A PROCESS ALGEBRAIC APPROACH TO TIME GRANULARITY SEMANTICS

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Abstract

In this paper we develop a theory for timed systems which relate to time granularity. We use the well known equivalence of bisimulation to study the effect of changing granularity. We identify situations where measuring time more accurately has no effect on the equivalence. Similarly we also present a few situations where measuring time less accurately has no effect on the equivalence. We also present properties of the situations where the semantics is indeed altered by a change in time granularity.

Keywords process algebra, real-time, time granularity

1 Introduction

It is quite well known in the real-time community that real-time computing is not synonymous with fast computing. What is of greater interest is that of predictability. That is, reproducibility of behaviour under limited variance of operating conditions is necessary. One such variance is the time resolution of the underlying clock available on different machines. One might expect that if all timing issues of a system are of the order of days, it should not matter if the underlying clock resolution is one millisecond or one microsecond. However, if the timing issues are of the order of microseconds the difference in the underlying clock resolution will be of importance. In this paper we present a formal study of the issues related to time resolution.

Studying timed behaviour in the context of process algebras [10, 4, 6] and timed automata [2, 1] is popular. It has been argued in the literature that a dense time domain results in a more flexible specification than one based on discrete time domain. Based on this various semantic relations ranging from untimed equivalences (where the observer cannot see time) to timed equivalences (where the observer has arbitrarily fine precision) have been studied.

In [6] it has been shown that for the purposes of decidability the user should be allowed to specify only integer time delays. The underlying transition system is still dense. But the guarantee of only integer delays induces a nice equivalence (timed region equivalence) which is decidable.

In [1] it has been argued that untimed equivalences are too weak and timed equivalences are too strong. They propose a k-clock equivalence. That is, they assume that the user has k integer clocks. There is a hierarchy of equivalences, i.e., larger the k the finer the equivalence.

In this paper we present an indexed operational semantics of a process language for general timed processes. Processes involving a dense time domain can be specified, but the underlying semantics will be indexed by the granularity of time of an implementation. The view is that of an observer having a fixed precision regarding time measurements. In general, specifications may contain non-integer (real) values which an implementation (as viewed by an observer) may only approximate by other reals.

In such cases it is useful to relate the behaviour required by the specification and the behaviour exhibited by implementations with different time granularity. In other words, the main purpose of this paper is to present results which describe scenarios where altering time granularity alters observed behaviour.

Consider for example a process which delays for 0.4s and then exhibits an action μ and another process which delays for 0.6s and then exhibits the action μ. If the time granularity is one second the observer cannot distinguish the two processes as both 0.4s and 0.6s are identified with zero. On the other hand if the granularity is 0.5 seconds the two processes can be distinguished. Intuitively, if the granularity is 0.5 the second process will exhibit a delay followed by μ, while the first process will not exhibit any delay.

Thus the same specification can exhibit different behaviours depending on the underlying granularity of time. This is very different from the approaches in [6, 2] where there is no real underlying time granularity and the measurement of time is assumed to have infinite precision. This is also different from the results presented in [1] where the power of the observer is increased by assigning more integer valued timers. The work presented here will assume a finite number of timers (the exact number will depend on the definition of the process) all using a single global clock whose precision can be varied.

In the rest of the paper we develop a calculus in which time granularity can be studied. The calculus is similar to the various timed calculi. But the operational semantics is indexed by the time granularity available to the observer. We also consider a step semantics [9] where more than one action can be exhibited in one step. The semantic equivalence we consider is bisimulation. We will show that for a fixed granularity, the time granularity bisimulation is in between strong timed bisimulation and time abstracted equivalence. We show that we do not get a hierarchy of bisimulations indexed by the granularity of observation. This is shown by distinguishing processes by increasing as well as decreasing the time granularity. The main results include proposition 8 and proposition 13 which state the conditions under which bisimilarity will be altered by altering the granularity. The key result is that if the associated delays with two distinct processes are not identical, one can always find a time granularity that will distinguish them. Other results relate to the step semantics. Proposition 7 presents an expansion like theorem which preserves bisimilarity under a fixed granularity. Proposition 11 identifies the delay under which a particular step under a specific time granularity cannot be exhibited while Proposition 12 states the size of possible steps could change with a change in time granularity but all actions will indeed be exhibited.

2 Timed Behaviours

Our study of timed behaviours is based on timed CCS; viz., CCS extended with a delay construct. As in CCS we assume a set of actions $\mathcal{A}$. For the sake of simplicity we do not consider communication and hence do not consider complementary actions or the special τ action.

The syntax for the times processes we consider is given below.

$$\mathrm{P} ::= \mathbf{0} \;\big|\; (\mu{\cdot}\mathrm{P}) \;\big|\; (\mathrm{t})\mathrm{P} \;\big|\; (\mathrm{P} + \mathrm{P}) \;\big|\; (\mathrm{P} \mid \mathrm{P}) \;\big|\; \mathrm{rec}(\mathrm{X}{:}\mathrm{P}) \;\big|\; \mathrm{X}$$

The intuitive semantics for the above syntax is as follows. The process **0** cannot perform any action, the process $(\mu{\cdot}\mathrm{P})$ exhibits μ and then behaves as P. The process

(t)P delays for t time units (t ranging over the non-negative reals) and then behaves as P. The combinator '+' represents choice while '|' represents parallelism. 'rec' and 'X' are used to define recursive processes. For the purposes of this paper we will assume that all recursive processes are well guarded. That is, we will assume that every occurrence of the recursive variable is prefixed by an action.

A few comments are in order before the operational semantics are defined.

Rather than focus on interleaving semantics for the | combinator we consider a step semantics [9]. That is, it is possible for different branches of the | combinator to evolve in one step. Hence our labels will be multisets. The reason for consider a step semantics is primarily to show that the detection of concurrence is dependent on the granularity of time. Consider for example the process $(1)\mu{\cdot}\mathbf{0}$ in parallel conjunction with $(0.7)\mu'{\cdot}\mathbf{0}$. If the time granularity was 1.5 time units, it would appear that the actions μ and μ' can occur in one step. A similar conclusion (i.e., that μ and μ' can be seen in one step after a delay) can be drawn if the time granularity is 0.6 time units. However, if the time granularity is 0.9 time units, the action μ' would necessarily appear before the action μ. Such distinctions will be formalised later.

The observable delays and actions will be influenced by the granularity of time. For the purposes of simplicity we will assume that the actions take no time.

As we wish the process (t)(s)P to behave as (t+s)P, the indexed operational semantics has to be developed with care. Let δ represent a clock tick related to the specified granularity. Let g be the granularity and $t < g$ be a time delay. A transition rule of the form $(t)\mathrm{P} \xrightarrow{\delta}_g \mathrm{P}$ is incorrect since if the delay t is less than the time grain, no idling can be detected. An alternate rule of the form (if ($\mathrm{P} \xrightarrow{\mu}_g \mathrm{P}'$) then $(t)\mathrm{P}$ $\xrightarrow{\mu}_g \mathrm{P}'$, where $t < g$) is also incorrect. Consider for example the process $(1)(1)\mu{\cdot}\mathbf{0}$. If the time granularity is 2 units, using the above rule one would conclude that $(1)(1)\mu{\cdot}\mathbf{0}$ can exhibit the action μ. This is because the process $(1)\mu{\cdot}\mathbf{0}$ has a delay less than 2 and hence can exhibit μ. This is clearly wrong as with a time resolution of 2 units, an idling action must be observed. This indicates that one needs to keep track of overall time at the semantic level, although at the observation level time granularity will play an important role. In other words a 'dense time domain' for the semantics is useful. Intuitively, the operational semantics will accumulate the delays from all the components. If this accumulated delay is larger than the granularity, a delay action can be exhibited. If the accumulated delay is less than the granularity and an action can be exhibited after the accumulated delay, the action is exhibited.

Following [8] the operational semantics is presented as two sets of rules. We

Termination	$\mathbf{0} \stackrel{t}{\Longrightarrow} \mathbf{0}$
Time Delay	$(t)\mathrm{P} \stackrel{s}{\Longrightarrow} (t-s)\ \mathrm{P}\ (t \geq s)$
Accumulated Delay	$\dfrac{\mathrm{P} \stackrel{t}{\Longrightarrow} \mathrm{P}' \stackrel{t'}{\Longrightarrow} \mathrm{P}''}{\mathrm{P} \stackrel{t+t'}{\Longrightarrow} \mathrm{P}''}$
Time Determinism	$\dfrac{\mathrm{P} \stackrel{t}{\Longrightarrow} \mathrm{P}' \quad \mathrm{Q} \stackrel{t}{\Longrightarrow} \mathrm{Q}'}{(\mathrm{P} + \mathrm{Q}) \stackrel{t}{\Longrightarrow} \mathrm{P}' + \mathrm{Q}'}$ $(\mathrm{P} \mid \mathrm{Q}) \stackrel{t}{\Longrightarrow} \mathrm{P}' \mid \mathrm{Q}'$
Recursion	$\dfrac{\mathrm{P} \stackrel{t}{\Longrightarrow} \mathrm{P}'}{\mathrm{rec(X{:}P)} \stackrel{t}{\Longrightarrow} \mathrm{P}'[\mathrm{X/rec(X{:}P)}]}$

Figure 1: Delay Rules

assume a step semantics for the untimed process indicated by $\longrightarrow$. The first set of rules is concerned with timing delays (shown in figure 1) while the second deals with external observation (shown in figure 2). The delay rules specifies the various delays that are possible for a given process. Thus a process of the form (t)P can delay for any time less than or equal to t. However if P is of the form (t′)P′, the process (t)(t′)P′ can delay for any time less than or equal to the sum of t and t′. This is specified by the accumulated delay rule. The time delay rules are not concerned with time granularity as the relation $\longrightarrow_g$ considers it. The observation rules are indexed by the granularity of time (denoted by g). We assume that, in general, g is a positive real number. The small delay rule ignores delays which are less than the time granularity while the large delay rules exhibits the appropriate number of delay actions.

Rather than observe the exact time that has elapsed we discretize it (which is the effect of time granularity) using a special δ action which denotes idling. For the observation rules we will assume that an observation of the form O, O_1, O_2 etc. does not include the idling action. Therefore, there will not be any interaction between the rules which actually exhibit actions and the rules dealing with time delay. This

Action Prefix	$(\mu{\cdot}\mathrm{P}) \xrightarrow{\{\mu\}}_g \mathrm{P}$
Small Delay	$\dfrac{\mathrm{P} \overset{t}{\Longrightarrow} \mathrm{P}' \xrightarrow{O} \mathrm{P}''}{\mathrm{P} \xrightarrow{O}_g \mathrm{P}'} \ (t < g)$
Large Delay	$\dfrac{\mathrm{P} \overset{g}{\Longrightarrow} \mathrm{P}'}{\mathrm{P} \xrightarrow{\{\delta\}}_g \mathrm{P}'}$
Binary Combinators	$\dfrac{\mathrm{P} \xrightarrow{O}_g \mathrm{P}'}{\begin{array}{c}\mathrm{P} \mid \mathrm{Q} \xrightarrow{O}_g \mathrm{P}' \mid \mathrm{Q} \\ \mathrm{Q} \mid \mathrm{P} \xrightarrow{O}_g \mathrm{Q} \mid \mathrm{P}' \\ \mathrm{P} + \mathrm{Q} \xrightarrow{O}_g \mathrm{P}' \\ \mathrm{Q} + \mathrm{P} \xrightarrow{O}_g \mathrm{P}'\end{array}}$
Step Semantics	$\dfrac{\mathrm{P} \xrightarrow{O_1}_g \mathrm{P}' \ , \ \mathrm{Q} \xrightarrow{O_2}_g \mathrm{Q}'}{\mathrm{P} \mid \mathrm{Q} \xrightarrow{O_1 \cup O_2}_g \mathrm{P}' \mid \mathrm{Q}'}$
Time Determinism	$\dfrac{\begin{array}{c}\mathrm{P} \xrightarrow{\{\delta\}}_g \mathrm{P}' \\ \mathrm{Q} \xrightarrow{\{\delta\}}_g \mathrm{Q}'\end{array}}{\begin{array}{c}(\mathrm{P} + \mathrm{Q}) \xrightarrow{\{\delta\}}_g \mathrm{P}' + \mathrm{Q}' \\ (\mathrm{P} \mid \mathrm{Q}) \xrightarrow{\{\delta\}}_g \mathrm{P}' \mid \mathrm{Q}'\end{array}}$
Recursion	$\dfrac{\mathrm{P} \xrightarrow{O}_g \mathrm{P}'}{\mathrm{rec(X{:}P)} \xrightarrow{O}_g \mathrm{P}'[\mathrm{X}/\mathrm{rec(X{:}P)}]}$
Recursion (Delay)	$\dfrac{\mathrm{P} \xrightarrow{\{\delta\}}_g \mathrm{P}'}{\mathrm{rec(X{:}P)} \xrightarrow{\{\delta\}}_g \mathrm{P}'[\mathrm{X}/\mathrm{rec(X{:}P)}]}$

Figure 2: Observation Rules

is consistent with our assumptions that actions take no time and maximal progress for all actions. Thus, if an action can be exhibited a delay cannot be exhibited as there is no delay rule for action prefix. For example, $a \cdot \mathbf{0} \mid (t)b \cdot \mathbf{0}$ with t greater than the granularity will exhibit the action a. It is not possible to exhibit a delay action from the subprocess $(t)b \cdot \mathbf{0}$.

In general an observation could either be a single idling action or a multiset of other actions. We do not keep track of the number of idling components as this is not relevant to the semantics we develop. Although a step semantics is permitted, it is not demanded. This permits asynchronous evolution of processes.

The execution model can be defined as follows. Given a timed system, there is an interface to the observer. It is the behaviour of the interface that simulates time granularity. It has a local timer and a buffer. The buffer can hold various actions which are exhibited by the system. For every tick of the local timer (whose behaviour could be quite different from the timer used by the system) the contents of the buffer are flushed. In between two ticks, the system fills the buffer with actions using its own timer.

Before we present the results, we present a few examples which illustrate the behaviour of various processes.

Example: Consider the process $((1)a \cdot \mathbf{0} \mid b \cdot \mathbf{0}\ ) + (3)c \cdot \mathbf{0}$ with a time granularity of two units. Due to maximal progress, the action b cannot be delayed and hence the process will evolve to $(1)a \cdot \mathbf{0}$. The option $(3)c \cdot \mathbf{0}$ is discarded.

Example: The behaviour of the process $(1)a \cdot \mathbf{0} + (3)c \cdot \mathbf{0}$ depends on the granularity of time. If the granularity is greater than 3, the process can exhibit either an a action or a c action. This is because of the "Small Delay" rule in figure 2. If the granularity is less than 3, say 1, only an a action is possible. After a delay of one time unit the process will evolve to $a \cdot \mathbf{0} + (2)c \cdot \mathbf{0}$. But the maximal progress of a will force the choice and the process $(2)c \cdot \mathbf{0}$ will be discarded.

Example: The behaviour of the process $(1)a \cdot \mathbf{0} \mid (2)b \cdot \mathbf{0}$ depends on the time granularity.

If the time granularity were less than one (say 0.9) the behaviour would be an δ action followed by the action a followed by another δ action which will be followed by b.

If the time granularity were between one and two (say 1.1) the behaviour would be the action a (by the "Small Delay" rule) followed by an δ action which would then be followed by the b action. Note that the delay of 1 to exhibit the action a does not reduce the delay associated with the action b. This is because with a granularity of 1.1, the term $(1)a$ is treated as having no delay.

If the time granularity were greater than 2 (say 2.1) the behaviour could include the observation $\{a,b\}$ due to the conjunction of the small delay rule and the step semantics rule. In this case the delays of one unit and two units are equated with zero.

Example: As a more concrete example consider a controller which issues open/close commands to a gate at a railway crossing specified as follows

$$train \cdot ((1)down \cdot (2) \cdot shut \cdot P \mid \ (3)at \cdot okay \cdot Q)$$

for some P and Q. The intuition is that the approaching of the train is indicated by the action *train*. After one time unit the controller can send the command to lower the gate and two seconds after that the controller can receive the locked signal from the gate. Concurrently, the train is moving and after three units of time it is at the gate and awaits the *okay* command. If the controller is actually very slow (that is, the interface has large time granularity), it is possible for the system to exhibit *down* and *at* in one step and the behaviour of the controller is not reliable. However if the time granularity of the controller is identical to the granularity of the other components, the system is safe in that the gate is lowered when the train is at the intersection.

In the above semantics, it can be shown that delay does not resolve choice. That is, in a process of the form (P + Q), if both P and Q delay, the choice is not made. This is similar to the semantics presented in [6] and can be formalised as follows.

Proposition 1 For every t, if $(\mathrm{P} \stackrel{t}{\Longrightarrow} \mathrm{P}')$ and $(\mathrm{P} \stackrel{t}{\Longrightarrow} \mathrm{P}'')$, P' and P'' are identical.

Proof: The proof proceeds by structural induction. If P is of the form (t')Q where $(t' \geq t)$, the result is obvious. If P is of the form (t')Q where $t' < t$, Q has to have the ability to delay for at least $(t - t')$. By induction Q will have a unique $(t - t')$ derivative and hence the result applies to P as well. If P is of the form $(\mathrm{Q}_1 + \mathrm{Q}_2)$

then both Q_1 and Q_2 have to delay. By induction on Q_1 and Q_2 the result holds. Similarly for the | combinator. □

The above result follows directly from the definitions as $\Longrightarrow$ represents the exact time one can delay.

Our semantics is different in two ways from the one presented in [6]. Firstly we insist on maximal progress for all actions. That is we do not have the rule of the form $(\mu \cdot P) \stackrel{t}{\Longrightarrow} (\mu \cdot P)$. This is motivated by the fact we wish to present a few results which characterise speed and predictability. Secondly we have presented a step semantics for the | combinator. By combining maximal progress for all actions and step semantics, we can consider the situation where enough resources are available to exhibit all the enabled actions in one step. This is the speed aspect of real-time computing. Some of the results show how speed can affect predictability.

Proposition 2 If there exists P′ such that $P \stackrel{O}{\longrightarrow}_g P'$ with $\delta \notin O$ then there cannot exist a P″ such that $(P \stackrel{\{\delta\}}{\longrightarrow}_g P'')$. Similarly, if there exists P″ such that $(P \stackrel{\{\delta\}}{\longrightarrow}_g P'')$ then there cannot exist a P′ such that $P \stackrel{O}{\longrightarrow}_g P'$ with $\delta \notin O$.

Proof: The proof uses induction.

The process $\mu \cdot P$ satisfies the above requirement as the action μ cannot delay.

Consider the process (t)P. If t greater than g cannot exhibit any non-idling action. If t is less than g, the behaviour is dependent on P. If P cannot delay for $g - t$ but can exhibit an action the small delay rule is applicable. Hence (t)P cannot exhibit a delay. Otherwise, the large delay rule is applicable in which case (t)P cannot exhibit any non-idling action.

Let P be of the form $(P_1 \mid P_2)$. Assume the property holds for both P_1 and P_2. If either P_1 or P_2 cannot exhibit a delay, P cannot exhibit a delay. That is, if $P_1 \stackrel{O}{\longrightarrow}_g P_1'$ with $\delta \notin O$ and $P_2 \stackrel{g}{\Longrightarrow} P_2'$, there is a Q such that $(P \stackrel{O}{\longrightarrow}_g Q)$ and there cannot be any R such that $(P \stackrel{g}{\Longrightarrow} R)$. Only if both P_1 and P_2 can exhibit an idling action, can P exhibit an idling action. However, P then cannot exhibit any non-idling action (due to the induction hypothesis P_1 and P_2 cannot exhibit any non-idling action).

The proof for the other cases follows from the above observation. □

The above proposition formalises the fact that maximal progress is demanded. That is, if a process can exhibit an action, it cannot delay. The following proposition shows that although we have considered a step semantics, interleaving is also possible.

Proposition 3 If P $\xrightarrow{O}_g$ P$'$ such that the cardinality of O is greater than 1, for every O_1 a subset of O, there is some P$''$ such that P $\xrightarrow{O_1}_g$ P$''$.

Proof: If P is of the form $\mu \cdot P'$ the result trivially holds as P cannot exhibit an observation whose cardinality is greater than one.

If P is of the form $(P_1 + P_2)$, either P_1 can exhibit O or P_2 can exhibit O. By an induction argument P_1 (or P_2) can exhibit the required O_1; hence P can exhibit O_1.

If P is of the form $(P_1 \mid P_2)$. If P_1 (or P_2) can exhibit O the result follows by an inductive argument. Let P_1 exhibit O' and P_2 exhibit O'' such that $O = O' \cup O''$. If for a particular O_1, $O_1 \subseteq O'$ or $O_1 \subseteq O''$, the result follows by an induction argument. If O_1 is neither of subset of O' or O'', P_1 can exhibit $O_1 \cap O'$ and P_2 can exhibit $O_2 \cap O''$. Hence the result holds. The other cases follow a similar analysis. □

The intuition is that only the parallel composition leads to an observation of multiple actions. But one could always use the interleaving semantics to break up the composite observation.

3 Time Granularity

The results presented thus far are not specific to time granularity. They hold for general timed behaviours with step semantics and maximal progress. In this section we present the results that are related to time granularity.

The abbreviations introduced in the following definition is useful in presenting properties about process with granularity semantics. In the rest of the paper we assume that, unless otherwise specified, an observation of the form O could consist of either the idling action or other actions.

Definition: 1 We say a process Q is a *derivative* of a process P iff there is a sequence of observations $O_1, O_2, \ldots, O_n$ such that P $\xrightarrow{O_1}_g \xrightarrow{O_2}_g \ldots \xrightarrow{O_n}_g$ Q.

We write (P $\stackrel{t}{\not\Longrightarrow}$) if $\neg$ ($\exists$ P$'$ such that P $\stackrel{t}{\Longrightarrow}$ P$'$).

Similarly we write (P $\not\longrightarrow_g$) if $\neg$ ($\exists$ O, P$'$ such that (P $\xrightarrow{O}_g$ P')).

We now present a few properties satisfied by the $\longrightarrow_g$ relation. Towards that we use the bisimulation relation [7]. We introduce a granularity based bisimulation and also present the definitions for time-abstracted and timed bisimulation.

Definition: 2 We let $\sim_g$ to be the largest relation satisfying the following properties. Given processes P and Q, we say (P $\sim_g$ Q) iff

(P $\xrightarrow{O}_g$ P$'$) implies $\exists$ Q$'$ such that Q $\xrightarrow{O}_g$ Q$'$ with P$'$ $\sim_g$ Q$'$ and

(Q $\xrightarrow{O}_g$ Q$'$) implies $\exists$ P$'$ such that P $\xrightarrow{O}_g$ P$'$ with P$'$ $\sim_g$ Q$'$.

Similarly we define (P $\sim_{ta}$ Q) the time-abstracted bisimulation to be the largest relation satisfying

(P $\stackrel{t_P}{\Longrightarrow}\stackrel{O}{\longrightarrow}$ P$'$) implies $\exists$ Q$'$ such that Q $\stackrel{t_Q}{\Longrightarrow}\stackrel{O}{\longrightarrow}$ Q$'$ with P$'$ $\sim_{ta}$ Q$'$ and

(Q $\stackrel{t_Q}{\Longrightarrow}\stackrel{O}{\longrightarrow}$ Q$'$) implies $\exists$ P$'$ such that P $\stackrel{t_P}{\Longrightarrow}\stackrel{O}{\longrightarrow}$ P$'$ with P$'$ $\sim_{ta}$ Q$'$.

Timed bisimulation, $\sim_t$, is defined to be the largest relation satisfying

(P $\stackrel{t}{\Longrightarrow}$ P$'$) implies $\exists$ Q$'$ such that (Q $\stackrel{t}{\Longrightarrow}$ Q$'$) P$'$ $\sim_t$ Q$'$

(Q $\stackrel{t}{\Longrightarrow}$ Q$'$) implies $\exists$ P$'$ such that (P $\stackrel{t}{\Longrightarrow}$ P$'$) P$'$ $\sim_t$ Q$'$

(P $\stackrel{O}{\longrightarrow}$ P$'$) implies $\exists$ Q$'$ such that (Q $\stackrel{O}{\longrightarrow}$ Q$'$) P$'$ $\sim_t$ Q$'$

(Q $\stackrel{O}{\longrightarrow}$ Q$'$) implies $\exists$ P$'$ such that (P $\stackrel{O}{\longrightarrow}$ P$'$) P$'$ $\sim_t$ Q$'$

The above definitions are only a recasting of the original definition of bisimulation. The main difference is the presence of multi-sets as observations. Granular bisimulation is related to the time abstracted bisimulation in [6] as the operational semantics has abstracted away some of the precise delays. However not all timing information is lost. One can observe delays via the idling action δ.

We assume that $rf(t, g)$ represents $((t \ div \ g) \times g)$ where $x = t \ div \ g$ implies there exists a $y < g$ such that $t = x \times g + y$. Intuitively $rf(t, g)$ represents the relevant component of t given a granularity of g.

Proposition 4

1. $\sim_g$ is a congruence for all combinators except the time delay.
2. (t)$\mu\cdot$P $\sim_g$ $(rf(t, g))\mu\cdot$P and (t)$\cdot\mathbf{0}$ $\sim_g$ $(rf(t, g))\cdot\mathbf{0}$.
3. (s)(t)P $\sim_g$ (s+t)P

Proof:

1. Let (P $\sim_g$ Q). We show that (P | R) $\sim_g$ (Q | R). If (P | R) $\xrightarrow{\{\delta\}}_g$ then both P and R have to delay for g units. Hence Q must also be able to delay for g units. Thus (Q | R) can exhibit an δ action. If (P | R) $\xrightarrow{O}_g$ then one of the cases to consider is P $\xrightarrow{O_1}_g$ and R $\xrightarrow{O_2}_g$ where $O = O_1 \cup O_2$. Thus, Q $\xrightarrow{O_1}_g$ and hence (Q | R) $\xrightarrow{O}_g$. The other cases are similar.

2. Follows directly from the definition of $rf(t, g)$. If t is less than g, the process (t)μ·P can exhibit μ. By definition $rf(t, g)$ is zero and hence $(rf(t, g))\mu$·P can also exhibit μ. If t is greater than g, let $rf(t, g)$ be n. Clearly (t)μ·P can exhibit n idling actions before μ as can $(rf(t, g))\mu$·P.

   The processes (t)·**0** and $(rf(t, g))$·**0** can only exhibit idling actions. The number of idling actions exhibited by both the processes is identical. Hence the result holds.

3. Follows directly from the accumulated delay rule.

□

It is easy to see that $(2)\mu\cdot\mathbf{0} \sim_3 (1)\mu\cdot\mathbf{0}$ but $(1)(2)\mu\cdot\mathbf{0} \not\sim_3 (1)(1)\mu\cdot\mathbf{0}$. For the second item in the above proposition, the action prefix is necessary so as to prevent delay accumulation. It is not true in general for any P that (t) P $\sim_g$ $(rf(t, g))$ P. For example $(2)\mu\cdot\mathbf{0} \sim_3 (1)\mu\cdot\mathbf{0}$ but $(2)(1)\mu\cdot\mathbf{0} \not\sim_3 (1)(1)\mu\cdot\mathbf{0}$. But if P cannot delay, replacing a delay of t units with a delay of $rf(t, g)$ units is valid. It is also valid for arbitrary positive values with an appropriate definition of 'div'. The last item in the proposition is true for all processes P and only represents the additive nature of time delays. It is easy to construct an equational characterisation of $\sim_g$. This is because the presence of time granularity converts continuous time into discrete time.

The next two propositions formally state that granularity based semantics is in between untimed semantics and fully timed semantics. Proposition 5 follows as the observer has no notion of time while proposition 6 is valid since P and Q have identical timed behaviour, the coarseness of time has no effect on the observed behaviour.

Proposition 5 If (P $\sim_g$ Q), then (P $\sim_{ta}$ Q).

Proposition 6 If (P $\sim_t$ Q), (P $\sim_g$ Q) is true for any g.

Although [3] proves that an expansion theorem for real-time calculus based on dense time cannot exist, the use of time granularity converts a dense time system to a 'discrete time' system. Hence we can present a form of expansion theorem for a fixed granularity. As we are considering step semantics we need to extend the syntax with multi-set prefix to capture the fact that more than one action can be performed in one step. The paper [5] contains all the details. As for CCS, we assume processes which are of a particular form (called standard form in CCS).

Proposition 7 Let P be $\sum_{i \in I}(t_i)a_i \cdot P_i$ and let Q be $\sum_{j \in J}(s_j)b_j \cdot Q_j$.

1. If no t_i or s_j is less than g, (P | Q) $\sim_g$ R where R is the following

$$(\mathrm{g})(\sum_{i \in I}(t_i - g)a_i \cdot P_i \mid \sum_{j \in J}(s_j - g)b_j \cdot Q_j)$$

2. If the set S = $\{$ k such that $(k \in I)$ and $t_k < g$ or $(k \in J)$ and $s_k < g$ $\}$ is not empty, (P | Q) $\sim_g$ R where R is the following

$$\sum_{i \in S} a_i \cdot (P_i \mid \mathrm{Q}) + \sum_{j \in S} b_j \cdot (\mathrm{P} \mid Q_j) + \sum_{i,j \in S} \{a_i, b_j\} \cdot (P_i \mid Q_j)$$

Proof:

1. If each of P_i's and Q_j's cannot exhibit any action the result holds. From proposition 2, (P | Q) can either exhibit an idling action or some other visible action. From time determinacy, (P | Q) can exhibit an idling action without resolving choice. The process R has the required property.

2. As the set S is not empty, one can assume without loss of generality that P $\xrightarrow{a_i}_g P_i$. This is matched by the first branch in R. If both P and Q can exhibit actions, say a_i and b_j, it is possible to exhibit the observation $\{a_i, b_j\}$ which is captured by the last branch in R.

□

In the rest of this paper we focus on results which are specific to time granularity.

The first result we present is perhaps counter intuitive. It states that increasing the granularity of time measurement does not always result in a finer bisimulation result.

Proposition 8 Let g and ϵ be positive numbers. If (P $\sim_g$ Q) but (P $\not\sim_{g+\epsilon}$ Q), there exists a derivative of P (say P_1) and a derivative of Q (say Q_1) such that (P_1 $\sim_g$ Q_1) and P_1 (or Q_1) cannot delay for time greater than or equal to $(g + \epsilon)$ while Q_1 (or P_1) can.

That is, $P_1 \stackrel{t}{\Longrightarrow}$ iff $Q_1 \stackrel{t}{\not\Longrightarrow}$ where $t \geq (g + \epsilon)$.

Proof: One can prove the above proposition using induction over the size of the processes for an arbitrarily fixed g and ϵ. We present two cases.

The first is a simple case while the second is the most complex case. If P is μ·P′ the result is obvious as Q's maximum delay is bounded by g and hence bounded by $g + \epsilon$. But Q's μ derivative must satisfy the required conditions as (μ·P′ $\sim_g$ Q).

Let P be of the form $(P_1 \mid P_2)$. Assume that $P_1 \stackrel{t_1}{\Longrightarrow} P_1'$ and $P_2 \stackrel{t_2}{\Longrightarrow} P_2'$ such that both P_1' and P_2' cannot delay. That is, t_1 and t_2 identifies the maximum delay. Break up t_1 and t_2 into multiples of g such that $P_1 \stackrel{ng}{\Longrightarrow} P_1'' \stackrel{k_1}{\Longrightarrow} P_1'$ and $P_2 \stackrel{ng}{\Longrightarrow} P_2'' \stackrel{k_2}{\Longrightarrow} P_2'$ such that either $k_1 < g$ or $k_2 < g$. This follows from the accumulation of time and proposition 1. By bisimilarity under the g granularity semantics Q $\stackrel{ng}{\Longrightarrow}$ Q$'$.

Apply the same argument under the $g + \epsilon$ semantics such that $P_1 \stackrel{m(g+\epsilon)}{\Longrightarrow} P_3 \stackrel{k_3}{\Longrightarrow}$ and $P_2 \stackrel{m(g+\epsilon)}{\Longrightarrow} P_4 \stackrel{k_4}{\Longrightarrow}$ such that either $k_3 < (g + \epsilon)$ or $k_4 < (g + \epsilon)$.

If Q cannot exhibit a delay of $m(g+\epsilon)$ we are done as P can exhibit the requisite delay while Q cannot. If $(P_3 \mid P_4)$ can exhibit O and Q can exhibit O we can apply the induction principle. Otherwise, it implies that Q has an extra delay while P does not. This is because the observation O could have come from either P_3 or P_4 or perhaps both depending on the values of k_3 and k_4.

Due to time determinism, the observational behaviour of $(P_1' \mid P_2')$ and $(P_3 \mid P_4)$ are related. There are various cases to consider. One of them is P_1' cannot delay while P_2' can while both P_3 and P_4 cannot delay. Hence there could no O transition from $(P_1' \mid P_2')$. However, there has to be an O_1 transition from P_1' which has to be matched by P_3. As step semantics is not demanded $(P_3 \mid P_4)$ can exhibit O_1 which will be a subset of O. This has been stated in proposition 3. Hence the induction step will be applied not to the O derivatives but rather to the O_1 derivatives. □

The result shows that it is possible to distinguish timed behaviour by increasing the coarseness of time measurement. For example $(2)\mu{\cdot}\mathbf{0} \sim_2 (3)\mu{\cdot}\mathbf{0}$ but $(2)\mu{\cdot}\mathbf{0} \not\sim_3 (3)\mu{\cdot}\mathbf{0}$. This is because with 3 units of time as the basic step, the process $(3)\mu{\cdot}\mathbf{0}$ will delay while $(2)\mu{\cdot}\mathbf{0}$ will not exhibit any delay. If the granularity is reduced to 2 units both processes will exhibit an idling action.

The following example shows when a different sequence of observation is required in the induction proof. Consider the processes $(4)a \cdot \mathbf{0} \mid (5)b \cdot \mathbf{0}$ and $(5)a \cdot \mathbf{0} \mid (5)b \cdot \mathbf{0}$. If the granularity is 2, both processes can exhibit two idling actions followed by $\{a, b\}$. Of course, after the two idling actions only $\{a\}$ is possible. If the granularity were increased to 2.5 the first process will exhibit an idling action followed by $\{a\}$.

Similarly $(1.1)\mu{\cdot}\mathbf{0} \sim_{0.5} (1.4)\mu{\cdot}\mathbf{0}$, but $(1.1)\mu{\cdot}\mathbf{0} \not\sim_{0.6} (1.4)\mu{\cdot}\mathbf{0}$. It is clear that $(0.5)\mu{\cdot}\mathbf{0}$ cannot delay for 0.6 time units while $(0.8)\mu{\cdot}\mathbf{0}$ can delay for 0.6 units thus exhibiting an δ action. Of course, it is possible to distinguish them by increasing the accuracy, but the above result shows that it is not always essential.

The above result also emphasises the fact that the clock is reset after an observation and that differences in granularities are not accumulated over a sequence of observations. This is because we require desired behaviour only from a derivative process. For example, consider the processes $(5)\mu_1 \cdot (3)\mu_2 \cdot \mathbf{0}$ (say P) and $(4)\mu_1 \cdot (4)\mu_2 \cdot \mathbf{0}$(say Q). It is clear that (P $\sim_3$ Q) but (P $\not\sim_4$ Q). Thus, although the total individual delays add up to 8, they are not bisimilar with 4 units granularity. This is because each individual time measurement has the granularity error and these errors do not accumulate across 'atomic' observations. This is intuitively correct as errors due to coarseness of time measurement occur at every significant measurement.

The following result is a direct consequence of proposition 8 and the examples described above.

Proposition 9 It is not the case that if $g < g'$, $\sim_g \subseteq \sim_{g'}$ or $\sim_{g'} \subseteq \sim_g$.

Thus unlike the k-clock case in [1], granularity based semantics does not lead to a nice hierarchy. This is also reflected in the engineering of real-time systems where changing the underlying clock system adversely affects the programs.

This is because the granularity based semantics is not characterising errors in measurement. Rather, it assumes a fixed granularity which can be measured accurately. If an error in measurement semantics is desired, the following definition can be used.

Definition: 3 We let $\sim_{g,\epsilon}$ to be the largest relation satisfying the following. Given processes P and Q, we say (P $\sim_{g,\epsilon}$ Q) iff both the following conditions are satisfied.

(P $\xrightarrow{O}_g$ P$'$) implies $\exists$ Q$'$ and δ such that Q $\xrightarrow{O}_{g+\delta}$ Q$'$ with P$' \sim_g$ Q$'$ and $(-\epsilon \leq \delta \leq \epsilon)$

(Q $\xrightarrow{O}_g$ Q$'$) implies $\exists$ P$'$ such that P $\xrightarrow{O}_{g+\delta}$ P$'$ with P$' \sim_g$ Q$'$ and $(-\epsilon \leq \delta \leq \epsilon)$

In the above definition ϵ represents the permitted error in measurement and hence to match a particular move, a certain error in the time measurement is permitted. This condition is very weak as all it demands is the existence of an appropriate error in measurement.

It is clear that if (P $\sim_{g,\epsilon}$ Q) and $(\epsilon' \geq \epsilon)$, (P $\sim_{g,\epsilon'}$ Q). That is, increasing the permitted error does not result in distinguishing equated processes. For a fixed g, varying ϵ establishes an hierarchy of bisimulations that is similar to the k-clock case. But as this semantics is too weak we do not explore this further. The rest of the paper focuses only on the granularity semantics.

The next result identifies the scenario where increasing the coarseness of timing does not result in any loss of information.

Proposition 10 Let g be a positive number. Assume the following: $\forall\ \epsilon \geq\ 0$, (P $\sim_g$ Q) implies (P $\sim_{g+\epsilon}$ Q). If the above holds, for every derivative P_1 of P and Q_1 of Q such that ($P_1 \sim_g Q_1$) the following is true.

If ($P_1 \stackrel{t}{\Longrightarrow} P_1'$ and $Q_1 \stackrel{t}{\not\Longrightarrow}$) then $t < g$

If ($Q_1 \stackrel{t}{\Longrightarrow} Q_1'$ and $P_1 \stackrel{t}{\not\Longrightarrow}$) then $t < g$

Proof: By proposition 2 there are two cases to consider. The first is that processes may be able to exhibit the idling action. Let $P \stackrel{t_1}{\Longrightarrow} P_1 \not\Longrightarrow$ and $Q \stackrel{t_2}{\Longrightarrow} Q_1 \not\Longrightarrow$. Assume without loss of generality that $t_1 > t_2$.

If both are less than g the result holds. It cannot be the case that one is less than g while the other is not as (P $\sim_g$ Q) and hence either both can or cannot exhibit an idling action. If both are greater than g, consider time granularity to be t_1. Now $P \stackrel{\{\delta\}}{\longrightarrow}_{t_1}$ which cannot be matched by Q and hence $P \not\sim_t Q$. Hence the condition holds. That is, if the the delays are not identical they are less than g and hence under any granularity greater than g they continue to remain irrelevant.

The second case is when the process cannot exhibit the idling action. If $P \stackrel{O}{\longrightarrow}_{g+\epsilon} P'$ then $Q \stackrel{O}{\longrightarrow}_{g+\epsilon} Q'$ such that $P' \sim_{g+\epsilon} Q'$. Now an inductive argument can be applied to P' and Q'. $\square$

It is easy to see that if there are non-identical delays (say $g_1 + k$ and $g_1 + n$), a granularity of the minimum of the two size will result in different behaviours. The intuition is that all non-identical maximal delays in all derivatives of P and Q are less than or equal to g. Hence increasing time granularity does not distinguish P and Q. Effectively the timed behaviour is reduced to an untimed behaviour. This is due to the granularity being larger than any actual delay.

The following propositions identify the nature of delays when time granularity is reduced, i.e., time is measured more accurately.

Proposition 11 Let P $\stackrel{O}{\longrightarrow}_g$ P$'$. If for a g_1 such that ($g_1 < g$), P $\stackrel{O}{\not\longrightarrow}_{g_1}$ P$'$, then there is a subterm (say P_1) of P such that $P_1 \stackrel{g_2}{\Longrightarrow}$ P$_2$ such that $g_2 \geq g_1$.

Proof: If the cardinality of O is 1 the result follows directly. That is, assume that P can delay for some time less than g and then exhibit O (a single action). If P cannot exhibit O if the granularity is g_1, it implies that the subterm in P which exhibit the O have to delay for some time between g_1 and g. This follows from the maximal progress assumption.

If the cardinality is greater than 1, the observation O can only be achieved using step semantics for parallel components. Hence if the composite collection of actions is not possible, one of the parallel components has be delayed by the required amount. That is, there must exist subterms P_1 and P_2 such that $P_1 \xrightarrow{O_1}_g$ and $P_2 \xrightarrow{O_2}_g$ and $(O = O_1 \cup O_2)$. If both P_1 and P_2 delay for time less than g_1, both P_1 and P_2 can exhibit O_1 and O_2 respectively. Clearly this is not allowed. Hence one of P_1 or P_2 must be unable to exhibit O_1 or O_2. Now one can apply an inductive argument to either P_1 or P_2. □

If measuring time more accurately (i.e., reducing the granularity) results in certain actions not being observed in one step, one can conclude that this is essentially due to a sufficiently large delay. Consider for example the process $((2)\mu_1 \cdot \mathbf{0} \mid (1)\mu_2 \cdot \mathbf{0})$. If the accuracy of time is only 3 units the actions μ_1 and μ_2 can be observed in one step. However, if the granularity is 2 units, the ability to exhibit μ_1 and μ_2 in one step cannot be observed. Hence one can conclude that there is a delay of at least 2 time units. Thus a finer measurement of time is useful in identifying the individual steps taken to form an observation.

Proposition 12 If P $\xrightarrow{O}_g$ but P $\not\xrightarrow{O}_{g_1}$ where $g_1 < g$, the following is valid.

P $\xrightarrow{\{\delta\}}_{g_1}^{*} \xrightarrow{O_1}_{g_1} \xrightarrow{\{\delta\}}_{g_1}^{*} \xrightarrow{O_2}_{g_1} \ldots \xrightarrow{\{\delta\}}_{g_1}^{*} \xrightarrow{O_n}_{g_1}$ where

$\xrightarrow{\{\delta\}}_{g_1}^{*}$ represents exhibiting zero or more idling actions

$O_1 \cup \ O_2 \cup \ \ldots \cup \ O_n = O$

Proof: As in the proof of proposition 11 let the cardinality of O be 1. As P $\not\xrightarrow{O}_{g_1}$, P has to delay for some time greater than or equal to g_1. Depending on the exact relationship between g and g_1 some finite non-zero idling actions will be observed after which the observation O can be performed. This follows as exhibiting the idling action does not resolve choice.

If the cardinality of O is greater than 1, then again as in proof of proposition 11 we can decompose P into various parallel components and apply the above argument to each component. Hence the observation O will be broken down into smaller observations. The components that have the least delay will contribute to O_1 and so on.

It is possible that after exhibit some O_i some observation larger than O_{i+1} is possible. But from proposition 3 the observation O_{i+1} is possible. □

The intuitive justification follows directly from proposition 11 and time determinism. As passing of time does not resolve choice, it is possible to break up a complex observation into various sub-components. The possibility of delays between the sub-components is essential by proposition 11.

Proposition 13 Let g and ϵ be positive numbers with $\epsilon < g$. If (P $\sim_g$ Q) but (P $\not\sim_{(g-\epsilon)}$ Q), there exists a derivative of P (say P_1) and a derivative of Q (say Q_1) with ($P_1 \sim_g Q_1$) such that one of the following is true.

$\exists\, t$, $(g-\epsilon) \leq t < g$ such that $P_1 \stackrel{t}{\Longrightarrow}$ and $Q_1 \stackrel{t}{\not\Longrightarrow}$

$\exists\, t$, $(g-\epsilon) \leq t < g$ such that $Q_1 \stackrel{t}{\Longrightarrow}$ and $P_1 \stackrel{t}{\not\Longrightarrow}$

Proof: The proof follows the line presented for proposition 8.

The intuitive justification for the result is very similar to the result in propositions 8 and 11. A delay of less than g time units will essentially be ignored under the g-granularity semantics but could contribute a δ action under a $(g-\epsilon)$ semantics.

4 Conclusion and Future Work

In this paper we have shown the effects of increasing or decreasing the accuracy of time measurements. This was presented in the context of a timed process algebra with step semantics and a bisimulation relation. The intriguing result is one where increasing the time granularity can make the resulting bisimulation relation more coarse. Other results which captured timed behaviour and concurrent behaviour were fairly natural. From a practical view point, a change in time granularity can be viewed as a change in resolution of the underlying clock. Such a change could arise because of changes to the software dealing with time. Increasing the accuracy of the clock could result in unpredictable results as processes which were identified could now be distinguished while processes which were distinguished could now be identified. If the granularity is altered from g_1 to g_2 it is important to ensure that for all delays of time t, $rf(t, g_1)$ should be equal to $rf(t, g_2)$. Apart from traditional real-time systems, a potential application is in the area of circuit design. Given a particular circuit, one needs to ensure that there are no race conditions. This could depend on the granularity of time. That is, the behaviour could depend on the jitter in the clock. Assuming a jitter tolerance, one can then check if the circuit implementation meets the specification for various granularities. This is still under investigation.

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