Reliability of Two-Stage Weighted-$\mathbf{k}$-out-of-n Systems With Components in Common

Yong Chen and Qingyu Yang

***Abstract*—This paper extends the existing one-stage weighted-k-out-of-n model to two-stage weighted-k-out-of-n models with components in common. Algorithms are developed to calculate the system reliability, and generate the minimal cuts & minimal paths of two-stage weighted-k-out-of-n systems. Reliability bounds for systems with s-dependent component failures are investigated based on the generated minimal cuts & minimal paths. Two types of two-stage weighted-k-out-of-n models, the SW-k-out-of-n model, and the PW-k-out-of-n model, can be applied to investigate reliability issues in network applications, such as the project management, and the shortest path problems. Examples are provided to demonstrate the developed models & algorithms.**

***Index Terms*—Minimal cuts, minimal paths, two-stage weighted-k-out-of-n, weighted-k-out-of-n.**

ACRONYMS[1]

PW-$\mathbf{k}$-out-of-n	Parallel-weighted-$\mathbf{k}$-out-of-n
SW-$\mathbf{k}$-out-of-n	series-weighted-$\mathbf{k}$-out-of-n

NOTATION

$MCS(z,s)$	cut polynomial of a one-stage weighted-z-out-of-s:F system, which is a logic function formed by writing terms for each minimal cut, separated by a plus sign (e.g., if a system of 4 components has two minimal cuts, {1, 2} & {2, 3, 4}, the cut polynomial is $\overline{x}_1\overline{x}_2 + \overline{x}_2\overline{x}_3\overline{x}_4$).
$MPS(z,s)$	path polynomial of a one-stage weighted-z-out-of-s:F system, which is a logic function formed by writing terms for each minimal path, separated by a plus sign (e.g., if a system of 4 components has two minimal paths, {1, 2, 3} & {2, 4}, the path polynomial is $x_1x_2x_3 + x_2x_4$).
m	Number of subsystems in a two-stage weighted-$\mathbf{k}$-out-of-n system
n	Number of components in the entire system
p_i	Probability that component i is functioning
$R(\mathbf{z},s)$	Reliability of a two-stage weighted-$\mathbf{z}$-out-of-s:F system; therefore $R(\mathbf{k},n)$ is the reliability of the two-stage weighted-$\mathbf{k}$-out-of-n:F system
$R(k,n)$	Reliability of a (one-stage) weighted-k-out-of-n:F system
w_{ij}	≥ 0 The j^{th} component's integer weight assigned to the i^{th} subsystem failure condition of a two-stage weighted-$\mathbf{k}$-out-of-n model, $i=1,\ldots,m$, $j=1,\ldots,n$
w_i	the i^{th} weight of component i for a one-stage system
$\mathbf{w}_s$	$\equiv [w_{1s}\ \ldots\ w_{ms}]^T$, $s=1,\ldots,n$
W_i	$= \sum_{j=1}^{n} w_{ij}$
$\boldsymbol{W}$	$\equiv [W_1\ \ldots\ W_m]^T$
x_i	a Boolean variable indicating functioning of component i, $i=1,\ldots,n$: $x_i=1$ if component i is functioning; $x_i=0$ if it is failed
$\overline{x}_i$	a Boolean variable indicating failure of component i, $i=1,\ldots,n$: $\overline{x}_i=1$ if component i is failed; $\overline{x}_i=0$ if it is functioning
$\overline{x}_{j_1}\overline{x}_{j_2}\cdots\overline{x}_{j_s}$	a term of a cut polynomial, which represents an s-component minimal cut consisting of components $j_1, j_2, \ldots, j_s$.
$x_{j_1}x_{j_2}\cdots x_{j_s}$	a term of a path polynomial, which represents an s-component minimal path consisting of components $j_1, j_2, \ldots, j_s$.
$\mathbf{x}$	$\equiv [x_1,\ldots,x_n]^T$
$\overline{\mathbf{x}}$	$\equiv [\overline{x}_1\ \ldots\ \overline{x}_n]$
y_i	functioning state variable of subsystem i, $i=1,\ldots,m$: $y_i=1$ if subsystem i is functioning; $y_i=0$ if it is failed
$\mathbf{z}$, $\mathbf{k}$	$m\times 1$ vectors $[z_1,\ldots,z_m]^T$, and $[k_1,\ldots,k_m]^T$, respectively
$\phi_I(\cdot)$	first-level structure function of a two-stage weighted-$\mathbf{k}$-out-of-n model

I. INTRODUCTION

THE weighted-k-out-of-n model was first proposed by [1] as an extension of the widely studied k-out-of-n model [2]–[4]. Similar concepts were also applied in introducing the weighted consecutive-k-out-of-n models [5], [6]. For a weighted-k-out-of-n:F system, an integer weight $w_i > 0$ is assigned to component i, $i=1,\ldots,n$. The system is failed iff the total weight of the failed components is *greater than or equal to* k, a pre-specified threshold value. In other words, the system is *functioning* iff the total weight of the failed components is *less than* k.

Let $R(z,s)$ represent the probability that the total weight of the failed components of a system with s components is *less than* z, i.e., $R(z,s)$ is the functioning probability of a system with s components, and a threshold equal to z. Then the reliability of a weighted-k-out-of-n:F system is $R(k,n)$. Similar to the result provided by [1] for the weighted-k-out-of-n:G system,

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[1]The singular and plural of an acronym are always spelled the same.

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the reliability of the weighted-k-out-of-n:F system when all component failures are mutually s-independent can be obtained based on the following recursive relationship:

$$R(z,s)=\begin{cases}0 & \text{for } z\le 0,\ s\ge 0\\ 1 & \text{for } z>0,\ s=0\\ (1-p_s)\cdot R(z-w_s,s-1) & \\ \quad +p_s\cdot R(z,s-1) & \text{otherwise}\end{cases} \tag{1}$$

where w_s denotes the weight of component s, $s=1,2,\ldots,n$.

Despite research on two-stage & multi-stage k-out-of-n structures in the literature [7]–[9], two-stage weighted-k-out-of-n structures have not been investigated. In this paper, the (one-stage) weighted-k-out-of-n model will be extended to a two-stage weighted-**k**-out-of-n model with components in common. Examples of potential applications of the two-stage weighted-**k**-out-of-n models will be also provided.

II. Two-Stage Weighted-k-Out-of-n Systems with Components in Common

This section will introduce the concepts, motivation applications, and main properties of the two-stage weighted-**k**-out-of-n model.

A. Two-Stage Weighted-k-Out-of-n Model

A two-stage weighted-**k**-out-of-n system consists of a number of subsystems. Let m denote the number of subsystems in a two-stage system. Each subsystem has a (one-stage) weighted-k-out-of-n structure, which is called the second-level structure. The interrelationship between the m subsystems follows a certain coherent structure, which is called the first-level structure, whose structure function is denoted by $\phi_I(\cdot)$. Let x_j denote the functioning state of component j : $x_j=1$ if component j is functioning, $x_j=0$ if it is failed, for $j=1,\ldots,n$. Let $\overline{x}_j$ be the complement of x_j, i.e., $\overline{x}_j=1$ iff component j is failed, $j=1,\ldots,n$. A two-stage weighted-**k**-out-of-n:F system is functioning iff

$$\phi_I(y_1,y_2,\ldots,y_m)=1$$

where y_i indicates the functioning state of subsystem i, i.e.,

$$y_i\equiv\begin{cases}1, & \text{if } \sum_{j=1}^{n} w_{ij}\cdot\overline{x}_j<k_i \quad \text{subsystem } i \text{ functioning}\\ 0 & \text{if } \sum_{j=1}^{n} w_{ij}\cdot\overline{x}_j\ge k_i \quad \text{subsystem } i \text{ failed}\end{cases}$$
$$i=1,\ldots,m. \tag{2}$$

Generally, a two-stage weighted-**k**-out-of-n system can be decomposed into two hierarchical levels: the first (higher) level can be of any coherent structure, whose structure function is $\phi_I(\cdot)$; and the second (lower) level has a weighted-k-out-of-n structure. Special cases of two-stage weighted-**k**-out-of-n systems include SW-**k**-out-of-n systems, whose first level has a series structure, and PW-**k**-out-of-n systems, whose first level has a parallel structure. Therefore, an SW-**k**-out-of-n:F system is failed iff *any* of the inequalities in (3) are satisfied:

$$\begin{aligned}&\text{subsystem } 1: w_{11}\cdot\overline{x}_1+w_{12}\cdot\overline{x}_2+\ldots+w_{1n}\cdot\overline{x}_n\ge k_1\\ &\text{subsystem } 2: w_{21}\cdot\overline{x}_1+w_{22}\cdot\overline{x}_2+\ldots+w_{2n}\cdot\overline{x}_n\ge k_2\\ &\cdots\\ &\text{subsystem } m: w_{m1}\cdot\overline{x}_1+w_{m2}\cdot\overline{x}_2+\ldots+w_{mn}\cdot\overline{x}_n\ge k_m\end{aligned} \tag{3}$$

where $\mathbf{k}\equiv[k_1 \quad \ldots \quad k_m]^T$ is a vector of thresholds for each subsystem. A PW-**k**-out-of-n:F system is failed iff *all* of the inequalities in (3) are satisfied. If the first-level structure of a system is a k-out-of-m:F structure, then the system is failed iff at least k of the m inequalities in (3) are satisfied. The i^{th} inequality in (3) is referred to as the i^{th} subsystem failure condition, $i=1,\ldots,m$. When m is equal to 1, the two-stage weighted-**k**-out-of-n system becomes a (one-stage) weighted-k-out-of-n system.

Remark 1: Although each subsystem failure condition in (2) & (3) is related to the same set of n components, the two-stage weighted-**k**-out-of-n model allows each subsystem to have different subsets of components. A subsystem does not need to contain all n components. If component j is not in subsystem i, for $1\le j\le n$, the weight w_{ij} will be set to zero. For example, if subsystem 1 has only two components, 1 & 2, then w_{11} & w_{12} are nonzero. All the other weights in the failure condition of subsystem 1, i.e., w_{1j}, for $3\le j\le n$, should be equal to zero. Based on this usage of zero weights for nonexistence components in a subsystem, the number of components in each subsystem in the model based on (2) & (3) can be different. The variable n in (2) & (3) can be understood as the *total* number of different components in the *entire* two-stage weighted-**k**-out-of-n system. The actual number of components in a subsystem can be found out by the number of nonzero weights in its failure condition. Including all n components in each subsystem failure condition, as in (2) & (3), can significantly simplify the notations in various results developed for two-stage weighted-**k**-out-of-n systems hereafter.

Remark 2: There is a significant difference between the two-stage model in this paper, and most of the two-stage models in the literature. Most two-stage systems studied in the literature are systems consisting of a number of *separate* identically structured subsystems. The subsystems are connected in a series structure, a parallel structure, or more complicated structures. Usually, it is assumed that there are *no components in common* between any two subsystems. With this assumption, the failures of the subsystems are s-independent if all components have s-independent failures. And the system reliability can be calculated easily based on the reliability of each individual subsystem. However, in the two-stage weighted-**k**-out-of-n model, subsystems typically have *some* (or *all* if all the weights w_{ij} are nonzero) *components in common*. Consequently, the system reliability cannot be evaluated by combining the failure probability of each subsystem in a straightforward way. From the discussion in Remark 1, the components in common between two subsystems can be found out based on the nonzero weights in each subsystem failure condition. For example, in (3), if w_{11} & w_{21} are nonzero, and $w_{i1}=0$ for all $2\le i\le m$, then component 1 is in both subsystems 1 & 2, and not in any of the other

TABLE I
ACTIVITIES OF ASSEMBLY PROJECT

Activity	Immediate Predecessors	Estimated Duration	Possible Delays
A = train workers	None	6 days	1 days
B = purchase raw materials	None	9 days	3 days
C = produce product 1	A, B	8 days	3 days
D = produce product 2	A, B	7 days	2 days
E = test product 2	D	10 days	2 days
F = assemble products 1 and 2	C, E	12 days	4 days

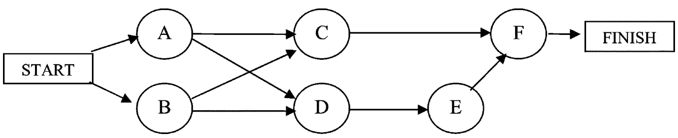

Fig. 1. Project network of the assembly project.

subsystems. So subsystems 1 & 2 share a common component, which is component 1.

B. Applications of Two-Stage Weighted-$\mathbf{k}$-out-of-n Model in Network Problems

Consider a directed network (N, A), where N is the set of nodes, and A is the set of directed arcs. In shortest path problems, the system can be considered as functioning iff there exists *at least one* path from the source node to the destination node with total distance/cost less than a specification (that is, the total distance of the shortest path is within specification). In project management problems, usually the system is considered as functioning iff *all* the paths from the source (start of a project) to the destination (end of a project) of the project network have total time less than a specified deadline. The distance/time of some arcs in A can be delayed (increased) by certain amounts upon corresponding arc failures. It will be illustrated in the following examples that the two-stage weighted-$\mathbf{k}$-out-of-n model can be applied to evaluate system reliability in the problems described above, where the shortest path problem fits in a PW-$\mathbf{k}$-out-of-n model, and the project management problem fits in an SW-$\mathbf{k}$-out-of-n model.

1) Project Management Example: Consider a project management example from [10] about the assembly of two products (products 1 & 2) to make a new product. This project consists of six activities as illustrated in Table I. The project network is illustrated in Fig. 1. There are four paths through this project network:

— 1. $\text{START} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{F} \rightarrow \text{FINISH}$ (total duration without delay $= 26$ days)
— 2. $\text{START} \rightarrow \text{A} \rightarrow \text{D} \rightarrow \text{E} \rightarrow \text{F} \rightarrow \text{FINISH}$ (total duration without delay $= 35$ days)
— 3. $\text{START} \rightarrow \text{B} \rightarrow \text{C} \rightarrow \text{F} \rightarrow \text{FINISH}$ (total duration without delay $= 29$ days)
— 4. $\text{START} \rightarrow \text{B} \rightarrow \text{D} \rightarrow \text{E} \rightarrow \text{F} \rightarrow \text{FINISH}$ (total duration without delay $= 38$ days)

The critical path, which is the longest path through the project network, is the one with total duration of 38 days. Therefore, the earliest finish time of the whole project, if there are no delays, is 38 days. Suppose each activity is subjected to delays with a probability $1-p$. The delay time for each activity is also listed in Table I. Let x_i denote the functioning state (delayed $= 0$; not delayed $= 1$) of activity i, $i = \text{A}, \text{B}, \ldots, \text{F}$. And assume the deadline of the whole project is 40 days. The project will *fail* to be completed within the deadline iff the actual completion time of at least one of the four paths from the START node to the FINISH node in Fig. 1 is greater than 40 days (i.e., at least 41 days). E.g., the total duration without delay for the activities on the second path: $\text{START} \rightarrow \text{A} \rightarrow \text{D} \rightarrow \text{E} \rightarrow \text{F} \rightarrow \text{FINISH}$ is $6+7+10+12 = 35$ days (please refer to Table I for individual activity durations). Therefore, a delay of $41 - 35 = 6$ days or more in completion time of this path will delay the whole project. The total delay due to possible failures of activities A, D, E, and F on this path is

$$\begin{aligned}&1 \cdot \overline{x}_A + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E + 4 \cdot \overline{x}_F \\ &\quad = 1 \cdot \overline{x}_A + 0 \cdot \overline{x}_B + 0 \cdot \overline{x}_C + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E + 4 \cdot \overline{x}_F\end{aligned}$$

where the coefficients/weights of $\overline{x}_A$, $\overline{x}_D$, $\overline{x}_E$, and $\overline{x}_F$ are the delay times of activities A, D, E, and F, respectively, as shown in Table I. Following Remark 1, the weights of $\overline{x}_B$ & $\overline{x}_C$ for the second path are set to zero because they are not in this path. As a result, the completion time of the second path is larger than 40 days iff

$$1 \cdot \overline{x}_A + 0 \cdot \overline{x}_B + 0 \cdot \overline{x}_C + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E + 4 \cdot \overline{x}_F \geq 41 - 35 = 6.$$

Based on similar analysis for all four paths in this project, the project will fail to meet the deadline iff *at least one* of the following failure conditions are satisfied:

$$\begin{aligned}&1 \cdot \overline{x}_A + 0 \cdot \overline{x}_B + 3 \cdot \overline{x}_C + 0 \cdot \overline{x}_D + 0 \cdot \overline{x}_E \\ &\quad + 4 \cdot \overline{x}_F \geq 41 - 26 = 15 \\ &1 \cdot \overline{x}_A + 0 \cdot \overline{x}_B + 0 \cdot \overline{x}_C + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E \\ &\quad + 4 \cdot \overline{x}_F \geq 41 - 35 = 6 \\ &0 \cdot \overline{x}_A + 3 \cdot \overline{x}_B + 3 \cdot \overline{x}_C + 0 \cdot \overline{x}_D + 0 \cdot \overline{x}_E \\ &\quad + 4 \cdot \overline{x}_F \geq 41 - 29 = 12 \\ &0 \cdot \overline{x}_A + 3 \cdot \overline{x}_B + 0 \cdot \overline{x}_C + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E \\ &\quad + 4 \cdot \overline{x}_F \geq 41 - 38 = 3. \end{aligned} \tag{4}$$

It can be seen that the failure conditions of (4) are exactly the failure conditions of an SW-$\mathbf{k}$-out-of-n:F system by considering each activity as a component, and each path as a subsystem. Because the maximal total weights in the left hand side of the first & the third failure conditions in (4) are less than the right hand side, these two failure conditions can be removed to simplify the computation. Consequently, we can focus on the following two failure conditions, where activity C is obviously irrelevant, and does not need to be included:

$$\begin{aligned}&1 \cdot \overline{x}_A + 0 \cdot \overline{x}_B + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E + 4 \cdot \overline{x}_F \geq 6 \\ &0 \cdot \overline{x}_A + 3 \cdot \overline{x}_B + 2 \cdot \overline{x}_D + 2 \cdot \overline{x}_E + 4 \cdot \overline{x}_F \geq 3.\end{aligned} \tag{5}$$

Hence this project management problem is formulated as a SW-weighted-$[6, 3]^T$-out-of-5:F system.

2) Example of a Shortest Path Problem: Consider a shortest path problem of taking a trip from one town (the origin), to another (the destination). The corresponding traffic network is

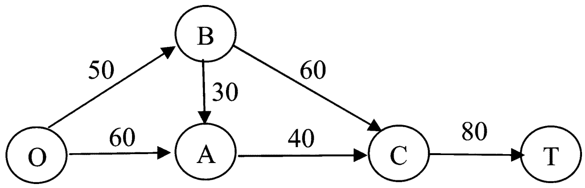

Fig. 2. Network of the shortest path problem.

illustrated in Fig. 2. The nodes denote towns ($\mathrm{O} = \mathrm{origin}$, and $\mathrm{T} = \mathrm{destination}$). The numbers on each arc represent the time (in minutes) to travel from one town to the next. Assume that each arc can allow one-way traffic only. There are 3 paths from the origin to the destination:

1. $\mathrm{OB} \rightarrow \mathrm{BC} \rightarrow \mathrm{CT}$ (total travel time without delay $= 190$ minutes)
2. $\mathrm{OB} \rightarrow \mathrm{BA} \rightarrow \mathrm{AC} \rightarrow \mathrm{CT}$ (total travel time without delay $= 200$ minutes)
3. $\mathrm{OA} \rightarrow \mathrm{AC} \rightarrow \mathrm{CT}$ (total travel time without delay $= 180$ minutes)

The shortest path (the third path above) has a total travel time of 180 minutes. Suppose each arc has a probability of $1 - p$ to be delayed by road constructions. Assume if there is a road construction underway on an arc, the travel time will be increased by 20%. We are interested in the probability that there exists at least one path with total travel time less than 195 minutes.

Each arc in the shortest path problem can be considered as a component. If there is no path with travel time within 195 minutes (considered as system failure), *all* of the following conditions must be satisfied (the second path needs not to be considered because its travel time is greater than 195 minutes no matter whether a delay occurs or not):

$$\begin{aligned} 10 \cdot \overline{x}_{OB} + 0 \cdot \overline{x}_{OA} + 0 \cdot \overline{x}_{BA} + 12 \cdot \overline{x}_{BC} \\ +0 \cdot \overline{x}_{AC} + 16 \cdot \overline{x}_{CT} \geq 195 - 190 = 5 \\ 0 \cdot \overline{x}_{OB} + 12 \cdot \overline{x}_{OA} + 0 \cdot \overline{x}_{BA} + 0 \cdot \overline{x}_{BC} \\ +8 \cdot \overline{x}_{AC} + 16 \cdot \overline{x}_{CT} \geq 195 - 180 = 15. \end{aligned} \quad (6)$$

It can be seen that the failure conditions in (6) are the same as the failure conditions of a PW-$[5, 15]^T$-out-of-6:F system.

C. Main Properties of Two-Stage Weighted-$\mathbf{k}$-out-of-n systems

Similarly to the k-out-of-n, and one-stage weighted-k-out-of-n systems, a one-to-one correspondence between F systems, and G systems exists for the two-stage weighted-$\mathbf{k}$-out-of-n systems. A two-stage weighted-$\mathbf{k}$-out-of-n:G system is functioning iff $\phi_I(\mathbf{y}') = 1$, where

$$\mathbf{y}' \equiv [y'_1 \quad \ldots \quad y'_m]^T \text{ with } y'_i \equiv \begin{cases} 0, & \text{if } \sum_{j=1}^{n} w_{ij} \cdot x_j < k_i \\ 1 & \text{if } \sum_{j=1}^{n} w_{ij} \cdot x_j \geq k_i. \end{cases} \quad (7)$$

That is, y'_i indicates the functioning state of subsystem i, where subsystem i has a weighted-k_i-out-of-n:G (not F) structure. The relationship between the F systems & G systems is presented in the following result.

Result 1: The two-stage weighted-$\mathbf{k}$-out-of-n:F (or G) system is equivalent to the two-stage weighted-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:G (or F) system with the same first level structure function $\phi_I(\cdot)$ where $\mathbf{1} \equiv [1 \quad \ldots \quad 1]^T$.

Please refer to Appendix A.1 for a proof of Result 1. With Result 1, all the algorithms for the F systems can be applied for the G systems correspondingly. Therefore, we will only focus on the F systems hereafter.

The dual structure $\phi^D(\mathbf{x})$ to a given structure $\phi(\mathbf{x})$ is defined in [11] as

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}).$$

By definition, the following result can be seen regarding the dual of a two-stage weighted-$\mathbf{k}$-out-of-n system.

Result 2: The two-stage weighted-$\mathbf{k}$-out-of-n:F system with the first-level structure function $\phi_I(\cdot)$, and the two-stage weighted-$\mathbf{k}$-out-of-n:G system with the first-level structure function $\phi_I^D(\cdot)$, are duals of each other.

Please refer to Appendix A.2 for a proof of Result 2.

The following result can be used to facilitate minimal path generation of a two-stage weighted-$\mathbf{k}$-out-of-n:F system.

Result 3: The *minimal paths* of a two-stage weighted-$\mathbf{k}$-out-of-n:F system with the first level structure function $\phi_I(\cdot)$ can be obtained by generating *minimal cuts* for the two-stage weighted-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:F system with the first level structure function $\phi_I^D(\cdot)$.

For example, the minimal paths of the SW-$\mathbf{k}$-out-of-n:F system, and the PW-$\mathbf{k}$-out-of-n:F system can be obtained by generating minimal cuts for the PW-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:F, and the SW-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:F system, respectively. Result 3 can be seen from the following reasoning. It is known that the minimal paths for a system are minimal cuts for its dual system [11]. Based on Result 2, the dual of a two-stage weighted-$\mathbf{k}$-out-of-n:F system with the first level structure function $\phi_I(\cdot)$ is the two-stage weighted-$\mathbf{k}$-out-of-n:G system with the first level structure function $\phi_I^D(\cdot)$. Therefore, the minimal paths of the two-stage weighted-$\mathbf{k}$-out-of-n:F system with the first level structure function $\phi_I(\cdot)$ are the minimal cuts of the two-stage weighted-$\mathbf{k}$-out-of-n:G system with the first level structure function $\phi_I^D(\cdot)$, which is equivalent to the two-stage weighted-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:F system with the first level structure function $\phi_I^D(\cdot)$, based on Result 1.

III. Algorithms for System Reliability Evaluation, and Generation of Minimal Cuts & Minimal Paths

This section focuses on the development of algorithms to evaluate system reliability, and generate the minimal cuts & minimal paths of two-stage weighted-$\mathbf{k}$-out-of-n systems.

A. Reliability Evaluation for Two-Stage Weighted-$\mathbf{k}$-out-of-n:F Systems

For a two-stage weighted-$\mathbf{k}$-out-of-n:F system with first-level structure function $\phi_I(\cdot)$, assuming s-independent component failures, the system reliability $R(\mathbf{k}, n)$ can be calculated based on the following recursive relationship:

$$R(\mathbf{z}, s) = \begin{cases} 0 & \text{if } \phi_I(\boldsymbol{\beta}_0) = 0 \\ 1 & \text{if } \phi_I(\boldsymbol{\beta}_1) = 1 \\ (1 - p_s) R(\mathbf{z} - \mathbf{w}_s, s - 1) & \\ \quad + p_s R(\mathbf{z}, s - 1) & \text{otherwise} \end{cases} \quad (8)$$

where

$$\boldsymbol{\beta}_0 \equiv [\beta_{01} \quad \ldots \quad \beta_{0m}]^T$$

with

$$\beta_{0i} = \begin{cases} 1, & \text{if } z_i > 0 \\ 0, & \text{if } z_i \leq 0 \end{cases}$$

and

$$\boldsymbol{\beta}_1 \equiv [\beta_{11} \quad \ldots \quad \beta_{1m}]^T$$

with

$$\beta_{1i} = \begin{cases} 1, & \text{if } \sum_{j=1}^{s} w_{ij} < z_i \\ 0, & \text{if } \sum_{j=1}^{s} w_{ij} \geq z_i. \end{cases}$$

Equation (8) is obtained by conditioning on the state of component s. Because the subsystems may have components in common, component s can be shared by more than one subsystems. Therefore, when component s is failed with probability $1 - p_s$, as in the first term of the third equation in (8), the system weight threshold $\mathbf{z}$ is reduced to $\mathbf{z} - \mathbf{w}_s$, i.e., all elements of $\mathbf{z}$ corresponding to all subsystems may be reduced. From Remark 1, if component s is not in a subsystem, the corresponding weight in $\mathbf{w}_s$ will be zero, and the corresponding element in $\mathbf{z}$ will not be affected. Based on the above discussion, conditioning on the state of component s in (8) will affect all subsystems sharing component s, and will not affect any subsystem without component s. It can be seen that the computational complexity of reliability evaluation in terms of the number of $R(\mathbf{z}, s)$, $z_i \in \{0, 1, \ldots, k_i\}$, $i = 1, \ldots, m$, $s \in \{0, 1, \ldots, n\}$, to be evaluated by (8) is $O(n \prod_{i=1}^{m} k_i)$.

*B. Minimal Cuts, and Minimal Paths of Two-Stage Weighted-**k**-out-of-n:F Systems*

Minimal paths & minimal cuts are useful toward the representation of coherent systems. Many classical methods use minimal paths or minimal cuts to efficiently evaluate or approximate system reliability based on methods, such as the inclusion-exclusion method, and the sum-of-disjoint-products method [11]–[13]. The generation of the minimal cuts & paths for two-stage weighted-**k**-out-of-n systems is based on the generation of minimal cuts & paths of one-stage systems, which will be first studied in this section. One important application of minimal paths & minimal cuts is to evaluate bounds of system reliability when component failures are s-dependent. This problem will also be addressed in this section following the procedure of minimal cut & path generation for two-stage systems.

1) Generation of Minimal Cuts & Paths for One-Stage Systems: The reliability evaluation of the one-stage weighted-k-out-of-n system has been studied by [1]. The minimal cut & minimal path generation of one-stage weighted-k-out-of-n systems, however, has not been investigated in the literature. We first assume that component weights have been sorted such that $w_1 \leq w_2 \leq \ldots w_n$. Let $MCS(z, s)$ denote the *cut polynomial*, which was defined in [14], of a one-stage weighted-z-out-of-s:F system. A cut polynomial is a logic function formed by writing terms for each minimal cut, separated by a plus sign (logical OR). E.g., if a system of 4 components has two minimal cuts, {1, 2} & {2, 3, 4}, the cut polynomial of this system is $\overline{x}_1\overline{x}_2 + \overline{x}_2\overline{x}_3\overline{x}_4$. The recursive relationship shown in (9) at the bottom of the page can be used to generate all the minimal cuts. The operations on cut polynomials in (9) are Boolean operations. It is worth to note that only additions & multiplications are used for the Boolean operations in (9). No other Boolean algebra, such as the absorption of terms $(a + aB = a)$, is needed to simplify the cut polynomials. The sorted component weights can guarantee that every term in the resulted cut polynomial represents a *minimal* cut without replications. It can be seen that the computational complexity of the minimal cut generation in terms of the number of $MCS(z, s), z \in \{0, 1, \ldots, k\}, s \in \{0, 1, \ldots, n\}$ to be evaluated is $O(n \cdot k)$.

Let $MPS(z, s)$ denote the *path polynomial* [14] of a one-stage weighted-z-out-of-s:F system. A path polynomial is a logic function formed by writing terms for each minimal path, separated by a plus sign (logical OR). E.g., if a system of 4 components has two minimal paths, {1, 2, 3} & {2, 4}, the path polynomial of this system is $x_1x_2x_3 + x_2x_4$. The recursive relationship shown in (10) at the bottom of the next page can be used to derive the set of minimal paths of a one-stage weighted-k-out-of-n:F system.The operations on path polynomials in (10) are Boolean operations. The relationship in (10) can be verified by (9), and the known result that $MPS = \overline{MCS}$. For example, for the last condition in (10), we have

$$\begin{aligned} MPS(z,s) &= \overline{MCS(z,s)} \\ &= \overline{MCS(z,s-1) + MCS(z-w_s,s-1)\cdot \overline{x}_s} \\ &= \overline{MCS(z,s-1)}\cdot\left(\overline{MCS(z-w_s,s-1)}+x_s\right) \\ &= \overline{MCS(z,s-1)}\cdot\overline{MCS(z-w_s,s-1)} \\ &\quad + \overline{MCS(z,s-1)}\cdot x_s \\ &= MPS(z,s-1)\cdot x_s + MPS(z,s-1) \\ &\quad \cdot MPS(z-w_s,s-1) \\ &= MPS(z,s-1)\cdot x_s + MPS(z-w_s,s-1). \end{aligned}$$

The last equation above is due to the fact that $MPS(z - w_s, s - 1) = 1 \Rightarrow MPS(z, s - 1) = 1$. Same as (9), the algorithm based on (10) has computational complexity of $O(n \cdot k)$.

$$MCS(z,s) = \begin{cases} 0 & \text{if } \sum_{i=1}^{s} w_i < z \left(\text{when } s = 0, \sum_{i=1}^{s} w_i = 0\right) \\ 1 & \text{if } z \leq 0 \\ \mathrm{MCS}(z,s-1) + MCS(z-w_s,s-1)\cdot\overline{x}_s & \text{otherwise} \end{cases} \tag{9}$$

In addition, the recursive relationship can be used to generate minimal cuts & minimal paths without requiring sorting the component weights as shown in (11) and (12) at the bottom of the page. An upper bound z_U, and a lower bound z_L are added to $MCS(z_L, s)$, and $MPS(z_U, s)$, respectively, to guarantee that the obtained cut sets & path sets are minimal without sorting the component weights. The minimal cuts of a one-stage weighted-k-out-of-n:F system can be obtained by evaluating $MCS(k, \infty, n)$; and its minimal paths can be obtained by evaluating $MPS(0, k, n)$. It can be seen that the computational complexity of (11) & (12) in terms of the number of $MCS(z_L, z_U, s)$ $(z_L \in \{0, \ldots, k\}, z_U \in \{0, \ldots, k, \infty\}, s \in \{0, \ldots, n\})$ & $MPS(z_L, z_U, s)$ $(z_L \in \{0, \ldots, k\}, z_U \in \{0, \ldots, k\}, s \in \{0, \ldots, n\})$ to be evaluated are both $O(n \cdot k^2)$, which is higher than that of (9) & (10). But the computational resources needed to sort all the component weights, which is required by (9) & (10), are saved when (11) & (12) are used.

2) Generation of Minimal Cuts & Paths for Two-Stage Systems: The minimal cuts & minimal paths of a two-stage weighted-$\mathbf{k}$-out-of-n system can be generated based on those of the one-stage weighted-k-out-of-n subsystems.

Result 3: Let $S_p(\cdot)$, and $S_c(\cdot)$ denote the path polynomial, and cut polynomial for the first-level structure of a two-stage weighted-$\mathbf{k}$-out-of-n:F system, respectively. Then the set of minimal paths of this two-stage weighted-$\mathbf{k}$-out-of-n:F system can be derived based on the following Boolean operations:

$$MPS(\mathbf{k}, n) = S_p\big(MPS(k_1, n), MPS(k_2, n), \ldots, MPS(k_m, n)\big). \quad (13)$$

And the set of minimal cuts of this two-stage weighted-$\mathbf{k}$-out-of-n:F system can be derived based on the following Boolean operations:

$$MCS(\mathbf{k}, n) = S_c\Big(\overline{MCS(k_1, n)}, \overline{MCS(k_2, n)}, \ldots, \overline{MCS(k_m, n)}\Big) \quad (14)$$

where $MPS(k_i, n)$, and $MCS(k_i, n)$ denote the path polynomial, and the cut polynomial, respectively, of the i^{th} subsystem, $i = 1, \ldots, m$.

Unlike the minimal cut & minimal path generation for one-stage systems, Boolean algebra such as the absorption of terms $(a + aB = a)$ is generally needed in (13) & (14) to simplify the path & cut polynomials, and guarantee the obtained path sets & cut sets are minimal. For example, consider a two-stage weighted-$\mathbf{k}$-out-of-n:F system $(n = 3)$ in which the first-level structure is a 2-out-of-3:F structure. The system is failed iff at least 2 of the 3 subsystems are failed, i.e., at least 2 of the following 3 failure conditions are satisfied:

$$\begin{aligned} &2 \cdot \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 3 \\ &0 \cdot \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2 \\ &0 \cdot \overline{x}_1 + 0 \cdot \overline{x}_2 + \overline{x}_3 \geq 1. \end{aligned} \quad (15)$$

Based on the weight coefficients in (15), and Remark 1, the first subsystem has 3 components: 1, 2, & 3; the second subsystem has 2 components: 2 & 3; and the third subsystem only has one component: 3. From direct observations, it is easy to see that the cut polynomial for the first subsystem is $MCS(k_1 = 3, n = 3) = \overline{x}_1\overline{x}_2 + \overline{x}_1\overline{x}_3$; that of the second subsystem is $MCS(k_2 = 2, n = 3) = \overline{x}_2\overline{x}_3$; and that of the third subsystem is $MCS(k_3 = 1, n = 3) = \overline{x}_3$. Let y_i denote the Boolean variable (0 or 1) indicating the functioning of subsystem i, $i = 1, 2, 3$. That is, $y_i = 1$ iff subsystem i is functioning, $i = 1, 2, 3$. Let $\overline{y}_i$ be the Boolean variable indicating failure of subsystem i, i.e., $\overline{y}_i$ is the complement of y_i. Because the set of any two subsystems is a minimal cut set of the first-level structure, which is a 2-out-of-3:F structure, the cut polynomial of the first-level structure is

$$S_c(y_1, y_2, y_3) = \overline{y}_1\overline{y}_2 + \overline{y}_2\overline{y}_3 + \overline{y}_1\overline{y}_3. \quad (16)$$

$$MPS(z, s) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 & \text{if } \sum_{i=1}^{s} w_i < z \left(\text{when } s = 0, \sum_{i=1}^{s} w_i = 0\right) \\ MPS(z, s-1) \cdot x_s + MPS(z - w_s, s-1) & \text{otherwise} \end{cases} \quad (10)$$

$$MCS(z_L, z_U, s) = \begin{cases} 0 & \text{if } s = 0 \text{ and } 0 < z_L \\ 0 & \text{if } z_0 \leq 0 \\ 1 & \text{if } s = 0 \text{ and } z_L 0 < z_U \\ MCS(z_L, z_U, s-1) \\ \quad + MCS\left(z_L - w_s, \min(z_L, z_U - w_s), s-1\right) \cdot \overline{x}_s & \text{otherwise} \end{cases} \quad (11)$$

$$MPS(z_L, z_U, s) = \begin{cases} 0 & \text{if } s = 0 \text{ and } 0 < z_L \\ 0 & \text{if } z_U \leq 0 \\ 1 & \text{if } s = 0 \text{ and } z_L \leq 0 < z_U \\ MPS\big(\max(z_L, z_U - w_s), z_U, s-1\big) \cdot x_s \\ \quad + MPS(z_L - w_s, z_U - w_s, s-1) & \text{otherwise} \end{cases} \quad (12)$$

Using (14) & (16), the cut polynomial of this system is

$$\begin{aligned}
&MCS\big([3,2,1]^T,3\big)\\
&= S_c\Big(y_1=\overline{MCS(k_1=3,n=3)}, y_2=\overline{MCS(k_2=2,n=3)},\\
&\qquad y_3=\overline{MCS(k_3=1,n=3)}\Big)\\
&= \overline{\overline{MCS(k_1=3,n=3)}}\cdot\overline{\overline{MCS(k_2=2,n=3)}}\\
&\quad + \overline{\overline{MCS(k_2=2,n=3)}}\cdot\overline{\overline{MCS(k_3=1,n=3)}}\\
&\quad + \overline{\overline{MCS(k_1=3,n=3)}}\cdot\overline{\overline{MCS(k_3=1,n=3)}}\\
&= MCS(k_1=3,n=3)\cdot MCS(k_2=2,n=3)\\
&\quad + MCS(k_2=2,n=3)\cdot MCS(k_3=1,n=3)\\
&\quad + MCS(k_1=3,n=3)\cdot MCS(k_3=1,n=3)\\
&= (\overline{x}_1\overline{x}_2+\overline{x}_1\overline{x}_3)\cdot\overline{x}_2\overline{x}_3+\overline{x}_2\overline{x}_3\cdot\overline{x}_3+(\overline{x}_1\overline{x}_2+\overline{x}_1\overline{x}_3)\cdot\overline{x}_3\\
&= \overline{x}_2\overline{x}_3+\overline{x}_1\overline{x}_3. \qquad (17)
\end{aligned}$$

Boolean algebra is applied in the last step of (17), to simplify the cut polynomial. From the result in (17), there are two minimal cuts for the two-stage weighted-$\mathbf{k}$-out-of-n:F system defined in (15): {2,3} & {1,3}. These can be verified by examining the failure conditions specified by (15).

3) Reliability Bounds for s-Dependent Component Failures Based on Minimal Paths & Cuts: In a great many situations with failure dependency, (nonnegative) *association* can be assumed for component states. Association means that the random variables $x_1, x_2, \ldots, x_n$, which represent the functioning states of the n components in a system, have nonnegative covariance [11], [15]. In other words, given a component is failed, the failure probabilities of some other components tend to be higher. Consequently, the probability that a set of components are in a specific state (functioning or failed) is then greater than or equal to the multiplication of the probabilities for each component to be in that state. Barlow & Proschan [11] suggested reliability situations where association can be assumed, such as components subjected to the same set of environmental stresses & shocks where a high stress will affect all the components adversely, components depending on common sources of power, and structures in which components share the load & the functioning (failure) of a component contributes to the functioning (failure) of the remaining components. It is commented in [15] that the (nonnegative) association assumption seems to cover most cases of s-dependent failures. Examples of negative failure dependence are not often observed in reliability applications. Please refer to [16] for examples of distributions exhibiting negative s-dependence. With (nonnegative) association assumed, it is known that the min-max bound [11] can be used to bound the system reliability:

$$\max_{1\le i\le r}\prod_{j\in MP_i} p_j \le R(\mathbf{k},n) \le \min_{1\le i\le l}\left\{1-\prod_{j\in MC_i}(1-p_j)\right\} \qquad (18)$$

where $MC_1, MC_2, \ldots, MC_l$ denote all the minimal cuts of the system, and $MP_1, MP_2, \ldots, MP_r$ denote all the minimal paths.

When we cannot assume nonnegative association/covariance among component states, a lower bound of system reliability can be derived for two-stage weighted-$\mathbf{k}$-out-of-n systems using a method similar to that used in [15]

$$1-\min_{1\le i\le r}\left(|MP_i|-\sum_{j\in MP_i} p_j\right)\le R(\mathbf{k},n). \qquad (19)$$

IV. Examples

In this section, the problem introduced in Section II-B-1 on project management, and that in Section II-B-2 on shortest path problem, will be investigated based on the algorithms developed in Section III.

A. Project Management Example

In Section II-B-1, the project management example is formulated as a SW-weighted-$[63]^T$-out-of-5:F system with failure conditions specified in (5). Using (8), the system reliability can be evaluated as follows:

$$\begin{aligned}
R\big([6,3]^T,1\big) &= 1 \ \big(\text{because } \boldsymbol{\beta}_1=[1,1]^T,\\
&\qquad \text{and } \phi_I(\boldsymbol{\beta}_1)=1\cdot 1=1\big)\\
R\big([6,0]^T,1\big) &= 0 \ \big(\text{because } \boldsymbol{\beta}_0=[1,0]^T,\\
&\qquad \text{and } \phi_I(\boldsymbol{\beta}_0)=1\cdot 0=0\big)\\
R\big([6,3]^T,2\big) &= (1-p)\cdot R\big([6,0]^T,1\big)+p\cdot R\big([6,3]^T,1\big)\\
&= p\\
R\big([4,0]^T,1\big) &= 0\\
R\big([4,1]^T,1\big) &= 1\\
R\big([4,1]^T,2\big) &= (1-p)\cdot R\big([4,0]^T,1\big)+p\cdot R\big([4,1]^T,1\big)\\
&= p\\
R\big([6,3]^T,3\big) &= (1-p)\cdot R\big([4,1]^T,2\big)+p\cdot R\big([6,3]^T,2\big)\\
&= p\cdot(1-p)+p^2\\
R\big([2,0]^T,2\big) &= 0\\
R\big([4,1]^T,3\big) &= (1-p)\cdot R\big([2,0]^T,2\big)+p\cdot R\big([4,1]^T,2\big)\\
&= p^2\\
R\big([6,3]^T,4\big) &= (1-p)\cdot R\big([4,1]^T,3\big)+p\cdot R\big([6,3]^T,3\big)\\
&= 2\cdot(1-p)\cdot p^2+p^3\\
R\big([2,0]^T,4\big) &= 0\\
R\big([6,3]^T,5\big) &= (1-p)\cdot R\big([2,0]^T,4\big)+p\cdot R\big([6,3]^T,4\big)\\
&= 2\cdot(1-p)\cdot p^3+p^4.
\end{aligned}$$

In the calculation above, all negative values of z_i are replaced with zero because from (8) obviously this will not affect the result. The system reliability, the probability that the project can still meet the deadline with the risk of delays, is

$$R\big([6,3]^T,5\big)=2\cdot(1-p)\cdot p^3+p^4=2\cdot p^3-p^4. \qquad (20)$$

Using (9), the set of minimal cuts for subsystem 1 can be derived as (first sorting the weights as $w_1 = w_B = 0$, $w_2 = w_A = 1$, $w_3 = w_D = 2$, $w_4 = w_E = 2$, $w_5 = w_F = 4$)

$$\begin{aligned} MCS(2,2) &= 0 \text{ (because } w_1 + w_2 = 1 < 2) \\ MCS(0,2) &= 1 \text{ (because } z = 0) \\ MCS(2,3) &= MCS(2,2) + MCS(0,2) \cdot \overline{x}_3 = \overline{x}_3 \\ MCS(0,3) &= 1 \\ MCS(2,4) &= MCS(2,3) + MCS(0,3) \cdot \overline{x}_4 = \overline{x}_3 + \overline{x}_4 \\ MCS(6,4) &= 0 \\ MCS(k_1 = 6,5) &= MCS(6,4) + MCS(2,4) \cdot \overline{x}_5 = \overline{x}_3\overline{x}_5 \\ &\quad + \overline{x}_4\overline{x}_5 = \overline{x}_D\overline{x}_F + \overline{x}_E\overline{x}_F. \end{aligned}$$

Therefore, there are two minimal cuts for subsystem 1: {D, F} & {E, F}. For the second subsystem, we use (11) to derive the set of minimal cuts without sorting the component weights

$$\begin{aligned} &MCS(3,\infty,0) = 0, \quad MCS(3,3,0) = 0, \\ &MCS(3,\infty,1) = MCS(3,\infty,0) + MCS(3,3,0) \cdot \overline{x}_1 = 0 \\ &MCS(0,3,0) = 1, \quad MCS(0,0,0) = 0, \\ &MCS(0,3,1) = MCS(0,3,0) + MCS(0,0,0) \cdot \overline{x}_1 = 1 \\ &MCS(3,\infty,2) = MCS(3,\infty,1) + MCS(0,3,1) \cdot \overline{x}_2 = \overline{x}_2, \\ &MCS(1,3,0) = 0, \quad MCS(1,1,0) = 0 \\ &MCS(1,3,1) = MCS(1,3,0) + MCS(1,1,0) \cdot \overline{x}_1 = 0, \\ &MCS(0,0,1) = 0, \\ &MCS(1,3,2) = MCS(1,3,1) + MCS(0,0,1) \cdot \overline{x}_2 = 0, \\ &MCS(3,\infty,3) = MCS(3,\infty,2) + MCS(1,3,2) \cdot \overline{x}_3 = \overline{x}_2, \\ &MCS(0,1,0) = 1 \\ &MCS(0,1,1) = MCS(0,1,0) + MCS(0,0,0) \cdot \overline{x}_1 = 1 \\ &MCS(0,1,2) = MCS(0,1,1) + MCS(0,0,1) \cdot \overline{x}_2 = 1 \\ &MCS(1,3,3) = MCS(1,3,2) + MCS(0,1,2) \cdot \overline{x}_3 = \overline{x}_3 \\ &MCS(3,\infty,4) = MCS(3,\infty,3) + MCS(1,3,3) \cdot \overline{x}_4 \\ &\qquad = \overline{x}_2 + \overline{x}_3\overline{x}_4 \\ &MCS(0,3,2) = MCS(0,3,1) + MCS(0,0,1) \cdot \overline{x}_2 = 1, \\ &MCS(0,0,2) = 0 \\ &MCS(0,3,3) = MCS(0,3,2) + MCS(0,0,2) \cdot \overline{x}_3 = 1, \\ &MCS(0,0,3) = 0 \\ &MCS(0,3,4) = MCS(0,3,3) + MCS(0,0,3) \cdot \overline{x}_4 = 1 \\ &\text{MCS}(k_2 = 3,5) = MCS(3,\infty,5) \\ &\qquad = MCS(3,\infty,4) + MCS(0,3,4) \cdot \overline{x}_5 \\ &\qquad = \overline{x}_2 + \overline{x}_3\overline{x}_4 + \overline{x}_5 = \overline{x}_B + \overline{x}_D\overline{x}_E + \overline{x}_F. \end{aligned}$$

Therefore, there are three minimal cuts for subsystem 2: {B}, {D, E} & {F}. From Result 4, the set of minimal cuts for the whole system can be obtained as

$$\begin{aligned} MCS\left([6,\ 3]^T,5\right) &= S_c\left(\overline{MCS(k_1=6,5)}, \overline{MCS(k_2=3,5)}\right) \\ &= MCS(k_1 = 6,5) + MCS(k_2 = 3,5) \\ &= \overline{x}_D\overline{x}_F + \overline{x}_E x_F + \overline{x}_D\overline{x}_E + \overline{x}_B + \overline{x}_F \\ &= \overline{x}_D\overline{x}_E + \overline{x}_B + \overline{x}_F. \end{aligned}$$

In the last equation, Boolean algebra $\overline{x}_D\overline{x}_F + \overline{x}_E\overline{x}_F + \overline{x}_F = \overline{x}_F$ has been used to simplify the cut polynomial. Hence, three minimal cuts exist for the whole system: $MC_1 = \{D, E\}$, $MC_2 = \{B\}$, and $MC_3 = \{F\}$.

Based on (10), the set of minimal paths for subsystem 1 can be derived as (the weights are still sorted as $w_1 = w_B = 0$, $w_2 = w_A = 1$, $w_3 = w_D = 2$, $w_4 = w_E = 2$, $w_5 = w_F = 4$)

$$\begin{aligned} MPS(2,2) &= 1 \text{ (because } w_1 + w_2 = 1 < 2) \\ MPS(0,2) &= 0 \text{ (because } z = 0) \\ MPS(2,3) &= MPS(2,2) \cdot x_3 + MPS(0,2) = x_3 \\ MPS(0,3) &= 0 \\ MPS(2,4) &= MPS(2,3) \cdot x_4 + MPS(0,3) = x_3x_4 \\ MPS(6,4) &= 1 \\ MPS(k_1 = 6,5) &= MPS(6,4) \cdot x_5 + MPS(2,4) \\ &= x_5 + x_3x_4 = x_F + x_Dx_E. \end{aligned}$$

Similarly, the minimal paths for the second subsystem can be obtained by either (10) or (12) as

$$MPS(k_2 = 3,5) = x_Dx_Bx_F + x_Ex_Bx_F.$$

From Result 4, the set of minimal paths for the whole system can be obtained as

$$\begin{aligned} MPS\left([6,3]^T,5\right) &= S_p\left(MPS(k_1=6,5), MPS(k_2=3,5)\right) \\ &= MPS(k_1 = 6,5) \cdot MPS(k_2 = 3,5) \\ &= (x_F + x_Dx_E) \cdot (x_Dx_Bx_F + x_Ex_Bx_F) \\ &= x_Ex_Bx_F + x_Dx_Bx_F. \end{aligned}$$

Hence, two minimal paths exist in the system: $MP_1 = \{\mathrm{E, B, F}\}$, and $MP_2 = \{\mathrm{D, B, F}\}$. The system reliability can also be evaluated from the minimal paths of the system based on the well-known inclusion-exclusion method [17] as

$$\begin{aligned} &R\left([6,\ 3]^T,5\right) \\ &\quad = \Pr(MP_1 \text{ functioning} \cup MP_2 \text{ functioning}) \\ &\quad = \Pr(MP_1 \text{ functioning}) + \Pr(MP_2 \text{ functioning}) \\ &\qquad - \Pr(MP_1 \text{ functioning} \cap MP_2 \text{ functioning}) \\ &\quad = 2 \cdot p^3 - p^4 \end{aligned}$$

which is the same as the result in (20). The bounds of system reliability when activity delays are s-dependent with (nonnegative) association can be obtained based on (18) as

$$p^3 \le R\left([6,\ 3]^T,5\right) \le p. \tag{21}$$

When association cannot be assumed, the lower bound of system reliability based on (19) is

$$1 - 3(1-p) \le R\left([6,\ 3]^T,5\right). \tag{22}$$

For example, when $p = 0.99$, the system reliability when component failures are s-independent is 0.980 002; when association exists among component failures, the bounds of system reliability using (21) is $0.970\,299 \leq R([6,\ 3]^T, 5) \leq 0.99$; and the lower bound of system reliability using (22) is $0.97 \leq R([6,\ 3]^T, 5)$.

B. Example of Shortest Path Problem

In Section II-B-2, the shortest path example is formulated as a PW-$[5, 15]^T$-out-of-6:F system with subsystem failure conditions specified in (6). Based on either (9) or (11), the set of minimal cuts for each subsystem can be derived as

$$MCS(k_1 = 5, n = 6) = \overline{x}_{OB} + \overline{x}_{BC} + \overline{x}_{CT}$$
$$MCS(k_2 = 15, n = 6) = \overline{x}_{CT} + \overline{x}_{OA}\overline{x}_{AC}.$$

From Result 4, the set of minimal cuts for the whole system can be obtained as

$$\begin{aligned} MCS&\big([5,15]^T, 6\big) \\ &= S_c\left(\overline{MCS(k_1 = 5, 6)}, \overline{MCS(k_2 = 15, 6)}\right) \\ &= MCS(k_1 = 5, 6) \cdot MCS(k_2 = 15, 6) \\ &= (\overline{x}_{OB} + \overline{x}_{BC} + \overline{x}_{CT}) \cdot (\overline{x}_{CT} + \overline{x}_{OA}\overline{x}_{AC}) \\ &= \overline{x}_{CT} + \overline{x}_{OB}\overline{x}_{OA}\overline{x}_{AC} + \overline{x}_{BC}\overline{x}_{OA}\overline{x}_{AC}. \end{aligned}$$

Based on either (10) or (12), the set of minimal paths for each subsystem can be derived as

$$MPS(k_1 = 5, n = 6) = x_{OB}x_{BC}x_{CT}$$
$$MPS(k_2 = 15, n = 6) = x_{CT}x_{OA} + x_{CT}x_{AC}.$$

From Result 4, the set of minimal paths for the whole system is

$$\begin{aligned} MPS\big([5,15]^T, 6\big) &= S_p\big(MPS(k_1{=}5, 6), MPS(k_2{=}15, 6)\big) \\ &= MPS(k_1{=}5, 6) + MPS(k_2{=}15, 6) \\ &= x_{OB}x_{BC}x_{CT} + x_{CT}x_{OA} + x_{CT}x_{AC}. \end{aligned}$$

Using the minimal paths, and the inclusion exclusion method, the system reliability can be evaluated as

$$\begin{aligned} R&\big([5,15]^T, 6\big) \\ &= \Pr(MP_1 \text{ functioning} \cup MP_2 \text{ functioning} \\ &\qquad \cup MP_3 \text{ functioning}) \\ &= \Pr(MP_1 \text{ functioning}) + \Pr(MP_2 \text{ functioning}) \\ &\quad + \Pr(MP_3 \text{ functioning}) \\ &\quad - \Pr(MP_1 \text{ functioning} \cap MP_2 \text{ functioning}) \\ &\quad - \Pr(MP_2 \text{ functioning} \cap MP_3 \text{ functioning}) \\ &\quad - \Pr(MP_1 \text{ functioning} \cap MP_3 \text{ functioning}) \\ &\quad + \Pr(MP_1 \text{ functioning} \cap MP_2 \text{ functioning} \\ &\qquad \cap MP_3 \text{ functioning}) \\ &= 2 \cdot p^2 - 2 \cdot p^4 + p^5. \end{aligned}$$

The bounds of system reliability when activity delays are s-dependent with (nonnegative) association can be obtained based on (18) as

$$p^2 \leq R\big([5,15]^T, 5\big) \leq p. \tag{23}$$

When nonnegative association between activity delays cannot be assumed, the lower bound of system reliability based on (19) is

$$1 - 2(1-p) \leq R\big([5,15]^T, 6\big). \tag{24}$$

For example, when $p = 0.99$, the system reliability when component failures are s-independent is 0.9900; the bounds of system reliability from (23) is $0.9801 \leq R([5,15]^T, 6) \leq 0.99$; and the lower bound of system reliability from (24) is $0.98 \leq R([5,15]^T, 6)$.

APPENDIX

A. Proof of Result 1

It can be seen that

$$\sum_{j=1}^{n} w_{ij} \cdot \overline{x}_j < k_i \Leftrightarrow \sum_{j=1}^{n} w_{ij} \cdot \overline{x}_j \leq k_i - 1 \Leftrightarrow W_i - \sum_{j=1}^{n} w_{ij} \cdot x_j \leq k_i - 1 \Leftrightarrow \sum_{j=1}^{n} w_{ij} \cdot x_j \geq W_i - k_i + 1.$$

Therefore, $y'_i = 1$ for a two-stage weighted-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:G system iff $y_i = 1$ for a two-stage weighted-$\mathbf{k}$-out-of-n:F system, where y_i, and y'_i are as defined in (2), and (7), respectively. The two-stage weighted-$\mathbf{k}$-out-of-n:F system is functioning iff $\phi_I(\mathbf{y}) = 1$, which is equivalent to $\phi_I(\mathbf{y}') = 1$, i.e., the functioning condition of the two-stage weighted-$(\mathbf{W}-\mathbf{k}+\mathbf{1})$-out-of-$n$:G system. So the two-stage weighted-$\mathbf{k}$-out-of-n:F system is equivalent to the two-stage weighted-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:G system with the same first-level structure function $\phi_I(\cdot)$. Replacing $\mathbf{k}$ with $\mathbf{W} - \mathbf{k} + \mathbf{1}$ in the above statement, it can be seen that the two-stage weighted-$\mathbf{k}$-out-of-n:G system is equivalent to the two-stage weighted-$(\mathbf{W} - \mathbf{k} + \mathbf{1})$-out-of-$n$:F system.

B. Proof or Result 2

Let $\phi_F(\cdot)$ denote the overall structure function of a two-stage weighted-$\mathbf{k}$-out-of-n:F system with a first-level structure function $\phi_I(\cdot)$, and $\phi_G(\cdot)$ denote the overall structure function of a two-stage weighted-$\mathbf{k}$-out-of-n:G system with a first-level structure function $\phi_I^D(\cdot)$. The subsystem state vectors $\mathbf{y}$, and $\mathbf{y}'$, as defined respectively in (2), and (7), can be considered as functions of component states $\mathbf{x}$. Let $\mathbf{y}(\mathbf{x})$, and $\mathbf{y}'(\mathbf{x})$ explicate this functional relationship. It can be seen that $\mathbf{y}(\mathbf{1} - \mathbf{x}) = \mathbf{1} - \mathbf{y}'(\mathbf{x})$. Therefore,

$$\begin{aligned} \phi_F(\mathbf{1} - \mathbf{x}) &= \phi_I\big(\mathbf{y}(\mathbf{1} - \mathbf{x})\big) = \phi_I\big(\mathbf{1} - \mathbf{y}'(\mathbf{x})\big) \\ &= 1 - \phi_I^D\big(\mathbf{y}'(\mathbf{x})\big) = 1 - \phi_G(\mathbf{x}) \end{aligned}$$

which shows $\phi_F(\cdot)$, and $\phi_G(\cdot)$ are duals of each other.

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