Tail Properties and Asymptotic Expansions for the Maximum of Logarithmic Skew-Normal Distribution

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Tail properties and asymptotic expansions for the maximum of logarithmic skew-normal distribution

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Abstract We discuss tail behaviors, subexponentiality and extreme value distribution of logarithmic skew-normal random variables. With optimal normalized constants, the asymptotic expansion of the distribution of the normalized maximum of logarithmic skew-normal random variables is derived. It shows that the convergence rate of the distribution of the normalized maximum to the Gumbel extreme value distribution is proportional to $1/(\log n)^{1/2}$.

Key words Extreme value distribution; Logarithmic skew normal distribution; Maximum; Pointwise convergence rate; Subexponentiality.

AMS 2000 subject classification Primary 62E20, 60G70; Secondary 60F15, 60F05.

1 Introduction

The major weakness of the normal distribution is its inability to model skewed data. Several skewed extensions of the normal distribution have been proposed in the literature. The most popular and the most widely used of these is the skew-normal distribution due to Azzalini (1985). The probability density function (pdf) of this distribution is given by

$$g_\lambda(x) = 2\phi(x)\Phi(\lambda x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\phi(x)$ is the standard normal pdf, and $\Phi(x)$ is the standard normal cumulative distribution function (cdf). Let $G_\lambda(x) = \int_{-\infty}^{x} g_\lambda(t)dt$ denote the cdf corresponding to (1.1). If a random variable, say X, has the pdf (1.1) then we write $X \sim \mathrm{SN}(\lambda)$. Clearly, $\mathrm{SN}(0)$ is a standard normal variable.

Liao et al. (2012) studied the tail behavior of the skew-normal distribution, establishing its extreme value distribution and associated convergence rates. The following expansion for the distribution of the normalized maximum of $\mathrm{SN}(\lambda)$ random variables was derived by Liao et al. (2012):

$$\bar{b}_n^2\left[\bar{b}_n^2\left(G_\lambda^n\left(\bar{a}_n x + \bar{b}_n\right) - \Lambda(x)\right) - \bar{\kappa}(x)\Lambda(x)\right] \to \left(\bar{\omega}(x) + \frac{\bar{\kappa}^2(x)}{2}\right)\Lambda(x)$$

as $n \to \infty$, where $\Lambda(x) = \exp\{-\exp(-x)\}$ denotes the Gumbel cdf and

$$\bar{\kappa}(x) = \left(x^2/2 + x\right)e^{-x},$$

$$\overline{\omega}(x) = -\left(x^4/8 + x^3/2 + x^2 + 2x\right) e^{-x}$$

with

$$1 - G_\lambda\left(\overline{b}_n\right) = n^{-1}, \qquad \overline{a}_n = \overline{b}_n^{-1}$$

for $\lambda \geq 0$; and

$$\overline{\kappa}(x) = \left(1+\lambda^2\right)^{-1}\left(x^2/2 + 2x\right) e^{-x},$$
$$\overline{\omega}(x) = -\lambda^{-2}\left(1+\lambda^2\right)^{-2}\Big(\lambda^2 x^4/8 + \lambda^2 x^3 + 3\lambda^2 x^2 + 2\left(1+3\lambda^2\right)x\Big)e^{-x}$$

with

$$1 - G_\lambda\left(\overline{b}_n\right) = n^{-1}, \qquad \overline{a}_n = \left(\left(1+\lambda^2\right)\overline{b}_n\right)^{-1}$$

for $\lambda < 0$.

The skew-normal distribution applies to data on the real line. Its version for positive data can be obtained by setting $X = \exp(\xi)$, where $\xi \sim \mathrm{SN}(\lambda)$. Then, we say that X follows the logarithmic skew-normal distribution, written $X \sim \mathrm{LSN}(\lambda)$. The pdf of $\mathrm{LSN}(\lambda)$ is given by

$$f_\lambda(x) = \frac{2}{x}\phi(\log x)\Phi(\lambda \log x), \qquad x > 0. \tag{1.2}$$

Let $F_\lambda(\cdot)$ denote the cdf corresponding to (1.2). Clearly, LSN(0) is standard log-normal random variable.

The logarithmic skew-normal distribution is relative more recent compared to the skew-normal distribution. But it has already received wide spread applications. Some selected applications and application areas have been: modeling of income data (Azzilini et al., 2003); analysis of auto insurance claim costs (Bolance et al., 2008); analysis of continuous data in a two-part stochastic model (Chai and Bailey, 2008); wireless communications (Wu et al., 2009, Li et al., 2011); model for particle size (Huang and Ku, 2010); cohort studies of paediatric respiratory symptoms (Mahmud et al., 2010); modeling of precipitation data (Marchenko and Genton, 2010). Some probabilistic properties of $\mathrm{LSN}(\lambda)$ have been studied by Lin and Stoyanov (2009).

The aim of this short note is to consider some further probabilistic properties of the logarithmic skew-normal distribution. The contents are organized as follows. Section 2 presents some preliminary results, including the tail behavior, the subexponentiality and the extreme value distribution of $\mathrm{LSN}(\lambda)$. Distributional expansions for the normalized maximum of $\mathrm{LSN}(\lambda)$ random variables are derived in Section 3. To the best of our knowledge, all of the properties presented are new.

2 Preliminary results

In this section, we derive Mills' inequalities, Mills' ratios, and an exact decomposition of the tail of $\mathrm{LSN}(\lambda)$. We also prove that $\mathrm{LSN}(\lambda)$ is strongly subexponential, denoted by $F_\lambda \in \mathcal{S}^*$.

For $\mathrm{LSN}(\lambda)$ and $\mathrm{SN}(\lambda)$, note that $1 - F_\lambda(x) = 1 - G_\lambda(\log x)$ and

$$\frac{1 - F_\lambda(x)}{f_\lambda(x)} = x\frac{1 - G_\lambda(\log x)}{g_\lambda(\log x)}.$$

So, by Proposition 1 in Liao et al. (2012) and by Mills' inequality and Mills' ratio of the standard normal distribution, we have the following two results.

Proposition 1. *Let $F_\lambda(x)$ and $f_\lambda(x)$ denote the cdf and the pdf of* $\mathrm{LSN}(\lambda)$*. For all $x > 1$, we have*

(i). if $\lambda > 0$,

$$\frac{x}{\log x}\left(1+(\log x)^{-2}\right)^{-1} < \frac{1-F_\lambda(x)}{f_\lambda(x)} < \frac{x}{\log x}\left(1-\frac{\phi(\lambda\log x)}{\lambda\log x}\right)^{-1};$$

(ii). if $\lambda = 0$,

$$\frac{x}{\log x}\left(1+(\log x)^{-2}\right)^{-1} < \frac{1-F_0(x)}{f_0(x)} < \frac{x}{\log x};$$

(iii). if $\lambda < 0$,

$$\begin{aligned}&\frac{x}{\log x}\left(1+(\log x)^{-2}\right)^{-1}\left(1-\frac{\lambda^2}{1+\lambda^2}\left(1+\frac{1}{\lambda^2(\log x)^2}\right)\right)\\ &\quad< \frac{1-F_\lambda(x)}{f_\lambda(x)} < \frac{x}{\log x}\left(1-\frac{\lambda^2}{1+\lambda^2}\left(1+\frac{1}{(1+\lambda^2)(\log x)^2}\right)^{-1}\right).\end{aligned}$$

Proposition 2. *Let $F_\lambda(x)$ and $f_\lambda(x)$ denote the cdf and the pdf of* $\mathrm{LSN}(\lambda)$*. For $\lambda \geq 0$, we have*

$$\frac{1-F_\lambda(x)}{f_\lambda(x)} \sim \frac{x}{\log x} \tag{2.1}$$

as $x \to \infty$. For $\lambda < 0$, we have

$$\frac{1-F_\lambda(x)}{f_\lambda(x)} \sim \frac{x}{(1+\lambda^2)\log x} \tag{2.2}$$

as $x \to \infty$.

The following result shows that $\mathrm{LSN}(\lambda)$ is strongly subexponential.

Corollary 1. *$F_\lambda \in \mathcal{S}^*$, so $F_\lambda \in \mathcal{S}$, the class of subexponential distributions.*

Proof. By Proposition 2, the hazard rate function $m_{F_\lambda}(x) = \frac{f_\lambda(x)}{1-F_\lambda(x)}$ is ultimately decreasing to zero as $x \to \infty$. If $\exp\left(x m_{F_\lambda}(x)\right)\overline{F}_\lambda(x)$ is integrable over $\mathbb{R}^+$, where $\overline{F}_\lambda(x) = 1 - F_\lambda(x)$, Theorem 3.32 in Foss et al. (2011) shows that $F_\lambda \in \mathcal{S}^*$. Combining with Theorem 3.27 in Foss et al. (2011), we have $F_\lambda \in \mathcal{S}$. So, we just need to check that $\exp\left(x m_{F_\lambda}(x)\right)\overline{F}_\lambda(x)$ is integrable over $\mathbb{R}^+$.

Consider the case of $\lambda \geq 0$. By (2.1), we know for arbitrary $\varepsilon > 0$ that there exist a sufficiently large $A > 0$ such that

$$(1-\varepsilon)\frac{x}{\log x} < \frac{1-F_\lambda(x)}{f_\lambda(x)} < (1+\varepsilon)\frac{x}{\log x}.$$

Hence, for $x > A$, we have

$$\begin{aligned}\exp\left(x m_{F_\lambda}(x)\right)\overline{F}_\lambda(x) &< (1+\varepsilon)\frac{x f_\lambda(x)}{\log x}\exp\left(\frac{1}{1-\varepsilon}\log x\right)\\ &< \frac{2(1+\varepsilon)}{\log A}\phi(\log x)\exp\left(\frac{1}{1-\varepsilon}\log x\right)\end{aligned}$$

$$= \frac{2(1+\varepsilon)}{\log A} \exp\left(\frac{1}{2(1-\varepsilon)^2}\right) \phi\left(\log x - \frac{1}{1-\varepsilon}\right).$$

So, one can check that $\lim_{x\to\infty} x^k \exp\left(x m_{F_\lambda}(x)\right) \overline{F}_\lambda(x) = 0$ for any $k > 1$, implying $\exp\left(x m_{F_\lambda}(x)\right)$ $\overline{F}_\lambda(x)$ is integrable over $\mathbb{R}^+$.

The same can be shown for the case of $\lambda < 0$ by using (2.2). The arguments are similar and are omitted here.

The desired result follows. □

In order to derive expansions for the distribution of the normalized maximum of LSN(λ) random variables, we need the following tail decomposition of LSN(λ).

Proposition 3. *Let $F_\lambda(x)$ denote the cdf of* LSN(λ)*. Then, for large x, if $\lambda \geq 0$ we have*

$$\begin{aligned} 1 - F_\lambda(x) &= \frac{f_\lambda(\log x)}{\log x}\Big(1 - (\log x)^{-2} + 3(\log x)^{-4} + O\left((\log x)^{-6}\right)\Big) \\ &= \sqrt{\frac{2}{\pi e}}\Phi(\lambda \log x)\Big(1 - (\log x)^{-2} + 3(\log x)^{-4} \\ &\quad + O\left((\log x)^{-6}\right)\Big) \exp\left(-\int_e^x \frac{\log s}{s}\left(1 + (\log s)^{-2}\right) ds\right). \end{aligned} \tag{2.3}$$

If $\lambda < 0$, we have

$$\begin{aligned} 1 - F_\lambda(x) &= \frac{\exp\left(-\frac{1+\lambda^2}{2}(\log x)^2\right)}{(-\lambda)\pi\left(1+\lambda^2\right)(\log x)^2}\Bigg(1 - \frac{1+3\lambda^2}{\lambda^2\left(1+\lambda^2\right)}(\log x)^{-2} \\ &\quad + \frac{15\lambda^4 + 10\lambda^2 + 3}{\lambda^4\left(1+\lambda^2\right)^2}(\log x)^{-4} + O\left((\log x)^{-6}\right)\Bigg) \\ &= \frac{\exp\left(-\frac{1+\lambda^2}{2}\right)}{(-\lambda)\pi\left(1+\lambda^2\right)}\Big(1 - \frac{1+3\lambda^2}{\lambda^2\left(1+\lambda^2\right)}(\log x)^{-2} + \frac{15\lambda^4 + 10\lambda^2 + 3}{\lambda^4\left(1+\lambda^2\right)^2}(\log x)^{-4} \\ &\quad + O\left((\log x)^{-6}\right)\Big) \exp\left(-\int_e^x \frac{\left(1+\lambda^2\right)\log s}{s}\left(1 + \frac{2}{\left(1+\lambda^2\right)(\log s)^2}\right) ds\right). \end{aligned}$$

Proof. Follows by integration by parts. □

Using Proposition 3, we can now derive the distributional tail representation of LSN(λ).

Proposition 4. *For large x,*

$$1 - F_\lambda(x) = c(x) \exp\left(-\int_e^x \frac{g(t)}{f(t)} dt\right),$$

where $c(x)$, $g(x)$ and $f(x)$ depend on λ as follows: In the case of $\lambda \geq 0$,

$$c(x) \to \sqrt{\frac{2}{\pi e}} \text{ as } x \to \infty,$$

$$f(x) = \frac{x}{\log x} > 0 \text{ with } f'(x) = -\frac{\log x - 1}{(\log x)^2} \to 0 \text{ as } x \to \infty$$

and

$$g(x) = 1 + \frac{1}{(\log x)^2} \to 1 \ \textit{as}\ x \to \infty;$$

In the case of $\lambda < 0$,

$$c(x) \to \frac{\exp\left(-\frac{1+\lambda^2}{2}\right)}{(-\lambda)\pi\left(1+\lambda^2\right)} \ \textit{as}\ x \to \infty,$$

$$f(x) = \frac{x}{\left(1+\lambda^2\right)\log x} > 0 \ \textit{with}\ f'(x) = -\frac{\log x - 1}{\left(1+\lambda^2\right)(\log x)^2} \to 0 \ \textit{as}\ x \to \infty$$

and

$$g(x) = 1 + \frac{2}{\left(1+\lambda^2\right)(\log x)^2} \to 1 \ \textit{as}\ x \to \infty.$$

In fact, Proposition 4 can also be obtained from Mills' ratio of LSN(λ). By Corollary 1.7 in Resnick (1987), we have $F_\lambda \in D(\Lambda)$ and the norming constants a_n and b_n are given by

$$n^{-1} = 1 - F_\lambda\left(b_n\right), \quad a_n = f\left(b_n\right) \tag{2.4}$$

such that

$$\lim_{n\to\infty} F_\lambda^n\left(a_n x + b_n\right) = \Lambda(x).$$

Remark 1. *The tail representation of* LSN(λ) *can be rewritten as:*

$$1 - F_\lambda(x) = c(x)\exp\left(-\int_e^x \frac{1}{f^*(t)}dt\right)$$

with $f^*(x) = f(t)/g(t)$ *eventually nondecreasing, where* $c(x)$, $f(t)$ *and* $g(t)$ *are those given by Proposition 4. By Corollary 2.5 in Goldie and Resnick (1988), we can easily check that* $F_\lambda \in \mathcal{S} \cap D(\Lambda)$ *since* $\lim_{x\to\infty} f^*(hx)/f^*(x) = h$ *for any constant* $h > 1$.

3 Expansion for the distribution of maximum

In this section, we derive an exact expansion for the distribution of the maximum of LSN(λ) random variables. This expansion is used to show that the convergence rate of $F_\lambda^n(a_n x + b_n)$ to $\Lambda(x)$ is of the order of $O\left((\log n)^{-1/2}\right)$.

Theorem 1. *For norming constants* a_n *and* b_n *given by* (2.4), *we have*

$$(\log b_n)\Big((\log b_n)\left(F_\lambda^n\left(a_n x + b_n\right) - \Lambda(x)\right) - \kappa(x)\Lambda(x)\Big) \to \left(\omega(x) + \frac{\kappa^2(x)}{2}\right)\Lambda(x)$$

as $n \to \infty$, *where* $\kappa(x)$ *and* $\omega(x)$ *depend on* λ *as follows: in the case of* $\lambda \geq 0$,

$$\begin{aligned}\kappa(x) &= -2^{-1}x^2e^{-x},\\ \omega(x) &= -24^{-1}\left(3x^4 - 8x^3 - 12x^2 - 24x\right)e^{-x};\end{aligned}$$

in the case of $\lambda < 0$,

$$\begin{aligned}\kappa(x) &= -2^{-1}\left(1+\lambda^2\right)^{-1}x^2e^{-x},\\ \omega(x) &= -24^{-1}\left(1+\lambda^2\right)^{-2}\left(3x^4 - 8x^3 - 12\left(1+\lambda^2\right)x^2 - 48\left(1+\lambda^2\right)x\right)e^{-x}.\end{aligned}$$

To prove Theorem 1, we need the following auxiliary result.

Lemma 1. *Let $H_\lambda(b_n;x)=F_\lambda(a_nx+b_n)$ and $h_\lambda(b_n;x)=n\log H_\lambda(b_n;x)+e^{-x}$, where the norming constants a_n and b_n are given by (2.4). Then*

$$\lim_{n\to\infty}(\log b_n)\Big((\log b_n)\,h_\lambda(b_n;x)-\kappa(x)\Big)=\omega(x),$$

where $\kappa(x)$ and $\omega(x)$ are those given by Theorem 1.

Proof. First, consider the case of $\lambda\geq 0$. It is easy to check the following two facts by (2.1) and $F_\lambda\in D(\Lambda)$:

$$\lim_{n\to\infty} n\left(1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)\right)=e^{-x} \tag{3.1}$$

and

$$\lim_{n\to\infty}\left(1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)\right)(\log b_n)^2=0. \tag{3.2}$$

Setting

$$\begin{aligned}A_\lambda(b_n) &= \Big[\Phi(\lambda\log b_n)\Big(1-(\log b_n)^{-2}+3(\log b_n)^{-4}\\ &\quad +O\Big((\log b_n)^{-6}\Big)\Big)\Big]\Big[\Phi\left(\lambda\log\left(\frac{b_n}{\log b_n}x+b_n\right)\right)\left(1-\left(\log\left(\frac{b_n}{\log b_n}x+b_n\right)\right)^{-2}\right.\\ &\quad \left.+3\left(\log\left(\frac{b_n}{\log b_n}x+b_n\right)\right)^{-4}+O\Big((\log b_n)^{-6}\Big)\right)\Big]^{-1},\end{aligned}$$

we have $\lim_{n\to\infty}A_\lambda(b_n)=1$ and

$$\lim_{n\to\infty}(A_\lambda(b_n)-1)(\log b_n)^2=0. \tag{3.3}$$

So, by (2.3), we have

$$\begin{aligned}&\frac{1-F_\lambda(b_n)}{1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)}e^{-x}\\ &= A_\lambda(b_n)\exp\left(\int_{b_n}^{b_n+\frac{b_n}{\log b_n}x}\frac{\log s}{s}\left(1+\frac{1}{(\log s)^2}\right)ds-x\right)\\ &= A_\lambda(b_n)\exp\left(\int_0^x\left(\frac{-t+\log\left(1+\frac{t}{\log b_n}\right)}{\log b_n+t}+\frac{1}{(\log b_n+t)\left(\log b_n+\log\left(1+\frac{t}{\log b_n}\right)\right)}\right)dt\right)\\ &= A_\lambda(b_n)\left(1+\int_0^x\left(\frac{-t+\log\left(1+\frac{t}{\log b_n}\right)}{\log b_n+t}+\frac{1}{(\log b_n+t)\left(\log b_n+\log\left(1+\frac{t}{\log b_n}\right)\right)}\right)dt\right.\\ &\quad\left.+\frac{1+o(1)}{2}\left(\int_0^x\left(\frac{-t+\log\left(1+\frac{t}{\log b_n}\right)}{\log b_n+t}+\frac{1}{(\log b_n+t)\left(\log b_n+\log\left(1+\frac{t}{\log b_n}\right)\right)}\right)dt\right)^2\right).\end{aligned}$$

$$(3.4)$$

Combining (2.1), (3.1), (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
& \lim_{n\to\infty} (\log b_n)\, h_\lambda(b_n;x) \\
= & \lim_{n\to\infty} \frac{n\log H_\lambda(b_n;x)+e^{-x}}{(\log b_n)^{-1}} \\
= & \lim_{n\to\infty} \frac{n\left(\log F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)+(1-F_\lambda(b_n))\,e^{-x}\right)}{(\log b_n)^{-1}} \\
= & \lim_{n\to\infty} \frac{n\left(-\left(1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)\right)-\frac{1}{2}\left(1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)\right)^2(1+o(1))+(1-F_\lambda(b_n))\,e^{-x}\right)}{(\log b_n)^{-1}} \\
= & \lim_{n\to\infty} \frac{n\left(1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)\right)\left(-1-\frac{1}{2}\left(1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)\right)(1+o(1))+\frac{1-F_\lambda(b_n)}{1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)}e^{-x}\right)}{(\log b_n)^{-1}} \\
= & \; e^{-x}\lim_{n\to\infty} \frac{-1+\frac{1-F_\lambda(b_n)}{1-F_\lambda\left(\frac{b_n}{\log b_n}x+b_n\right)}e^{-x}}{(\log b_n)^{-1}} \\
= & \; e^{-x}\lim_{n\to\infty} \frac{-1+A_\lambda(b_n)\left(1+\int_0^x\left(\frac{-t+\log\left(1+\frac{t}{\log b_n}\right)}{\log b_n+t}+\frac{1}{(\log b_n+t)\left(\log b_n+\log\left(1+\frac{t}{\log b_n}\right)\right)}\right)dt\,(1+o(1))\right)}{(\log b_n)^{-1}} \\
= & \; e^{-x}\lim_{n\to\infty} \frac{A_\lambda(b_n)-1+A_\lambda(b_n)\int_0^x\left(\frac{-t+\log\left(1+\frac{t}{\log b_n}\right)}{\log b_n+t}+\frac{1}{(\log b_n+t)\left(\log b_n+\log\left(1+\frac{t}{\log b_n}\right)\right)}\right)dt\,(1+o(1))}{(\log b_n)^{-1}} \\
= & \; e^{-x}\lim_{n\to\infty}\int_0^x\left(\frac{-t+\log\left(1+\frac{t}{\log b_n}\right)}{1+\frac{t}{\log b_n}}+\frac{1}{\left(1+\frac{t}{\log b_n}\right)\left(\log b_n+\log\left(1+\frac{t}{\log b_n}\right)\right)}\right)dt \\
= & -\frac{x^2}{2}e^{-x} := \kappa(x),
\end{aligned}$$

where the final step follows by the dominated convergence theorem. Similarly, one can show that

$$\lim_{n\to\infty} (\log b_n)\Big((\log b_n)\, h_\lambda(b_n;x)-\kappa(x)\Big)=\omega(x).$$

The same results hold for $\lambda>0$ by (2.2) and Proposition 4. The arguments are similar and are omitted here.

The proof is complete. □

Proof of Theorem 1. Note that $\lim_{n\to\infty} h_\lambda(b_n;x)=0$ by Lemma 1. Using Lemma 1 again, we have

$$\begin{aligned}
& (\log b_n)\left((\log b_n)\left(F_\lambda(a_n x+b_n)-\Lambda(x)\right)-\kappa(x)\Lambda(x)\right) \\
= & (\log b_n)\left((\log b_n)\left(\exp\left(h_\lambda(b_n;x)-1\right)\right)-\kappa(x)\right)\Lambda(x)
\end{aligned}$$

$$
\begin{aligned}
&= (\log b_n)\left((\log b_n)\left(h_\lambda(b_n;x) + \frac{h_\lambda^2(b_n;x)}{2} + \frac{h_\lambda^3(b_n;x)}{3!}(1+o(1))\right) - \kappa(x)\right) \\
&= \left((\log b_n)((\log b_n) h_\lambda(b_n;x) - \kappa(x)) + (\log b_n)^2 h_\lambda^2(b_n;x)\left(\frac{1}{2} + \frac{h_\lambda(b_n;x)}{3!}(1+o(1))\right)\right)\Lambda(x) \\
&\to \left(\omega(x) + \frac{\kappa^2(x)}{2}\right)\Lambda(x)
\end{aligned}
$$

as $n \to \infty$. The desired result follows. □

Remark 2. *By the definition of b_n, it is easy to check that $1/\log b_n = O\left(1/(\log n)^{1/2}\right)$. So, Theorem 1 shows that the pointwise convergence rate of $F_\lambda^n(a_n x + b_n)$ to its limit is proportional to $1/(\log n)^{1/2}$. Further, the pointwise convergence rate of $(\log b_n)(F_\lambda^n(a_n x + b_n) - \Lambda(x))$ to its limit is also proportional to $1/(\log n)^{1/2}$ by Theorem 1.*

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