

A note on the degree condition of completely independent spanning trees*

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Abstract

Given a graph G , a set of spanning trees of G are completely independent if for any vertices x and y , the paths connecting them on these trees have neither vertex nor edge in common, except x and y . In this paper, we prove that for graphs of order n , with $n \geq 6$, if the minimum degree is at least $n - 2$, then there are at least $\lfloor n/3 \rfloor$ completely independent spanning trees.

Keyword: Completely independent spanning trees; Dirac's condition; Ore's condition;

1 Introduction

In a graph G , a set of spanning trees are said to be *completely independent* if for any vertices x and y , the paths connecting them on the spanning trees have neither vertex nor edge in common, except x and y . Graphs discussed in this paper are assumed to be connected and *simple*, where a graph is simple if it contains neither parallel edge nor loop. The concept of completely independent spanning trees was introduced by Hasunuma [3] in 2001. He showed that determining whether a graph admits k completely independent spanning trees is NP-complete, even for $k = 2$ [4]. The construction of k completely independent spanning trees was then investigated on some graph classes, with connectivity at least $2k$ [3–5]. In addition, Hasunuma [4] gave a conjecture, which states that being $2k$ -connected is sufficient for having k completely independent spanning trees, but that was

disproved by Péterfalve [7]. Counterexamples were also given by Pai et al. [6], via a necessary condition to a k -connected and k -regular graph with $\lfloor k/2 \rfloor$ completely independent spanning trees.

In 2013, Araki [1] gave a new characterization, which is summarized later in Section 2, for graphs admitting k completely independent spanning trees. Based on this, he proved that the following condition is sufficient for a graph G , with enough vertices, admitting two completely independent spanning trees.

$$\forall x \in V \quad \deg(x) \geq n/2, \quad (1)$$

where V is the vertex set of G , $n = |V|$, and $\deg(x)$ is the degree of x . Usually, (1) is called the *Dirac's condition*, due to the well-known sufficient condition for a graph being Hamiltonian. Later on, Fan et al. [2] generalized the result by *Ore's condition*, which is

$$\forall x, y \in V \quad \deg(x) + \deg(y) \geq n/2. \quad (2)$$

Both (1) and (2) deal with the existence of two completely independent spanning trees via the constraint on vertex degree. This motivates us to relate the number of completely independent spanning trees and some possible constraints on vertex degree.

The main result of this paper is the following theorem.

Theorem 1. *Let G be a graph of order n with minimum degree at least $n - 2$. If $n \geq 6$, there are $\lfloor n/3 \rfloor$ completely independent spanning trees.*

The proof of Theorem 1 is given in Section 3, and some preliminaries are given in Section 2. In the discussion below, we assume that V is the vertex set of the graph being considered and $|V| = n \geq 6$.

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2 Preliminaries

In a graph G , the minimum degree $\delta(G)$ of G is defined as $\min\{\deg(x) \mid x \in V\}$. A vertex x is said to be *universal* if $\deg(x) = n - 1$. The open neighborhood of a vertex x consists of all the vertices adjacent to x , denoted by $N(x)$. We use P_i and C_i to denote a path and a cycle with i vertices, respectively. For $U \subseteq V$, the subgraph induced by U is denoted by $G[U]$. For $x \in V$, we denoted the subgraph induced by $V \setminus \{x\}$ by $G - x$, and analogously, we use $G - U$ for $G[V \setminus U]$. For an edge with end vertices x and y , we denote it by xy , and the graph obtained by removing xy is denoted by $G - xy$. Let E be the edge set of G . For disjoint subsets $U_1, U_2 \subseteq V$, the graph $B(U_1, U_2)$ is defined to be a bipartite graph with partite sets U_1 and U_2 , and edge set $\{xy \in E \mid x \in U_1, y \in U_2\}$. Araki [1] gave the following characterization for a graph admitting k completely independent spanning trees.

Theorem 2 (See [1]). *A graph G has k completely independent spanning trees if and only if there is a partition of V into V_1, V_2, \dots, V_k such that*

- for $i \in \{1, 2, \dots, k\}$, $G[V_i]$ is connected.
- for distinct $i, j \in \{1, 2, \dots, k\}$, $B(V_i, V_j)$ has no tree component.

The partition is called a *CIST-partition* of G .

3 Main results

To simply the presentation, we elaborate the results for even n and odd n , separately, in Sections 3.1 and 3.2. The proofs for both cases are very similar.

3.1 For even n

The idea for proving Theorem 1 for a graph G of even order is to show the following.

- There is an $(n - 2)$ -regular spanning subgraph of G .
- All $(n - 2)$ -regular graphs are isomorphic.
- There are $\lfloor n/3 \rfloor$ completely independent spanning trees in an $(n - 2)$ -regular graph.

Details are given in Lemmas 1, 2, and 3.

Lemma 1. *Let G_1 and G_2 be simple graphs of order n , where n is even. If G_1 and G_2 are $(n - 2)$ -regular, then G_1 and G_2 are isomorphic.*

Proof. Let $n = 2k$, for $k \geq 1$. We prove this lemma by induction on k . For $k = 1$ the lemma holds trivially. Consider the case where $k > 1$. Since G_1 is $(2k - 2)$ -regular, there are two vertices x_1 and x_2 which are not adjacent. It follows immediately that $N(x_1) = N(x_2)$, and $G_1[N(x_1)]$ is a $(2k - 4)$ -regular graph. Similarly, for G_2 , there are two non-adjacent vertices y_1 and y_2 such that $G_2[N(y_1)]$ is $(2k - 4)$ -regular. By the induction hypothesis, $G_1[N(x_1)]$ and $G_2[N(y_1)]$ are isomorphic, and let $f: V(G_1[N(x_1)]) \rightarrow V(G_2[N(y_1)])$ be an isomorphism. It can be easily derived that G_1 and G_2 are isomorphic via the isomorphism $f': V(G_1) \rightarrow V(G_2)$, defined as

$$f'(v) = \begin{cases} y_1 & \text{if } v = x_1, \\ y_2 & \text{if } v = x_2, \\ f(v) & \text{otherwise.} \end{cases}$$

This proves the lemma. □

Lemma 2. *Let n be a positive even integer, and let G be a simple graph with $\delta(G) \geq n - 2$. There exists an $(n - 2)$ -regular spanning subgraph of G .*

Proof. Since $\delta(G) \geq n - 2$, we may assume that there are i vertices with degree $n - 1$ and $n - i$ vertices with degree $n - 2$, where $0 \leq i \leq n$. Considering the sum of vertex degrees of G , we have

$$\sum_{v \in V(G)} \deg(v) = i(n-1) + (n-i)(n-2) = n(n-2) + i,$$

which leads to the fact that i is even. For $i \neq 0$, there are at least two adjacent vertices which are of degree $n - 1$. By iteratively removing these edges, we have an $(n - 2)$ -regular subgraph. □

Remark 1. *If n is even, the $(n - 2)$ -regular graph can be obtained by removing a perfect matching from K_n .*

Remark 2. *Graphs with minimum degree k do not always contain a k -regular subgraph, even for $k = n - 3$.*

Lemma 3. *Let G be an $(n - 2)$ -regular graph. If $n \geq 6$, then there are $\lfloor n/3 \rfloor$ completely independent spanning trees.*

Proof. By Theorem 2, it suffices to show that there is a size $\lfloor n/3 \rfloor$ CIST-partition. Let $k = \lfloor n/3 \rfloor$, and let V be partitioned into V_1, V_2, \dots, V_k as even as possible, i.e.,

$$\forall i \in \{1, \dots, k\} \quad 3 \leq |V_i| \leq 4.$$

Without loss of generality, let j be the index such that $|V_i| = 3$ for $1 \leq i \leq j$, and $|V_i| = 4$ for $i > j$.

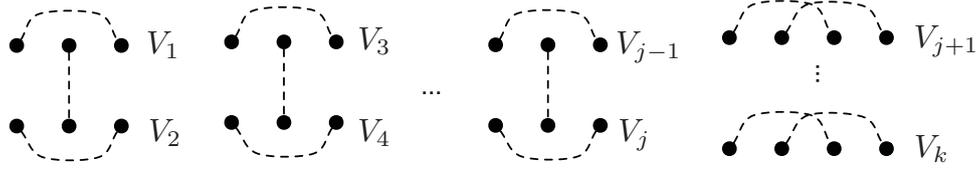


Figure 1: The CIST-partition of G defined in Lemma 3. The vertices are pairwise adjacent, except those connected by dotted lines. In other words, dotted lines denoted the perfect matching removed from K_n .

Notice that when $j = 0$, there is no V_i of size 3. Clearly, j is even, and we can further define the partition so that

- $G[V_i]$ is P_3 , for $1 \leq i \leq j$.
- $G[V_{2i-1} \cup V_{2i}]$ is a 4-regular graph, for $1 \leq i \leq j/2$.
- $G[V_i]$ is C_4 , for $i > j$.

An illustration is given in Figure 1. By definition, V_i induces either P_3 or C_4 on G . Moreover, for any two subsets V_l and V_r , $B(V_l, V_r)$ is either $K_{3,3} - e$ or a complete bipartite graph other than a star. It follows that V_1, V_2, \dots, V_k is a CIST-partition of G . This proves the lemma. \square

3.2 For odd n

For odd n , any graph G with minimum degree at least $n - 2$ cannot be $(n - 2)$ -regular since otherwise $\sum_v \deg(v)$, which is $n(n - 2)$, is odd. Thus, at least one vertex x of G has degree $n - 1$. Consider the graph $G - x$. Obviously, $G - x$ is a graph on which lemmas in Section 3.1 hold. Therefore, the “odd n version” of Lemmas 1 and 2 can be derived in a straightforward manner. We elaborate in more detail as follows.

For convenience, we define a graph G to be k^+ -regular if

- there is exactly one vertex x with $\deg(x) = k + 1$, and
- for each vertex y other than x , $\deg(y) = k$.

Lemma 4. *Let G_1 and G_2 be simple graphs of order n , where n is odd. If G_1 and G_2 are $(n - 2)^+$ -regular, then G_1 and G_2 are isomorphic.*

Lemma 5. *Let n be a positive odd integer, and let G be a simple graph with $\delta(G) \geq n - 2$. There exists an $(n - 2)^+$ -regular spanning subgraph of G .*

Similar to Lemma 3, we can show the existence of $\lfloor n/3 \rfloor$ completely independent spanning trees of G by defining a CIST-partition of size $\lfloor n/3 \rfloor$. The partition is similar to that defined in Lemma 3. The only modification we need is to deal with the odd

number of subsets of size three. By letting the universal vertex be in one of these subsets, we get the requested CIST-partition. More precisely, let x be the universal vertex, and let yz be a non-edge. The graph $G - \{x, y, z\}$ is of order $n - 3$ and is $(n - 5)$ -regular. We can partition the vertices of $G - \{x, y, z\}$ as we do in Lemma 3. Together with the subset $\{x, y, z\}$, we have the CIST-partition of G . This yields the following theorem.

Lemma 6. *Let G be an $(n - 2)^+$ -regular graph with order n . If $n \geq 6$, then there are $\lfloor n/3 \rfloor$ completely independent spanning trees in G .*

3.3 Proof of Theorem 1

Proof. Consider the case where n is even. By Lemmas 1 and 2, it suffices to show that “the” $(n - 2)$ -regular graph contains $\lfloor n/3 \rfloor$ CISTs, and the result is given by Lemma 3. Similarly, for n being odd, the result can be derived from Lemmas 4, 5 and 6. Thus, Theorem 1 is proved. \square

4 Concluding remarks

In this paper, we show that for a graph G of order $n \geq 6$ with $\delta(G) \geq n - 2$, there are $\lfloor n/3 \rfloor$ completely independent spanning trees. As mentioned in Section 1, the following results are known:

- If $\delta(G) \geq n - 1$, there are $\lfloor (n - 1)/2 \rfloor$ completely independent spanning trees [6].
- If $\delta(G) \geq n/2$, there are 2 completely independent spanning trees [1].

A straightforward generalization of the above two statements is “If $\delta(G) \geq a$, there are b completely independent spanning trees.” How a and b are related is what we are interested in, and this problem will be conducted as future work.

References

- [1] T. Araki, Dirac's condition for completely independent spanning trees, *J. Graph Theor.* 77 (2014), 171–179.
- [2] G. Fan, Y. Hong, Q. Liu, Ore's condition for completely independent spanning trees, *Discrete Appl. Math.* 177 (2014), 95–100.
- [3] T. Hasunuma, Completely independent spanning trees in the underlying graph of a line graph. *Discrete Math.* 234 (2001), 149–157.
- [4] T. Hasunuma, Completely independent spanning trees in maximal planar graphs, *Proceedings of 28th Graph Theoretic Concepts of Computer Science (WG2002)*, LNCS 2573, 235–245.
- [5] T. Hasunuma, C. Morisaka, Completely independent spanning trees in torus networks, *Networks* 60 (2012), 60–69.
- [6] K.-J. Pai, S.-M. Tang, J.-M. Chang, and J.-S. Yang, Completely independent spanning trees on complete graphs, complete bipartite graphs and complete tripartite graphs, *IEICE Trans. Inf. Syst.* 20 (2013), 107–113.
- [7] F. Péterfalvi, Two counterexamples on completely independent spanning trees, *Discrete Math.* 312 (2012), 808–810.