Relations Between Hidden Regular Variation and Tail Order of Copulas

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Abstract

We study the relations between tail order of copulas and hidden regular variation (HRV) on subcones generated by order statistics. Multivariate regular variation (MRV) and HRV deal with extremal dependence of random vectors with Pareto-like univariate margins. Alternatively, if one uses copula to model the dependence structure of a random vector, then upper exponent and tail order functions can be used to capture the extremal dependence structure. After defining upper exponent functions on a series of subcones, we establish the relation between tail order of a copula and tail indexes for MRV and HRV. We show that upper exponent functions of a copula and intensity measures of MRV/HRV can be represented by each other, and the upper exponent function on subcones can be expressed by a Pickands-type integral representation. Finally, a mixture model is given with the mixing random vector leading to the finite directional measure in a product-measure representation of HRV intensity measures.

Key words: Multivariate regular variation, tail dependence, upper exponent function, tail order function, intermediate tail dependence.

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1 Introduction

Extremal dependence of a random vector can be described by relative decay rate of certain joint tail probabilities of the random vector with respect to that of its margins. Such extremal dependence can be analyzed by using Multivariate regular variation (MRV) or Hidden Regular Variation (HRV) (Resnick, 2007), or alternatively, by using tail dependence or tail order functions of copulas (Hua and Joe, 2011; Joe et al., 2010; Li and Sun, 2009). With the MRV or HRV methods, univariate marginal distributions are usually transformed to Pareto-like distributions, whereas with the copula method, univariate marginal distributions are transformed to the standard uniform distribution over $[0,1]$. In this paper, we aim at finding the relation between these two approaches.

To explain the two approaches, the following notations will be used throughout the paper. Let

$$I_d := \{1,\dots,d\},\ \overline{\mathbb{R}}_+^d := [0,\infty]^d \text{ and } \mathbb{R}_+^d := [0,\infty)^d.$$

For any d-dimensional real vector $\boldsymbol{x}$, $x_{[i]}$ denotes the ith largest component, and $x_{(i)}$ denotes the ith smallest component. For a subset $I \subseteq I_d$, $|I|$ is the cardinality of I; $\boldsymbol{x}_I := (x_i, i \in I)$. If $(x_j, j \in J)$, $J \subseteq I_d$, is a collection of variables x_j's, then $(x_J)_{[i]}$ is the ith largest of these x_j's, and similarly, $(x_J)_{(i)}$ is the ith smallest of these x_j's. For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the sum $\boldsymbol{x}+\boldsymbol{y}$, product $\boldsymbol{x}\boldsymbol{y}$, quotient $\boldsymbol{x}/\boldsymbol{y}$ and vector power and vector inequalities such as $\boldsymbol{x} \le \boldsymbol{y}$ are all operated component-wise.

For a measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$, if for any $t > 0$, $\lim_{x\to\infty} g(xt)/g(x) = t^\alpha$ with $\alpha \in \mathbb{R}$, then g is said to be regularly varying at ∞ with variation exponent α, and we denote this as $g \in \mathrm{RV}_\alpha$; if $\alpha = 0$, then g is said to be slowly varying at ∞ and we often specifically use ℓ as a slowly varying function. For two positive functions g and h, $g(t) \sim h(t)$ as $t \to t_0$ means $\lim_{t\to t_0}[g(t)/h(t)] = 1$. For $l, m \in I_d$, $E^{(l)} := \{\boldsymbol{x} \in \overline{\mathbb{R}}_+^d : x_{[l]} > 0\}$ and thus $E^{(m)} \subseteq E^{(l)}$ if $l \le m$. Also, define $E_0^{(l)} := \{\boldsymbol{x} \in \mathbb{R}_+^d : x_{[l]} > 0\}$. Let $\nu_l(\cdot)$ denote the corresponding limit measure (aka. intensity measure) for Multivariate Regular Variation (MRV) on subcone $E^{(l)}$. In what follows, we use $-\alpha_l$ with $\alpha_l > 0$ as the corresponding exponent of regular variation on $E^{(l)}$ and call $\alpha_l > 0$ its *tail index*.

Let $\boldsymbol{X}$ be a random vector with identical univariate margins F_i, $1 \le i \le d$, that are regularly varying with tail index $\alpha > 0$ (i.e., $\overline{F}_i \in \mathrm{RV}_{-\alpha}$). Roughly speaking, if the decay rate of certain joint tail probabilities of $\boldsymbol{X}$ is comparable to the tail decay rate of the marginal survival function $1 - F_i$, then the usual tail dependence (see, e.g., Joe (1997), page 33) appears. However, if the dependence in the upper tail is not sufficiently strong, then the joint tail probability decays at a faster rate that may be comparable to that of a function $g \in \mathrm{RV}_{-\alpha_2}$ with a larger tail index $\alpha_2 > \alpha$, then we need to use HRV to capture the dependence structure hidden in the upper tail interior. Alternatively, we can use a copula C to capture the dependence structure of $\boldsymbol{X}$. Let $\widehat{C}$ be the corresponding survival copula of $\boldsymbol{X}$. If $\widehat{C}(u, \dots, u) \sim u^{\kappa}\ell(u)$ as $u \to 0^+$, then κ is referred to as the upper tail order of C and $\kappa \geq 1$ (Hua and Joe, 2011). Here, a larger κ tends to lead to a weaker dependence in the upper tail. In this paper, we will show that for the HRV case, $\kappa = \alpha_2/\alpha$. We will study the relation between the intensity measures of MRV or HRV and the tail dependence/order functions of copulas; moreover, through defining an upper exponent function on a subcone for copulas, we will obtain explicitly the expression that connects the HRV intensity measure and tail order functions.

Recently, Mitra and Resnick (2010, 2011) employ order statistics to construct a product-measure decomposition for characterizing HRV. Their approach overcomes the issue of infinite angular measures for HRV that is studied earlier in Maulik and Resnick (2004) or Section 9.4.1 of Resnick (2007). We find that the finite directional measure studied in Mitra and Resnick (2010, 2011) can also be used to characterize the upper exponent functions for copulas. Furthermore, a mixture model for HRV is studied, in which the distribution function of a mixing random vector will be related to the finite directional measure in the product-measure decomposition for HRV.

The remainder of this paper is organized as follows: in Section 2, the relation between MRV and the usual tail dependence or the upper tail order $\kappa = 1$ will be discussed; in Section 3, the relation will be extended to the comparison between HRV and the upper tail order κ; the relation between tail order function and a Pickands-type representation with finite directional measure for HRV will be investigated in Section 4; a mixture model for HRV is presented in Section 5; finally, some remarks in Section 6 conclude the paper.

2 Tail dependence for MRV

Let $\boldsymbol{X} := (X_1, \dots, X_d)$ be a random vector with joint cumulative distribution function (cdf) F and continuous univariate margins $F_1, \dots, F_d$. Without loss of generality, we may assume that $\boldsymbol{X}$ is non-negative componentwise. Consider the standard case in which the survival functions $\overline{F}_i(x) := 1 - F_i(x)$, $1 \leq i \leq d$, of the univariate margins are right tail equivalent; that is,

$$\frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \frac{1-F_i(x)}{1-F_1(x)} \to 1, \text{ as } x \to \infty, \ 1 \le i \le d. \tag{2.1}$$

The distribution F or the random vector $\boldsymbol{X}$ is said to be of MRV at ∞ with intensity measure ν if there exists a scaling function $h(t) \uparrow \infty$ and a non-zero Radon measure $\nu(\cdot)$ such that as $t \uparrow \infty$, the following vague convergence holds,

$$t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{h(t)} \in \cdot\right) \xrightarrow{v} \nu(\cdot), \quad \text{in cone } \ \overline{\mathbb{R}}_+^d \setminus \{\boldsymbol{0}\}; \tag{2.2}$$

that is, $t\,\mathbb{P}\left(\boldsymbol{X}/h(t) \in B\right) \to \ \nu(B)$, for any relatively compact set $B \subseteq \overline{\mathbb{R}}_+^d \setminus \{\boldsymbol{0}\}$, with $\nu(\partial B) = 0$. Note that, the MRV discussed in this section is actually MRV on cone $E^{(1)}$, and MRV can also be defined on subcones in the sense of (3.1). The extremal dependence information of $\boldsymbol{X}$ is encoded in the intensity measure ν that satisfies $\nu(tB) = t^{-\alpha}\nu(B)$, for all relatively compact subsets B that are bounded away from the origin, where $\alpha > 0$ is known as the *tail index* for MRV (i.e., $-\alpha$ is the exponent of variation for MRV). Since the set $B_1 = \{\boldsymbol{x} \in \overline{\mathbb{R}}_+^d : x_1 > 1\}$ is relatively compact within the cone $\overline{\mathbb{R}}_+^d \setminus \{\boldsymbol{0}\}$ with $\nu(\partial B_1) = 0$ and $\nu(B_1) > 0$ under (2.1), it follows from (2.2) that the scaling function $h(t)$ can be chosen to satisfy that $\overline{F}_1(h(t)) = t^{-1}$, $t > 0$, after appropriately normalizing the intensity measure by $\nu(B_1)$. That is, $h(t)$ can be chosen as $h(t) = \overline{F}^{\leftarrow}(t^{-1}) = F_1^{\leftarrow}(1-t^{-1})$ under the condition (2.1), where $F^{\leftarrow}(t) := \inf(x : F(x) \ge t)$ are the quantile functions of F, and $F^{\leftarrow}$ are left-continuous. Therefore, (2.2) can be expressed equivalently as

$$\lim_{t\to\infty} \frac{\mathbb{P}(\boldsymbol{X} \in tB)}{\mathbb{P}(X_1 > t)} = \nu(B), \ \forall \text{ relatively compact sets } B \subseteq \overline{\mathbb{R}}_+^d \setminus \{\boldsymbol{0}\}, \tag{2.3}$$

satisfying that $\mu(\partial B) = 0$. It follows from (2.3) and (2.1) that for $1 \le i \le d$,

$$\lim_{t\to\infty} \frac{\mathbb{P}(X_i > ts)}{\mathbb{P}(X_i > t)} = \nu((s,\infty] \times \overline{\mathbb{R}}_+^{d-1}) = s^{-\alpha}\nu((1,\infty] \times \overline{\mathbb{R}}_+^{d-1}), \ \forall\ s > 0.$$

That is, univariate margins have regularly varying right tails and $\overline{F}_i \in \mathrm{RV}_{-\alpha}$. In general, a Borel-measurable function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is regularly varying with exponent $\rho \in \mathbb{R}$, denoted as $g \in \mathrm{RV}_\rho$, if and only if

$$g(t) = t^\rho \ell(t), \text{ with } \ell(\cdot) \ge 0 \text{ satisfying that } \lim_{t\to\infty} \frac{\ell(ts)}{\ell(t)} = 1, \text{ for } s > 0. \tag{2.4}$$

The function $\ell(\cdot)$ is slowly varying, that is, $\ell \in \mathrm{RV}_0$. Since $\overline{F}_1 \in \mathrm{RV}_{-\alpha}$, $1/\overline{F}_1 \in \mathrm{RV}_\alpha$, by Proposition 2.6 (v) of Resnick (2007), the scaling function h in (2.2) satisfies that $h \in \mathrm{RV}_{\alpha^{-1}}$. Since all the margins are tail equivalent as assumed in (2.1), one has

$$\overline{F}_i(t) = t^{-\alpha}\ell_i(t), \text{ where } \ell_i \in \mathrm{RV}_0, \text{ and } \ell_i(t)/\ell_j(t) \to 1 \text{ as } t \to \infty, \text{ for any } i \neq j, \tag{2.5}$$

which, together with $\overline{F}_1(h(t)) = t^{-1}$, imply that

$$\lim_{t\to\infty} t\,\mathbb{P}(X_i > h(t)s) = \frac{\mathbb{P}(X_i > h(t)s)}{\overline{F}_i(h(t))}\frac{\overline{F}_i(h(t))}{\overline{F}_1(h(t))} = s^{-\alpha}, \ s > 0, \ 1 \le i \le d. \tag{2.6}$$

More detailed discussions on univariate regular variation and MRV can be found in Bingham et al. (1987); Resnick (1987, 2007). The extension of MRV beyond the non-negative orthant can be done by using the tail probability of $||\boldsymbol{X}||$, where $||\cdot||$ denotes a norm on $\mathbb{R}^d$, in place of the marginal tail probability in (2.3) (Resnick, 2007, Section 6.5.5). The case that the limit in (2.1) is any non-zero constant can be easily converted into the standard tail equivalent case by properly rescaling margins. If the limit in (2.1) is zero or infinity, then some margins have heavier tails than the others. One way to overcome this problem is to standardize the margins via marginal monotone transforms.

A copula C is a multivariate distribution with standard uniformly distributed margins on $[0,1]$. Sklar's theorem (see, e.g., Joe (1997), Section 1.6) states that every multivariate distribution F with univariate margins $F_1,\dots,F_d$ can be written as $F(x_1,\dots,x_d) = C(F_1(x_1),\dots,F_d(x_d))$ for some d-dimensional copula C. In the case of continuous univariate cdfs, C is unique and

$$C(u_1,\dots,u_d) = F(F_1^{\leftarrow}(u_1),\dots,F_d^{\leftarrow}(u_d)).$$

Let $(U_1,\dots,U_d)$ denote a random vector with $U_i, 1 \le i \le d$, being uniformly distributed on $[0,1]$. The survival copula $\widehat{C}$ is defined as

$$\widehat{C}(u_1,\dots,u_n) = \mathbb{P}(1-U_1 \le u_1,\dots,1-U_n \le u_n) = \overline{C}(1-u_1,\dots,1-u_n), \tag{2.7}$$

where $\overline{C} := 1 + \sum_{\emptyset\neq I\subseteq I_d}(-1)^{|I|}C_I$, is the joint survival function of C, where C_I is the copula for the I-margin. The lower and upper tail dependence functions, introduced in Jaworski (2006); Joe et al. (2010); Klüppelberg et al. (2008); Nikoloulopoulos et al. (2009), are defined as follows,

$$\begin{aligned} b^L(\boldsymbol{w};C) &:= \lim_{u\to 0^+} \frac{C(uw_i, 1 \le i \le d)}{u}, \ \forall \boldsymbol{w} > \boldsymbol{0} \\ b^U(\boldsymbol{w};C) &:= \lim_{u\to 0^+} \frac{\overline{C}(1-uw_i, 1 \le i \le d)}{u}, \ \forall \boldsymbol{w} > \boldsymbol{0} \end{aligned} \tag{2.8}$$

provided that the limits exist. Since $b^L(\boldsymbol{w};\widehat{C}) = b^U(\boldsymbol{w};C)$, a result on upper tail dependence can be easily translated into a similar result for lower tail dependence, and thus we only focus on upper tail dependence in this paper. Instead of upper orthants used in (2.8), it is often more convenient to work with the complements of lower orthants, leading to the *upper exponent function* (Joe et al., 2010; Nikoloulopoulos et al., 2009):

$$a^U(\boldsymbol{w};C) := \lim_{u\to 0^+} \frac{\mathbb{P}(\bigcup_{i=1}^d \{U_i > 1-uw_i\})}{u}, \quad \boldsymbol{w} \in E_0^{(1)} \tag{2.9}$$

provided that the limits exist. Note that exponent functions and tail dependence functions are related through inclusion-exclusion relations. If the exponent function $a^U(\cdot;C)$ exists for a d-dimensional copula C, then the exponent function $a^U(\boldsymbol{w}_I;C_I)$ of any multivariate margin $C_I(u_i, i\in I)$ of C, also exists. Therefore, the existence of the exponent function $a^U(\cdot;C)$ implies that the upper tail dependence function $b^U(\cdot;C_I)$ of any multivariate margin $C_I(u_i, i\in I)$ of C exists (may be 0).

With the copula approach, the intensity measure ν in (2.3) can be decomposed, as was shown in Li and Sun (2009), into the scale-invariant tail dependence and tail index.

Theorem 2.1. Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector with distribution F and copula C, satisfying (2.1).

1. If F is MRV as defined in (2.3) with intensity measure ν, then for all continuity points $\boldsymbol{w} > \mathbf{0}$,
$$b^U(\boldsymbol{w};C) = \nu\Big(\prod_{i=1}^d (w_i^{-1/\alpha}, \infty]\Big), \text{ and } a^U(\boldsymbol{w};C) = \nu\Big(\big(\prod_{i=1}^d [0, w_i^{-1/\alpha}]\big)^c\Big).$$
2. If the upper exponent function defined in (2.9) exists, and marginal distributions $F_1, \ldots, F_d$ are regularly varying, then $F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d))$ is MRV on $E^{(1)}$.

Example 2.1. (Tail comonotonicity (Hua and Joe, 2012b,c)) Let $X_1,\ldots,X_d$ are non-negative random variables with survival functions $\overline{F}_i$'s satisfying (2.1) and $\overline{F}_1 \in \mathrm{RV}_{-\alpha}$. Suppose $\boldsymbol{X} := (X_1,\ldots,X_d)$ is upper tail comonotonic with copula C; that is, the upper tail dependence function is $b^U(w_1,\ldots w_d) = \lim_{u\to 0^+} \overline{C}(1-uw_1,\ldots,1-uw_d)/u = \min\{w_1,\ldots,w_d\}$ for $w_i > 0, i = 1,\ldots,d$.

On one hand, by definition, $a^U(\boldsymbol{w};C) = \sum_{\emptyset\neq I\subseteq I_d} (-1)^{|I|-1} b(\boldsymbol{w}_I;C_I)$. By Proposition 2 of Hua and Joe (2012c), $b(\boldsymbol{w}_I;C_I) = \min\{w_i, i\in I\}$. Without loss of generality, let $w_1 \le \cdots \le w_d$. Then $a^U(\boldsymbol{w};C) = \sum_{j=0}^{d-1}\sum_{i=1}^{d-j}(-1)^j w_i \binom{d-i}{j} = w_d = \max\{w_1,\ldots,w_d\}$. On the other hand, by Resnick (2007) (page 196), the intensity measure ν satisfies $\nu([\mathbf{0},\mathbf{x}]^c) = (\min\{x_1,\ldots,x_d\})^{-\alpha}$. Then, by Theorem 2.1, we shall have $a^U(\boldsymbol{w};C) = \Big(\min\{w_1^{-1/\alpha},\ldots,w_d^{-1/\alpha}\}\Big)^{-\alpha} = \max\{w_1,\ldots,w_d\}$, which is consistent to what we have derived.

3 Tail order for HRV on subcones

The regular variation property (2.3) defined on the cone $\overline{\mathbb{R}}_+^d \setminus \{\mathbf{0}\}$ employs the relatively faster scaling $h(t)$ that is necessary for convergence on the margins (see (2.6)), but such a coarse normalization fails to reveal the finer dependence structure that may be present in the interior. A scaling of smaller order is necessary for any regular variation properties resided or hidden in a smaller cone $E^{(l)}$ for $l = 2, 3, \ldots, d$. Precisely speaking, the MRV discussed in Section 2 is MRV on the cone $E^{(1)}$. MRV can also be defined on subcones (Mitra and Resnick, 2010, 2011). Let $\boldsymbol{X}$ be a non-negative random vector. Then $\boldsymbol{X}$ possesses MRV on a subcone $E^{(l)}$ if there exists a scaling function $h_l(t) \uparrow \infty$ and a non-zero Radon measure ν_l such that

$$t\mathbb{P}\left(\frac{\boldsymbol{X}}{h_l(t)} \in \cdot\right) \xrightarrow{v} \nu_l(\cdot), \text{ in } E^{(l)}, \quad \text{as } t \to \infty. \tag{3.1}$$

Moreover, if $\boldsymbol{X}$ also has MRV on a subcone $E^{(m)}$ that is a proper subset of $E^{(l)}$, then $\boldsymbol{X}$ is said to possess HRV on $E^{(m)}$.

In order to illustrate the relation between tail order of a copula and HRV, we first focus on the simpler case where HRV is defined on the subcone $E^{(2)}$. A random vector $\boldsymbol{X}$ is said to have HRV on $E^{(2)}$, if, in addition to (2.2), there exists an increasing scaling function $h_2(t) \uparrow \infty$ such that $h(t)/h_2(t) \to \infty$ as $t \to \infty$, and there exists a non-zero Radon measure ν_2 on $E^{(2)}$ such that

$$t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{h_2(t)} \in B\right) \to \nu_2(B), \text{ as } t \to \infty \tag{3.2}$$

for all relatively compact sets $B \subseteq E^{(2)}$ satisfying $\nu_2(\partial B) = 0$. See Section 9.4 of Resnick (2007) for more details on HRV on $E^{(2)}$.

Note that ν_2 is necessarily homogeneous of order $-\alpha_2$ on $E^{(2)}$ with $\alpha_2 \geq \alpha$. Consider the set $B_\wedge = \{\boldsymbol{x} \in \mathbb{R}_+^d : \wedge_{i=1}^d x_i > 1\}$, which is relatively compact within $E^{(2)}$. Since $\nu_2(\cdot)$ is non-zero and homogeneous, we must have $\nu_2(B_\wedge) > 0$. Since (3.2) implies that $t\mathbb{P}(\wedge_{i=1}^d X_i > h_2(t)) \to \nu_2(B_\wedge)$, then

$$\frac{\mathbb{P}(\wedge_{i=1}^d X_i > h_2(t)s)}{\mathbb{P}(\wedge_{i=1}^d X_i > h_2(t))} \to \frac{\nu_2(sB_\wedge)}{\nu_2(B_\wedge)} = s^{-\alpha_2}, \text{ as } t \to \infty. \tag{3.3}$$

Let $\overline{F}_\wedge(t) := \mathbb{P}(\wedge_{i=1}^d X_i > t)$, and the above limit shows that $\overline{F}_\wedge \in \mathrm{RV}_{-\alpha_2}$. The scaling function $h_2(\cdot)$ can be chosen to satisfy that $\overline{F}_\wedge(h_2(t)) = t^{-1}$, $t > 0$, after appropriately normalizing the hidden intensity measure by $\nu_2(B_\wedge)$. That is, $h_2(t)$ can be chosen as $h_2(t) = F_\wedge^{\leftarrow}(1 - t^{-1})$, where $F_\wedge^{\leftarrow}(\cdot)$ denotes the left-continuous inverse of $F_\wedge(\cdot)$, and thus $h_2 \in \mathrm{RV}_{\alpha_2^{-1}}$.

Remark 3.1. 1. The typical relatively compact sets in $E^{(2)}$ include subsets $\{x_i > h_2(t)w_i, x_j > h_2(t)w_j\}$, $w_i > 0, w_j > 0$, $i \neq j$, whereas the topology on $E^{(1)}$ makes marginal events such as $\{x_i > h(t)w_i\}$ relatively compact. In order for $\mathbb{P}(X_i > h_2(t)w_i, X_j > h_2(t)w_j)$ and $\mathbb{P}(X_i > h(t)w_i)$ to decay to zero at a comparable speed, $h(t)$ has to grow relatively faster than $h_2(t)$ does to accommodate the relatively large marginal tail probability. Moreover, if both (2.2) and (3.2) hold, then for any $w > 0$, and $B := (w, \infty]^2 \times \overline{\mathbb{R}}_+^{d-2} \subseteq E^{(2)}$,

$$\begin{aligned} \nu(B) &= \lim_{t\to\infty} t\,\mathbb{P}(\boldsymbol{X} \in h(t)B) = \lim_{t\to\infty} t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{h_2(t)} \in \frac{h(t)}{h_2(t)}B\right) \\ &\sim \nu_2\left(\frac{h(t)}{h_2(t)}B\right) \to 0, \text{ as } t \to \infty, \end{aligned}$$

because $h(t)/h_2(t) \to \infty$ implies that $h(t)B/h_2(t) \to \emptyset$ as $t \to \infty$. Thus HRV on $E^{(2)}$ implies asymptotic independence in the sense that $\nu(E^{(2)}) = 0$ (see Proposition 5.27 of Resnick (1987) and Property 9.1 of Resnick (2007)).

2. If F has HRV on $E^{(2)}$, then univariate margins F_i, $1 \le i \le d$, have regularly varying right tails with tail index α. In contrast, F could have lighter multivariate tails in the interior with tail index α_2. Geometrically, if $\alpha = \alpha_2$ with $h(t)/h_2(t) \to c > 0$ (c is a constant), marginal tails are in comparable magnitude with tails in the interior that hang the marginal tails together, resulting in dependence among multivariate extremes. If $\alpha < \alpha_2$, tails in the interior are lighter and decay faster than marginal tails, resulting in lack of tail dependence among random variables. In such a case, scaling that is comparable to lighter tails in the interior must be used to reveal the extremal dependence structure in the interior.

HRV can be also analyzed using the copula method. Since the copula C of distribution F satisfies

$$\overline{C}(1 - uw_i, 1 \le i \le d) \le u \min\{w_1, \dots, w_d\},\ \forall\ 0 \le u \le 1,\ (w_1, \dots, w_d) \in \mathbb{R}_+^d,$$

if the decay rate of the left hand side with respect to u is faster than u as $u \to 0$, then the usual tail dependence function would be 0 (i.e., $b^U(\boldsymbol{w}, C) \equiv 0$), and in this case a higher order approximation with scaling u^κ, $\kappa > 1$ must be used to reveal finer information about extremal dependence. For this sake, an upper tail order function (Hua and Joe, 2011) is introduced as follows:

$$b^U(\boldsymbol{w}; \kappa) := \lim_{u\to 0^+} \frac{\overline{C}(1 - uw_i, 1 \le i \le d)}{u^\kappa \ell(u)},\ w_i > 0, 1 \le i \le d, \tag{3.4}$$

if the limit exists for $\kappa \geq 1$ and some non-negative function $\ell(u)$ that is slowly varying at 0 (i.e., $\ell(t^{-1}) \in \mathrm{RV}_0$). The lower tail order function can be similarly defined.

The idea of the tail order function is to explore higher order approximations to extremal dependence in the upper tail from the tail of the uniform margin. If

$$\overline{C}(1-uw_i, 1 \leq i \leq d) \sim u^{\kappa}\ell(u)b^U(\boldsymbol{w};\kappa) = [\mathbb{P}(F_i(X_i) > 1 - u\,\ell^{1/\kappa}(u))]^{\kappa} b^U(\boldsymbol{w};\kappa),$$

then as u is small, $b^U(\boldsymbol{w};\kappa)$ can be used to capture the dependence structure emerged from joint tails at the rate of κ times the rate of the marginal tail. The constant $\kappa \geq 1$ is referred to as *tail order*. The case where $\kappa = 1$ and $\lim_{u\to 0}\ell(u) = \lambda > 0$ corresponds to the usual tail dependence (2.8), and the case where $d > \kappa > 1$ may lead to intermediate tail dependence (Hua and Joe, 2011). More precisely, when certain positive dependence assumptions hold (Hua and Joe, 2011), a copula C is said to have intermediate tail dependence if the limit (3.4) exists and is non-zero for a scaling function $u^{\kappa}\ell(u)$ satisfying that

$$u^{\kappa-1}\ell(u) = \frac{u^{\kappa}\ell(u)}{u} \to 0, \text{ and } u^{\kappa-d}\ell(u) = \frac{u^{\kappa}\ell(u)}{u^d} \to \infty, \text{ as } u \to 0. \tag{3.5}$$

That is, the scaling function $u^{\kappa}\ell(u)$ decays to zero at a faster rate than that of the linear scaling u used in (2.8) but slower than that of u^d. When $\kappa = d$ and $\ell(u) \to k$ a finite non-zero value, we refer to this case as *tail orthant independence*. In this paper we will show that HRV may not only lead to intermediate tail dependence or tail orthant independence, but also may give rise to tail negative dependence (see Remark 4.3 for a quick impression).

Similar to the situation for a tail dependence function (see Joe et al. (2010)), the existence of the upper tail order function of a copula C does not in general ensure the existence of upper tail order functions of its multivariate margins. Note that the upper tail order function describes the relative decay rate of joint probabilities on upper orthants, whereas for HRV on different subcones $E^{(l)}$, $1 \leq l < d$, the corresponding intensity measure may have masses on complements of lower orthants. That is, upper orthant sets may not contain all compact subsets in $E^{(l)}$, $1 \leq l < d$, with an exception of the smallest subcone $E^{(d)}$, in which any compact subset is contained in an upper orthant set. This is a key to establishing the relation between the measure-theoretic MRV method and orthant-based copula approach (see Lemma 6.1 in Resnick (2007)). To this end, we introduce upper exponent functions on subcones. Let $\boldsymbol{U} := (U_1, \ldots, U_d)$ have the copula C with

U_i being uniformly distributed on $[0,1]$ for each i. Define the *upper exponent function on* $E^{(2)}$ as follows,

$$a^U(\boldsymbol{w};2,\kappa) := \lim_{u\to 0^+} \frac{\mathbb{P}\big(\bigcup_{i\neq j}\{U_i > 1-uw_i, U_j > 1-uw_j\}\big)}{u^\kappa \ell(u)}, \quad \boldsymbol{w} \in E_0^{(2)}, \tag{3.6}$$

provided that the limit exists, where $\ell(u)$, slowly varying at 0, satisfies (3.5). Note that the upper exponent function a^U (on $E^{(1)}$) in (2.9) describes the tail dependence among univariate margins on $[0,1]^d$, whereas $a^U(\boldsymbol{w};2,\kappa)$ describes the tail dependence among bivariate margins hidden in $E^{(2)}$. Similarly, we can define the upper exponent function on a subcone $E^{(l)}$ as follows, for $2 \le l \le d$,

$$a^U(\boldsymbol{w};l,\kappa) := \lim_{u\to 0^+} \frac{\mathbb{P}\left(\bigcup_{i_1\neq i_2 \neq \cdots \neq i_l}\{U_{i_1} > 1-uw_{i_1}, \dots, U_{i_l} > 1-uw_{i_l}\}\right)}{u^\kappa \ell(u)}, \quad \boldsymbol{w} \in E_0^{(l)}, \tag{3.7}$$

provided that the limit exists. Observe that for $I \subseteq I_d$ with $|I| = l$, $a^U((\boldsymbol{w}_I, \mathbf{0}_{I^c}); l, \kappa) = b_I^U(\boldsymbol{w}_I; \kappa)$, the tail order function of the multivariate marginal copula $C_I(u_i, i \in I)$.

The upper exponent function on $E^{(2)}$ and upper tail order functions for any multivariate marginal copula C_I with $|I| \ge 2$ are related via inclusion-exclusion relations.

Proposition 3.1. *If the exponent function* $a^U(\boldsymbol{w};2,\kappa)$ *defined in* (3.6) *exists, then the upper tail order function* $b_I^U(\cdot;\kappa)$ *of any multivariate margin* C_I *with* $I \subseteq I_d$ *and* $2 \le |I| \le d$ *also exists.*

Proof. We prove the statement by induction. When $d = 2$, $a^U(\cdot;2,\kappa) = b^U(\cdot;\kappa)$. Suppose that the statement is true for dimension $d-1$ (≥ 1) or less, we need to show that the statement is true for dimension d. If $a^U(\boldsymbol{w};2,\kappa)$ exists, then the upper exponent function $a_I^U(\boldsymbol{w}_I;2,\kappa)$ for any multivariate margin $C_I(u_i, i \in I)$ with $I \subseteq I_d$ and $2 \le |I| < d$ also exists. The induction hypothesis implies that the upper tail order function $b_I^U(\cdot;\kappa)$ of any multivariate margin $C_I(u_i, i \in I)$ of C, $I \subset \{1,\dots,d\}$, $|I| \ge 2$, exists. We now need to show that the tail order function $b^U(w;\kappa)$ of C exists.

When $d \ge 3$, for $\boldsymbol{w} \in E_0^{(2)}$,

$$\begin{aligned}
&\mathbb{P}\left(\cup_{i\neq j}\{U_i > 1-uw_i, U_j > 1-uw_j\}\right) \\
&= \mathbb{P}\left(\cup_{i=1}^d \{U_i > 1-uw_i\}\right) - \sum_{i=1}^d \mathbb{P}\left(\{U_i > 1-uw_i\} \cap \left(\cap_{j\neq i}\{U_j \le 1-uw_j\}\right)\right).
\end{aligned}$$

For any given $i \in I_d$,

$$
\begin{aligned}
&\mathbb{P}\left(U_i > 1 - uw_i, U_j \le 1 - uw_j \text{ for all} j \ne i\right) \\
&= \mathbb{P}\left(U_i > 1 - uw_i\right) - \mathbb{P}\left(\{U_i > 1 - uw_i\} \cap \left(\cup_{j \in I_d \setminus \{i\}} \{U_j > 1 - uw_j\}\right)\right) \\
&= \mathbb{P}\left(U_i > 1 - uw_i\right) - \sum_{\emptyset \ne J \subseteq I_d \setminus \{i\}} (-1)^{|J|-1} \mathbb{P}\left(\cap_{j \in J} \{U_i > 1 - uw_i, U_j > 1 - uw_j\}\right) \\
&= \mathbb{P}\left(U_i > 1 - uw_i\right) - \sum_{j \ne i} \mathbb{P}(U_i > 1 - uw_i, U_j > 1 - uw_j) \\
&\quad + \cdots + (-1)^{d-1} \mathbb{P}(U_1 > 1 - uw_1, \ldots, U_d > 1 - uw_d).
\end{aligned}
$$

In addition,

$$
\mathbb{P}\left(\cup_{i=1}^d \{U_i > 1 - uw_i\}\right) = \sum_{\emptyset \ne J \subseteq I_d} (-1)^{|J|-1} \mathbb{P}\left(U_j > 1 - uw_j, j \in J\right).
$$

Therefore,

$$
\begin{aligned}
&\mathbb{P}\big(\cup_{i \ne j} \{U_i > 1 - uw_i, U_j > 1 - uw_j\}\big) = H(uw_1, \ldots, uw_d) \\
&- (-1)^{d-1}(d-1)\mathbb{P}(U_1 > 1 - uw_1, \ldots, U_d > 1 - uw_d),
\end{aligned}
$$

where $H(uw_1, \ldots, uw_d)$ is a linear function of $\mathbb{P}(U_j > 1 - uw_j, j \in I)$, $I \subset \{1, \ldots, d\}$ with $d > |I| \ge 2$. Since the upper tail order function $b_I^U(\cdot;\kappa)$ of any multivariate margin $C_I(u_i, i \in I)$ of C, $I \subset \{1, \ldots, d\}$, $d > |I| \ge 2$, exists, the limit $\lim_{u \to 0^+} H(uw_1, \ldots, uw_d)/(u^\kappa \ell(u))$ exists. Thus

$$
b^U(w;\kappa) = (-1)^d (d-1)^{-1} \left[a^U(w; 2, \kappa) - \lim_{u \to 0^+} \frac{H(uw_1, \ldots, uw_d)}{u^\kappa \ell(u)} \right]
$$

exists. □

Remark 3.2. 1. Similarly, if the upper exponent function $a^U(\boldsymbol{w}; l, \kappa)$ on $E^{(l)}$ defined in (3.7) exists, then the upper tail order function $b_I^U(\cdot;\kappa)$ of any marginal copula C_I with $I \subseteq I_d$ and $l \le |I| \le d$ also exists. In particular, $a^U(\boldsymbol{w}; d, \kappa) = b^U(\boldsymbol{w};\kappa)$ in the smallest subcone $E^{(d)}$.

2. In general, the upper exponent function $a^U(\boldsymbol{w}; l, \kappa) \not\equiv 0$ on $E^{(l)}$ if and only if the upper tail order functions $b_I^U(\cdot;\kappa)$ are non-zero for at least one $|I|$-dimensional margin C_I with $|I| = l$.

Relations between tail order and HRV can be established by Propositions 3.3 and 3.5 below. The proofs of the two propositions need repeated use of some operating properties of regularly varying functions (see, e.g., Proposition 2.6 in Resnick (2007)). These properties are usually proved for increasing regularly varying functions, but we also need these properties for decreasing regularly varying functions. We list these properties for decreasing regularly varying functions in the following lemma, and their proofs can be obtained from Proposition 2.6 in Resnick (2007) by using simple variable substitutions.

Lemma 3.2. Let $g, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel-measurable. Let $g^{\leftarrow}, g_1^{\leftarrow}, g_2^{\leftarrow}$ denote, if exist, the left-continuous inverses of g, g_1, g_2 respectively.

1. If $g_1 \in \mathrm{RV}_{\rho_1}$ and $g_2 \in \mathrm{RV}_{\rho_2}$, $\rho_2 < \infty$, and $\lim_{t\to\infty} g_2(t) = \infty$, then $g_1 \circ g_2 \in \mathrm{RV}_{\rho_1\rho_2}$.
2. If $g_1(t^{-1}) \in \mathrm{RV}_{\rho_1}$ and $g_2 \in \mathrm{RV}_{\rho_2}$, $\rho_2 > -\infty$ and $\lim_{t\to\infty} g_2(t) = 0$, then $g_1 \circ g_2 \in \mathrm{RV}_{-\rho_1\rho_2}$.
3. Suppose that g is increasing, $\lim_{t\to\infty} g(t) = \infty$, and $g \in \mathrm{RV}_\rho$, $\rho \ge 0$. Then $g^{\leftarrow} \in \mathrm{RV}_{\rho^{-1}}$.
4. Suppose that g is decreasing, $\lim_{t\to\infty} g(t) = 0$, and $g \in \mathrm{RV}_{-\rho}$, $\rho \ge 0$. Then $g^{\leftarrow}$ is regularly varying at 0 with exponent $-\rho^{-1}$, or equivalently, $g^{\leftarrow}(t^{-1}) \in \mathrm{RV}_{\rho^{-1}}$.
5. Suppose that g_1 and g_2 are increasing and regularly varying at ∞ with exponent ρ, $\rho > 0$. Then for $0 < \gamma < \infty$
$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = \gamma \ \text{ if and only if } \ \lim_{t\to\infty} \frac{g_1^{\leftarrow}(u)}{g_2^{\leftarrow}(u)} = \gamma^{-\rho^{-1}}.$$
6. Suppose that g_1 and g_2 are decreasing and regularly varying at ∞ with exponent $-\rho$, $\rho > 0$. Then for $0 < \gamma < \infty$
$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = \gamma \ \text{ if and only if } \ \lim_{u\to 0} \frac{g_1^{\leftarrow}(u)}{g_2^{\leftarrow}(u)} = \gamma^{\rho^{-1}}.$$

The following result establishes the relation between HRV on $E^{(2)}$ and the upper exponent function $a^U(\cdot; 2, \kappa)$ on $E^{(2)}$, where the κ is determined by the tail index α for MRV on $E^{(1)}$ and the tail index α_2 for MRV on $E^{(2)}$.

Proposition 3.3. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a non-negative random vector with distribution F, continuous margins $F_1, \dots, F_d$ satisfying (2.1), and copula C. Assume that F is regularly varying with intensity measure ν and tail index α. If F has HRV on $E^{(2)}$ with intensity measures ν_2 and tail index α_2, then the upper exponent function $a^U(\boldsymbol{w}; 2, \kappa)$ of C exists, where $\kappa = \alpha_2/\alpha$, and*
$$a^U(\boldsymbol{w}; 2, \kappa) = \nu_2\Big(\Big\{\boldsymbol{x} \ge \boldsymbol{0} : (\boldsymbol{w}^{1/\alpha}\boldsymbol{x})_{[2]} > 1\Big\}\Big), \ \textit{for all continuity points } \boldsymbol{w} \in E^{(2)}.$$

Thus the upper tail order function $b_I^U(\cdot;\kappa)$ for each copula C_I exists, where C_I is the copula for the I-margin with $2 \le |I| \le d$, and in particular, $b^U(\boldsymbol{w};\kappa) = \nu_2\Big(\prod_{i=1}^d (w_i^{-1/\alpha}, \infty]\Big)$.

Proof: HRV on $E^{(2)}$ implies that there exists a function $h_2 \in \mathrm{RV}_{\alpha_2^{-1}}$ that satisfies (3.2). Then the left-continuous inverse $h_2^{\leftarrow}(\cdot) \in \mathrm{RV}_{\alpha_2}$, and thus the reciprocal of the inverse can be written as $[h_2^{\leftarrow}(t)]^{-1} = t^{-\alpha_2}\ell_0(t)$, for some $\ell_0 \in \mathrm{RV}_0$. Combining this expression with (3.2) yields

$$\frac{\mathbb{P}\,(\boldsymbol{X} \in t\,B)}{[h_2^{\leftarrow}(t)]^{-1}} = \frac{\mathbb{P}\,(\boldsymbol{X} \in t\,B)}{t^{-\alpha_2}\ell_0(t)} \to \nu_2(B), \text{ as } t \to \infty, \tag{3.8}$$

for all relatively compact sets $B \subseteq E^{(2)}$ satisfying $\nu_2(\partial B) = 0$.

Since $\overline{F}_1 \in \mathrm{RV}_{-\alpha}$, by Lemma 3.2 (4), $\overline{F}_1^{\leftarrow}(u)$ is regularly varying at 0 with exponent $-1/\alpha$. That is,

$$t_u := \overline{F}_1^{\leftarrow}(u) = u^{-1/\alpha}\ell_1(u), \tag{3.9}$$

for some function $\ell_1(\cdot)$ that is slowly varying at 0. Note that $u \to 0$ if and only if $t_u \to \infty$.

Let $\ell(u) := [\ell_1(u)]^{-\alpha_2}\ell_0(u^{-1/\alpha}\ell_1(u))$. It follows from Lemma 3.2 (1) that $\ell_0(u^{-1/\alpha}\ell_1(u))$ is slowly varying at 0, and thus $\ell(u)$ is slowly varying at 0. For any fixed $\boldsymbol{w} \in E_0^{(2)}$, $\kappa = \alpha_2/\alpha$, consider,

$$\begin{aligned} a_u^U(\boldsymbol{w};2,\kappa) &= \frac{\mathbb{P}\left(\bigcup_{i\neq j}\{U_i > 1 - uw_i, U_j > 1 - uw_j\}\right)}{u^{\kappa}\ell(u)} \\ &= \frac{\mathbb{P}\left(\bigcup_{i\neq j}\{F_i(X_i) > 1 - w_iu, F_j(X_j) > 1 - w_ju\}\right)}{[u^{-1/\alpha}\ell_1(u)]^{-\alpha_2}\ell_0(u^{-1/\alpha}\ell_1(u))} \\ &= \frac{\mathbb{P}\left(\bigcup_{i\neq j}\{X_i > \overline{F}_i^{\leftarrow}(w_iu), X_j > \overline{F}_j^{\leftarrow}(w_ju)\}\right)}{t_u^{-\alpha_2}\ell_0(t_u)}. \end{aligned} \tag{3.10}$$

Since the margins are continuous and $\overline{F}_i(x)/\overline{F}_1(x) \to 1$ as $x \to \infty$, we have from Lemma 3.2 (6) that $\overline{F}_i^{\leftarrow}(u)/\overline{F}_1^{\leftarrow}(u) \to 1$ as $u \to 0^+$. Since $\overline{F}_i(x) = x^{-\alpha}\ell_i(x)$ for $x \ge 0$, $\overline{F}_i^{\leftarrow}(u)$ is regularly varying at 0 with exponent $-\alpha^{-1}$ (see Lemma 3.2 (4)), or more precisely, $\overline{F}_i^{\leftarrow}(uc)/\overline{F}_i^{\leftarrow}(u) \to c^{-1/\alpha}$ as $u \to 0^+$ for any $c > 0$. Thus

$$\frac{\overline{F}_i^{\leftarrow}(w_iu)}{\overline{F}_1^{\leftarrow}(u)} = \frac{\overline{F}_i^{\leftarrow}(w_iu)}{\overline{F}_i^{\leftarrow}(u)}\frac{\overline{F}_i^{\leftarrow}(u)}{\overline{F}_1^{\leftarrow}(u)} \to w_i^{-1/\alpha}, \text{ as } u \to 0^+. \tag{3.11}$$

The limit (3.11) implies that for any small $\epsilon > 0$, when u is sufficiently small,

$$(1-\epsilon)w_i^{-1/\alpha} < \frac{\overline{F}_i^{\leftarrow}(w_iu)}{\overline{F}_1^{\leftarrow}(u)} < (1+\epsilon)w_i^{-1/\alpha}, \text{ for any } 1 \le i \le d.$$

Combining these inequalities with (3.10) and (3.9), we have that for any $\epsilon > 0$, when u is sufficiently small,

$$\begin{aligned}&\frac{\mathbb{P}(\bigcup_{i\neq j}\{X_i > t_u w_i^{-1/\alpha}(1-\epsilon), X_j > t_u w_j^{-1/\alpha}(1-\epsilon)\})}{t_u^{-\alpha_2}\ell_0(t_u)}\\ &\geq a_u^U(\boldsymbol{w}; 2, \kappa) \geq \frac{\mathbb{P}(\bigcup_{i\neq j}\{X_i > t_u w_i^{-1/\alpha}(1+\epsilon), X_j > t_u w_j^{-1/\alpha}(1+\epsilon)\})}{t_u^{-\alpha_2}\ell_0(t_u)}. \end{aligned} \tag{3.12}$$

When $w_i = 0$, $w_i^{-1/\alpha}$ is treated as ∞. Let

$$A := \left\{\boldsymbol{x} \geq \boldsymbol{0} : x_i > w_i^{-1/\alpha}, x_j > w_j^{-1/\alpha},\ \text{for some } i \neq j\right\} = \left\{\boldsymbol{x} \geq \boldsymbol{0} : (\boldsymbol{w}^{1/\alpha}\boldsymbol{x})_{[2]} > 1\right\},\ \boldsymbol{w} \in E^{(2)},$$

which is relatively compact on $E^{(2)}$. Then for any continuity set A with $\nu_2(\partial A)$=0 (3.8) implies that

$$\nu_2((1+\epsilon)A) \leq \liminf_{u\to 0^+} a_u^U(\boldsymbol{w}; 2, \kappa) \leq \limsup_{u\to 0^+} a_u^U(\boldsymbol{w}; 2, \kappa) \leq \nu_2((1-\epsilon)A),$$

for any small $\epsilon > 0$. Since ν_2 is homogeneous of order $-\alpha_2$, the above chain of inequalities become

$$(1+\epsilon)^{-\alpha_2}\nu_2(A) \leq \liminf_{u\to 0^+} a_u^U(\boldsymbol{w}; 2, \kappa) \leq \limsup_{u\to 0^+} a_u^U(\boldsymbol{w}; 2, \kappa) \leq (1-\epsilon)^{-\alpha_2}\nu_2(A),$$

which yields, with $\epsilon \to 0^+$, that $\lim_{u\to 0^+} a_u^U(\boldsymbol{w}; 2, \kappa)$ exists, and

$$\begin{aligned} a^U(\boldsymbol{w}; 2, \kappa) &= \lim_{u\to 0^+} a_u^U(\boldsymbol{w}; 2, \kappa)\\ &= \nu_2(A) = \nu_2\Big(\left\{\boldsymbol{x} \geq \boldsymbol{0} : (\boldsymbol{w}^{1/\alpha}\boldsymbol{x})_{[2]} > 1\right\}\Big),\ \boldsymbol{w} \in E^{(2)} \end{aligned} \tag{3.13}$$

as desired. Then Proposition 3.1 implies that the upper tail order functions b_I^U exists for $|I| \geq 2$, which completes the proof. ☐

Proposition 3.5 proves that existence of $a^U(\boldsymbol{w}; 2, \kappa)$ and asymptotically equivalent regularly varying univariate margins will lead to HRV on $E^{(2)}$. The following characterization of HRV will be useful for proving the result.

Lemma 3.4. (Resnick (2007), page 326) Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a non-negative random vector with distribution F, continuous margins $F_1, \dots, F_d$ satisfying (2.1) with

$$\lim_{t\to\infty} t\mathbb{P}\big(X_i > h(t)s\big) = s^{-\alpha},\ s > 0,$$

where $h \in \mathrm{RV}_{\alpha^{-1}}$, $\alpha > 0$. Suppose that $h_2 \in \mathrm{RV}_{\alpha_2^{-1}}$ with $\alpha_2 \geq \alpha$ and $h(t)/h_2(t) \to \infty$ as $t \to \infty$. Then F has regular variation on $E^{(1)}$ with scaling function $h(\cdot)$, intensity measure ν and tail index $\alpha > 0$ and hidden regular variation on $E^{(2)}$ with scaling function $h_2(\cdot)$, hidden intensity measures ν_2 and hidden tail index $\alpha_2 \geq \alpha$ if and only if the following convergences hold.

1. For all $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{R}_+^d \backslash \{\mathbf{0}\}$,

$$\lim_{t\to\infty} t\mathbb{P}\Big(\max_{1\le i\le d}\{z_i X_i\} > h(t)s\Big) = g(\boldsymbol{z}) s^{-\alpha}, s > 0 \tag{3.14}$$

for some function $g(\boldsymbol{z}) > 0$.

2. For all $\boldsymbol{z} = (z_1, \ldots, z_d) \in \overline{\mathbb{R}}_+^d$, where $z_i > 0$, $1 \le i \le d$, such that at least two components are finite,

$$\lim_{t\to\infty} t\mathbb{P}\Big(\min_{1\le i\le d}\{z_i X_i\} > h_2(t)s\Big) = d(\boldsymbol{z}) s^{-\alpha_2}, s > 0 \tag{3.15}$$

for some function $d(\boldsymbol{z})$.

Proposition 3.5. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a non-negative random vector with distribution F, continuous margins $F_1, \ldots, F_d$ satisfying (2.1), and copula C. Assume that marginal distributions F_i, $1 \le i \le d$, are regularly varying with tail index α. If for some $\kappa \ge 1$, the limit (3.6) exists with $u^\kappa \ell(u)/u \to 0$, then the distribution of $\boldsymbol{X}$, $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$, has HRV on $E^{(2)}$.*

Proof. We need to establish convergences (3.14) and (3.15). For any $\boldsymbol{w} \in E_0^{(1)} \backslash E_0^{(2)}$, $a^U(\boldsymbol{w}; C)$ is clearly well-defined. For any $\boldsymbol{w} \in E_0^{(2)}$, let $A_1 := \bigcup_{i=1}^d \{U_i > 1 - uw_i\}$ and $A_2 := \bigcup_{i\ne j}\{U_i > 1 - uw_i, U_j > 1 - uw_j\}$. It follows from the inclusion-exclusion relation that

$$\mathbb{P}(A_1) = \sum_{i=1}^d \mathbb{P}(U_i > 1 - uw_i) - \mathbb{P}(A_2) = u\big(\sum_{i=1}^d w_i\big) - \mathbb{P}(A_2). \tag{3.16}$$

Then, (3.6) together with (3.16) imply that

$$a^U(\boldsymbol{w}; C) = \lim_{u\to 0^+} \frac{\mathbb{P}(A_1)}{u} = \sum_{i=1}^d w_i - \lim_{u\to 0^+} \frac{\mathbb{P}(A_2)}{u^\kappa \ell(u)} \frac{u^\kappa \ell(u)}{u} = w_1 + \cdots + w_d,$$

exists for any $\boldsymbol{w} \in E_0^{(2)}$. Therefore, Theorem 2.1 implies that F is MRV on $E^{(1)}$ with tail index α, and the corresponding intensity measure ν satisfies $\nu(A_2) = 0$ (asymptotic independence). The convergence (3.14) follows from Theorem 2.1: for all $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{R}_+^d \backslash \{\mathbf{0}\}$,

$$\lim_{t\to\infty} t\mathbb{P}\big(\max_{1\le i\le d}\{z_i X_i\} > h(t)s\big) = s^{-\alpha} a^U(\boldsymbol{z}^\alpha; C), s > 0.$$

It remains to prove that the convergence (3.15) holds. It follows from Proposition 3.1 that the upper tail order function $b_I^U(\cdot; \kappa)$ of any multivariate margin C_I with $I \subseteq I_d$ and $2 \le |I| \le d$ also exists. For any $\boldsymbol{w} \ge \mathbf{0}$, $I \subseteq I_d$ and $2 \le |I| \le d$, consider

$$b_u^U(\boldsymbol{w}_I; \kappa) := \frac{\mathbb{P}\left(\bigcap_{i\in I}\{U_i > 1 - uw_i\}\right)}{u^\kappa \ell(u)} = \frac{\mathbb{P}(\bigcap_{i\in I}\{X_i > \overline{F}_i^{\leftarrow}(uw_i)\})}{\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(u^\kappa \ell(u)))}, \tag{3.17}$$

and $b_I^U(\boldsymbol{w}_I;\kappa) = \lim_{u\to 0^+} b_u^U(\boldsymbol{w}_I;\kappa)$ exists for $\kappa \ge 1$. It follows from (3.11) that for any small $\epsilon > 0$, $1 \le i \le d$,

$$\frac{\overline{F}_i^{\leftarrow}((1-\epsilon)w_i u)}{\overline{F}_1^{\leftarrow}(u)} \to (1-\epsilon)^{-1/\alpha} w_i^{-1/\alpha},\ \frac{\overline{F}_i^{\leftarrow}((1+\epsilon)w_i u)}{\overline{F}_1^{\leftarrow}(u)} \to (1+\epsilon)^{-1/\alpha} w_i^{-1/\alpha}$$

as $u \to 0^+$. For any small $\epsilon > 0$, when u is sufficiently small,

$$\overline{F}_i^{\leftarrow}((1+\epsilon)w_i u) \le w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}(u) \le \overline{F}_i^{\leftarrow}((1-\epsilon)w_i u),\ 1 \le i \le d,$$

which, together with (3.17), implies that (When $w_i = 0$, $w_i^{-1/\alpha}$ is treated as ∞).

$$\begin{aligned} b_I^U((1-\epsilon)\boldsymbol{w}_I;\kappa) &\le \liminf_{u\to 0^+} \frac{\mathbb{P}(\bigcap_{i\in I}\{X_i > w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}(u)\})}{\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(u^\kappa \ell(u)))} \\ &\le \limsup_{u\to 0^+} \frac{\mathbb{P}(\bigcap_{i\in I}\{X_i > w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}(u)\})}{\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(u^\kappa \ell(u)))} \le b_I^U((1+\epsilon)\boldsymbol{w}_I;\kappa). \end{aligned}$$

Since $b_I^U(\cdot;\kappa)$ is homogeneous of order κ, the above chain of inequalities become

$$\begin{aligned} (1-\epsilon)^\kappa b_I^U(\boldsymbol{w}_I;\kappa) &\le \liminf_{u\to 0^+} \frac{\mathbb{P}(\bigcap_{i\in I}\{X_i > w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}(u)\})}{\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(u^\kappa \ell(u)))} \\ &\le \limsup_{u\to 0^+} \frac{\mathbb{P}(\bigcap_{i\in I}\{X_i > w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}(u)\})}{\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(u^\kappa \ell(u)))} \le (1+\epsilon)^\kappa b_I^U(\boldsymbol{w}_I;\kappa). \end{aligned}$$

Let $t := u^{-\kappa}$. As $\epsilon \to 0$, the following limit,

$$\lim_{t\to\infty} \frac{\mathbb{P}(\bigcap_{i\in I}\{X_i > w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}(t^{-1/\kappa})\})}{\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(t^{-1}\ell(t^{-1/\kappa})))} = b_I^U(\boldsymbol{w}_I;\kappa)$$

exists for all $\boldsymbol{w} \ge \boldsymbol{0}$. Let $g(t^{-1}) := t^{-1}\ell(t^{-1/\kappa})$, and $g(t^{-1})$ is eventually decreasing to zero. Observe that $\ell(t^{-1/\kappa}) \in \mathrm{RV}_0$, and thus we have $g(t^{-1}) \in \mathrm{RV}_{-1}$. Set $s^{-1} = g(t^{-1})$, leading to $t^{-1} \sim g^{\leftarrow}(s^{-1})$ as $s \to \infty$, where $g^{\leftarrow}(\cdot)$ denotes the left-continuous inverse of $g(\cdot)$. Thus we have

$$\lim_{s\to\infty} \frac{s\mathbb{P}(\bigcap_{i\in I}\{X_i > w_i^{-1/\alpha}\overline{F}_1^{\leftarrow}([g^{\leftarrow}(s^{-1})]^{1/\kappa})\})}{s\mathbb{P}(X_1 > \overline{F}_1^{\leftarrow}(s^{-1}))} = b_I^U(\boldsymbol{w}_I;\kappa).$$

Let $h_2(s) := \overline{F}_1^{\leftarrow}([g^{\leftarrow}(s^{-1})]^{1/\kappa})$ and $h(s) := \overline{F}_1^{\leftarrow}(s^{-1})$. Define $\alpha_2 := \kappa\alpha \ge \alpha$. and then it follows from Lemma 3.2 that $h(s) \in \mathrm{RV}_{1/\alpha}$, and $h_2(s) \in \mathrm{RV}_{1/\alpha_2}$. On the other hand, clearly, $s\mathbb{P}(X_1 > h(s)) \to 1$, as $s \to \infty$. Thus, for $\boldsymbol{w} \ge \boldsymbol{0}$,

$$\lim_{t\to\infty} t\,\mathbb{P}\left(\frac{\boldsymbol{X}}{h_2(t)} \in \prod_{i\in I}(w_i^{-1}, \infty]\right) = b_I^U(\boldsymbol{w}_I^\alpha;\kappa).$$

Rephrase this limit differently, we have for any $I \subseteq I_d$ and $2 \le |I| \le d$,

$$\lim_{t\to\infty} t\mathbb{P}\Big(\min_{i\in I}\{w_i X_i\} > h_2(t)s\Big) = b_I^U((\boldsymbol{w}_I/s)^\alpha;\kappa) = s^{-\alpha_2} b_I^U(\boldsymbol{w}_I^\alpha;\kappa),\ s > 0,$$

where $\alpha_2 = \kappa\alpha$, leading to (3.15) when only components with indexes in I are finite. By Lemma 3.4, F is MRV on $E^{(2)}$. $\square$

Remark 3.3. The proof of Proposition 3.5 also yields the interpretation for functions $g(\cdot)$ and $d(\cdot)$ of Lemma 3.4 in terms of upper exponent and tail order functions of the underlying copula C. That is, if $\kappa = \alpha_2/\alpha$, then

1. $g(\boldsymbol{z}) = a^U(\boldsymbol{z}^\alpha; C)$, for all $\boldsymbol{z} \in \mathbb{R}^d_+\backslash\{0\}$;

2. $d(\boldsymbol{z}) = b^U_I(\boldsymbol{z}^\alpha_I; \kappa)$ if only components of $\boldsymbol{z}$ with indexes in I are finite, $I \subseteq I_d$ and $2 \le |I| \le d$.

For a subspace $E \subseteq \overline{\mathbb{R}}^d_+$, let $M_+(E)$ denote the class of the non-negative Radon measures on E. Mitra and Resnick (2011) provide a following representation of MRV on a general subcone $E^{(l)}$ for $l = 1, \dots, d$. In contrast to the previous study on norm-based polar transforms for MRV on subcones, the method in Mitra and Resnick (2011) fixes directions on an order-statistics-based unit envelope $\delta\mathbb{N}_l := \{\boldsymbol{x} \in E^{(l)} : x_{[l]} = 1\}$ that wraps all open portions of the boundaries of subcone $E^{(l)}$ from within. Note that $\delta\mathbb{N}_l$ is always compact within $E^{(l)}$, and this leads to a product-measure representation for the intensity measure $\nu_l(\cdot)$ of MRV on $E^{(l)}$, where the spectral or directional measure $S_l(\cdot)$ is always finite.

Lemma 3.6 (Proposition 3.1 of Mitra and Resnick (2011))**.** A random vector $\boldsymbol{X}$ has MRV on $E^{(l)}$ in the sense of (3.1) if and only if

$$t\mathbb{P}\left((X_{[l]}/h_l(t), \boldsymbol{X}/X_{[l]}) \in \cdot\right) \xrightarrow{v} \nu_{\alpha_l} \times S_l(\cdot), \quad \text{in } M_+((0,\infty] \times \delta\mathbb{N}_l).$$

The intensity measure ν_l in (3.1) and the finite directional measure S_l are related by

$$\nu_l(\{\boldsymbol{x} \in E^{(l)} : x_{[l]} \ge r, \boldsymbol{x}/x_{[l]} \in \Lambda\}) = r^{-\alpha_l} S_l(\Lambda), \tag{3.18}$$

where $r > 0$ and $\Lambda \in \mathcal{B}(\delta\mathbb{N}_l)$, the Borel σ-field of $\delta\mathbb{N}_l$.

The order-statistics-based homogeneous transform used in Lemma 3.6 also yields a representation of Pickands type for the upper exponent function on $E^{(l)}$. The following result illustrates the idea with the subcone $E^{(2)}$.

Proposition 3.7. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a non-negative random vector with distribution F, continuous marginal cdfs $F_1, \dots, F_d$ satisfying* (2.1)*. Assume that F has HRV on $E^{(2)}$ with intensity measure ν_2 and* *tail index α_2. Then*

$$a^U(\boldsymbol{w}; 2, \kappa) = \int_{\delta\mathbb{N}_2} \left[(\boldsymbol{w}^{1/\alpha}\boldsymbol{s})_{[2]}\right]^{\alpha_2} S_2(d\boldsymbol{s}),$$

where $\alpha_2 = \kappa\alpha$, and $S_2(\cdot)$ is the directional measure in the representation (3.18)*.*

Proof: By Proposition 3.3, for any $\boldsymbol{w} \in E^{(2)}$,

$$a^U(\boldsymbol{w}; 2, \kappa) = \nu_2\Big(\Big\{\boldsymbol{x} \geq \boldsymbol{0} : (\boldsymbol{w}^{1/\alpha}\boldsymbol{x})_{[2]} > 1\Big\} \Big) =: \nu_2(A).$$

Noticing that $\delta\mathbb{N}_2 := \{\boldsymbol{x} \in E^{(2)} : x_{[2]} = 1\}$, let $T : E^{(2)} \to (0, \infty) \times \delta\mathbb{N}_2$ be a transform such that

$$T(\boldsymbol{y}) = (y_{[2]}, \boldsymbol{y}/y_{[2]}) =: (r, \boldsymbol{s}), \tag{3.19}$$

with the left inverse $T^{-1}(r, \boldsymbol{s}) = r\boldsymbol{s}$. Consider

$$\nu_2(A) = \nu_2 \circ T^{-1}(T(A)) = \nu_2 \circ T^{-1}\Big(\Big\{(r, \boldsymbol{s}) : r > 1/(\boldsymbol{w}^{1/\alpha}\boldsymbol{x})_{[2]}, \boldsymbol{s} \in \delta\mathbb{N}_2\Big\} \Big).$$

By (3.18), $\nu_2 \circ T^{-1}(dr, d\boldsymbol{s}) = \alpha_2 r^{-\alpha_2 - 1} dr S_2(d\boldsymbol{s})$. Since both measures are finite, we can apply Fubini's theorem to get

$$\nu_2(A) = \int_{\delta\mathbb{N}_2} \int_{1/(\boldsymbol{w}^{1/\alpha}\boldsymbol{x})_{[2]}}^{\infty} \alpha_2 r^{-\alpha_2 - 1} dr S_2(d\boldsymbol{s}) = \int_{\delta\mathbb{N}_2} \left[(\boldsymbol{w}^{1/\alpha}\boldsymbol{s})_{[2]}\right]^{\alpha_2} S_2(d\boldsymbol{s}),$$

which completes the proof. □

All of the above results on $E^{(2)}$ can be extended to subcone $E^{(l)}$, $3 \leq l \leq d$. For example, if F has HRV on $E^{(l)}$ with intensity measure ν_l and tail index α_l, then

$$a^U(\boldsymbol{w}; l, \kappa) = \int_{\delta\mathbb{N}_l} \left[(\boldsymbol{w}^{1/\alpha}\boldsymbol{s})_{[l]}\right]^{\alpha_l} S_l(d\boldsymbol{s}), \tag{3.20}$$

where $\alpha_l = \kappa\alpha$, and $S_l(\cdot)$ is the directional measure in the representation (3.18).

4 Tail order functions

Tail order functions are directly related to upper exponent functions (see Remark 3.2). Based on the representation (3.20), the upper exponent function can be derived from a finite measure S_l. Since the upper tail order function of a d-dimensional copula C coincides with its upper exponent function on $E^{(d)}$, we can construct the upper tail order function from the finite directional measure S_d as well; this will be studied in

Section 4.1 with the bivariate case. However, for HRV on $E^{(l)}$ with $1 < l < d$, there is an inclusion-exclusion relation between $a^U(\boldsymbol{w}; l, \kappa)$ and upper tail order functions, and the related discussions and examples will be given in Section 4.2.

4.1 Bivariate cases

When $d = 2$, $a^U(w_1, w_2; 2, \kappa) = b(w_1, w_2; \kappa)$ where b is the upper tail order function that is homogeneous of order κ. The support of the measure S_2 is $\{(s_2, 1), (1, s_1) : 1 \le s_1, s_2 \le \infty\}$, which is the union of a horizontal line and a vertical line that meet at the corner point $(1, 1)$. Let H_1 be a finite measure defined on the vertical line $[(1, 1), (1, \infty)]$, and H_2 be a finite measure defined on the horizontal line $[(1, 1), (\infty, 1)]$. We can write (since $\kappa = \alpha_2/\alpha$)

$$\begin{aligned} &a^U(w_1, w_2; 2, \kappa) \\ &= \int_{(1,\infty]} \min\{w_1^\kappa, s^{\alpha_2} w_2^\kappa\} H_1(ds) + \int_{(1,\infty]} \min\{w_2^\kappa, s^{\alpha_2} w_1^\kappa\} H_2(ds) + \min\{w_1^\kappa, w_2^\kappa\} S(\{(1, 1)\}). \end{aligned} \tag{4.1}$$

For $w_1 = 1$, $w_2 = w > 1$,

$$\begin{aligned} &b(1, w; \kappa) = a^U(1, w; 2, \kappa) \\ &= H_1((1, \infty]) + \int_{(1,w^{1/\alpha}]} s^{\alpha_2} H_2(ds) + w^\kappa H_2((w^{1/\alpha}, \infty]) + S(\{(1, 1)\}). \end{aligned} \tag{4.2}$$

If H_2 is absolutely continuous with respect to the Lebesgue measure and h_2 is the density, then $b(1, w; \kappa)$ is differentiable in w, and by Leibniz's rule for integral,

$$\frac{\partial b(1, w; \kappa)}{\partial w} = \kappa w^{\kappa-1} H_2((w^{1/\alpha}, \infty]).$$

Therefore, H_2 has the following representation

$$H_2((w^{1/\alpha}, \infty]) = \kappa^{-1} w^{1-\kappa} \frac{\partial b(1, w; \kappa)}{\partial w} =: \kappa^{-1} w^{1-\kappa} g_2(w), \quad 1 < w \le \infty. \tag{4.3}$$

By symmetry, if H_1 is absolutely continuous with respect to the Lebesgue measure, then

$$H_1((w^{1/\alpha}, \infty]) = \kappa^{-1} w^{1-\kappa} \frac{\partial b(w, 1; \kappa)}{\partial w} =: \kappa^{-1} w^{1-\kappa} g_1(w), \quad 1 < w \le \infty. \tag{4.4}$$

Now we study how to relate the tail order function to strength of dependence.

Example 4.1. Let $b(w_1, w_2; \kappa) = w_1^{\xi} w_2^{\kappa-\xi}$, where $1 < \kappa < 2$ and $0 < \xi < \kappa$. It can be an upper tail order function from the survival copula of a bivariate extreme value copula. Then

$$b(1, w; \kappa) = w^{\kappa-\xi}, \quad g_2(w) = (\kappa - \xi)w^{\kappa-\xi-1}$$
$$b(w, 1; \kappa) = w^{\xi}, \quad g_1(w) = \xi w^{\xi-1}.$$

Therefore, by (4.3) and (4.4),

$$H_2((w^{1/\alpha}, \infty]) = \kappa^{-1} w^{1-\kappa}(\kappa - \xi)w^{\kappa-\xi-1} = \kappa^{-1}(\kappa - \xi)w^{-\xi},$$
$$H_1((w^{1/\alpha}, \infty]) = \kappa^{-1} w^{1-\kappa} \xi w^{\xi-1} = \kappa^{-1} \xi w^{\xi-\kappa}.$$

Then, the densities of the measures H_2 and H_1 are

$$h_2(w) = \alpha\xi(\kappa - \xi)\kappa^{-1} w^{-\alpha\xi-1},$$
$$h_1(w) = \alpha\xi(\kappa - \xi)\kappa^{-1} w^{\alpha(\xi-\kappa)-1},$$

respectively. Moreover, (4.2) implies that

$$\begin{aligned} S(\{(1,1)\}) &= b(1, w; \kappa) - H_1((1, \infty]) - \int_{(1,w^{1/\alpha}]} s^{\alpha_2} H_2(ds) - w^{\kappa} H_2((w^{1/\alpha}, \infty]) \\ &= w^{\kappa-\xi} - \kappa^{-1}\xi - \kappa^{-1}\xi(w^{\kappa-\xi} - 1) - \kappa^{-1}(\kappa - \xi)w^{\kappa-\xi} \equiv 0. \end{aligned}$$

Example 4.2. Let $b(w_1, w_2; \kappa) = (w_1 + w_2)^{\kappa} - w_1^{\kappa} - w_2^{\kappa}$, where $1 < \kappa < 2$. It can be an upper tail order function from a bivariate Archimedean copula (Hua and Joe, 2011).

$$b(1, w; \kappa) = b(w, 1; \kappa) = (1 + w)^{k} - 1 - w^{\kappa}, \quad g_1(w) = g_2(w) = \kappa(1 + w)^{\kappa-1} - \kappa w^{\kappa-1}.$$

Therefore, by (4.3) and (4.4),

$$H_1((w^{1/\alpha}, \infty]) = H_2((w^{1/\alpha}, \infty]) = \kappa^{-1} w^{1-\kappa}[\kappa(1 + w)^{\kappa-1} - \kappa w^{\kappa-1}] = (1 + w^{-1})^{\kappa-1} - 1,$$

and $h_1(w) = h_2(w) = \alpha(\kappa - 1)(w^{-\alpha} + 1)^{\kappa-2} w^{-\alpha-1}$. Therefore, by (4.2)

$$S(\{(1,1)\}) = [(1 + w)^{k} - 1 - w^{\kappa}] - (2^{\kappa-1} - 1) - [(1 + w)^{\kappa-1} - 2^{\kappa-1}] - w^{\kappa}[(1 + w^{-1})^{\kappa-1} - 1] \equiv 0.$$

Remark 4.1. Choosing $\xi = \kappa/2$ in Example 4.1, the densities $h_1, h_2 \in \mathrm{RV}_{-1-\alpha_2/2}$. For Example 4.2, the densities $h_1, h_2 \in \mathrm{RV}_{-1-\alpha}$. Also, $1 < \kappa < 2$ and $\kappa = \alpha_2/\alpha$ imply that $-1 - \alpha_2/2 > -1 - \alpha$. Note from

(4.1) that, with the same corresponding tail order κ, the upper tail of a bivariate Archimedean copula may be more dependent than the lower tail of a bivariate extreme value copula in the sense that the tail order function for the Archimedean copula is relatively larger.

Example 4.3 (Geometric mixtures of comonotonicity and independence)**.** Let $b(w_1, w_2; \kappa) = (w_1 \wedge w_2)^{2-\kappa}(w_1 w_2)^{\kappa-1}$, $1 < \kappa < 2$. This is a geometric mixture of tail order functions of comonotonicity and independence. Then

$$b(1, w; \kappa) = b(w, 1; \kappa) = w^{\kappa-1}, \quad g_1(w) = g_2(w) = (\kappa - 1)w^{\kappa-2}.$$

Equations (4.3) and (4.4) imply that

$$H_1((w^{1/\alpha}, \infty]) = H_2((w^{1/\alpha}, \infty]) = \kappa^{-1} w^{1-\kappa}[(\kappa - 1)w^{\kappa-2}] = \kappa^{-1}(\kappa - 1)w^{-1},$$

and $h_1(w) = h_2(w) = \alpha\kappa^{-1}(\kappa - 1)w^{-\alpha-1}$. Therefore, by (4.2)

$$S(\{(1, 1)\}) = [w^{\kappa-1}] - [\kappa^{-1}(\kappa - 1)] - [\kappa^{-1}(w^{\kappa-1} - 1)] - w^{\kappa}[\kappa^{-1}(\kappa - 1)w^{-1}] = 2\kappa^{-1} - 1.$$

Remark 4.2. Based on Example 4.3, we find that the mass assigned on the point $(1, 1)$ by the measure S may affect the strength of tail dependence: larger mass on $(1, 1)$ leads to stronger positive tail dependence.

Given any finite measure on $\delta\mathbb{N}_2$ with sufficient regularity conditions (or equivalently given that H_1, H_2 and $S(\{(1, 1)\})$ are finite, and H_1, H_2 are absolutely continuous), we now study whether the b function based on (4.2) and a parallel expression for $b(w, 1; \kappa)$ is always an appropriate tail order function; more specifically, whether

$$b(w_1, w_2; \kappa) = \begin{cases} w_1^{\kappa} b(1, w_2/w_1; \kappa) & 0 < w_1 < w_2 \\ w_2^{\kappa} b(w_1/w_2, 1; \kappa) & 0 < w_2 \le w_1 \end{cases} \tag{4.5}$$

is a tail order function with tail order κ. Note from (4.1) that $b((1, 1); \kappa)$ is consistent for $b(1, w; \kappa)$ and $b(w, 1; \kappa)$, so we can consider the expression (4.5). If such a function b is a tail order function for the bivariate case, then $b(w_1, w_2)$ must be positively homogeneous of order 2, increasing in each argument and also 2-increasing. By construction, it is homogeneous of order κ. It is easy to verify that $b(w_1, w_2)$ is increasing in w_1 and w_2 by (4.3) and (4.4). Now we check the 2-increasing requirement. Differentiate (4.5) with respect to w_1 and w_2. We show the details only for $0 < w_1 < w_2$ and the other case is symmetric.

$$\frac{\partial b(w_1, w_2)}{\partial w_2} = w_1^{\kappa-1} g_2(w_2/w_1)$$

and

$$\frac{\partial^2 b(w_1,w_2)}{\partial w_1 \partial w_2} = (k-1)w_1^{\kappa-2} g_2(w_2/w_1) - w_2 w_1^{\kappa-3} g_2'(w_2/w_1).$$

Note that $\frac{\partial b(w_1,w_2)}{\partial w_2}$ is homogeneous of order $\kappa-1$ and $h(w_1,w_2) := \frac{\partial^2 b(w_1,w_2)}{\partial w_1 \partial w_2}$ is homogeneous of order $\kappa-2$. Let $w := w_2/w_1 > 1$, then without loss of generality,

$$\begin{aligned} h(w_1,w_2) = h(1, w_2/w_1)w_1^{\kappa-2} &= w_1^{\kappa-2}[(k-1)g_2(w_2/w_1) - (w_2/w_1)g_2'(w_2/w_1)] \\ &= w_1^{\kappa-2}[(k-1)g_2(w) - w g_2'(w)]. \end{aligned}$$

By (4.3), $g_2(w) = \kappa w^{\kappa-1} H_2((w^{1/\alpha}, \infty])$. Therefore,

$$\begin{aligned} &(\kappa-1)g_2(w) - w g_2'(w) \\ &= (\kappa-1)\kappa w^{\kappa-1} H_2((w^{1/\alpha},\infty]) - (\kappa-1)\kappa w^{\kappa-1} H_2((w^{1/\alpha},\infty]) + (\kappa/\alpha) h_2(w^{1/\alpha}) w^{\kappa+1/\alpha-1} \\ &= (\kappa/\alpha) h_2(w^{1/\alpha}) w^{\kappa+1/\alpha-1} \geq 0. \end{aligned}$$

Therefore, $h(w_1,w_2) \geq 0$; that is, the function b being 2-increasing is proved, and thus b is a tail order function.

Moreover, depending on the measures of H_1 and H_2, $b(1,w;\kappa)$ and $b(w,1;\kappa)$ can be bounded or unbounded as $w \to \infty$. For example, based on (4.2), if H_2 is a probability measure defined on $(1,\infty]$ with $F_{H_2}(\cdot)$ as the distribution, and $F_{H_2} \in \mathrm{RV}_{-\alpha_2-\epsilon}$, then $b(1,w;\kappa)$ is bounded (unbounded), as $w \to \infty$, when $\epsilon > 0$ ($\epsilon < 0$).

4.2 Multivariate cases

For multivariate cases with dimension $d \geq 3$, Proposition 2 (2) in Hua and Joe (2011) implies that $\kappa_I \leq \kappa_J$ if $I \subseteq J \subseteq I_d$. It follows from the proof in Proposition 3.1 that when we analyze HRV on $E^{(2)}$, we can determine the tail order κ and the slowly varying function ℓ used in (3.6) based on all the bivariate margins. The smallest κ among the tail orders for all the bivariate margins and its associated slowly varying functions would be used for (3.6); if more than one bivariate margins have smallest κ, then the tail order κ and the largest slowly varying function for these bivariate margins would be used for (3.6). In this case, the upper exponent function on $E^{(2)}$ exists, and by Proposition 3.1, all the upper tail order functions exist but some of them could always be 0.

If for all multivariate margins C_I with $I \subset I_d$ and $1 < |I|$, the corresponding upper tail orders are the same and the associated slowly varying functions are tail equivalent up to some finite constants at 0^+, then HRV on $E^{(2)}$ suffices; see Example 4.4 for such a case with an Archimedean copula. Otherwise, one needs to seek HRV on the other subcones $E^{(i)}, i = 3, \dots, d$; Example 4.5 shows such a case with a Gaussian copula.

Example 4.4. (Archimedean copula based on LT of Inverse Gamma (aka. ACIG copula, Hua and Joe (2011))) A positive continuous random vector $\boldsymbol{X} := (X_1, \dots, X_d)$ that has an ACIG copula with the dependence parameter $\beta > 1$, and tail equivalent univariate margins $\overline{F}_i \in \mathrm{RV}_{-\alpha}, i \in I_d$, has HRV on $E^{(2)}$, because of Proposition 3.5 and the fact that all bivariate margins have upper tail order $\kappa = \min\{2, \beta\}$. The interesting thing is, if $1 < \beta < 2$, then for any marginal copula C_I with $I \subseteq I_d$, the upper tail order is $\kappa(C_I) = \beta$. Moreover, by reviewing the proof of Proposition 6 of Hua and Joe (2012a), the associated slowly varying functions for upper tail order functions of the ACIG copula are constants. Therefore, by Proposition 3.1, all the upper tail order functions for C_I exist and not 0, so there are no HRV on $E^{(i)}, i = 3, \dots, d$.

Example 4.5. (Gaussian copula) For the following Gaussian copula

$$C(u_1, \dots, u_d) := \Phi_\Sigma(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)),$$

we assume that all the correlation coefficients are ρ. From Hua and Joe (2011), the tail order for such a copula is $\kappa = \mathbf{1}_d \Sigma^{-1} \mathbf{1}_d^{\mathrm{T}} = d/[1 + (d-1)\rho]$, and all bivariate margins have upper tail order $\kappa_{\{ij\}} = 2/(1+\rho)$. If a positive, continuous random vector $\boldsymbol{X}$ has the Gaussian copula and the univariate margins $X_i \in \mathrm{RV}_{-\alpha}, i \in I_d$, then $\boldsymbol{X}$ has HRV on $E^{(2)}$ by Proposition 3.5. The HRV structure here is different than Example 4.4. The Gaussian copula has different upper tail orders for margins with different dimensions, and there are still HRVs on $E^{(3)}, \dots, E^{(d)}$. The upper exponent function on $E^{(2)}$ leads to the upper tail order functions being always 0 for C_I with $3 \le |I| \le d$, and cannot provide useful information for the interior of $E^{(2)}$, and one needs to seek HRV on the other subcones.

Remark 4.3. Note from the bivariate Gaussian copula with a negative ρ that HRV on $E^{(2)}$ can even lead to $\kappa = \alpha_2/\alpha > 2$. That is, HRV may lead to upper tail negative dependence. It is the ratio between the tail index for HRV and the tail index for MRV on $E^{(1)}$ that determines the value of the upper tail order, and thus the pattern of dependence in the upper tail. Therefore, one needs to pay more attention on the actual meaning of the notion of "asymptotic independence" used in the literature.

5 A mixture representation

In this section, we present a general mixture representation that can be used to generate tail order functions. One special case of the mixture representation corresponds to the Pickands type representation (3.20). In comparison, the previous section just has tail order functions from some known copula families. If we were to use tail order function for tail risk analysis or inferences on joint tail probabilities, it is important to have large classes of tail order functions as potential models.

Let $B \sim \text{Bernoulli}(\pi)$, $0 < \pi < 1$, $Z \sim \text{Pareto}(\alpha_2)$, $\boldsymbol{R} := (R_1, \ldots, R_d)$ be a random vector with cdf $F_{\mathbf{R}}$ with each margin being defined on $[1, \infty]$, and let $X_1, \ldots, X_d$ be independent Pareto(α_2) random variables. Suppose $B, Z, X_1, \ldots, X_d$, and $(R_1, \ldots, R_d)$ are all mutually independent. Moment assumptions on the R_j are given below. Consider

$$Y_j := BR_jZ + (1-B)X_j^{\gamma}, \quad j = 1, \ldots, d; \quad 1 < \gamma. \tag{5.1}$$

We will show that the random vector $\boldsymbol{Y} := (Y_1, \ldots, Y_d)$ has HRV on $E^{(d)}$ and the upper tail order is $\kappa = \gamma$. Moreover, for the bivariate case, if the probability measure generated by $F_{\boldsymbol{R}}$ only puts mass on $\delta\mathbb{N}_2$, then F_{R_1} and F_{R_2} correspond to the measures of H_1 and H_2 in Section 4, respectively, up to some normalization constants.

For each univariate margin,

$$\mathbb{P}(Y_j > y) = \pi \int_1^{\infty} (1 + y/r)^{-\alpha_2} dF_{R_j}(r) + (1-\pi)(1 + y^{1/\gamma})^{-\alpha_2}.$$

Assume that $\int_1^{\infty} r^{\alpha_2} dF_{R_j}(r) < \infty$, that is, $\mathbb{E}[R_j^{\alpha_2}]$ is finite. Then $\overline{F}_{Y_j} \in \text{RV}_{-\alpha_2/\gamma}$, and as $u \to 0$, $\overline{F}_{Y_j}^{-1}(u) \sim [u/(1-\pi)]^{-\gamma/\alpha_2}$. Let $\alpha := \alpha_2/\gamma$, then for each univariate margin Y_j, $\overline{F}_{Y_j} \in \text{RV}_{-\alpha}$.

For the joint survival probability,

$$\mathbb{P}(Y_j > y_j, j = 1, \ldots, d) = \pi \int_1^{\infty} [1 + \max_j \{y_j/r_j\}]^{-\alpha_2} dF_{\mathbf{R}}(\mathbf{r}) + (1-\pi) \prod_{j=1}^{d} (1 + y_j^{1/\gamma})^{-\alpha_2}.$$

The survival copula for $\boldsymbol{Y}$ is

$$\begin{aligned} C(\mathbf{u}) &= \mathbb{P}(Y_j > \overline{F}_{Y_j}^{-1}(u), j = 1, \ldots, d) \\ &= \pi \int_{\mathbf{1}}^{\boldsymbol{\infty}} [1 + \max_j \{\overline{F}_{Y_j}^{-1}(u)/r_j\}]^{-\alpha_2} dF_{\mathbf{R}}(\mathbf{r}) + (1-\pi) \prod_{j=1}^{d} (1 + [\overline{F}_{Y_j}^{-1}(u)]^{1/\gamma})^{-\alpha_2}. \end{aligned}$$

Then

$$\lim_{u\to 0^+}\frac{C(u\boldsymbol{w})}{u^\gamma}$$
$$= \lim_{u\to 0^+}\frac{\pi\int_{\mathbf{1}}^{\boldsymbol{\infty}}[1+\max_j\{\overline{F}_{Y_j}^{-1}(uw_j)/r_j\}]^{-\alpha_2}dF_{\mathbf{R}}(\mathbf{r})+(1-\pi)\prod_{j=1}^d(1+[\overline{F}_{Y_j}^{-1}(uw_j)]^{1/\gamma})^{-\alpha_2}}{u^\gamma}$$

$$= \lim_{u\to 0^+}\frac{\pi\int_{\mathbf{1}}^{\boldsymbol{\infty}}[1+\max_j\{\overline{F}_{Y_j}^{-1}(uw_j)/r_j\}]^{-\alpha_2}dF_{\mathbf{R}}(\mathbf{r})}{u^\gamma} \tag{5.2}$$
$$= \pi(1-\pi)^{-\gamma}\int_{\mathbf{1}}^{\boldsymbol{\infty}}\left[\min_j\{w_j^{\gamma/\alpha_2}r_j\}\right]^{\alpha_2}dF_{\mathbf{R}}(\mathbf{r})$$
$$= \pi(1-\pi)^{-\gamma}\int_{\mathbf{1}}^{\boldsymbol{\infty}}\left[\min_j\{w_j^{1/\alpha}r_j\}\right]^{\alpha_2}dF_{\mathbf{R}}(\mathbf{r}). \tag{5.3}$$

In (5.2), we require that the limit and integration can be exchanged; a sufficient condition is that $\mathbb{E}\left[(\min_{j\in\{1,\dots,d\}}R_j)^{\alpha_2}\right]<\infty$. Therefore, the tail order κ satisfies $1<\kappa=\gamma$, and by Proposition 3.5, $\boldsymbol{Y}$ has HRV on $E^{(2)}$.

Comparing (5.3) to (3.20) with $l=d$, we find that R_i's here may play a similar role as what the measure S_d does. For the bivariate case, the upper tail order function coincides with the upper exponent function on $E^{(2)}$. Then let $\kappa=\gamma>1$,

$$b(w_1,w_2;\kappa)=\pi(1-\pi)^{-\kappa}\int_1^\infty\int_1^\infty\min\{w_1^\kappa r_1^{\alpha_2},w_2^\kappa r_2^{\alpha_2}\}dF_{\mathbf{R}}(\mathbf{r}). \tag{5.4}$$

Note that in the mixture representation (5.1), we do not specify the dependence structure between R_j's. If in the bivariate case, the probability measure generated by the distribution $F_{\boldsymbol{R}}$ of R_1 and R_2 only puts mass on the L-shape line $\delta\mathbb{N}_2$, then F_{R_1} and F_{R_2} correspond to the H_1 and H_2 measures in (4.4) and (4.3), respectively, up to some finite normalization constants. More generally, consider polar coordinates with angle $\theta\in[0,\pi/2]$, radial variable z with $r_1=z\cos\theta$ and $r_2=z\sin\theta$, and random variables (Z,Θ) obtained from (R_1,R_2). The moment condition on $\mathbb{E}\left[(\min\{R_1,R_2\})^{\alpha_2}\right]$ implies a moment condition on Z. Also, note that

$$w_1^\kappa r_1^{\alpha_2}\le w_2^\kappa r_2^{\alpha_2}\iff r_1^{\alpha_2}\le(w_2/w_1)^\kappa r_2^{\alpha_2}\iff r_2^{\alpha_2}\ge r_1^{\alpha_2}(w_1/w_2)^\kappa\iff(\tan\theta)^{\alpha_2}\ge(w_1/w_2)^\kappa.$$

Let $\vartheta(w_1,w_2):=\arctan[(w_1/w_2)^{\kappa/\alpha_2}]$, $E_2(\theta):=\mathbb{E}\left(Z^{1+\alpha_2}|\Theta=\theta\right)$, F_Θ be the cdf of the angle Θ obtained from $F_{\mathbf{R}}$. Then (5.4) implies

$$\begin{aligned}\frac{b(w_1,w_2;\kappa)}{\pi(1-\pi)^{-\kappa}} &= \int_0^{\vartheta(w_1,w_2)}\int_{z=1}^\infty w_2^\kappa z^{\alpha_2}(\sin\theta)^{\alpha_2}\,z\,dF_{Z,\Theta}(z,\theta)+\int_{\vartheta(w_1,w_2)}^{\pi/2}\int_{z=1}^\infty w_1^\kappa z^{\alpha_2}(\cos\theta)^{\alpha_2}\,z\,dF_{Z,\Theta}(z,\theta)\\ &= w_2^\kappa\int_0^{\vartheta(w_1,w_2)}E_2(\theta)(\sin\theta)^{\alpha_2}\,dF_\Theta(\theta)+w_1^\kappa\int_{\vartheta(w_1,w_2)}^{\pi/2}E_2(\theta)(\cos\theta)^{\alpha_2}\,dF_\Theta(\theta).\end{aligned}$$

Hence $b(\cdot)$ depends on $F_{\mathbf{R}}$ only through $E_2(\theta)$ and F_Θ.

If R_1 and R_2 are comonotonic, then (5.4) provides an upper bound for all bivariate upper tail order functions for random vectors that have the representation (5.1); that is, after appropriate normalization, $\min\{w_1^\kappa, w_2^\kappa\}$ can be an upper bound for all such bivariate upper tail order functions. Interesting tail order functions can result by taking specific parametric families for R_1 and R_2; for example, if R_1, R_2 are independent Weibull random variables, then the tail order function (hence, the associated intensity measure ν_2 in Proposition 3.3) could incorporate negative as well as positive dependence, in the sense that X_1 and X_2, with the induced survival function $\overline{F}_{X_1,X_2}(x_1, x_2) := b(x_1^{-1}, x_2^{-1})/b(1,1) = \nu_2((x_1,\infty)\times(x_2,\infty))/\nu_2((1,\infty)\times(1,\infty))$, can be positively or negatively dependent.

For dimension $d \geq 3$, the upper tail order function coincides with the upper exponent function on $E^{(d)}$. Then (5.3) can still be used to explain the measure S_d for HRV on $E^{(d)}$. That is, if the probability measure $F_{\boldsymbol{R}}$ only puts mass on $\delta\mathbb{N}_d$, then $F_{\boldsymbol{R}}$ corresponds to the measure S_d, up to some normalization constants. However, for HRV on $E^{(l)}$ with $2 \leq l < d$ and $d \geq 3$, the mixture representation (5.1) is not suitable to explain the measure S_d, and a related discussion for this case is already given in Section 4.2.

6 Concluding Remarks

Depending on the tail index of univariate margins and the tail index for MRV on subcones of $E^{(i)}$, $i = 2, \dots, d$, the upper tail order κ of the underlying copula can take any values ≥ 1. This means, in addition to the usual tail dependence, MRV may also incorporate the cases of intermediate tail dependence, tail orthant independence and even tail negative dependence. If a d-dimensional random vector $\boldsymbol{X}$ has MRV on $E^{(1)}$ with copula C and tail equivalent Pareto-like univariate margins with tail index α, then the upper tail order for the marginal copula C_I with $I \subseteq I_d$ and $2 \leq |I| \leq d$ is determined by the ratio of the tail index $\alpha_{|I|}$, if exists, of MRV on $E^{(|I|)}$ and the tail index α for MRV on $E^{(1)}$.

Upper exponent functions on subcones for copulas are introduced, and connections between those upper exponent functions and the intensity measures ν_i's for HRV on the subcones are established. The representations of Pickands type for upper exponent functions on subcones are obtained by using an order-statistics-based product-measure decomposition. A mixture model for HRV on $E^{(d)}$ is constructed, for which, after appropriate normalization, the upper tail order function (i.e., the upper exponent function on $E^{(d)}$)

can be represented by an integral with respect to a mixing probability measure that resembles the finite directional measure in the Pickands representation.

We have built in this paper a bridge between the measure-theoretic MRV/HRV theory and the copula method for analyzing extremal dependence. On one hand, the methods already developed in the MRV/HRV theory can be used to analyzing tail dependence of copulas. On the other hand, there are many existing parametric copula families, and it is also relatively easier to create new parametric copula families. Therefore, the copula approach provides rich parametric distribution families to facilitate statistical analysis for multivariate heavy tail phenomena.

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