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Monopoly Price Discrimination and Demand Curvature

By Iñaki Aguirre, Simon Cowan, and John Vickers*

This paper presents a general analysis of the effects of monopolistic third-degree price discrimination on welfare and output when all markets are served. Sufficient conditions—involving straightforward comparisons of the curvatures of the direct and inverse demand functions in the different markets—are presented for discrimination to have negative or positive effects on social welfare and output. (*JEL* D42)

This paper develops general conditions that determine whether third-degree price discrimination by a monopolist serving all markets reduces or raises output and social welfare, defined as the sum of consumer surplus and profit. A firm practicing third-degree price discrimination uses an exogenous characteristic, such as the age or location of the consumer or the time of purchase, to divide customers into separate markets. The monopoly price can then be set in each market if discrimination is allowed. Moving from nondiscrimination to discrimination raises the firm's profits, harms consumers in markets where prices increase and benefits the consumers who face lower prices. The overall effect on welfare can be positive or negative. The main aim of this paper is to provide conditions based on the shapes of the demand functions to determine the sign of the welfare effect. We also address the classic question of the effect of discrimination on total output, and the paper combines new findings with existing results in a unified framework.

The effect of discrimination on welfare can be divided into a misallocation effect and an output effect. With discrimination output is inefficiently distributed because consumers face different prices in different markets. This negative feature of discrimination may, however, be offset if there is an increase in total output, which is socially valuable since prices exceed marginal costs. Arthur Pigou (1920) proved that if all demand functions are linear and all markets are served at the nondiscriminatory price then total output remains at the no-discrimination level, in which case discrimination is bad for welfare. Joan Robinson's (1933) pioneering analysis, taken forward by Richard Schmalensee (1981), showed how the curvature of demands determines the sign of output effect. Hal R. Varian (1985) proved very generally that a necessary condition for welfare to rise with discrimination is that total output increases (see also Marius Schwartz 1990).

In this paper we explore the welfare effect *directly* using the technique developed by Schmalensee (1981) and Thomas J. Holmes (1989) to analyze the output effect.[1] Throughout it is assumed that at the nondiscriminatory price all markets are served with positive quantities, so

*Aguirre: Departamento de Fundamentos del Análisis Económico I, University of the Basque Country, Avda. Lehendakari Aguirre 83, 48015-Bilbao, Spain (e-mail: inaki.aguirre@ehu.es); Cowan: Department of Economics, University of Oxford, Manor Road Building, Manor Road, Oxford OX1 3UQ, UK (e-mail: simon.cowan@economics.ox.ac.uk); Vickers: All Souls College, Oxford OX1 4AL, UK (e-mail: john.vickers@all-souls.ox.ac.uk). Financial support from the Ministerio de Ciencia y Tecnología and FEDER (SEJ2006-05596), and from the Departamento de Educación, Universidades e Investigación del Gobierno Vasco (IT-223-07) is gratefully acknowledged. We are grateful to a referee who, along with very helpful comments, recommended that we combine three working papers (available on request) into one, as well as to three referees for comments on one of the papers. We would also like to thank Mark Armstrong, Christopher Bliss, Ian Jewitt, Meg Meyer, Volker Nocke, Ignacio Palacios-Huerta, Joe Perkins, John Thanassoulis, and Michael Whinston for their comments.

[1] Earlier applications of the method are by Wassily Leontief (1940) and Eugene Silberberg (1970).

price discrimination does not open up new markets.[2] To simplify the exposition, but without loss of generality, we explore the case with two markets. The firm is supposed initially to be required to set the same price in both markets—i.e., the price difference is constrained to be zero. As this constraint is relaxed, the firm moves towards the *laissez-faire* outcome with price discrimination. As this happens, output and welfare will increase in what Robinson (1933) termed the "weak" market, where the discriminatory price is below the nondiscriminatory price, and decrease in the "strong" market. The question is how overall welfare and total output vary as the price-difference constraint is relaxed.

Central to our analysis is a (commonly met) condition on demand functions—the *increasing ratio condition*—which ensures that welfare varies monotonically with the price-difference constraint, or else has a single interior peak. Given the increasing ratio condition, discrimination is shown to reduce welfare if the direct demand function in the strong market is at least as convex as that in the weak market at the nondiscriminatory price. Second, welfare is higher with discrimination if the discriminatory prices are not far apart and the inverse demand function in the weak market is locally more convex than that in the strong market: total output then rises while the misallocation effect is relatively small. Outside these cases, welfare first rises but then falls as the price-difference constraint is relaxed, so an intermediate degree of discrimination would be optimal, and the overall effect on welfare of unfettered discrimination can be positive or negative. Its sign can however be determined in important special cases: (i) when inverse demand curvature is constant, welfare falls with discrimination if curvature is sufficiently below unity and rises if curvature is sufficiently above unity, and (ii) when demands have constant elasticities, although total output rises with discrimination (Aguirre 2006), welfare falls if the difference between the elasticities is no more than one. In parallel to the welfare analysis, we also obtain new results on how discrimination affects total output, which rises if both inverse and direct demand in the weak market are more convex than those functions in the strong market, but not if both inverse and direct demand in the strong market are at least as convex as those in the weak market.[3]

The broad economic intuition for why the difference between the curvatures of demand in weak and strong markets is important for welfare and output is as follows. A price increase when demand is concave has relatively little effect on welfare (the extreme form of concavity is when the demand function is rectangular and there is no deadweight loss from monopoly pricing). If at the same time price falls in a market with relatively convex demand, there is a large increase in output and thus in welfare in that market. This is the insight of Robinson (1933), who showed that total output rises when discrimination causes prices to rise in markets with concave demands and prices to fall in markets with convex demands, and David A. Malueg (1994) explored further the relationship between the curvature of the demand function and the deadweight loss from monopoly pricing.[4]

The paper is organized as follows. Section I presents the model of monopoly pricing with and without third-degree price discrimination. Section II contains the welfare analysis using the price-difference technique. The effect of discrimination on total output is considered in Section III. Section IV presents the results of the welfare analysis using a restriction on how far quantities can vary from their nondiscriminatory levels, and considers the important special case where demands have constant elasticities. Conclusions are in Section V.

[2] See Jerry A. Hausman and Jeffrey K. Mackie-Mason (1988); Stephen K. Layson (1994); and Victor Kaftal and Debashis Pal (2008); for analyses of price discrimination that opens new markets.

[3] These output results build on, and encompass, those of Robinson (1933); Schmalensee (1981); Jun-ji Shih, Chao-cheng Mai, and Jung-chao Liu (1988); and Francis Cheung and Xinghe Wang (1994).

[4] See Glen E. Weyl and Michal Fabinger (2009) for a general analysis of demand curvature and social welfare with imperfect competition.

I. The Model of Monopoly Pricing

A monopolist sells its product in two markets and has a constant marginal cost, $c \geq 0$. The assumption of two markets is made for simplicity—all the results can be generalized to the case of more than two markets and the method for doing this is discussed later. Utility functions are quasi-linear. Demand in a representative market with price p is $q(p)$, which is twice-differentiable, decreasing and independent of the price in the other market. (To avoid notational clutter we omit subscripts where it is not necessary to indicate which market is which.) The price elasticity of demand is $\eta \equiv -pq'/q$. The profit function in a market is $\pi = (p - c)q(p)$. Assume that

$$\pi''(p) \;=\; 2q' \;+\; (p - c)q'' \;=\; \left[2 + (p - c)\;\frac{q''}{q'}\right]q' \;<\; 0,$$

so the expression in square brackets is positive and the profit function is strictly concave.[5] With strict concavity the second-order conditions hold for the maximization problems that follow. Define $\alpha(p) \equiv -pq''/q'$ as the convexity (or curvature) of direct demand, which is analogous to relative risk aversion for a utility function and is the elasticity of the slope of demand. The Lerner index, the mark-up of price over marginal cost, is $L(p) \equiv (p - c)/p$ and $2 + (p - c)q''/q' = 2 - L\alpha > 0$ by strict concavity. Similarly the curvature or convexity of the inverse demand function $p(q)$ is $\sigma(q) \equiv -qp''/p' = qq''/[q']^2$. The two curvature measures are related to the price elasticity by $\sigma = \alpha/\eta$. The values of σ and of α play key roles in the analysis.

When the firm discriminates, the first-order condition for its problem in each market is

$$\pi'(p^*) \;=\; q(p^*) \;+\; (p^* - c)q'(p^*) \;=\; 0,$$

where $p^* > c$ is the profit-maximizing price and the star denotes the value that applies with full discrimination. From the first-order condition comes the Lerner condition for monopoly pricing $L^* = 1/\eta^*$. Thus $L^*\alpha^* = \alpha^*/\eta^* = \sigma^*$ and, with strict concavity, $2 - L^*\alpha^* = 2 - \sigma^* > 0$. The subscript w denotes the weak market, where the discriminatory price is below the nondiscriminatory one (see below), and subscript s denotes the strong market, where the price is higher with discrimination. The classification of a market as strong or weak is endogenous. It is assumed that both markets are served at the nondiscriminatory price—a sufficient condition for this is that $q_w(p_s^*) > 0$.

When the firm cannot discriminate it chooses the single price $\overline{p}$ that maximizes aggregate profit, which is defined by the first-order condition $\pi_w'(\overline{p}) + \pi_s'(\overline{p}) = 0$. The first-order condition and the assumption that both markets are served at the nondiscriminatory price imply that $\pi_w'(\overline{p}) = q_w(\overline{p})[1 - L(\overline{p})\eta_w(\overline{p})] < 0$ and $\pi_s'(\overline{p}) = q_s(\overline{p})[1 - L(\overline{p})\eta_s(\overline{p})] > 0$, so $\eta_w(\overline{p}) > \eta_s(\overline{p})$. The weak market has the higher elasticity at the nondiscriminatory (or uniform) price. With strict concavity of each profit function it follows that $p_s^* > \overline{p} > p_w^*$. Social welfare W is the sum of consumer surplus and producer surplus (or gross utility minus cost) so the marginal effect of price on social welfare in a market is $dW/dp = (p - c)q'(p)$, i.e., the effect on the quantity multiplied by the price-cost margin.

[5] Part A of the Appendix discusses conditions that ensure strict concavity. See Babu Nahata, Krzysztof Ostaszewski, and Prasanna K. Sahoo (1990) for an analysis of price discrimination when profit functions are not concave in prices.

II. The Effect of Discrimination on Welfare

The method used by Schmalensee (1981), Holmes (1989), and Lars A. Stole (2007) to consider the output effect is adopted here to analyze the welfare effect. In the following section we use it to reexamine the output effect. Initially the firm is not allowed (or is unable) to discriminate and thus sets the uniform price $\overline{p}$. Then the constraint on the firm's freedom to discriminate is gradually relaxed until the firm can discriminate as much as it likes. Our approach is to calculate the marginal effect on welfare of relaxing the constraint; if this keeps the same sign as more discrimination is allowed, then the overall effect of discrimination can be found.

In particular, we assume that the firm chooses its prices to maximize profit subject to the constraint that $p_s - p_w \leq r$ where $r \geq 0$ is the degree of discrimination allowed. The objective function is $\pi_w(p_w) + \pi_s(p_w + r)$ and the first-order condition is $\pi_w'(p_w) + \pi_s'(p_w + r) = 0$ when the constraint binds. When $r = 0$ the firm sets the nondiscriminatory price. As r rises more discrimination is allowed, the price in the weak market falls and that in the strong market rises:

$$(1) \qquad p_w'(r) = \frac{-\pi_s''}{\pi_w'' + \pi_s''} < 0; \; p_s'(r) = \frac{\pi_w''}{\pi_w'' + \pi_s''} > 0.$$

When the constraint does not bind the firm sets the discriminatory prices. The marginal change in social welfare W as more price discrimination is allowed is

$$(2) \qquad W'(r) = (p_w - c)q_w'(p_w)p_w'(r) + (p_s - c)q_s'(p_s)p_s'(r).$$

A relaxation of the constraint alters prices and thus the quantities demanded, and each additional unit of output has social value equal to the price-cost margin in that market. For $r > r^* = p_s^* - p_w^*$ the marginal welfare effect is zero because the prices remain at the discriminatory levels. Define $W'(0)$ and $W'(r^*)$ as right- and left-derivatives respectively. The marginal effect on total output is $Q'(r) \equiv q_w'p_w' + q_s'p_s'$ so, following Schmalensee (1981), (2) may be written as:

$$(3) \qquad W'(r) = \underbrace{(p_w - \overline{p})q_w'(p_w)p_w'(r) + (p_s - \overline{p})q_s'(p_s)p_s'(r)}_{\text{Misallocation effect}} + \underbrace{(\overline{p} - c)Q'(r)}_{\text{(Value of) output effect}}.$$

The first two terms equal zero at $r = 0$ and are negative for $r > 0$. Together they represent the marginal misallocation effect. The final term is the value of the change in output. At the nondiscriminatory price, because there is no misallocation effect, the marginal welfare effect is proportional to the marginal change in aggregate output. Integrating (3) over $[0, r^*]$ gives the total welfare effect as two negative terms (the total misallocation effect) plus $(\overline{p} - c)$ times the total output change. This confirms that an output increase is necessary for social welfare to rise.[6]

In our analysis a crucial role is played by

$$(4) \qquad z(p) \equiv \frac{(p - c)q'(p)}{2q' + (p - c)q''} = \frac{p - c}{2 - L\alpha},$$

[6] It should be noted that the decomposition of the total welfare effect into an output effect and a misallocation effect is not unique. See Aguirre (2008) for a graphical analysis based on a different decomposition.

the ratio of the marginal effect of a price increase on social welfare to the second derivative of the profit function. Substituting the comparative statics results for prices, (1), into (2) and using (4) gives the marginal welfare effect:

$$W'(r) = \underbrace{\left(\frac{-\pi''_w \pi''_s}{\pi''_w + \pi''_s}\right)}_{>\,0} \left[z_w(p_w(r)) - z_s(p_s(r))\right]. \tag{5}$$

The marginal welfare effect thus has the same sign as $\left[z_w(p_w(r)) - z_s(p_s(r))\right]$. The following assumption is made for the three propositions in this section.

The increasing ratio condition (**IRC**): $z(p)$ is increasing in p in each market.

This holds for a very large set of demand functions. These include: functions that are linear; inverse demands with constant positive curvature, including the exponential and constant-elasticity functions; direct demand functions with constant curvature (whether positive or negative); probits and logits (derived from the normal and logistic distributions respectively); and demand functions derived from the lognormal distribution. Part B of the Appendix presents sufficient conditions for the condition to hold and gives a fuller list of the demand functions to which it applies. While the increasing ratio condition holds very commonly—and always it holds locally in the region around marginal cost—it is not universally applicable. For example when inverse demand has constant negative curvature the condition does not hold for high enough prices, and different techniques are necessary to deal with this case.

LEMMA: *Given the IRC, if there exists $\hat{r}$ such that $W'(\hat{r}) = 0$ then $W''(\hat{r}) < 0$.*

PROOF:

From (5).

$$W''(r) = \left(\frac{-\pi''_w \pi''_s}{\pi''_w + \pi''_s}\right) \left[z'_w p'_w - z'_s p'_s\right] + \left[z_w - z_s\right] \frac{d}{dr}\left(\frac{-\pi''_w \pi''_s}{\pi''_w + \pi''_s}\right),$$

which is negative if $W' = 0$ because $z'_w p'_w < 0$ and $z'_s p'_s > 0$, and $z_w = z_s$ where $W' = 0$.

The IRC therefore implies that $W(r)$ is strictly quasi-concave and thus is monotonic in r or has a single interior peak. Only three outcomes are possible: either welfare, as a function of r, is everywhere decreasing, or everywhere increasing, or it first rises then falls. Which holds depends on the signs of $W'(0)$ and $W'(r^*)$. First, if $W'(0) \leq 0$, then $W(r)$ is decreasing for $r > 0$ and discrimination therefore reduces welfare.

PROPOSITION 1: *Given the IRC, if the direct demand function in the strong market is at least as convex as that in the weak market at the nondiscriminatory price then discrimination reduces welfare.*

PROOF:

The Lemma implies that discrimination reduces welfare if $W'(0) \leq 0$. At the nondiscriminatory price, where $r = 0$, $p_w - c = p_s - c$ and $L_w = L_s$. So from (5), $[z_w(\overline{p}) - z_s(\overline{p})]$ and hence $W'(0)$ have the sign of $[\alpha_w(\overline{p}) - \alpha_s(\overline{p})]$, the difference in curvatures of direct demand, which is non-positive under the condition stated in the proposition.

The condition on the difference in the demand curvatures implies that locally output does not increase, and since at the nondiscriminatory price the marginal misallocation effect is zero, a local output effect that is negative or zero implies that the welfare effect has the same sign. The IRC then extends this local result to all additional increases in the amount of discrimination and thus acts as a sign preserver.

Proposition 1 encompasses the results of Cowan (2007), who has demand in the strong market being an affine transformation of demand in the weak market, i.e., $q_s(p) = M + Nq_w(p)$ where M and N are positive (and demand in both markets is zero at a sufficiently high price). At the same price the direct demand functions, by construction, have the same curvature. This is analogous to the result in expected utility theory that the coefficients of absolute and relative risk aversion, at a given income level, are invariant to positive affine transformations of the utility function. An example is when the direct demand functions have constant and common curvature, α, and a special case is when both demand functions are linear ($\alpha = 0$). Proposition 1 is more general because it allows the demand functions to have different parameters or different functional forms, as in the following example.

Example 1: Exponential and linear demands. Demand in market 1 is $q_1(p) = Be^{-p/b}$(with B and b positive), so $\sigma_1 = 1$, $\alpha_1 = \eta_1 = p/b > 0$ and $p_1^* = b + c$. Demand in market 2 is $q_2(p) = a - p$ so $\eta_2(p) = p/(a - p)$, $\alpha_2 = \sigma_2 = 0$ and $p_2^* = (a + c)/2$. Proposition 1 applies if $b > (a - c)/2$, which is the condition for market 1 to be the strong one. The weak market is served with nondiscriminatory pricing if (but not only if) $a > b + c$.

If discrimination is to raise welfare, given the IRC, direct demand in the weak market must be strictly more convex than demand in the strong market at the nondiscriminatory price. Only then does a small amount of discrimination cause total output to rise. This is a local version of the condition that for welfare to rise total output must increase.

Figure 1 shows, in a standard monopoly diagram, that as demand in the weak market becomes more convex the welfare gain in this market from discrimination rises. Initially inverse demand is the linear function $p_1(q)$ and its associated marginal revenue curve is $MR_1(q) = p_1(q) + qp_1'(q)$. The nondiscriminatory quantity is $\overline{q}$, and the discriminatory quantity is q_1. Suppose that demand becomes more convex, while retaining the same slope and position at the non-discriminatory price (which thus will be unchanged). The new inverse demand is $p_2(q)$. At $\overline{q}$ the marginal revenues intersect (because the price, demand slope, and quantity are the same).[7] For $q > \overline{q}$, however, $MR_2(q) > MR_1(q)$ because the price is higher, and a given output increase causes a smaller price reduction. Thus discriminatory output, q_2, is higher with the new demand than with the old demand function. The change in welfare in this market when discrimination occurs is then larger with the transformed function for two reasons: the output increase is greater, and at every quantity above $\overline{q}$ the price, and thus the marginal social value of output, is higher. Discrimination is more likely to raise welfare overall when demand in the weak market becomes more convex.

We now present a sufficient condition for welfare to be higher with discrimination than without. This is found by examining what happens at the discriminatory prices.

PROPOSITION 2: *Given the IRC, if* $(p_w^* - c)/(2 - \sigma_w^*) \geq (p_s^* - c)/(2 - \sigma_s^*)$ (*so inverse demand in the weak market is more convex than that in the strong market at the discriminatory prices, which are close together*) *then welfare is higher with discrimination.*

[7] See Jeremy I. Bulow and Paul Pfleiderer (1983) for a discussion of the relationship between the marginal revenue curves associated with demand curves that have a point of tangency, and also Malueg (1994).

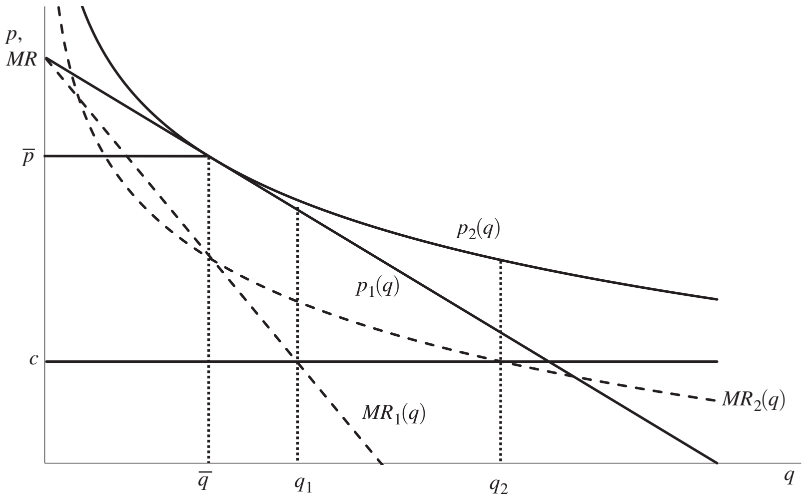

FIGURE 1. THE EFFECT OF INCREASED DEMAND CURVATURE IN THE WEAK MARKET

PROOF:

The Lemma implies that discrimination increases welfare if $W'(r^*) \geq 0$. From (5) and the fact that $L^*\alpha^* = \alpha^*/\eta^* = \sigma^*$ the left-derivative of $W(r)$ at r^* has the same sign as

$$z_w(p_w^*) - z_s(p_s^*) = \frac{p_w^* - c}{2 - \sigma_w^*} - \frac{p_s^* - c}{2 - \sigma_s^*},$$

which is non-negative under the stated condition.

The condition in Proposition 2 ensures that the price difference with full discrimination is small enough that the benefit to welfare from the output increase exceeds the misallocation effect, and this holds for all marginal increases in discrimination. The proposition may be useful if discrimination is currently practiced and a ban on discrimination is being considered. More generally it characterizes conditions for discriminatory welfare to be higher in a way that is both intuitive and straightforward to apply. We use the demand functions of Example 1 to illustrate. Proposition 2 applies if $(a - c)/2 > b \geq (a - c)/4$. The first inequality states that the discriminatory margin in the exponential market, b, is below that in the linear market, so the exponential market is now the weak one. The second inequality is the condition for $W'(r^*) \geq 0$. The multiplicative market-size parameter, B, in the exponential demand function determines the nondiscriminatory price but does not affect the discriminatory prices. Since Proposition 2 depends only on the discriminatory prices it is independent of such multiplicative market-size parameters.

The marginal effect on the discriminatory price of a small change in marginal cost is $1/(2 - \sigma^*)$, i.e., the pass-through rate.[8] This suggests a potential empirical strategy for determining whether the condition in Proposition 2 holds when the firm initially discriminates. A regression of the monopoly price on marginal cost yields a slope coefficient which is the estimate of $1/(2 - \sigma^*)$. The demand function itself does not need to be estimated in this reduced-form approach. Alternatively it may be feasible to estimate the demand functions directly. David Genesove and

[8] Weyl and Fabinger (2009) show, among other results, how the division of the surplus between a monopolist and consumers depends on the cost pass-through rate, which in turn depends on inverse demand curvature.

Wallace P. Mullin (1998) estimate four inverse demand functions with constant curvature in their analysis of oligopoly in the sugar industry.

The next proposition gives conditions for welfare to rise initially, and then to fall, as discrimination increases.

PROPOSITION 3: *Given the IRC, if* (i) *direct demand in the weak market is more convex than demand in the strong market at the nondiscriminatory price, and* (ii) *inverse demand in the strong market is at least as convex as that in the weak market at the discriminatory prices, then welfare rises initially as the degree of discrimination increases, and then falls.*

PROOF:

Condition (i), which is the opposite of the condition in Proposition 1, implies $W'(0) > 0$. Condition (ii), which negates the condition in Proposition 2, implies $W'(r^*) < 0$. From the Lemma, there is a unique $\hat{r} \in (0, r^*)$ at which $W'(\hat{r}) = 0$. For $r < \hat{r}$ we have $W'(r) > 0$, while $W'(r) < 0$ for $r > \hat{r}$.

Under the conditions of Proposition 3, some discrimination is better than none, but *laissez-faire* leads to too much. Of course it would be difficult in practice to identify the "optimal" amount of discrimination at which welfare reaches its peak. Proposition 3 does not determine whether the effect on welfare of full discrimination is positive or negative. We show in Section IV how other techniques can resolve this question for two applications of Proposition 3. The first is where σ is strictly positive and the same in the two markets, so condition (ii) holds (weakly), as does condition (i) because $\alpha = \sigma\eta$ and the elasticity is higher in the weak market at the nondiscriminatory price. The second case is that of constant-elasticity demand functions, with η_w and η_s being the constant elasticities in the weak and strong markets respectively (so $\eta_w > \eta_s$). The generic constant-elasticity demand function is $q = Ap^{-\eta}$ for $A > 0$. The curvature of direct demand is $\alpha = 1 + \eta$, which is higher in the weak market so condition (i) holds, and the curvature of inverse demand is $\sigma = 1 + 1/\eta$, which is higher in the strong market so (ii) also holds.

The results in this section can be generalized to the case of more than two markets using the technique of Schmalensee (1981). The firm chooses its prices subject to a constraint on the weighted sum of the differences between the prices and the uniform price, with the weight on each price difference being marginal profitability at the uniform price, $\pi'(\overline{p})$. With two markets the weights are of equal size and opposite sign (by the first-order condition for the nondiscrimination problem), so the technique is the same as constraining the price difference, as above. The propositions generalize naturally. For example Proposition 1 becomes "If the direct demand functions in the strong markets are at least as convex as those in the weak markets at the nondiscriminatory price then discrimination reduces welfare."

III. The Effect of Discrimination on Output

The same method is now used to determine the output effect. In doing this we use the approach of Holmes (1989), which follows Schmalensee (1981), but the analysis is taken further. The increasing ratio condition is no longer necessary. Total output, as a function of the allowed amount of discrimination, is $Q(r) = q_w(p_w(r)) + q_s(p_s(r))$. Using the comparative statics formulae for prices, (1), the marginal output effect is:

$$Q'(r) = \underbrace{\left(\frac{-q'_w q'_s}{\pi''_w + \pi''_s}\right)}_{>0} [L_w\alpha_w - L_s\alpha_s], \tag{6}$$

which has the sign of

$$L_w\alpha_w - L_s\alpha_s \;=\; L_w\eta_w(\sigma_w - \sigma_s) \;+\; (L_w\eta_w - L_s\eta_s)\sigma_s. \tag{7}$$

At the nondiscriminatory price, $Q'(0) \overset{sgn}{=} \alpha_w(\overline{p}) - \alpha_s(\overline{p})$. With discriminatory pricing $L\eta = 1$ in each market so $Q'(r^*) \overset{sgn}{=} \sigma_w(q_w^*) - \sigma_s(q_s^*)$.

From the LHS of (7) and the fact that $L_w \leq L_s$, it is evident that discrimination increases output if, throughout the relevant range of prices, $\alpha_w \geq 0 \geq \alpha_s$ (strictly if one inequality is strict), or if $0 \geq \alpha_w \geq \alpha_s$. Output decreases if $\alpha_s \geq 0 \geq \alpha_w$ or $\alpha_s \geq \alpha_w \geq 0$. Further, the mild assumption that $(d/dp)(L\eta) > 0$, which holds very generally (see Appendix, part B), implies that $L_w\eta_w > L_s\eta_s$ for $r < r^*$. We can then see from the RHS of (7) that discrimination increases output if $\sigma_w \geq \sigma_s \geq 0$ and decreases output if $\sigma_w \leq \sigma_s \leq 0$. Summarizing these observations the following result encompasses much of the literature on the effect of discrimination on output that has developed from the analysis of Robinson (1933).

PROPOSITION 4: (i) *If demand is concave in the strong market and less concave, or convex, in the weak market then output increases with discrimination.* (ii) *If demand is convex in the strong market and concave, or less convex, in the weak market then output decreases with discrimination.* (iii) *If inverse demands are convex, and more so in the weak market, then output increases with discrimination, while* (iv) *if inverse demands are concave, and more so in the weak market, output falls with discrimination.*

Pigou's 1920 result for linear demands, that output does not change, is on the boundary between all four cases, so fits all interpreted weakly. Robinson's (1933) result that output does not rise if demand in the weak market is concave and demand in the strong market is convex is an instance of (ii); she derived a parallel condition for output to rise that is within (i). Cheung and Wang (1994), generalizing Shih, Mai, and Liu (1988), show that output increases if $\sigma_w \geq \sigma_s > 0$, as in (iii), and falls if $\sigma_w \leq \sigma_s < 0$, as in (iv). Cowan (2007) has $\alpha_w = \alpha_s$ so this is an example of (i) if $\alpha < 0$ and of (ii) if $\alpha > 0$.

The four results in Proposition 4 are usefully summarized in the two following statements. If both direct demand and inverse demand are more convex in the weak market than in the strong market, so (i) or (iii) holds, total output rises. If both direct demand and inverse demand are more (or equally) convex in the strong market than in the weak market, so (ii) or (iv) applies, total output does not increase. Thus Robinson (1933) was almost right when she stated that for third-degree price discrimination to increase total output the demand in the weak market should be "in some sense" (p. 193) more convex than the demand in the strong market.

In cases of Proposition 4 where output does not rise with discrimination, welfare falls. The next question is whether we can find conditions, beyond those of Proposition 2, for welfare to increase. For example, when $\sigma_w = \sigma_s > 0$ and is constant Proposition 3 holds and the welfare effect is ambiguous, but output certainly rises. The critical value of σ in this case turns out to be 1. To show this we use another analytical approach.

IV. Constant Curvature of Inverse Demand

Further insight into the welfare effects of discrimination can be gained using two additional techniques that are useful when inverse demand curvature is constant. The first is a quantity-restriction technique that is analogous to the price-restriction method of Sections II and III. Let $\overline{q}_i \equiv q_i(\overline{p})$ be the quantity sold in market i when the uniform price is charged, and define $\Pi_i(q_i) = \pi_i(p_i(q_i))$ as profit as a function of quantity. To ensure concavity of the profit function assume that marginal revenue is declining in output in each market, which holds when $\sigma < 2$. Consider the problem of maximizing $\Sigma_i \Pi_i(q_i)$ subject to a limit $t \geq 0$ on how much quantities can vary relative to their nondiscriminatory levels:

$$-p_w'(\overline{q}_w)(q_w - \overline{q}_w) \leq -p_s'(\overline{q}_s)(q_s - \overline{q}_s) + t.$$

Constrained-optimal quantities then satisfy

$$\frac{\Pi_w'(q_w(t))}{p_w'(\overline{q}_w)} + \frac{\Pi_s'(q_s(t))}{p_s'(\overline{q}_s)} = 0. \tag{8}$$

When $t = 0$ the firm chooses the quantities that are sold at the uniform price $\overline{p}$. As t rises the firm increases the quantity in the weak market and cuts supply to the strong market so prices move toward their discriminatory levels. From (8) it follows that, as more quantity variation is allowed, $W'(t)$ has the sign of

$$\delta(t) \equiv \left(\frac{p_w - c}{2 - \sigma_w}\right) \frac{p_w'(\overline{q}_w)}{p_w'(q_w)} - \left(\frac{p_s - c}{2 - \sigma_s}\right) \frac{p_s'(\overline{q}_s)}{p_s'(q_s)}. \tag{9}$$

Note that $\delta(0)$ simply has the sign of $\sigma_w - \sigma_s$ evaluated at the quantities at price $\overline{p}$. To sign the marginal welfare effect for $t > 0$ we assume that the curvatures of inverse demand σ_i are constant in each market.

PROPOSITION 5: *With constant curvature of inverse demand:* (i) *If* $1 > \sigma_s \geq \sigma_w$ *and* $L(\overline{p})\eta_w(\overline{p})\sigma_w \leq 1$, *discrimination reduces welfare.* (ii) *If* $\sigma_w \geq \sigma_s > 1$ *and* $L(\overline{p})\eta_s(\overline{p})\sigma_s \geq 1$, *discrimination raises welfare.*

PROOF:
See Appendix, part C.

The condition in part (i) of the proposition that inverse demand in the strong market is at least as convex as in the weak market implies that $W'(0) \leq 0$. The other condition implies that $W'(t)$ is decreasing and so preserves the sign of $W'(t)$. If the discriminatory prices are not far apart, then $L\eta \approx 1$, in which case the second condition in (i) follows from the first. Likewise for part (ii) of the proposition.

Proposition 5 goes beyond Propositions 3 and 4 when both demands are strictly convex. For example, when $\sigma_w = \sigma_s > 0$, Proposition 3 applies and Proposition 4(iii) tells us that output increases. Proposition 5 shows that, if the discriminatory prices are not far apart, whether welfare falls or rises depends on whether σ is below or above 1. If $\sigma < 1$ the misallocation effect outweighs the positive output effect, while if $\sigma > 1$ the output effect is strong enough to exceed the misallocation effect.

A special case of constant inverse demand curvature is when both demand functions have constant elasticities. Proposition 3 applies, so welfare first rises and then falls as the price difference is increased, but none of the other results so far presented yields a welfare or output conclusion (because the weak market has higher curvature of direct demand but lower inverse demand curvature). For this case Richard A. Ippolito (1980) conducts an extensive numerical analysis, finding in his examples that output always rises with discrimination but welfare can fall or rise, depending on the difference in the elasticities and the relative sizes of the markets.[9] Aguirre (2006) proves that total output rises with discrimination using an inequality due to Bernoulli. Here we present a simpler proof of Aguirre's output result and go on to find a welfare result. The key is to define the "harmonic mean" price ρ_i by:

$$\frac{1}{\rho_i} = \frac{m}{p_i^*} + \frac{1-m}{\bar{p}}, \tag{10}$$

for $i \in \{w, s\}$ and for $0 \leq m \leq 1$. Thus ρ_i is a weighted harmonic mean of the discriminatory and nondiscriminatory prices, with weight m on the former. As before we look for conditions that ensure monotonicity, here as m goes from 0 (no discrimination) to 1 (full discrimination). Total output and welfare are respectively convex and (under the stated condition) concave in m, with zero derivatives at $m = 0$, allowing us to state:

PROPOSITION 6. *When demand functions have constant elasticities:* (i) *total output is higher with discrimination, and* (ii) *social welfare is lower with discrimination if the difference between the elasticities is at most* 1.

PROOF:

See Appendix, part D.

The condition in part (ii) of the proposition can be relaxed to the elasticity difference not exceeding the reciprocal of the share of the weak market in total output at the nondiscriminatory price. (However, the condition can fail when elasticities differ substantially—as Ippolito's (1980) examples illustrate—and welfare can increase with discrimination.) The welfare result is somewhat puzzling: it is natural to think that when the elasticities are close the discriminatory prices will be close and thus the misallocation effect will be small. But so too is the output effect, and it is far from obvious intuitively which is smaller. Proposition 6 provides a conditional answer.

V. Conclusion

This paper provides direct analysis of the classic problem of whether monopolistic third-degree price discrimination raises or reduces social welfare when all markets are served. The conditions that determine the welfare effects depend on simple curvature properties of demand functions. The main results are that welfare is higher with discrimination when inverse demand in the low-price market is more convex than that in the other market and the price difference with discrimination is small, and discrimination reduces welfare when the direct demand function is more convex in the high-price market. We also present new analysis of how discrimination affects total output and provide a synthesis of existing results on welfare and output. It is well known that the effect of price discrimination can be positive or negative for welfare, and the paper provides conditions under which the sign of the welfare effect can be predicted. In many cases discrimination reduces welfare, but our analysis has shown that the conditions for discrimination to raise welfare are not implausible.

[9] See also Melvin L. Greenhut and Hiroshi Ohta (1976) and John P. Formby, Stephen K. Layson, and James W. Smith (1983).

Appendix

A. *Concavity of the Profit Function*

Concavity of the demand function implies that the profit function is concave in the price. When demand is strictly convex, however, restrictions on the amount of convexity are required for the profit function to be strictly concave, i.e., for $2 - L\alpha > 0$. One sufficient condition is that $\alpha < 2$ everywhere since $L \leq 1$. An alternative sufficient condition works when inverse demand has constant positive curvature, which includes the special cases of constant-elasticity and exponential demand functions. In this case if all prices are below $2p^* - c$, where p^* is the profit-maximizing price, the profit function is strictly concave. To show this use the generic inverse demand function with constant σ, $p = a - bq^{1-\sigma}/(1-\sigma)$ for $\sigma \neq 1$ or $p = a - b\ln(q)$ for $\sigma = 1$. It follows that $p^* = [(1-\sigma)a + c]/(2-\sigma)$ when $\sigma \neq 1$ and $p^* = b + c$ when $\sigma = 1$. The second derivative of the profit function is negative if and only if $2p^* - c > p$. This holds automatically in the strong market, but for the weak market it must be assumed. Effectively this is an assumption that the uniform price is not too far above the discriminatory price in the weak market.

B. *Sufficient Conditions for the Increasing Ratio Condition*

We may write $z(p)$ as:

$$z(p) \equiv \frac{p-c}{2 + (p-c)\dfrac{q''}{q'}} = \frac{p-c}{2 - L\alpha} = \frac{p-c}{2 - L\eta\sigma}.$$

Using the first version $z(p)$ is seen to be increasing if the absolute curvature of direct demand, $\alpha/p \equiv -q''/q'$, is non-decreasing in p. This is equivalent to the slope of demand, $-q'(p)$, being log-concave, and existing results on log-concave density functions in probability theory can be used, since the slope of the demand function can be thought of as a density function (see Andrew Caplin and Barry Nalebuff 1991; Mark Bagnoli and Ted Bergstrom 2005; and Cowan 2007). Many demand functions have log-concave slopes including linear demand, probit demand (from a normal distribution), demands derived from the extreme value and logistic distributions, demands with constant negative α, and functions with constant σ in $[0,1]$, including exponential demand. From the second version $z(p)$ is increasing if $(d/dp)(L\alpha) > 0$. In turn there are two sets of sufficient conditions. First, if α is non-decreasing in p and is positive then $(d/dp)(L\alpha) > 0$. This is useful when the slope of demand is not log-concave. Examples are: α that is constant and positive, a special case of which is the constant-elasticity demand curve with $\alpha = 1 + \eta$ (where $\eta > 1$ is the elasticity); demands derived from the lognormal, F, Weibull and Gamma distributions (when their slopes are log-convex), demand that comes from the t distribution with two or more degrees of freedom (see Cowan 2007) and single-commodity AIDS demand (where $q = (a + b\log(p))/p$ for $b < 0$). Second, use the fact that $L\alpha = L\eta\sigma$, which gives the third version of $z(p)$. If σ is positive and non-decreasing, and $(d/dp)(L\eta) > 0$, then $z(p)$ is increasing. Differentiating $L\eta = -(p-c)q'/q$ gives $(d/dp)(L\eta) \overset{sgn}{=} 1 + (1/L\eta) - \sigma$ which is positive if (but not only if) $\sigma \leq 1$. For constant $\sigma > 1$, with inverse demand $p = a - bq^{1-\sigma}/(1-\sigma)$, $(d/dp)(L\eta) > 0$ if and only if $c > a$, which is necessary for demand to be finite and downward-sloping. The derivative of $z(p)$ evaluated at c is 0.5 for all demand functions, so $z(p)$ cannot be everywhere decreasing. When σ is constant and negative $z(p)$ is increasing when prices are close to marginal cost but is decreasing at high enough prices.

C. *Proof of Proposition 5*

When the σ_i do not vary with outputs, we have from (9) that

$$\delta'(t) = \frac{1 - L_w \eta_w \sigma_w}{2 - \sigma_w} p_w'(\overline{q}_w) q_w'(t) - \frac{1 - L_s \eta_s \sigma_s}{2 - \sigma_s} p_s'(\overline{q}_s) q_s'(t).$$

If, over the relevant range of prices, $(1 - L\eta\sigma)$ has the same sign in both markets, then $\delta'(t)$ has the opposite sign to that because $q_w'(t) > 0 > q_s'(t)$. So if $\sigma_w \leq \sigma_s$ and $L\eta\sigma \leq 1$ in both markets, then $\delta(0) \leq 0$ and $\delta' < 0$, so $\delta(t) < 0$ for all $t > 0$. Given that $L\eta$ is increasing in p, which is certainly the case if $\sigma \leq 1$, then $L\eta\sigma \leq 1$ in the weak market for all $p \in [p_w^*, \overline{p}]$ if $L(\overline{p})\eta_w(\overline{p})\sigma_w \leq 1$, and $L\eta\sigma \leq 1$ in the strong market too because $L\eta \leq 1$ for all $p \in [\overline{p}, p_s^*]$. It follows that $W'(t) < 0$ for all t, and discrimination is therefore bad for welfare. By the same argument the opposite is true if $\sigma_w \geq \sigma_s$ and $L\eta\sigma \geq 1$ in both markets.

D. *Proof of Proposition 6*

(i) Define the function $\rho_i(m)$ by

$$\frac{1}{\rho_i(m)} \equiv \frac{m}{p_i^*} + \frac{1 - m}{\overline{p}},$$

where the right-hand side is the weighted average of the inverse discriminatory price in market i and the inverse nondiscriminatory price. As m ranges from 0 to 1, $\rho_i(m)$ moves from $\overline{p}$ to p_i^*. Define

$$\tilde{Q}(m) \equiv \sum_{i \in \{w,s\}} q_i(\rho_i(m)) = \sum_{i \in \{w,s\}} A_i [\rho_i(m)]^{-\eta_i} = \sum_{i \in \{w,s\}} A_i \left(\frac{m}{p_i^*} + \frac{1-m}{\overline{p}} \right)^{\eta_i}$$

as total output at prices $\rho_i(m)$. Since $\eta > 1$, $\tilde{Q}(m)$ is the sum of strictly convex functions that are linear in m and is thus itself strictly convex. At $m = 0$:

$$\tilde{Q}'(0) = \overline{p} \sum q_i(\overline{p}) \eta_i \left(\frac{1}{p_i^*} - \frac{1}{\overline{p}} \right) = \frac{\overline{p}}{c} \sum q_i(\overline{p}) \eta_i \left(\frac{\overline{p} - c}{\overline{p}} - \frac{p_i^* - c}{p_i^*} \right)$$

$$= -\frac{\overline{p}}{c} \sum [q_i'(\overline{p})(\overline{p} - c) + q_i(\overline{p})] \text{ using } \frac{p_i^* - c}{p_i^*} = \frac{1}{\eta_i}$$

$$= 0 \text{ by the profit-maximizing first-order condition for } \overline{p}.$$

Because $\tilde{Q}'(0) = 0$ and $\tilde{Q}(m)$ is strictly convex it follows that $\tilde{Q}(1) > \tilde{Q}(0)$, so output increases with discrimination.

(ii) We now show that welfare falls if $\eta_w - \eta_s \leq 1$. Define welfare $\tilde{W}(m)$ as a function of m as above. Note first that $\tilde{W}'(0) = (\overline{p} - c)\tilde{Q}'(0) = 0$. More generally

$$\tilde{W}'(m) = \sum (\rho_i - c) q_i'(\rho_i) \rho_i'(m)$$

and

$$\tilde{W}''(m) = \sum\{(\rho_i - c)q_i'' + q_i'](\rho_i')^2 + (\rho_i - c)q_i'\rho_i''\}$$

$$= \sum - q_i'(\rho_i')^2[L_i(\rho_i)(1 + \eta_i) - 1 - 2L_i(\rho_i)] \text{ since } \rho\rho'' = 2(\rho')^2$$

$$= \sum - q_i'(\rho_i')^2[L_i(\rho_i)(\eta_i - 1) - 1$$

$$\text{(A1)} \qquad < 0 \text{ if } \eta_w - 1 \leq \frac{1}{L(\overline{p})} = s_w\eta_w + (1 - s_w)\eta_s,$$

where $s_w \equiv q_w(\overline{p})/Q(\overline{p})$ is the weak market's share of total output at the nondiscriminatory price. Condition (A1) ensures that $L_i(\rho_i)(\eta_i - 1) \leq 1$ for all ρ_i because $L_s \leq 1/\eta_s$ always and L_w is increasing in p_w. Condition (A1) is equivalent to

$$(\eta_w - \eta_s)(1 - s_w) \leq 1,$$

a condition certainly met if elasticities are not more than one apart. With $\tilde{W}'(0) = 0$ and $\tilde{W}'(m)$ decreasing in m, $\tilde{W}(0) > \tilde{W}(1)$, so welfare is reduced by discrimination when (A1) holds.

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