Dynamic topological logic of the real line

Maria Nogin

California State University, Fresno

Spring AMS Western Section Meeting

3 May 2008

Subset interpretation

Let X be a set.

Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$
- $(P \to Q) \equiv ((\neg P) \vee Q)$

Given a mapping from propositional variables (P, Q, *etc.*) to subsets of X, every formula is mapped to a subset X.

e.g. $P \wedge Q \quad \mapsto \quad P \cap Q$

$P \vee \neg P \quad \mapsto \quad P \cup \overline{P}$

Definition. Formulas that are always mapped to the whole set X are called valid with respect to interpretation in X.

Soundness and completeness

Let X be a set.

1. All tautologies (= formulas derivable from axioms) of the classical logic are valid with respect to interpretation in X. The classical logic is sound with respect to this interpretation.
2. If X is non-empty, the tautologies (= derivable formulas) of the classical logic are the only formulas valid with respect to interpretation in X. The classical logic is complete with respect to this interpretation.

The language of classical logic does not distinguish different non-empty sets X.

S4: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\Box$

- Axioms of classical logic
- $\Box P \rightarrow P$
- $\Box P \rightarrow \Box\Box P$
- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

Rules of inference:

(1) $\dfrac{P,\ P \rightarrow Q}{Q}$

(2) $\dfrac{P}{\Box P}$

Topological interpretation of $\Box$:

$$\Box P \quad \mapsto \quad \mathrm{interior}(P)$$

Theorem. Let X be a topological space. Then S4 is sound with respect to interpretation in X.

Completeness

Theorem. S4 is complete with respect to all interpretations in all topological spaces X, i.e. for any formula F, the following statements are equivalent:

1. F is derivable in S4
2. F is valid in each interpretation (for each topological space X)
3. F is valid in each interpretation for each $\mathbb{R}^n$
4. F is valid in each interpretation for some $\mathbb{R}^n$

Corollary. The modal logic (with operations $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\Box$) does not distinguish $\mathbb{R}^n$'s for different n.

Dynamic topological systems

Definition. A dynamic topological system is a topological space X with a continuous function $f\colon X \to X$.

New modal operator $\bigcirc$:

$\bigcirc P$ is interpreted as $f^{-1}(P)$.

S4C

- Axioms of classical logic
- $\Box P \to P$
- $\Box P \to \Box\Box P$
- $\Box(P \to Q) \to (\Box P \to \Box Q)$
- $\bigcirc(P \to Q) \to (\bigcirc P \to \bigcirc Q)$
- $(\bigcirc\neg P) \leftrightarrow (\neg\bigcirc P)$
- $(\bigcirc\Box P) \leftrightarrow (\Box\bigcirc\Box P)$

Rules of inference:

(1) $\dfrac{P,\ P \to Q}{Q}$

(2) $\dfrac{P}{\Box P}$

(3) $\dfrac{P}{\bigcirc P}$

Completeness

Theorem. Let F be a formula. The following are equivalent:

1. F is derivable in S4C
2. F is valid with respect to every interpretation in every $\mathbb{R}^n$

However, the above statements are not equivalent to

3. F is valid with respect to every interpretation in $\mathbb{R}$

Corollary. The language of S4C distinguishes $\mathbb{R}$ from $\mathbb{R}^n$ for $n > 1$.

Logic of $\mathbb{R}$

Open question. Which formulas are valid with respect to interpretation in $\mathbb{R}$?

New axioms:

- $\bigcirc Q \wedge \Diamond(\bigcirc\neg Q \wedge \bigcirc\Diamond\neg P \wedge \Box\bigcirc P) \longrightarrow$
  $\Diamond(\bigcirc\neg Q \wedge \Diamond\bigcirc\neg P \wedge \Diamond\Box\bigcirc P)$
- $\bigcirc\neg P \wedge \bigcirc\neg Q \wedge \Diamond\Box\bigcirc P \wedge \Diamond\bigcirc(\neg P \wedge Q) \wedge$
  $\Box\bigcirc S \longrightarrow$
  $\Diamond(\Diamond\Box\bigcirc P \wedge \Diamond\bigcirc\neg P \wedge \bigcirc\Box S)$

(where $\Diamond = \neg\Box\neg$)

Maria Nogin, Aleksey Nogin:
"On dynamic topological logic of the real line"
submitted to JLC

mnogin@csufresno.edu

http://zimmer.csufresno.edu/~mnogin/