance is the base change used in LOOP as described in Lemma 4.1. If we are not only interested in the minimal length of a vector of a pseudo-base but also want to reduce the maximal length, "size reduction" becomes important. For fixed i Condition 3 of Definition 4.3 is achieved for decreasing j (without affecting the former steps). The inequality (10) guarantees that this step-by-step strategy leads to a value of $T(x_i, x_i)$ not "to far" from $\mu_T(x_i + \mathfrak{a}_i^{-1} L_{1,i-1})$. We may improve this by "looking ahead" k_1 steps in each step, i.e. replacing Condition 3 in Definition 4.3 by

3'. $\forall\, 1 \le j < i \le n$ *with* $j' := \max(1, j - k_1)$*:*

$$q_1\, \mu_T\Big(p_{j'}\big(x_i + \mathfrak{a}_i^{-1} L_{j',j-1}\big)\Big) \le \mu_T\Big(p_{j'}\big(x_i + \mathfrak{a}_i^{-1} L_{j',j}\big)\Big)$$

and making the appropriate change in sub-algorithm S-RED(i).
We plan to investigate the complexity of the algorithm in a general setting.

5 Results

The neighbour method in the case of positive definite hermitian spaces over imaginary quadratic fields over $\mathbb{Q}$ was realized as a C++ program. We make use of the C-programs [Sou95] as subroutines and of some datatypes from the C++-library [LED]. Nearly all calculations are done with integer arithmetic. Critical subroutines use the dynamic integers from [LED].

Classification:
In its main mode the program generates the neighbour graph of an integral lattice L at a suitable prime $\mathfrak{P} \nmid \mathfrak{d}(L)$. If $\mathfrak{P} \mid 2$ and L is even, it is possible to compute only the connected subgraph of the even lattices, i.e. those lattices M with $2 \mid h(x,x)$ for $x \in M$. We used the program to classify all unimodular genera in hermitian spaces of small dimensions over some imaginary quadratic fields. It is known that for any $n \in \mathbb{N}$ and any imaginary quadratic field E there exist exactly 2^{t-1} isomorphism classes of positive definite hermitian spaces of dimension n over E containing unimodular lattices. Here, t is the number of distinct prime factors in the discriminant of E. These spaces can be characterized by the Steinitz classes of such lattices as described in [Hof91].
Table 1 and 2 show those genera of unimodular lattices over some fields E that were classified by the program. The discriminant and number of ideal classes of E are denoted by Δ and h. A hermitian space H belonging to the standard form is denoted by I and the positive definite space with determinant $d\,\mathrm{N}_{E/\mathbb{Q}}(E^*)$ is denoted by $I - \langle d \rangle$. In each box of the tables the number h_n of classes of unimodular

hermitian forms in an n-dimensional quadratic space H over $\mathbb{Q}(\sqrt{\Delta})$ is given. The number in parenthesis is the largest minimum attained by a class in the genus. If there is an even and an odd genus of unimodular forms in a given space H, the class numbers $h_{n,\mathrm{odd}}$ and $h_{n,\mathrm{even}}$ of even and odd classes are given in the form $h_{n,\mathrm{odd}} + h_{n,\mathrm{even}}$ in one box and the largest minima are printed below.
The complete lists of all classes in those genera with the order of their automorphism groups and some coefficients of their theta series and the masses of the genera can be obtained at `http://www.math.uni-sb.de/~schulzep/`. Most of the masses (including all genera of the standard form) were checked by the mass formula given in [HK89]. The following construction, that was suggested by Schulze-Pillot, gives another check of the results in some cases: Let L be an integral n-dimensional $\mathcal{O}_E$-lattice in a hermitian space (V, h) over an imaginary quadratic extension $E/\mathbb{Q}$ and let $\mathfrak{a} \subseteq \mathcal{O}_E$ be an ideal. By multiplying L with $\mathfrak{a}$ and scaling h by the positive generator of $(\mathfrak{a}\overline{\mathfrak{a}})^{-1}$ we obtain an integral lattice L' in the hermitian space (V, h'). It is immediate that an E-linear endomorphism of V is an h-isometry of L and M if and only if it is an h'-isometry of L' and M'. Furthermore, we have $\mathfrak{d}(L) = \mathfrak{d}(L')$. This construction yields a 1-1 correspondence of the classes in $\mathrm{gen}^0(L)$ and those in $\mathrm{gen}^0(L')$, possibly in a different hermitian space. The Steinitz class of L' is the one of L multiplied by $\mathfrak{a}^n$. If $\mathfrak{a}^n$ is not principal this can be used to construct $\mathrm{gen}^0(L')$ from $\mathrm{gen}^0(L)$. In some large genera this construction was used to reduce the computations, and in the other cases to check the results. Our calculations reproduce older results by Feit [Fei78] (unimodular genera over $\mathbb{Q}(\sqrt{-3})$ up to dimension 12), Iyanaga [Iya69] (odd unimodular genera over $\mathbb{Q}(\sqrt{-1})$ up to dimension 7) and Hoffmann [Hof91] (dimension 2,3 over fields with discriminant $-\Delta \leq 20$).
Limitations of the computation arise from the number of classes in the genus and from the number of $\mathfrak{P}$-neighbours of each lattice. This last number grows rapidly with the dimension and with $\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{P})$. We must expand (compute the neighbours of) all classes in the genus to be sure that all classes have been generated. Even if we have a test for completeness (via the mass formula), it may theoretically be necessary to expand all but the last class. We experimented with different strategies to generate the neighbour graph, but none of them was significantly better than the others. Testing pairs of lattices for isometry takes most of the time of the computations. The time for each test, too, grows rapidly with the dimension of the lattices.

Isometric trace forms:
Let E/F be a quadratic extension as above. Two hermitian spaces (V, h) and (V', h') over E are isometric, if and only if the F-spaces V and V' with the quadratic forms $\mathrm{tr}_{E/F} \circ h$ and $\mathrm{tr}_{E/F} \circ h'$ are isometric ([Sch85] Chap.10 Theo.1.1). The analogous assertion on lattices is wrong. In the cases we examined, non-isometric hermitian lattices frequently are isometric in the quadratic space with the trace-form.

Extremal lattices:
The notion of extremality in this context was introduced by Quebbemann in [Que95], [Que97]. A $\mathbb{Z}$-lattice M in an m-dimensional quadratic space is called l-modular in this context, iff the l-scaled dual of M is isometric to M. This implies, that the discriminant of M is $l^{m/2}$. If l is a prime, the theta series of such lattices are modular forms belonging to the Fricke group. Of special interest are spaces of modular forms where we have a unique "extremal" form $f(z) = 1 + \sum_{m \geq k} e^{2\pi i m z}$ with maximal $k \geq 1$ among such modular forms. This holds for $l{+}1 \mid 24$. Lattices whose theta series are extremal modular forms are called (analytically) extremal. For a survey on extremal lattices see [SSP].
Let $E = \mathbb{Q}(\sqrt{-l})$ for a squarefree $l > 0$. Unimodular hermitian lattices L in a hermitian space (E^n, h) are l-modular lattices in the quadratic space (E^n, t) over $\mathbb{Q}$, where $t := 1/2\, \mathrm{tr}_{E/\mathbb{Q}} \circ h$ if $l \equiv 2, 3 \pmod 4$ or $t := \mathrm{tr}_{E/\mathbb{Q}} \circ h$ if $l \equiv 1 \pmod 4$,

respectively. Integral lattices remain integral via this transfer, and in the first case even lattices remain even, in the second case all unimodular lattices give rise to even $\mathbb{Z}$-lattices. The dual $L_t^\#$ of L with respect to t is $L_t^\# = 1/\sqrt{-l}\,L^\#$ (using (4) in Sec. 1). Thus, for unimodular L we have $L_t^\# = 1/\sqrt{-l}\,L$ and the isometry between this lattice in the l-scaled space and L in (E^n, t) is simply multiplication with $\sqrt{-l}$. The neighbour method can be used to search for lattices with large minima in genera that are too large to be classified completely. To achieve this we construct all neighbours at a small prime $\mathfrak{P}$ of a given lattice and choose the neighbour with the smallest number of vectors with lengths up to a certain bound to restart the process (Instead of the smallest number of "short vectors" we use a somewhat more sophisticated heuristic). In all cases we checked, the lattice with the largest minimum is constructed after very few iterations of this process. Using this heuristic we found some examples of extremal lattices in cases where the existence was not known before, e.g. strongly 6-modular lattices with minimum 8 (6) in dimension 28 (20) and 3-modular lattices with minimum 6 in dimension 30 and 34. The complete classification gives a negative answer to the existence of analytically extremal lattices coming from hermitian lattices over $\mathcal{O}_E$ in some cases, where the existence of an arbitrary extremal lattice is not known. This is the case for 11-modular lattices of rank 14, and for 7-modular lattices of rank 14 and 18.

Table 1:

Δ	h	H	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}
-3	1	I	1	1	1	1	2 *(2)*	2 *(1)*	3 *(2)*	4 *(2)*	6 *(2)*
-4	1	I	1	1	1+1 *(1)(2)*	2 *(1)*	3 *(2)*	4 *(2)*	6+3 *(2)(2)*	12 *(2)*	25 *(2)*
-7	1	I	1	2 *(2)*	3 *(2)*	5 *(2)*	11 *(2)*	26 *(2)*	71 *(2)*	291 *(3)*	2225 *(4)*
-8	1	I	1+1 *(1)(2)*	2 *(1)*	3+2 *(2)(2)*	7 *(2)*	15+5 *(2)(2)*	38 *(2)*	142+26 *(3)(4)*		
-11	1	I	2 *(2)*	2 *(1)*	6 *(2)*	10 *(3)*	39 *(3)*	112 *(3)*	1027 *(4)*		
-15	2	I	2 *(2)*	5 *(2)*	14 *(2)*	48 *(3)*	238 *(3)*	2120 *(4)*			
		$I-\langle 2\rangle$	2 *(2)*	5 *(2)*	14 *(3)*	48 *(3)*	240 *(4)*	2120 *(4)*			
-19	1	I	2 *(2)*	3 *(2)*	12 *(3)*	32 *(3)*	290 *(4)*	5225 *(4)*			
-20	2	I	3 *(2)*	6 *(3)*	18+13 *(3)(4)*	98 *(3)*	879 *(4)*				
		$I-\langle 2\rangle$	1+2 *(1)(2)*	6 *(2)*	21 *(2)*	98 *(3)*	773+158 *(4)(4)*				
-23	3	I	9 *(3)*	30 *(3)*	126 *(4)*	768 *(4)*	8895 *(4)*				
-24	2	I	2+1 *(2)(2)*	7 *(2)*	28+9 *(3)(4)*	167 *(3)*	2196+161 *(4)(4)*				
		$I-\langle 2\rangle$	2+1 *(2)(2)*	7 *(3)*	25+7 *(3)(4)*	167 *(3)*	2231+154 *(4)(4)*				

Table 2:

Δ	h	H	h_2	h_3	h_4	h_5	h_6
-31	3	I	9 *(3)*	39 *(3)*	216 *(4)*	2064 *(4)*	
-35	2	I	4 *(3)*	10 *(3)*	76 *(4)*	677 *(5)*	
		$I-\langle 3\rangle$	4 *(2)*	10 *(3)*	72 *(4)*	677 *(5)*	
-39	4	I	8 *(4)*	42 *(4)*	326 *(4)*	4754 *(5)*	
		$I-\langle 2\rangle$	8 *(3)*	42 *(4)*	304 *(5)*	4754 *(5)*	
-40	2	I	3+1 *(3)(2)*	12 *(3)*	83+19 *(4)(4)*	1346 *(5)*	− +2531 *(6)*
		$I-\langle 2\rangle$	2+1 *(2)(2)*	12 *(3)*	78+14 *(4)(4)*	1346 *(5)*	
-43	1	I	3 *(3)*	8 *(3)*	69 *(5)*	1212 *(5)*	
-56	4	I	8+4 *(3)(4)*	62 *(4)*	718+134 *(6)(6)*		
		$I-\langle 3\rangle$	8+4 *(4)(4)*	62 *(4)*	696+104 *(5)(6)*		
-67	1	I	4 *(4)*	17 *(4)*	301 *(6)*	19580 *(6)*	
-163	1	I	8 *(7)*	111 *(7)*	11438 *(11)*		
-455	20	I	160 *(13)*	12700 *(14)*			
		$I-\langle 2\rangle$	110 *(12)*	12700 *(16)*			
		$I-\langle 3\rangle$	150 *(13)*	12700 *(14)*			
		$I-\langle 5\rangle$	170 *(12)*	12700 *(15)*			

References

[AD66] *Algebraic groups and discontinuous groups*, volume 9 of *Proc. Symposia in Pure Math.* Amer. Math. Soc., 1966.

[Coh95] Henri Cohen. *A course in computational algebraic number theory.* Springer-Verlag Berlin, 1995.

[Fei78] Walter Feit. Some lattices over $\mathbb{Q}(\sqrt{-3})$. *Journal of Algebra*, 52:248–263, 1978.

[Fie97] Claus Fieker. *Über relative Normgleichungen in algebraischen Zahlkörpern.* PhD thesis, Techn. Univ. Berlin, 1997.

[HK89] Ki-ichiro Hashimoto and Harutaka Koseki. Class numbers of definite unimodular hermitian forms over the rings of imaginary quadratic fields. *Tohoku Math. Journal II*, 41:1–30, 1989.

[Hof91] Detlev W. Hoffmann. On positive definite hermitian forms. *manuscripta math.*, 71:399–429, 1991.

[HS94] Boris Hemkemeier and Rudolf Scharlau. Classification of integral lattices with large class number. Universität Bielefeld, SFB 343, Preprint 94-103, 1994.

[Hum40] Pierre Humbert. Théorie de la réduction des formes quadratiques définies positives dans un corps algébrique K fini. *Comment. Math. Helv.*, 12:263–306, 1940.

[Iya69] Kenichi Iyanaga. Class numbers of definite hermitian forms. *J. Math. Soc. Japan*, 21:359–374, 1969.

[Jac62] Ronald Jacobowitz. Hermitian forms over local fields. *Amer. J. Math.*, 84:441–465, 1962.

[Kne57] Martin Kneser. Klassenzahlen definiter quadratischer Formen. *Arch. Math*, 8:241–250, 1957.

[Kne87] Martin Kneser. *Quadratische Formen, Vorlesung 1973/74*. Mathematisches Institut Göttingen, 1987.

[Lan36] W. Landherr. Äquivalenz Hermitscher Formen über einem beliebigen algebraischen Zahlkörper. *Abh. Math. Sem. Univ. Hamburg*, 11:245–248, 1936.

[LED] Kurt Mehlhorn and Stefan Näher and Christian Uhrig. LEDA: Library for efficient data types and algorithms, Version 3.5.1. see `http://www.mpi-sb.mpg.de/LEDA/leda.html`.

[LLL82] A. K. Lenstra, H. W. Lenstra, and L. Lovász. Factoring polynomials with rational coefficients. *Mathematische Annalen*, 261:515–534, 1982.

[O'M63] O. T. O'Meara. *Introduction to quadratic forms*. Springer Verlag, 1963.

[PS97] W. Plesken and B. Souvignier. Computing isometries of lattices. *J. Symbolic Comp.*, 24(3/4):327–334, 1997.

[Que95] H.-G. Quebbemann. Modular lattices in euclidean spaces. *Journal of Number Theory*, 54:190–202, 1995.

[Que97] H.-G. Quebbemann. Atkin-Lehner eigenforms and strongly modular lattices. *L'Enseignement Mathématique*, 43:55–65, 1997.

[Sch85] Winfried Scharlau. *Quadratic and hermitian forms*. Springer Verlag, 1985.

[Sch94] C. P. Schnorr. Block Korkine-Zolotarev bases and successive minima. *Comb. Probab. Comput.*, 3(4):507–522, 1994.

[Shi64] Goro Shimura. Arithmetic of unitary groups. *Annals of Mathematics*, 79:369–409, 1964.

[Sou95] Bernd Souvignier. AUTO / ISOM: C-program to compute the automorphism group of a $\mathbb{Z}$-lattice / isometries of lattices., 1995.

[SSP] Rudolf Scharlau and Rainer Schulze-Pillot. Extremal lattices. to appear.