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LOCAL STRUCTURE WHEN ALL MAXIMAL INDEPENDENT SETS HAVE EQUAL WEIGHT*

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Abstract. In many combinatorial situations there is a notion of independence of a set of points. Maximal independent sets can be easily constructed by a greedy algorithm, and it is of interest to determine, for example, if they all have the same size or the same parity. Both of these questions may be formulated by weighting the points with elements of an abelian group, and asking whether all maximal independent sets have equal weight. If a set is independent precisely when its elements are pairwise independent, a graph can be used as a model. The question then becomes whether a graph, with its vertices weighted by elements of an abelian group, is *well-covered*, i.e., has all maximal independent sets of vertices with equal weight. This problem is known to be co-NP-complete in general. We show that whether a graph is well-covered or not depends on its local structure. Based on this, we develop an algorithm to recognize well-covered graphs. For graphs with n vertices and maximum degree Δ, it runs in linear time if Δ is bounded by a constant, and in polynomial time if $\Delta = O(\sqrt[3]{\log n})$. We mention various applications to areas including hypergraph matchings and radius k independent sets. We extend our results to the problem of determining whether a graph has a weighting which makes it well-covered.

Key words. well-covered graph, maximal independent set, local structure, polynomial time algorithm, recognition algorithm, hypergraph matching, independence system

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1. Introduction.

In many situations in mathematics there is a notion of "independence" for subsets of a set. We can "greedily" construct a "large" independent subset by repeatedly adding elements until there are no further feasible elements to add. We often want to know if some property of our final independent set is the same, regardless of how we choose the elements to add. For example, we may wish to determine whether we always get an independent set of maximum size. Or, in a game where players take turns adding elements independent of the previously chosen ones until no further moves can be made, we may ask whether one player inevitably wins. In this paper we show that many natural problems of this kind can be answered by examining their "local structure" and that problems whose local structures satisfy certain size bounds can be solved by polynomial time algorithms. Our results have applications to well-covered graphs, hypergraphs in which maximum or perfect matchings can be found greedily, graphs where certain vertex packings can be found greedily, graph games whose outcome depends on the parity of a maximal independent set, and other related problems.

Our fundamental problem may be formulated in three equivalent ways. In what follows, A is an arbitrary abelian group represented additively, which we use to assign weights to elements of a structure. Our weighted problems all have unweighted

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counterparts, which may be considered as the case where $A = \mathbf{Z}$ and all weights are 1; the weight of a set then becomes its cardinality.

Our first formulation is in terms of graphs. An *A-weighted graph G* is a graph whose vertices are weighted by elements of A. Given $S \subseteq V(G)$, the *weight of S* is the sum of the weights of its elements. If all maximal independents sets in G have the same weight, we call it the *independence weight* $\mathrm{iw}(G)$ of G and say that G is *well-covered*.

Graph problem. *Given an A-weighted graph, is it well-covered?*

As multiple edges do not affect the independent sets of vertices and vertices with loops can never appear in an independent set, we assume that our graphs have no loops or multiple edges.

Well-covered (unweighted) graphs were introduced by Plummer [20]. One of the definitions is that a graph is well-covered exactly when all maximal independent sets have the same cardinality, which is just the unweighted version of our more general definition. Plummer [21] recently surveyed known results on well-covered graphs (note that [21, Theorem 5.7] does not correctly summarize the results of [2]). Another special case of the Graph Problem is considered by Finbow and Hartnell [9, 10], who examine the problem of recognizing graphs in which all maximal independent sets have the same parity, i.e., $\mathbf{Z}_2$-weighted graphs with all vertices of weight 1 that are well-covered.

Our second formulation involves independence systems. An *A-weighted independence system* consists of a (finite) set of *points*, a nonempty collection of sets of points known as *independent sets*, which is closed under taking subsets and a function assigning each point a weight in A. The weight of a set of points is the sum of the weights of its elements. A maximal independent set is called a *base*, and a minimal dependent set is called a *circuit*.

Independence system problem. *Given an A-weighted independence system with all circuits of cardinality 2, do all bases have the same weight?*

The Independence System Problem is equivalent to the Graph Problem, via the following one-to-one correspondence between independence systems with all circuits of cardinality 2 and graphs. Given a graph, its independent sets of vertices form an independence system, and a circuit is a set containing both ends of any one edge, which has cardinality 2. Conversely, given an independence system with circuits of cardinality 2, we may construct a graph where points become vertices and each circuit becomes the set of ends of an edge. We may also allow circuits of cardinality 1 without essentially changing the problem, as a vertex in a circuit of cardinality 1 can never appear in a base.

Unweighted independence systems with all bases of the same cardinality are the "greedy hereditary systems" of Caro, Sebő, and Tarsi [5], which properly include matroids.

At first glance, considering only independence systems with circuits of cardinality exactly 2 seems very restrictive. However, all circuits are of cardinality 2 when independence of a set of points means pairwise independence of its elements, a very common situation. In particular, the points may be sets, and independence may mean pairwise disjointness. This motivates our third formulation below.

The Independence System Problem expresses our problem in a more abstract way, which allows us easily to recognize many problems as fitting into our framework. For example, Gunther, Hartnell, and Whitehead [15, 16] have considered *radius 2 independent sets*, also known as *2-packings*, in graphs, i.e., sets of vertices which are pairwise at distance greater than 2. They examine the problem of determining when all maximal radius 2 independent sets have the same cardinality or when they all have the same parity. More generally, Hartnell and Whitehead [18] have examined the cardinality problem for radius k independent sets or k-packings. Clearly these sets form an independence system in which independence means pairwise independence, so our results can provide some information about them. The problems discussed here correspond to $\mathbf{Z}$- and $\mathbf{Z}_2$-weighted independence systems with all weights 1. (This example may also be simply formulated as an instance of the Graph Problem, since a radius k independent set is just an independent set in G^k, the kth power of G.)

Our third formulation involves hypergraphs. An *A-edge-weighted hypergraph* consists of a (finite) set of *vertices*, a collection (with repetitions allowed) of (possibly empty) sets of vertices known as *edges*, and a function giving each edge a weight from A. The weight of a set of edges is the sum of the weights of its elements. A *matching* is a set of mutually disjoint edges.

HYPERGRAPH MATCHING PROBLEM. *Given an A-edge-weighted hypergraph, do all maximal matchings have the same weight?*

The Hypergraph Matching Problem is also equivalent to the Graph Problem, via the following correspondence. Given a graph G, construct a hypergraph H whose vertex set is the edge set of G. For every vertex v of G, the set of edges incident with v gives an edge e_v of H; the weight of e_v is just the weight of v. Conversely, given a hypergraph H, construct a graph G whose vertex set is the edge set of H with two vertices of G adjacent if the corresponding edges of H intersect; the weight of a vertex in G is just the weight of the corresponding edge in H.

Many aspects of the unweighted version of the Hypergraph Matching Problem are discussed in [5, Section 2], including applications to two graph decomposition problems studied by Ruiz [24] and Caro, Ruiz, and Rojas [4]. As an example of the use of weights in the Hypergraph Matching Problem, suppose we assign each edge a weight from $\mathbf{Z}$ equal to its cardinality. Then our question is equivalent to asking whether all maximal matchings use the same number of vertices. In particular, we may ask if they are all *perfect*, i.e., use all the vertices.

Since all three problems above are equivalent, we may work with whichever is most convenient for a particular application. For the purposes of this paper it is easiest to work with the Graph Problem. The following is known about the computational complexity of this problem. In the unweighted case, recognition of well-covered graphs is a co-NP-complete problem, as shown by Chvátal and Slater [8] and, independently, Sankaranarayana and Stewart [25]. It is co-NP-complete even when a graph has no induced $K_{1,4}$'s [5, Theorem 2.1], although polynomial algorithms have been found for line graphs [19] and claw-free graphs—graphs with no induced $K_{1,3}$ [26, 27]. Moreover, structural characterizations of certain classes of well-covered graphs easily yield recognition algorithms, such as for cubic graphs [2] and for graphs of girth at least 5 [11] (the *girth* is the length of a shortest cycle in the graph). Other structural work on well-covered graphs includes work by Ravindra [23] on bipartite graphs, Ramey on graphs of maximum degree at most 3 [22], and by various authors on graphs with no 4-cycles [12, 14, 17].

If a fixed abelian group A can be represented by finite strings that can be added in polynomial time, then the graph problem, i.e., the recognition problem for well-covered A-weighted graphs, is co-NP-complete, even for $K_{1,4}$-free graphs: the unweighted proof of [5, Theorem 2.1] is easily modified by giving each vertex the same nonzero weight. In particular, the problem of recognizing graphs in which all maximal independent sets have the same parity, as studied by Finbow and Hartnell [9, 10] is co-NP-complete for $K_{1,4}$-free graphs.

We will examine minimal non-well-covered graphs, which arise in characterizing well-covered graphs, and use them to give an algorithm for recognizing well-covered graphs. For n-vertex graphs this runs in polynomial time if the maximum degree is bounded by $O(\sqrt[3]{\log n})$ and in linear time if the maximum degree is bounded by a constant. This answers a question posed in [2], as to whether well-covered graphs of bounded degree can be recognized in polynomial time.

2. Minimal non-well-covered graphs. In this section we discuss ways to characterize well-covered A-weighted graphs, leading to the notion of a minimal non-well-covered graph. When no confusion can result, we just refer to an A-weighted graph as a *graph*. Unless explicitly stated otherwise, vertices in subgraphs of a graph G inherit their weights from G, and when we combine graphs $G_1, G_2, \ldots, G_k$ to obtain G, in such a way that $V(G)$ is the disjoint union of $V(G_1), V(G_2), \ldots, V(G_k)$, then each vertex of G inherits its weight from the appropriate G_i.

If S is a set of vertices in a graph G, the *closed neighborhood* of S is the set $N_G[S]$, or just $N[S]$, containing S and all neighbors of vertices in S. We abbreviate $N[\{v\}]$ to $N[v]$. If S is independent, the graph $G \backslash N[S]$ is said to be obtained from G by *neighborhood deletion*. The following important observation seems to have been first stated by Campbell [1] for unweighted graphs. A similar result in the special context of randomly decomposable graphs was observed by Caro, Rojas, and Ruiz [4].

OBSERVATION 2.1. *Suppose $S \subseteq V(G)$ is independent. If G is well-covered, then $G \backslash N[S]$ is well-covered. Equivalently, if any component of $G \backslash N[S]$ is not well-covered, then G is not well-covered.*

This observation is very useful in characterizing classes of well-covered graphs. If we can show that a certain structure in a graph G contains an independent set S of G such that $G \backslash N[S]$ has a component which is a non-well-covered graph L, then that structure cannot occur in a well-covered graph. We are, therefore, interested in generating (isomorphism classes of) non-well-covered graphs L, which can be used to restrict the possible structure of well-covered graphs.

Some non-well-covered graphs L are not needed to characterize well-covered graphs, because any structures they eliminate can also be eliminated by smaller non-well-covered graphs. In particular, suppose that L contains a nonempty independent set T for which $M = L \backslash N_L[T]$ is non-well-covered. Then, whenever S is an independent set in G such that $G \backslash N[S]$ contains L as a component, $S \cup T$ is also an independent set in G such that $G \backslash (S \cup T)$ contains M as a component. In other words, M can eliminate any structures that L can. The essential non-well-covered graphs L, those that cannot be replaced in this way, are those for which $L \backslash N_L[T]$ is well-covered for all nonempty independent T in L. Thus, they are the non-well-covered graphs which are minimal with respect to the neighborhood deletion operation; we call them simply *minimal non-well-covered graphs*. In the unweighted case, translated into the hypergraph matching form of our problem, they correspond exactly to the *critical nongreedy hypergraphs* investigated by Caro, Sebő, and Tarsi [5, Section 2.7], and they generalize an idea developed by Caro, Rojas, and Ruiz [4] in the context of randomly decomposable graphs. We may summarize the usefulness of minimal non-well-covered graphs as follows.

OBSERVATION 2.2. *G is non-well-covered if and only if there exists some* (*possibly empty*) *independent set S in G such that $G \backslash N[S]$ has a component which is minimal non-well-covered.*

We now characterize minimal non-well-covered graphs. Our characterization is a natural generalization of the characterizations for the unweighted case obtained independently by Ramey [22, Theorems 2.11, 2.12] and by Caro, Sebő, and Tarsi (using the hypergraph matching form of the problem) [5, Theorem 2.8]. Our proof adapts that of [5]. Note that $G_1 + G_2 + \cdots + G_k$ denotes the *join* of G_1, G_2, …, G_k, obtained from their disjoint union by adding an edge between every pair of vertices in different graphs.

THEOREM 2.3. *An A-weighted graph G is minimal non-well-covered if and only if there exist well-covered A-weighted graphs* $G_1, G_2, \ldots, G_k$ *such that* $G = G_1 + G_2 + \cdots + G_k$ *and* $\mathrm{iw}(G_i) \neq \mathrm{iw}(G_j)$ *for some i and j.*

Proof. Suppose G is minimal non-well-covered. For each $v \in V(G)$, the graph $G \backslash N[v]$ is well-covered, and therefore every maximal independent set of G containing v has the same weight, which we denote $t(v)$. Let $\{t(v) : v \in V(G)\}$ be $\{t_1, t_2, \ldots, t_k\}$; since G is not well-covered, $k \geq 2$. Let G_i be the subgraph of G induced by $\{v \in V(G) : t(v) = t_i\}$. If $v \in V(G_i)$ and $w \in V(G_j)$, then v and w must be adjacent, for otherwise there would be a maximal independent set including $\{v, w\}$, and we would have $t(v) = t(w)$. Thus, $G = G_1 + G_2 + \cdots + G_k$, and a set of vertices of G is independent if and only if it is an independent set in some G_i. Therefore, any maximal independent set in G_i is a maximal independent set in G and has weight t_i, so that G_i is well-covered with $\mathrm{iw}(G_i) = t_i$. Since $k \geq 2$, $\mathrm{iw}(G_i) \neq \mathrm{iw}(G_j)$ for some i and j.

Suppose now that $G = G_1 + G_2 + \cdots + G_k$, where $G_1, G_2, \ldots, G_k$ are all well-covered. Clearly, any set $S \subseteq V(G)$ is independent if and only if S is an independent set in G_i for some i. Therefore, the maximal independent sets of G are the maximal independent sets of the individual G_i's, and G is well-covered if and only if $\mathrm{iw}(G_i)$ is the same for all i. Thus, if $\mathrm{iw}(G_i) \neq \mathrm{iw}(G_j)$ for some i and j, then G is non-well-covered. Moreover, it is minimal with respect to neighborhood deletion, because for any nonempty independent S in G_i, we have $G \backslash N_G[S] = G_i \backslash N_{G_i}[S]$, which is well-covered by Observation 2.1. ∎

The following corollary will be very important in the next section.

COROLLARY 2.4. *A minimal non-well-covered A-weighted graph has diameter at most* 2.

Proof. Such a graph is a join, and any join has diameter at most 2. ∎

Some special cases of Theorem 2.3 are of interest. Suppose G is unweighted and minimal non-well-covered. If G is bipartite, or, in fact, if G has girth 4 or more, then G must be a complete bipartite graph $K_{m,n}$ with $1 \leq m < n$. If G has girth 5 or more, then G must be a star $K_{1,n}$ with $n \geq 2$. The fact that the minimal non-well-covered graphs in these situations can easily be described seems to be reflected in the fact that well-covered unweighted graphs that are bipartite or have girth 5 or more have relatively simple characterizations [11, 23]. It also suggests that the problem of characterizing well-covered graphs of girth 4, which has been considered difficult, may in fact be tractable. Note also that a consequence of the unweighted version of Theorem 2.3 has been used by Tankus and Tarsi [27] to find a simple proof of their earlier result [26] that well-covered claw-free graphs can be recognized in polynomial time.

Theorem 2.3 also has some implications for the characterization of well-covered graphs of bounded degree.

COROLLARY 2.5. *Let G be a minimal non-well-covered A-weighted graph. If G has maximum degree* Δ*, then G contains at most* 2Δ *vertices.*

Proof. By Theorem 2.3, we know that $V(G)$ can be partitioned into two sets (e.g., $V(G_1)$ and $V(G_2) \cup V(G_3) \cup \cdots \cup V(G_k)$) such that every possible edge between the sets is in $E(G)$. Since the maximum degree is Δ, each set cannot contain more than Δ vertices; otherwise, the vertices in the other set would have degree greater than Δ. ∎

In the unweighted case, this corollary is close to the best possible, as shown by the complete bipartite graph $K_{\Delta-1,\Delta}$, which is minimal non-well-covered on $2\Delta - 1$ vertices with maximum degree Δ. It is best possible in situations where $K_{\Delta,\Delta}$ with bipartition (V_1, V_2) can be assigned weights so that V_1 and V_2 have different weights.

Corollary 2.5 implies immediately that in the unweighted case, or if A is finite, there are finitely many minimal non-well-covered graphs of maximum degree at most Δ. All may be constructed as joins of well-covered graphs, not all with the same independence weight, on at most Δ vertices. We summarize some results for the unweighted case. There is only one minimal non-well-covered graph with maximum degree 2, namely, $P_3 = K_{1,2} = K_1 + 2K_1$. The four minimal graphs with maximum degree 3 are $K_{1,3} = K_1 + 3K_1$, $K_1 + (K_2 \cup K_1)$, $K_{1,1,2} = K_1 + K_1 + 2K_1$, and $K_{2,3} = 2K_1 + 3K_1$. There are 14 minimal graphs with maximum degree 4 and 43 minimal graphs with maximum degree 5. The minimal graphs with maximum degree at most 3 were used, without realizing their nature, in [2] in characterizing well-covered cubic graphs, and were consciously used by Ramey [22] to characterize the well-covered graphs with maximum degree at most 3.

3. Testing well-coveredness. In this section, we use Corollaries 2.4 and 2.5 to prove that well-coveredness depends on the local, rather than the global, structure of a graph. We then show how this can be used to test whether a graph is well-covered, resulting in polynomial time algorithms under certain circumstances. Let $N_k[v]$ denote the set of vertices at distance at most k from a vertex v in the graph G.

THEOREM 3.1. *Consider an A-weighted graph G with maximum degree Δ. The following are equivalent.*

(i) *G is non-well-covered.*

(ii) *There exist $v \in V(G)$ and an independent set S in the subgraph $Q = Q(v)$ of G induced by $N_4[v]$, such that $Q \backslash N_Q[S]$ has a minimal non-well-covered component containing v.*

(iii) *There exist $v \in V(G)$ and an independent set S in the subgraph $Q = Q(v)$ of G induced by $N_4[v]$, such that $Q \backslash N_Q[S]$ has a non-well-covered component with at most 2Δ vertices and of diameter at most 2 containing v.*

Proof. (i) ⇒ (ii): If G is non-well-covered, by Observation 2.2 we can choose an independent set S, minimal with respect to inclusion, for which $G \backslash N[S]$ has a minimal non-well-covered component L. A vertex of S cannot be at distance 0 or 1 from L, and if it is at distance 3 or more from L then we may delete it from S without changing the fact that L is a component of $G \backslash N[S]$. Thus, by minimality of S, every vertex of S is at distance 2 from L. Let v be an arbitrary vertex of L. Since L has diameter at most 2 by Corollary 2.4, every vertex of S is in $N_4[v]$ and L is a component of $Q \backslash N_Q[S]$, where Q is induced by $N_4[v]$.

(ii) ⇒ (iii): By Corollaries 2.4 and 2.5, a minimal non-well-covered component has the properties specified in (iii).

(iii) ⇒ (i): If v, S, and Q exist as stated in (iii), let L be the non-well-covered component of $Q \backslash N_Q[S]$ containing v. Since L has diameter at most 2, no vertex of L is adjacent in G to a vertex outside Q. Therefore, L is also a component of $G \backslash N_G[S]$ and G is non-well-covered. ∎

For creating theoretical characterizations of classes of well-covered graphs, condition (ii) is very useful. Condition (iii) is easier to check by computer than (ii), so it is useful in constructing algorithms.

THEOREM 3.2. *Let G be an A-weighted graph with n vertices and maximum degree Δ, represented by a list of neighbors for each vertex. Then we may determine whether or not G is well-covered in $O(ne^{2\Delta}\Delta^{2\Delta+5/2}2^{2\Delta^3-4\Delta^2})$ operations (or, more roughly, in $O(n2^{2\Delta^3})$ operations). (Each addition or comparison in A is counted as one operation.)*

Proof. We check condition (iii) of Theorem 3.1 by brute force. Given a vertex v, we try to find a non-well-covered component L, containing v, of some $Q\backslash N_Q[S]$, by first constructing all possible graphs L and then trying to find S.

Fix v. There are at most $\Delta + \Delta(\Delta-1) = \Delta^2$ vertices at distance 1 or 2 from v, which we locate in $O(\Delta^2)$ operations. The number of sets of vertices of cardinality at most 2Δ that include v and otherwise contain only vertices at distance 1 or 2 from v is at most

$$\binom{\Delta^2}{0} + \binom{\Delta^2}{1} + \cdots + \binom{\Delta^2}{2\Delta-1} = O(e^{2\Delta}2^{-2\Delta}\Delta^{2\Delta-3/2})$$

(using Stirling's formula and other standard estimations). The subgraph L induced by such a set has at most 2Δ vertices and maximum degree at most Δ. Thus, L may be generated in $O(\Delta^2)$ operations (including the generation of its vertex set), and checked for diameter 2 in $O(\Delta^3)$ operations. Each of the $O(2^{2\Delta})$ subsets of $V(L)$ may be generated, be checked to see if it is a maximal independent set in L, and have its weight calculated in $O(\Delta^2)$ operations, so we require $O(\Delta^2 2^{2\Delta})$ operations to determine whether L is non-well-covered.

Now, given a non-well-covered diameter 2 graph L, we must see if S exists with $Q\backslash N_Q[S]$ having L as a component. As in the proof of Theorem 3.1, we may assume that every vertex of S is at distance 2 from L. Given a set S of vertices at distance 2 from L, we need check only that S is independent and that every vertex at distance 1 from L is covered by, i.e., adjacent to, a vertex of S. Since L has at most 2Δ vertices, there are at most $2\Delta(\Delta-1)$ vertices at distance 1 from L, and at most $2\Delta(\Delta-1)^2$ vertices at distance 2 from L. Therefore, there are $O(2^{2\Delta(\Delta-1)^2})$ potential sets S, each of which can be generated in $O(\Delta^3)$ operations and checked for independence and covering in $O(\Delta^4)$ operations.

Combining the above estimates, we come up with a bound on the number of operations in which the dominant term comes from the number of vertices v, the number of sets giving a possible L, the number of possible sets S, and the time to check each S for independence and covering. Thus the total number of operations required is

$$O(n \cdot e^{2\Delta}2^{-2\Delta}\Delta^{2\Delta-3/2} \cdot 2^{2\Delta(\Delta-1)^2} \cdot \Delta^4) = O(ne^{2\Delta}\Delta^{2\Delta+5/2}2^{2\Delta^3-4\Delta^2}),$$

which is clearly $O(n2^{2\Delta^3})$. □

In some cases, where the maximum degree is large but there are only a few vertices of maximum degree and they are widely separated, it may be more useful to observe that the above method also uses at most $O(nq^2 2^{2q})$ operations, where q is the maximum of $|N_4[v]|$ for $v \in V(G)$. For example, if most vertices have degree 3 or less, with any two vertices of degree 4 or more always being separated by distance 5 or more, then $q \leq 15\Delta + 1$, and so we obtain a bound on the number of operations of $O(n\Delta^2 2^{30\Delta})$, which is better than a bound involving $2^{2\Delta^3}$.

Theorem 3.1, in fact, implies that in a non-well-covered graph G, there is some vertex for which the subgraph $Q = Q(v)$ induced by $N_4[v]$ is non-well-covered. If the converse of this statement were true, it would give a very simple local characterization of well-covered graphs. Unfortunately, the converse is false, as shown by the graph G obtained from a 9-cycle by joining one pendant vertex to each original vertex: this graph is well-covered, but every $Q(v)$ is non-well-covered.

We can now answer the question posed in [2], of whether well-covered graphs of bounded degree can be recognized in polynomial time.

COROLLARY 3.3. *Suppose that addition and comparison in A can be done in polynomial time. For graphs, let n denote the number of vertices and Δ the maximum degree.*

(i) *Suppose $\mathcal{F}$ is a family of A-weighted graphs such that $\Delta = O(\sqrt[3]{\log n})$. Then we may determine whether a graph in $\mathcal{F}$ is well-covered in polynomial time.*

(ii) *Suppose $\mathcal{F}$ is a family of A-weighted graphs such that $\Delta = O(1)$, i.e., Δ is bounded by a constant. Then we may determine whether a graph in $\mathcal{F}$ is well-covered in linear time* ($O(n)$ *operations*).

We now mention some applications of the above, to some of the specific problems mentioned earlier and to some classes of graphs derived from (unweighted) well-covered graphs. We expect that our results will also apply to many other interesting situations involving "greedy" or "random" processes.

COROLLARY 3.4. *The following questions may be settled in polynomial time:*

(a) *Given a family of graphs with $\Delta = O(\sqrt[3]{\log n})$, do all maximal independent sets have the same parity?*

(b) *Given a family of graphs with $\Delta = O(\sqrt[3k]{\log n})$, do all radius k independence sets* (k*-packings*) *have the same cardinality? Or do they all have the same parity?*

(c) *Given a family of hypergraphs where the number of edges is m and each edge intersects at most $O(\sqrt[3]{\log m})$ other edges, do all matchings have the same cardinality? Or are all matchings perfect?*

(d) *Given a family of graphs with $\Delta = O(\sqrt[3]{\log n})$, is a graph in the class W_2* (*also known as* 1*-well-covered*) *or is it strongly well-covered?* (*See* [21] *for definitions and work on these concepts. One of us* (*Caro*) *has shown that the first of these two problems is co-NP-complete for general graphs or even $K_{1,4}$-free graphs* [3].)

Is it possible to significantly improve Theorem 3.2? For example, is there $\epsilon > 0$ such that determining whether a graph (weighted or unweighted) is well-covered can be done in polynomial time when $\Delta = O(n^\epsilon)$? For any $\epsilon > 0$, this problem is co-NP-complete, because for any graph G with n vertices, determining whether G is well-covered is equivalent to determining whether the union of $n^{1/\epsilon - 1}$ disjoint copies of G is well-covered and for this union, $\Delta = O(n^\epsilon)$.

Theorem 3.2 and the previous paragraph refer to graphs which are sparse. Some observations may also be made for dense graphs. Let $\overline{\Delta}$ denote the maximum degree of the complement of a graph, equal to $n - 1 - \delta$, where δ is the minimum degree. If $\overline{\Delta} = O(\log n)$, then any vertex has at most $O(\log n)$ nonneighbors, and all independent sets may be constructed and tested for maximality in polynomial time, so determining whether a graph (weighted or unweighted) is well-covered may be done in polynomial time. However, for any $\epsilon > 0$, the problem of determining whether graphs with $\overline{\Delta} = O(n^\epsilon)$ are well-covered is co-NP-complete. For any G, determining whether G is well-covered is equivalent to determining whether the join of $n^{1/\epsilon - 1}$ disjoint copies of G is well-covered and for this join, $\overline{\Delta} = O(n^\epsilon)$.

Algorithmically, it may be possible to take advantage of the ease of finding a maximum independent set in an unweighted well-covered graph, without explicitly being able to recognize well-covered graphs. Jerry Spinrad (personal communication) has posed the following problem: Is there a polynomial time algorithm which, given any graph, either finds a maximum independent set for the graph or reports that the graph is not well-covered? A weighted version of this makes sense only in the context of nonnegative real weights.

Finally, it is interesting to contrast the behavior of two closely-related problems: determining whether a graph is well-covered and finding a maximum independent set. The recognition problem for well-covered graphs is co-NP-complete [8, 25], but for any constant Δ, we can recognize well-covered graphs of maximum degree at most Δ in linear time. The maximum independent set problem, however, remains NP-complete even for cubic planar graphs (see [13, p. 194] for references).

4. Well-covered weightings. In this section we consider the following concept. Given an abelian group A and an unweighted graph G, a function $x : V(G) \to A$ is called a *well-covered weighting* of G if it makes G into a well-covered A-weighted graph. The zero function is always a well-covered weighting, but does a graph have any nonzero (i.e., nonzero for at least one vertex) well-covered weighting? Sometimes we may wish to add stronger restrictions, such as that the weighting must be nonzero for *all* vertices, or that it must take nonnegative or positive values if A is an additive subgroup of $\mathbf{R}$.

Not every graph has a nonzero well-covered weighting. To give two arbitrary examples, the Petersen graph and every cycle of length 8 or more have no nonzero well-covered weighting over any abelian group. It is not difficult to prove this by using Observation 2.1 to derive properties of a well-covered weighting and then deduce that it must be zero.

Let $\mathcal{B}(G)$ denote the set of maximal independent sets, i.e., bases, of G. If x is a well-covered weighting, then $x(B)$ must be the same for all $B \in \mathcal{B}(G)$, i.e., if B_0 is a fixed element of $\mathcal{B}(G)$, then

$$x(B) - x(B_0) = 0 \quad \forall B \in \mathcal{B}(G) \backslash \{B_0\}.$$

We call this system of equations in the variables $x(v)$, $v \in V(G)$, a *global well-covering system* for G. When A is a commutative ring with identity, as well as an additive abelian group, it is a linear system over A. It is a finite system, but, in general, it will have exponentially many equations, and so we will not be able to determine in polynomial time if there is a nonzero well-covered weighting of G.

However, we can replace a global system by another system using Theorem 3.1. In the following discussion, we use the notation from that theorem. For each $v \in V(G)$, let $\mathcal{L}(v)$ denote the set of all subgraphs L which (i) are obtained as a component of $Q(v) \backslash N_{Q(v)}[S]$ for some independent S in $Q(v)$, (ii) contain v, (iii) contain at most $2\Delta(G)$ vertices, and (iv) have diameter at most 2. By Theorem 3.1, G is well-covered if and only if each element of $\mathcal{L}(G) = \cup_{v \in V(G)} \mathcal{L}(v)$ is well-covered. Therefore, a weight function makes G into a well-covered graph if and only if it makes each element of $\mathcal{L}(G)$ well-covered. Thus, the well-covering system for G has the same solution set as the union of a global well-covering system for each element of $\mathcal{L}(G)$: we will call this union a *local well-covering system* for G. Constructing a local well-covering system requires only a minor modification in the algorithm of Theorem 3.2. Instead of a step to compute and compare its weight for a maximal independent set in a graph $L \in \mathcal{L}(G)$, we have a step to set up its equation. Therefore, a local well-covering system can be constructed in $O(n2^{2\Delta^3})$ operations. In particular, if $\Delta = O(\sqrt[3]{\log n})$, we obtain a polynomial time algorithm to set up a local well-covering system. By Gaussian elimination, we can then find a basis for the solutions of that system in polynomial time, and so we have a polynomial time algorithm to determine a basis for the space of well-covered weightings of G.

As a minor modification of the above, a local well-covering system may be used to compute the rank over a given field of the maximal independent set incidence matrix X of a graph (rows are indexed by maximal independent sets S, columns by vertices v, with an entry being 1 if $v \in S$ and 0 otherwise). Since a local well-covering system is equivalent to any global well-covering system, the rows of the matrix of a local well-covering system together with the incidence vector of any one maximal independent set span precisely the rowspace of X. If $\Delta = O(\sqrt[3]{\log n})$ this gives a polynomial time algorithm for finding the rank of X. In the case where the graph is (unweighted) well-covered, every row of X has the same number of 1's and so X has some special structure; however, the algorithm is valid regardless of whether the graph is well-covered, which is a little surprising.

A case which appears to have special interest is the case of positive real, rational, or integral weightings. Working with real or rational numbers is essentially the same, as in either case we have a basis for the solution set of a (global or local) well-covering system using only rational numbers, because all coefficients in the system are integral (in fact, 0 or ± 1). And working with rational numbers or integers is essentially the same, because to obtain an integral solution we merely multiply by a constant to clear the denominator in a rational solution. Therefore, we assume that we are working with rational numbers. Not all graphs have a positive well-covered weighting. The simplest example known to us is P_5; the central vertex must have weight 0 in any well-covered weighting over any A. We would guess that determining whether a positive well-covered weighting exists is, in general, a difficult problem.

If we have a positive well-covered weighting for a graph, then by multiplying by a suitable positive constant, we know that there is a positive well-covered weighting for which all weights are at least 1. Therefore, we can formulate the question as to whether a graph has a positive well-covered weighting as follows: Is there a weighting which satisfies a (global or local) well-covering system, and for which all weights are at least 1? This feasibility problem is solvable via linear programming, and thus there is a polynomial algorithm to solve it when a well-covering system can be found in polynomial time, e.g., when $\Delta = O(\sqrt[3]{\log n})$. The existence problem for nonnegative nonzero well-covered weightings can be solved in a similar way.

As a final remark, note that the idea of the space of well-covered weightings (vertex weightings with a uniform sum on the maximal independent sets of a graph) has been extended by Caro and Yuster [6, 7] to the idea of a *uniformity space* (the vertex weightings with a uniform sum on the edges of an arbitrary hypergraph).

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