SIAM REVIEW
Vol. 44, No. 2, pp. 237–256

The Breakdown of Superconductivity Due to Strong Fields for the Ginzburg–Landau Model*

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Abstract. We study the behavior of a superconducting material subjected to a constant applied magnetic field, $\mathbf{H}_a = h\mathbf{e}$ with $|\mathbf{e}| = 1$, using the Ginzburg–Landau theory. We analytically show the existence of a critical field $\overline{h}$, for which, when $h > \overline{h}$, the normal states are the only solutions to the Ginzburg–Landau equations. We estimate $\overline{h}$. As $\kappa \downarrow 0$ we derive $\overline{h} = O(1)$, while as $\kappa \to \infty$ we obtain $\overline{h} = O(\kappa)$.

Key words. superconductivity, Ginzburg–Landau equations, upper critical fields, normal state

AMS subject classifications. 35J60, 35J65, 35Q40

PII. S0036144501399512

1. Introduction.

If a superconducting body is subjected to a sufficiently strong applied magnetic field, its ability to act as a superconductor breaks down and only the normally conducting (resistive) state is observed. In this paper we consider superconductivity as modeled by the Ginzburg–Landau theory and establish this type of phenomenon. Here, superconductivity is characterized in terms of a complex valued order parameter ψ (where $|\psi|^2$ represents the density of superconducting electron pairs) and a vector field $\mathbf{A}$—the magnetic potential.

Consider a superconducting body given by a bounded domain $\boldsymbol{\mathcal{D}} \subset \mathbb{R}^n$, where $n = 2$ or 3 and $\partial\boldsymbol{\mathcal{D}}$ is of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. Assume the body has constant permeability normalized to 1 and that the exterior consists of a second material with constant permeability $\mu_e > 0$. Define the permeability density as follows:

$$\begin{aligned} \mu(\mathbf{x}) &= 1 \qquad \text{for } \mathbf{x} \in \boldsymbol{\mathcal{D}}, \\ &= \mu_e \quad\ \text{for } \mathbf{x} \in \mathbb{R}^n \setminus \overline{\boldsymbol{\mathcal{D}}}. \end{aligned}$$

A magnetic field is applied to all space in the form $\mathbf{H}_a = h\mathbf{e}$, where h is a positive constant and $\mathbf{e} \in \mathbb{R}^3$ is a fixed unit vector. The presence of $\boldsymbol{\mathcal{D}}$ produces an

*Published electronically May 1, 2002. This paper originally appeared in *SIAM Journal on Mathematical Analysis*, Volume 30, Number 2, 1999, pages 341–359. This research was sponsored in part by NSF grant 9622305-DMS and also supported by the Department of Energy grant DE-FG02-90ER45427 through the Midwest Superconductivity Consortium.
http://www.siam.org/journals/sirev/44-2/39951.html
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induced magnetic field, $\frac{1}{\mu}\operatorname{curl}\mathbf{A}$ in $\mathbb{R}^3$, and a supercurrent density $\mathbf{j} := \frac{-i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \mathbf{A}|\psi|^2$ in $\boldsymbol{\mathcal{D}}$. Here $\kappa > 0$ is the Ginzburg–Landau constant determined from the superconducting material, and the superscript $*$ denotes complex conjugation. According to this theory, the pair $(\psi, \mathbf{A})$ is an equilibrium state for the Gibbs free energy

$$\begin{aligned} G(\psi,\mathbf{A}) := &\int_{\boldsymbol{\mathcal{D}}} \left(\left| \frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi \right|^2 + \frac{1}{2}(1-|\psi|^2)^2 \right) d\mathbf{x} \\ &+ \int_{\mathbb{R}^n} \mu \left| \frac{1}{\mu}\operatorname{curl}\mathbf{A} - h\mathbf{e} \right|^2 d\mathbf{x} + \frac{\gamma}{\kappa}\int_{\partial\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{s} \end{aligned} \tag{1.1}$$

(see [6], [16]). The constant $\gamma \geq 0$ reflects the retarding effect of the material in the exterior domain on the density $|\psi|^2$ at $\partial\boldsymbol{\mathcal{D}}$; γ is taken to be zero if $\mathbb{R}^n \setminus \overline{\boldsymbol{\mathcal{D}}}$ is a vacuum and large if the exterior is a magnetic material. Thus, we consider pairs $(\psi, \mathbf{A})$ such that

$$\psi \in H^1(\boldsymbol{\mathcal{D}};\mathbb{C}) \equiv \mathcal{H}^1(\boldsymbol{\mathcal{D}}), \quad \mathbf{A} \in H^1_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n),$$

which are weak solutions to

$$\begin{cases} \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0 & \text{in } \boldsymbol{\mathcal{D}}, \\ \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbf{A}\right) + \left(\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2\right)\chi_{\boldsymbol{\mathcal{D}}} = 0 & \text{in } \mathbb{R}^n, \\ \mathbf{n}\cdot\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi = -i\gamma\psi & \text{on } \partial\boldsymbol{\mathcal{D}}, \\ \left(\frac{1}{\mu}\operatorname{curl}\mathbf{A} - h\mathbf{e}\right) \in L^2(\mathbb{R}^n;\mathbb{R}^3). \end{cases} \tag{1.2}$$

Here $\mathbf{n}$ is the outward normal to $\boldsymbol{\mathcal{D}}$ at $\partial\boldsymbol{\mathcal{D}}$ and $\chi_{\boldsymbol{\mathcal{D}}}$ is the characteristic function for $\boldsymbol{\mathcal{D}}$.

A principal feature of the energy (1.1) and the solutions to (1.2) is that they are invariant under the gauge transformation

$$(\psi, \mathbf{A}) \to (\psi', \mathbf{A}'),$$

where

$$\psi' = \psi e^{i\kappa\eta}, \quad \mathbf{A}' = \mathbf{A} + \nabla\eta$$

for an arbitrary real valued function $\eta \in H^2_{\text{loc}}(\mathbb{R}^n)$. Moreover, the intrinsic quantities for a solution are preserved under this transformation: its density $|\psi'|^2 = |\psi|^2$, magnetic field $\frac{1}{\mu}\operatorname{curl}\mathbf{A}' = \frac{1}{\mu}\operatorname{curl}\mathbf{A}$, current $\mathbf{j}' = \mathbf{j}$, and the modulus of the derivative $|(\frac{i}{\kappa}\nabla + \mathbf{A}')\psi'| = |(\frac{i}{\kappa}\nabla + \mathbf{A})\psi|$.

A solution is in the *normal phase* if $\psi \equiv 0$ in $\boldsymbol{\mathcal{D}}$. This is written as $(\psi, \mathbf{A}) = (0, h\mathbf{a}_N)$, where $\mathbf{a}_N$ satisfies

$$\begin{aligned} &\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbf{a}_N\right) = 0 \quad \text{in } \mathbb{R}^n, \\ &\left(\frac{1}{\mu}\operatorname{curl}\mathbf{a}_N - \mathbf{e}\right) \in L^2(\mathbb{R}^n;\mathbb{R}^3). \end{aligned} \tag{1.3}$$

Such a solution is called a *normal state.* It is uniquely determined by μ and $\boldsymbol{\mathcal{D}}$ up to a gauge transformation; that is, (1.3) uniquely determines $\operatorname{curl}\mathbf{a}_N$.

Let κ be fixed. We denote $\overline{h}$ as the *upper critical field* for the body, where

$$\overline{h} := \inf\{h' : \text{ normal states are the only solutions to (1.2) for all } h > h'\}.$$

For the case in which the body is a bounded domain $\boldsymbol{\mathcal{D}} \subset \mathbb{R}^3$, we prove the following statement:

Let $\boldsymbol{\mathcal{D}} \subset \mathbb{R}^3$. Given κ, μ_e, and γ we have $\overline{h} = \overline{h}(\kappa, \mu_e, \gamma, \boldsymbol{\mathcal{D}}) < \infty$ (see Theorem 3.12).

We show that the normal induction is continuous on $\overline{\boldsymbol{\mathcal{D}}}$. In the case that it does not vanish on $\overline{\boldsymbol{\mathcal{D}}}$, we can estimate $\overline{h}$.

If $\operatorname{curl}\mathbf{a}_N \neq \mathbf{0}$ in $\overline{\boldsymbol{\mathcal{D}}} \subset \mathbb{R}^3$, then there are constants m, $\phi \geq 0$, depending on μ_e and $\boldsymbol{\mathcal{D}}$ so that

$$\overline{h}(\kappa, \mu_e, \gamma, \boldsymbol{\mathcal{D}}) \leq \max\left(\frac{m}{\kappa}, \phi\kappa\right) \tag{1.4}$$

(see Theorem 3.9).

In the classic case where $\mu_e = 1$, it follows that $\operatorname{curl}\mathbf{a}_N \equiv \mathbf{e}$, and as such (1.4) applies.

If $\mu \equiv 1$, then there are constants m and ϕ such that $\overline{h} \leq \max(\frac{m}{\kappa}, \phi\kappa)$ (see Corollary 3.10).

We also consider the case of a cylindrical domain of the form $\boldsymbol{\mathcal{D}} \times \mathbb{R}$, where the cross section $\boldsymbol{\mathcal{D}}$ is a bounded domain in $\mathbb{R}^2$ with a $C^{2,\alpha}$ boundary and the applied field $\mathbf{H}_a = h\mathbf{e} = h\mathbf{e}_3$ is perpendicular to the cross section. From symmetry the problem reduces to one in two dimensions. We consider $\psi(x, y)$ for $(x, y) \in \boldsymbol{\mathcal{D}}$ and $\mathbf{A} = (A_1(x, y), A_2(x, y))$ for $(x, y) \in \mathbb{R}^2$. The functional (1.1) then represents the Gibbs free energy per unit length for the cylinder. We prove the following theorem.

Let $\boldsymbol{\mathcal{D}} \times \mathbb{R}$ be a cylindrical body in a parallel applied field $h\mathbf{e}_3$. Given κ, μ_e, and γ there is a finite upper critical field $\overline{h}$, so that if $h > \overline{h}$, then the only solution to (1.2) *with $n = 2$ is normal. Moreover, there is a constant $\phi(\mu_e, \boldsymbol{\mathcal{D}})$ so that* $\overline{h}(\kappa, \mu_e, \gamma, \boldsymbol{\mathcal{D}}) \leq \max(\frac{1}{\kappa}, \phi\kappa)$ (see Theorem 2.9).

Finally, we consider the case of small κ. We prove the following result.

Let $n = 2$ with $\mu_e > 0$ or $n = 3$ with $\mu_e = 1$. Then $\overline{h} = O(1)$ as $\kappa \downarrow 0$ (see Theorem 4.1).

It is of interest to compare these results with conjectures made by physicists. For κ fixed, de Gennes and Saint-James have studied the local problem of determining the smallest value of h for which all normal states are stable for $h' \geq h$. The infimum, denoted as h_{c_3}, is the value for which it is possible to have a family of superconducting solutions bifurcate away from the normal state. In [15] Saint-James and de Gennes discussed the case of an infinite slab $-d < x < d$, $-\infty < y$, $z < \infty$ in $\mathbb{R}^3$. The symmetry of the domain reduced the linear analysis to a one-dimensional problem. They gave an ansatz for determining h_{c_3} and predicted $\lim_{\kappa\to\infty} h_{c_3}/\kappa = c_0$ for some constant $1 < c_0 < 2$. This can be compared with our estimates for $\overline{h}$ from Theorems 2.9 and 3.9. We have $h_{c_3} \leq \overline{h} = O(\kappa)$ as $\kappa \to \infty$. For small κ, physicists have predicted that $\overline{h} = O(1)$ as $\kappa \to 0$ for a slab of finite thickness $-d < x < d$, $-\infty < y$, $z < \infty$ and that $\overline{h} = O(\kappa^{-\frac{1}{2}})$ as $\kappa \to 0$ for the infinitely thick slab $-\infty < x < 0$, $-\infty < y$, $z < \infty$ (see [5], [8], and [14]). Our estimate from Theorem 4.1 gives the result $\overline{h} = O(1)$ as $\kappa \to 0$ for our domain $\boldsymbol{\mathcal{D}}$.

We next comment on past analytic work. In [3] and [4], Bolley and Bolley and Helffer made the ansatz for the slab rigorous and proved asymptotic estimates for h_{c_3}.

In [5] they obtained partial results for estimating an upper critical field for the slab. They considered a particular family of one-dimensional functions. For each fixed κ, they showed there is a finite upper critical field when considering only solutions in this family. In [2] Bauman, Phillips, and Tang estimated h_{c_3} for the case of a circular cylinder, $B_r \times \mathbb{R}$. This estimate is relevant here as it plays a central role in our analysis of $\overline{h}$ for general domains.

The central estimates from this paper were applied by Bauman et al. in [1] to investigate phase transitions in liquid crystals. A strong analogy exists between the normal-superconducting phase transition characterized by (1.1) and the nematic-smectic phase transition in liquid crystals. The phases for the latter are described in terms of a complex-valued wave function Ψ and the molecular director field $\mathbf{n}$. These are the analogues of the order parameter ψ and the vector potential $\mathbf{A}$, respectively. Smectic structure is observed through Ψ, and one has $\Psi \equiv 0$ in the nematic phase. The free energy characterizing the nematic-smectic phase transition is the Landau–de Gennes energy introduced by de Gennes in [9]. The de Gennes model was motivated by the formal analogies between the phase transitions for liquid crystals and superconductivity. There are, however, significant differences between the two theories. For example, the Ginzburg–Landau energy (1.1) is gauge invariant, whereas the Landau–de Gennes energy is not. Furthermore, the director field $\mathbf{n}$ is required to satisfy the constraint $|\mathbf{n}| = 1$, whereas the corresponding term from superconductivity, $\mathbf{A}$, is not.

In section 2 we consider cylindrical domains and establish Theorem 2.9. In section 3 we extend these ideas to treat bounded domains in $\mathbb{R}^3$. In section 4 we estimate $\overline{h}$ for small κ.

2. Superconductivity within an Infinitely Long Cylinder in a Parallel Field.

Let $(\psi, \mathbf{A})$ be a weak solution to (1.2) with $n = 2$ and $\boldsymbol{\mathcal{D}} \subset \mathbb{R}^2$. Recall that $\mathbf{H}_a = h\mathbf{e}_3$ is perpendicular to the cross section. We first examine the magnetic induction, $\operatorname{curl}\mathbf{A}$, in $\boldsymbol{\mathcal{D}}^c$.

LEMMA 2.1. *Let* $(\psi, \mathbf{A})$ *satisfy* (1.2). *Then* $\operatorname{curl}\mathbf{A}$ *is constant in each component of* $\overline{\boldsymbol{\mathcal{D}}}^c$. *Moreover,* $\operatorname{curl}\mathbf{A} = \mu_e h\mathbf{e}_3$ *in the unbounded component.*

Proof. From (1.2) we see that $\operatorname{curl}(\operatorname{curl}\mathbf{A}) = \mathbf{0}$ in each component of $\overline{\boldsymbol{\mathcal{D}}}^c$. Since

$$\operatorname{curl}(\operatorname{curl}\mathbf{A}) = (D_y(D_xA_2 - D_yA_1), -D_x(D_xA_2 - D_yA_1), 0),$$

we have that $\operatorname{curl}\mathbf{A} = (D_xA_2 - D_yA_1)\mathbf{e}_3$ is constant in each of these components. The last assertion follows from the fourth equation in (1.2). □

We now determine $\operatorname{curl}\mathbf{a}_N$.

LEMMA 2.2. *A normal state exists. Moreover, any normal state* $(0, h\mathbf{a}_N)$ *satisfies* $\operatorname{curl}\mathbf{a}_N = \mu\mathbf{e}_3$.

Proof. Consider $w = \Gamma_2 * (\mu - \mu_e)$, where $\Gamma_2(\mathbf{x}) = \frac{1}{2\pi}\ln(|\mathbf{x}|)$, $\mathbf{x} = (x, y)$. The function $\mu - \mu_e$ has bounded support. As a result, w is well defined with $w \in H^2_{\text{loc}}(\mathbb{R}^2)$ and $\Delta w = (\mu - \mu_e)$ in $\mathbb{R}^2$. Set $\mathbf{b}_N = (-w_y, w_x) + \frac{\mu_e}{2}(-y, x)$. Then $\operatorname{curl}\mathbf{b}_N = \mu\mathbf{e}_3$ and we see that $\mathbf{b}_N$ is a weak solution to (1.3), that is,

$$\int_{\mathbb{R}^2} \frac{1}{\mu} \operatorname{curl}\mathbf{b}_N \operatorname{curl}\varphi \, d\mathbf{x} = 0 \text{ for all } \varphi \in H^1(\mathbb{R}^2; \mathbb{R}^2)$$

such that φ has bounded support. Thus, a normal state exists.

Suppose $\mathbf{a}_N$ is another weak solution. Taking the difference of the equations for $\mathbf{b}_N$ and $\mathbf{a}_N$, we get

$$\int_{\mathbb{R}^2} \frac{1}{\mu} \operatorname{curl}(\mathbf{b}_N - \mathbf{a}_N) \cdot \operatorname{curl} \varphi \, d\mathbf{x} = 0. \tag{2.1}$$

Let $\boldsymbol{\mathcal{E}}$ be the unbounded component of $\overline{\boldsymbol{\mathcal{D}}}^c$. From Lemma 2.1 we have $\operatorname{curl} \mathbf{b}_N = \operatorname{curl} \mathbf{a}_N$ outside the bounded set $\boldsymbol{\mathcal{E}}^c$. As a result we can take φ such that $\varphi = \mathbf{b}_N - \mathbf{a}_N$ in a neighborhood of $\boldsymbol{\mathcal{E}}^c$ in (2.1). Whence, $\operatorname{curl} \mathbf{b}_N \equiv \operatorname{curl} \mathbf{a}_N$. □

We can now show that a weak solution has a gauge-equivalent representative that satisfies a Sobolev estimate.

LEMMA 2.3. *Let* $(\zeta, \mathbf{B})$ *and* $(0, h\mathbf{a}_N)$ *be weak solutions to* (1.2). *Then there is a weak solution* $(\psi, \mathbf{A})$ *that is gauge equivalent to* $(\zeta, \mathbf{B})$ *such that*

$$\int_{\boldsymbol{\mathcal{D}}} |\mathbf{A} - h\mathbf{a}_N|^2 d\mathbf{x} \le C_0 \int_{\mathbb{R}^2} |\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N)|^2 d\mathbf{x}, \tag{2.2}$$

where C_0 *depends only on* $\boldsymbol{\mathcal{D}}$.

Proof. Set $\operatorname{curl}(\mathbf{B} - h\mathbf{a}_N) = f\mathbf{e}_3$. From Lemmas 2.1 and 2.2, we have that the support of f is contained in the bounded set $\boldsymbol{\mathcal{E}}^c$ and $f \in L^2(\mathbb{R}^2)$. Set $w = \Gamma_2 * f$; then standard estimates on the Newtonian potential give $w \in H^2_{\text{loc}}(\mathbb{R}^2)$, $\nabla w = \nabla\Gamma_2 * f$, $\|\nabla w\|_{L^2(\boldsymbol{\mathcal{E}}^c)} \le C_0(\boldsymbol{\mathcal{D}})\|f\|_{L^2(\boldsymbol{\mathcal{E}}^c)}$, and $\Delta w = f$ (see [11]). Thus, setting $\widetilde{\mathbf{A}} = (-w_y, w_x)$ we have $\widetilde{\mathbf{A}} \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and $\operatorname{curl} \widetilde{\mathbf{A}} = \Delta w \mathbf{e}_3 = \operatorname{curl}(\mathbf{B} - h\mathbf{a}_N)$. Let $\mathbf{A} = \widetilde{\mathbf{A}} + h\mathbf{a}_N$. Then $\operatorname{curl}(\mathbf{B} - \mathbf{A}) = \mathbf{0}$. Hence, $\mathbf{A} = \mathbf{B} + \nabla\eta$ for some $\eta \in H^2_{\text{loc}}(\mathbb{R}^2)$ and

$$\int_{\boldsymbol{\mathcal{D}}} |\mathbf{A} - h\mathbf{a}_N|^2 d\mathbf{x} \le \int_{\boldsymbol{\mathcal{E}}^c} |\nabla w|^2 d\mathbf{x} \le C_0 \int_{\boldsymbol{\mathcal{E}}^c} |f|^2 d\mathbf{x} = C_0 \int_{\mathbb{R}^2} |\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N)|^2 d\mathbf{x}. \qquad \square$$

We need the following property for weak solutions.

PROPOSITION 2.4. (see [10]). *Let* $(\psi, \mathbf{A})$ *be a weak solution to* (1.2); *then* $|\psi| \le 1$ *almost everywhere in* $\boldsymbol{\mathcal{D}}$.

Next we write the weak formulation of (1.2):

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{D}}} \left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right) \cdot \left(\frac{i}{\kappa}\nabla\varphi + \mathbf{A}\varphi\right)^* d\mathbf{x} + \int_{\boldsymbol{\mathcal{D}}} (|\psi|^2 - 1)\psi\varphi^* d\mathbf{x} \\ &\qquad = -\frac{\gamma}{\kappa}\int_{\partial\boldsymbol{\mathcal{D}}} \psi\varphi^* d\mathbf{s} \qquad \text{for any } \varphi \in \mathcal{H}^1(\boldsymbol{\mathcal{D}}), \\ &\int_{\mathbb{R}^2} \frac{1}{\mu} \operatorname{curl}\mathbf{A} \cdot \operatorname{curl}\mathbf{B}\, d\mathbf{x} + \int_{\boldsymbol{\mathcal{D}}} \left[\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2\right] \cdot \mathbf{B}\, d\mathbf{x} = 0 \end{aligned} \tag{2.3}$$

for any $\mathbf{B} \in H^1(\mathbb{R}^2; \mathbb{R}^2)$ with bounded support. Considering $\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi$ for $(\psi, \mathbf{A}) \in \mathcal{H}^1(\boldsymbol{\mathcal{D}}) \times H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, we have

$$\Re e\left[\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\psi^*\right] = \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2,$$

$$\Im m\left[\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\psi^*\right] = \frac{1}{2\kappa}\nabla|\psi|^2.$$

Thus,

$$\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi = \left\{\left[\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2\right]|\psi|^{-1} + i\left[\frac{1}{\kappa}\nabla|\psi|\right]\right\}\frac{\psi}{|\psi|} \tag{2.4}$$

for almost every $\mathbf{x}$ such that $\psi \neq 0$. Moreover, since $\nabla\psi = 0$ almost everywhere on the set $\{\psi = 0\}$, it is consistent to define the term in braces equal to zero on this set. We conclude that (2.4) holds almost everywhere.

LEMMA 2.5. *Let* $(\psi, \mathbf{A})$ *and* $(0, h\mathbf{a}_N)$ *be weak solutions satisfying* (2.2). *Then there is a constant* $C_1 = C_1(\boldsymbol{\mathcal{D}}, \mu_e)$ *such that*

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + \kappa h\mathbf{a}_N)\psi|^2 d\mathbf{x} \leq C_1\kappa^2 \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x}. \tag{2.5}$$

Proof. Let $\varphi = \psi$ in the first equation of (2.3). Using (2.4), Proposition 2.4, and $\gamma \geq 0$, we obtain

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{D}}} \left(\left|\frac{1}{\kappa}\nabla|\psi|\right|^2 + \left| \left[\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2\right] |\psi|^{-1}\right|^2 \right) d\mathbf{x} \\ &\quad = \int_{\boldsymbol{\mathcal{D}}} \left| \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right|^2 d\mathbf{x} \leq \int_{\boldsymbol{\mathcal{D}}} (1 - |\psi|^2)|\psi|^2 d\mathbf{x} \leq \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x}. \end{aligned} \tag{2.6}$$

This inequality is also valid for $n = 3$. Consider the second equation in (1.2) for the solutions $(\psi, \mathbf{A})$ and $(0, h\mathbf{a}_N)$. Taking the difference of their respective weak equations we have

$$\int_{\mathbb{R}^2} \frac{1}{\mu} \operatorname{curl}(\mathbf{A} - h\mathbf{a}_N) \cdot \operatorname{curl} \mathbf{B}\, d\mathbf{x} = -\int_{\boldsymbol{\mathcal{D}}} \left[\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2\right] |\psi|^{-1} \cdot |\psi|\mathbf{B}\, d\mathbf{x}.$$

Using (2.6) and the Cauchy–Schwarz inequality, we see

$$\int_{\mathbb{R}^2} \frac{1}{\mu} \operatorname{curl}(\mathbf{A} - h\mathbf{a}_N) \cdot \operatorname{curl} \mathbf{B}\, d\mathbf{x} \leq \varepsilon^{-1} \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x} + \varepsilon \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 |\mathbf{B}|^2 d\mathbf{x}$$

for any $\varepsilon > 0$. Let $\mathbf{B}$ be such that $\mathbf{B} = \mathbf{A} - h\mathbf{a}_N$ in $\boldsymbol{\mathcal{E}}^c$, where $\boldsymbol{\mathcal{E}}$ is the unbounded component of $\boldsymbol{\mathcal{D}}^c$. Then since $\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N) = \mathbf{0}$ in $\boldsymbol{\mathcal{E}}$ and $|\psi| \leq 1$ we derive

$$\int_{\mathbb{R}^2} \frac{1}{\mu} |\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N)|^2 d\mathbf{x} \leq \varepsilon^{-1} \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x} + \varepsilon \int_{\boldsymbol{\mathcal{D}}} |A - h\mathbf{a}_N|^2 d\mathbf{x}.$$

Combining this inequality with (2.2), we see that we can take ε sufficiently small so that

$$\int_{\boldsymbol{\mathcal{D}}} |\mathbf{A} - h\mathbf{a}_N|^2 d\mathbf{x} \leq M \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x} \tag{2.7}$$

for some constant $M = M(\operatorname{diam} \boldsymbol{\mathcal{D}}, \mu_e)$.

Next we write

$$\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi = \left(\frac{i}{\kappa}\nabla + h\mathbf{a}_N\right)\psi + (\mathbf{A} - h\mathbf{a}_N)\psi. \tag{2.8}$$

We will use the elementary inequality

$$\frac{1}{2}|\mathbf{c}|^2 - |\mathbf{b}|^2 \leq |\mathbf{c} + \mathbf{b}|^2 \quad \text{for } \mathbf{c}, \mathbf{b} \in \mathbb{C}. \tag{2.9}$$

Let $(\frac{i}{\kappa}\nabla + \mathbf{A})\psi = \mathbf{b}$ and $-(\frac{i}{\kappa}\nabla + h\mathbf{a}_N)\psi = \mathbf{c}$. Then using (2.6) and (2.8) we derive

$$\frac{1}{2}\int_{\mathcal{D}}\left|\left(\frac{i}{\kappa}\nabla + h\mathbf{a}_N\right)\psi\right|^2 d\mathbf{x} \le \int_{\mathcal{D}}|\psi|^2 d\mathbf{x} + \int_{\mathcal{D}}|\mathbf{A} - h\mathbf{a}_N|^2|\psi|^2 d\mathbf{x}.$$

Since $|\psi| \le 1$, we can apply (2.7) to obtain

$$\int_{\mathcal{D}}|(i\nabla + \kappa h\mathbf{a}_N)\psi|^2 d\mathbf{x} \le 2(1+M)\kappa^2\int_{\mathcal{D}}|\psi|^2 d\mathbf{x}.$$

We set $C_1 = 2(1+M)$ and the lemma is proved. □

We see that if a superconducting state (i.e., a solution with $\psi \not\equiv 0$) exists, then (2.5) implies that the principal eigenvalue for $(i\nabla + \kappa h\mathbf{a}_N)^2$ on $\mathcal{D}$ is bounded by $C_1\kappa^2$. We will show that there exists a constant ϕ such that if $h > \max(\frac{1}{\kappa}, \phi\kappa)$, then the principal eigenvalue is greater than $C_1\kappa^2$. It then follows for such κ and h that there are only normal solutions to (1.2).

The corresponding eigenfunctions are expected to take the form of a boundary layer. The following lemma gives a way of measuring to what extent functions can concentrate near the boundary.

For a set $\mathcal{O}$ we define the *τ-neighborhood in $\mathcal{O}$ of $\partial\mathcal{O}$* by

$$\mathcal{O}_\tau = \{\mathbf{x} \in \mathcal{O} : \operatorname{dist}(\mathbf{x}, \partial\mathcal{O}) < \tau\}.$$

LEMMA 2.6. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ with a C^1 boundary. Given $\lambda_0 > 0$, there is a constant $d(\lambda_0, \mathcal{O}) > 0$ such that whenever*

$$\int_{\mathcal{O}}|\nabla f|^2 d\mathbf{x} \le \lambda^2\int_{\mathcal{O}}|f|^2 d\mathbf{x} \tag{2.10}$$

for some $f \in H^1(\mathcal{O})$ with $\lambda \ge \lambda_0$, then

$$\frac{1}{2}\int_{\mathcal{O}}|f|^2 d\mathbf{x} \le \int_{\mathcal{O}\setminus\mathcal{O}_{\frac{d}{\lambda}}}|f|^2 d\mathbf{x}. \tag{2.11}$$

Proof. Let $\cup_{k=0}^{N}F_k$ be an open cover for $\overline{\mathcal{O}}$ such that $\overline{F}_0 \subset \mathcal{O}$ and such that for each k, $1 \le k \le N$, we have

$$F_k \cap \mathcal{O} = \{(x', x_n) : g_k(x') < x_n < g_k(x') + \delta_1, |x'| < \delta_2\},$$

where δ_1 and δ_2 are positive constants, (x', x_n) are suitably rotated and translated coordinates, and $g_k(x') = x_n$ characterizes $\partial\mathcal{O} \cap F_k$. We can further assume without loss of generality that $g_k(\cdot)$ is defined for $|x'| \le 2\delta_2$, $|\nabla g_k| < 1$, and

$$\begin{aligned} &\{(x', x_n) : g_k(x') < x_n < g_k(x') + 4\delta_1, |x'| < 2\delta_2\} \subset \mathcal{O}, \\ &\{(x', x_n) : g_k(x') - 4\delta_1 < x_n < g_k(x'), |x'| < 2\delta_2\} \subset \mathbb{R}^n \setminus \overline{\mathcal{O}}. \end{aligned} \tag{2.12}$$

Let $f \in H^1(\mathcal{O})$, $0 \le t$, $v \le \delta_1$, and fix $k \ge 1$. We have

$$\begin{aligned} &\int_{\{x_n - g_k(x') = v\}\cap F_k}|f|^2 d\mathbf{s} - \int_{\{x_n - g_k(x') = t\}\cap F_k}|f|^2 d\mathbf{s} \\ &\quad \le \int_{\{|x'| \le \delta_2\}}|f^2(x', g_k(x') + v) - f^2(x', g_k(x') + t)|\sqrt{1 + |\nabla g_k|^2}\, dx' \\ &\quad \le \int_{\mathcal{O}\cap F_k}\left|\frac{\partial(f^2)}{\partial x_n}\right| d\mathbf{x}. \end{aligned}$$

Integrating in t from 0 to δ_1 and then dividing by δ_1 gives

$$\int_{\{x_n-g_k(x')=v\}\cap F_k} |f|^2 d\mathbf{s} \le \frac{1}{\delta_1}\left(\int_{\mathcal{O}} |f|^2 d\mathbf{x} + 2\delta_1 \int_{\mathcal{O}} |f|\,|\nabla f| d\mathbf{x}\right).$$

Next integrate v from 0 to $\frac{2d}{\lambda}$, where $0<d<\lambda_0\delta_1/2$ is to be determined. We derive

$$\int_{\{0<x_n-g_k(x')\le\frac{2d}{\lambda}\}\cap F_k} |f|^2 d\mathbf{x} \le \frac{2d}{\lambda\delta_1}\left(\int_{\mathcal{O}} |f|^2 d\mathbf{x} + 2\delta_1 \int_{\mathcal{O}} |f|\,|\nabla f| d\mathbf{x}\right). \tag{2.13}$$

From the assumptions on g_k, for $1\le k\le N$ we have

$$F_k \cap \mathcal{O}_{\frac{d}{\lambda}} \subset F_k \cap \left\{0 < x_n - g_k(x') \le \frac{2d}{\lambda}\right\} \tag{2.14}$$

if $\frac{d}{\lambda}$ is small enough. Indeed, if this is false for some k we can find $\mathbf{x}=(x',x_n)$ such that $|x'|\le\delta_2$, $x_n > g_k(x')+\frac{2d}{\lambda}$, and $\mathbf{y}=(y',y_n)\in\partial\mathcal{O}$ such that $|\mathbf{x}-\mathbf{y}|<\frac{d}{\lambda}$. If $\frac{d}{\lambda}<\min(\delta_1,\delta_2)$, we have $|x'-y'|<\frac{d}{\lambda}<\delta_2$, implying that $|y'|<2\delta_2$ and $|y_n-x_n|<\frac{d}{\lambda}<\delta_1$. We claim that $y_n=g_k(y')$. In fact,

$$|y_n - g_k(y')| \le |y_n - x_n| + |x_n - g_k(x')| + |g_k(x') - g_k(y')|,$$

and each term on the right is bounded by δ_1. We have shown this for the first one. This is true for the second since $\mathbf{x}\in F_k\cap\mathcal{O}$. For the last term we use $|\nabla g_k|<1$ and $|x'-y'|<\frac{d}{\lambda}<\delta_1$. Thus, $|y_n-g_k(y')|<3\delta_1$. From (2.12) we see that the only possibility for such a $\mathbf{y}\in\partial\mathcal{O}$ is $y_n=g_k(y')$. As a result,

$$|g_k(x') - g_k(y')| \ge |g_k(x') - x_n| - |g_k(y') - x_n| > \frac{2d}{\lambda} - \frac{d}{\lambda} = \frac{d}{\lambda}.$$

On the other hand, since $|\nabla g_k|<1$ we have $|g_k(x')-g_k(y')|<|x'-y'|<\frac{d}{\lambda}$, and this is a contradiction.

Using (2.13) and (2.14) and summing on k for $1\le k\le N$, we obtain

$$\int_{\mathcal{O}_{\frac{d}{\lambda}}} |f|^2 d\mathbf{x} \le M_1\frac{d}{\lambda}\left(\int_{\mathcal{O}} |f|^2 d\mathbf{x} + \lambda\int_{\mathcal{O}} |f|^2 d\mathbf{x} + \lambda^{-1}\int_{\mathcal{O}} |\nabla f|^2 d\mathbf{x}\right),$$

where $M_1=M_1(\delta_1,N)$.

Using (2.10) we have

$$\int_{\mathcal{O}_{\frac{d}{\lambda}}} |f|^2 d\mathbf{x} \le M_2 d \int_{\mathcal{O}} |f|^2 d\mathbf{x},$$

where $M_2=M_2(\delta_1,N,\lambda_0)$. Setting $d=\min(\frac{1}{2M_2},\frac{\lambda_0\delta_1}{2},\frac{\lambda_0\delta_2}{2})$, we conclude that

$$\int_{\mathcal{O}_{\frac{d}{\lambda}}} |f|^2 d\mathbf{x} \le \frac{1}{2}\int_{\mathcal{O}} |f|^2 d\mathbf{x}.$$

The assertion (2.11) follows from this inequality. □

We will use the following result from [2] for $B_r(\mathbf{0})\subset\mathbb{R}^2$.

PROPOSITION 2.7. *There is a continuous function $\sigma(\cdot) : t \in [0,\infty) \to \mathbb{R}$ with $\sigma(t) > 0$ for $t > 0$ for which $\lim_{t\to\infty} \sigma(t)$ exists with $0 < \lim_{t\to\infty} \sigma(t) < 1$, and such that*

$$\int_{B_r(\mathbf{0})} \left| \left(i\nabla + \frac{\omega^2}{2}(-y,x) \right) \zeta \right|^2 d\mathbf{x} \geq \omega^2 \sigma(\omega r) \int_{B_r(\mathbf{0})} |\zeta|^2 d\mathbf{x} \tag{2.15}$$

for all $\zeta \in \mathcal{H}^1(B_r(\mathbf{0}))$ and $\omega \geq 0$.

Indeed, in [2, section 2], it is shown that

$$\inf_{\substack{\|\zeta\|_{L^2}=1 \\ \zeta \in W^{1,2}(B_r;\mathbb{C})}} \int_{B_r} \left| \left(i\nabla + \frac{\omega^2}{2}(-y,x) \right) \zeta \right|^2 d\mathbf{x} \equiv \omega^2 \sigma,$$

where $\sigma = \sigma(\omega r)$. Furthermore, σ is characterized by $\sigma(t) = \inf_{n \in \mathbf{z}} \sigma(t,n)$, where for each n, $\sigma(t,n)$ is analytic and positive on $0 < t < \infty$. Moreover, $\lim_{t\to 0} \sigma(t,0) = 0$. In [2, section 6], it is also shown that

$$\sigma(t) = \min_{0 \leq n \leq n_0 - 1} \sigma(t,n) \qquad \text{for } 0 \leq t \leq n_0.$$

As a result, it follows that $\sigma(t)$ is positive and continuous. The $\lim_{t\to\infty} \sigma(t)$ is analyzed in [2, section 6], as well.

Remark. If $\mathbf{b}$ is another vector field such that $\mathbf{b} \in H^1(B_r(\mathbf{0});\mathbb{R}^2)$ with $\operatorname{curl} \mathbf{b} = \mathbf{e}_3$, then (2.15) is also valid with $\frac{1}{2}(-y,x)$ replaced by $\mathbf{b}$. Indeed, we can define a function $q \in H^2(B_r(\mathbf{0}))$ such that $\nabla q = \mathbf{b} - \frac{1}{2}(-y,x)$. With this we can define a local gauge transformation

$$\zeta' = \zeta e^{i\omega^2 q}, \qquad \mathbf{b} = \frac{1}{2}(-y,x) + \nabla q,$$

for which $\zeta' \in \mathcal{H}^1(B_r(\mathbf{0}))$ provided $\zeta \in \mathcal{H}^1(B_r(\mathbf{0}))$. Moreover,

$$|(i\nabla + \omega^2 \mathbf{b})\zeta'| = \left| \left(i\nabla + \frac{\omega^2}{2}(-y,x) \right) \zeta \right| \text{ and } |\zeta'| = |\zeta|$$

so that

$$\int_{B_r(\mathbf{0})} |(i\nabla + \omega^2 \mathbf{b})\zeta'|^2 d\mathbf{x} \geq \omega^2 \sigma(\omega r) \int_{B_r(\mathbf{0})} |\zeta'|^2 d\mathbf{x} \tag{2.16}$$

for all $\zeta' \in \mathcal{H}^1(B_r(\mathbf{0}))$, $\mathbf{b} \in H^1(B_r(\mathbf{0});\mathbb{R}^2)$ such that $\operatorname{curl} \mathbf{b} = \mathbf{e}_3$.

We now derive an estimate similar to (2.16) for $\boldsymbol{\mathcal{D}}$ provided ω is bounded away from zero.

LEMMA 2.8. *Given $m > 0$, there is a constant $C_2 = C_2(m, \boldsymbol{\mathcal{D}})$, $0 < C_2 \leq 1$, such that if $\omega^2 \geq m$, then*

$$C_2 \omega^2 \int_{\boldsymbol{\mathcal{D}}} |\zeta|^2 d\mathbf{x} \leq \int_{\boldsymbol{\mathcal{D}}} |(i\nabla + \omega^2 \mathbf{b})\zeta|^2 d\mathbf{x} \tag{2.17}$$

for all $\zeta \in \mathcal{H}^1(\boldsymbol{\mathcal{D}})$ and $\mathbf{b} \in H^1(\boldsymbol{\mathcal{D}};\mathbb{R}^2)$ for which $\operatorname{curl} \mathbf{b} = \mathbf{e}_3$.

Proof. Let $\zeta \in \mathcal{H}^1(\boldsymbol{\mathcal{D}})$ such that $\int_{\boldsymbol{\mathcal{D}}} |\zeta|^2 d\mathbf{x} > 0$ and

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + \omega^2 \mathbf{b})\zeta|^2 d\mathbf{x} \leq \omega^2 \int_{\boldsymbol{\mathcal{D}}} |\zeta|^2 d\mathbf{x}$$

for some ω, $\omega^2 \geq m$. If no such ζ exists, then (2.17) is valid with $C_2 = 1$ and we are finished. From (2.4) we see $|\nabla|\zeta|\,| \leq |(i\nabla + \omega^2\mathbf{b})\zeta|$. Thus,

$$\int_{\mathcal{D}} |\nabla|\zeta|\,|^2 d\mathbf{x} \leq \omega^2 \int_{\mathcal{D}} |\zeta|^2 d\mathbf{x}.$$

As a result, we can apply Lemma 2.6 to conclude that

$$\frac{1}{2}\int_{\mathcal{D}} |\zeta|^2 d\mathbf{x} \leq \int_{\mathcal{D}\setminus\mathcal{D}_{\frac{d}{\omega}}} |\zeta|^2 d\mathbf{x}. \tag{2.18}$$

Next we choose a cover for $\mathcal{D}\setminus\mathcal{D}_{\frac{d}{\omega}}$ consisting of a finite collection of disks $\{B_{\frac{d}{\omega}}(\mathbf{x}_k)$, $k = 1, \ldots, N(\omega)\}$, each contained in $\mathcal{D}$ in such a way that $\sum_{k=1}^{N(\omega)} \chi_{B_{\frac{d}{\omega}}(\mathbf{x}_k)} \leq K_1$, where K_1 is independent of ω.

We see

$$\int_{\mathcal{D}} |(i\nabla + \omega^2\mathbf{b})\zeta|^2 d\mathbf{x} \geq \int_{\cup_{k=1}^{N} B_{\frac{d}{\omega}}(\mathbf{x}_k)} |(i\nabla + \omega^2\mathbf{b})\zeta|^2 d\mathbf{x}$$

$$\geq K_1^{-1} \sum_{k=1}^{N} \int_{B_{\frac{d}{\omega}}(\mathbf{x}_k)} |(i\nabla + \omega^2\mathbf{b})\zeta|^2 d\mathbf{x}.$$

Using Proposition 2.7, the last term bounds

$$K_1^{-1}\omega^2\sigma(d) \sum_{k=1}^{N} \int_{B_{\frac{d}{\omega}}(\mathbf{x}_k)} |\zeta|^2 d\mathbf{x} \geq K_2(d)\omega^2 \int_{\mathcal{D}\setminus\mathcal{D}_{\frac{d}{\omega}}} |\zeta|^2 d\mathbf{x} \geq \frac{K_2}{2}\omega^2 \int_{\mathcal{D}} |\zeta|^2 d\mathbf{x},$$

where the final inequality follows from (2.18). Set $C_2 = K_2/2$. This chain of inequalities establishes the lemma. □

We now establish the principal result in this section. Here we prove the existence of an upper critical field $\overline{h}$ and obtain a bound for it as $\kappa \to \infty$ and $\kappa \to 0$.

THEOREM 2.9. *There is a constant* $\phi = \phi(\mu_e, \mathcal{D})$ *so that if* $h > \max(\frac{1}{\kappa}, \phi\kappa)$, *then any weak solution for* (1.2) *with* $n = 2$ *is normal.*

Proof. Let $(0, \mathbf{a}_N)$ be a normal state for (1.3) and $(\psi, \mathbf{A})$ be a weak solution for (1.2). A state is normal if and only if its entire gauge equivalence class is normal. Therefore, we can assume without loss of generality that $(\psi, \mathbf{A})$ and $(0, h\mathbf{a}_N)$ satisfy (2.2). Set $\omega^2 = h\kappa$. Then $\omega^2 \geq 1$ by hypothesis. We apply (2.17) with $m = 1$ to derive

$$C_2 h\kappa \int_{\mathcal{D}} |\psi|^2 d\mathbf{x} \leq \int_{\mathcal{D}} |(i\nabla + h\kappa\mathbf{a}_N)\psi|^2 d\mathbf{x}, \tag{2.19}$$

and by Lemma 2.5, the right-hand side of (2.19) is bounded by $C_1\kappa^2 \int_{\mathcal{D}} |\psi|^2 d\mathbf{x}$. Let $\phi = C_1/C_2$. We have

$$h\int_{\mathcal{D}} |\psi|^2 d\mathbf{x} \leq \phi\kappa \int_{\mathcal{D}} |\psi|^2 d\mathbf{x}.$$

By assumption, $h > \phi\kappa$. Hence it must hold that $\int_{\mathcal{D}} |\psi|^2 d\mathbf{x} = 0$. □

3. Three-Dimensional Bodies. In this section, we consider a superconducting body given by a bounded domain $\boldsymbol{\mathcal{D}} \subset \mathbb{R}^3$ subjected to a uniform applied field $\mathbf{H}_a = h\mathbf{e}$. We will assume without loss of generality that $\mathbf{e} = \mathbf{e}_3$ throughout this section.

Denote by $\check{H}^1(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with respect to the norm

$$\|\mathbf{B}\|_{\check{H}^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla \mathbf{B}|^2 d\mathbf{x}\right)^{\frac{1}{2}}.$$

One can show that elements $\mathbf{B} \in \check{H}^1(\mathbb{R}^3)$ satisfy the following relationships:

$$\|\mathbf{B}\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} \leq \theta \|\mathbf{B}\|_{\check{H}^1(\mathbb{R}^3)}, \tag{3.1}$$

where θ is independent of $\mathbf{B}$, and

$$\|\mathbf{B}\|^2_{\check{H}^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\operatorname{div} \mathbf{B}|^2 + |\operatorname{curl} \mathbf{B}|^2) d\mathbf{x} \tag{3.2}$$

(see [12]).

In order to represent magnetic fields we need the following lemma.

LEMMA 3.1. *Let* $\mathbf{g} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ *such that* $\operatorname{div} \mathbf{g} = 0$ *in* $\mathcal{D}'(\mathbb{R}^3)$. *Then there is a unique* $\mathbf{u} \in \check{H}^1(\mathbb{R}^3)$ *such that* $\operatorname{curl} \mathbf{u} = \mathbf{g}$ *and* $\operatorname{div} \mathbf{u} = 0$.

Proof. Consider $D_k \Gamma * \mathbf{g}$, where $\Gamma(\mathbf{x}) = \Gamma_3(\mathbf{x}) = \frac{-1}{3\omega_3 |\mathbf{x}|}$ is the Newtonian potential for $\mathbb{R}^3$ and $1 \leq k \leq 3$. We claim that $D_k \Gamma * \mathbf{g} \in \check{H}^1(\mathbb{R}^3)$. To see this, let us first assume that $\mathbf{g}$ has bounded support. Then $D_k \Gamma * g$ exists as a weakly singular integral and

$$|D_k \Gamma * \mathbf{g}| = O(|\mathbf{x}|^{-2})$$

and

$$|\nabla(D_k \Gamma * \mathbf{g})| = O(|\mathbf{x}|^{-3}) \text{ as } |\mathbf{x}| \to \infty.$$

Let $\varphi_R(\mathbf{x})$ be a standard C^∞ cutoff function such that $\varphi_R = 1$ for $|\mathbf{x}| \leq R$ and $\varphi_R = 0$ for $|\mathbf{x}| \geq R+1$. It follows directly that $\{\varphi_{R(n)}(D_k \Gamma * \mathbf{g})\}$ is a Cauchy sequence in $\check{H}^1(\mathbb{R}^3)$ that converges to $D_k \Gamma * \mathbf{g}$ pointwise for any sequence $R(n) \to \infty$. Thus, $D_k \Gamma * \mathbf{g} \in \check{H}^1(\mathbb{R}^3)$, assuming $\mathbf{g}$ has bounded support. Finally, by standard L^2-singular integral theory,

$$\|D_k \Gamma * \mathbf{g}\|_{\check{H}^1(\mathbb{R}^3)} \leq \|\mathbf{g}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}, \tag{3.3}$$

and as a consequence, $D_k \Gamma * \mathbf{g} \in \check{H}^1(\mathbb{R}^3)$ for all $\mathbf{g} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$.

Define $\mathbf{u} : \mathbf{g} \in L^2(\mathbb{R}^3;\mathbb{R}^3) \to \check{H}^1(\mathbb{R}^3)$ by

$$\mathbf{u}(\mathbf{g}) = -(D_2\Gamma * g_3 - D_3\Gamma * g_2, D_3\Gamma * g_1 - D_1\Gamma * g_3, D_1\Gamma * g_2 - D_2\Gamma * g_1).$$

Let $\mathbf{g}_\varepsilon$ be a mollification of $\mathbf{g}$; then $\mathbf{g}_\varepsilon \to \mathbf{g}$ in L^2 as $\varepsilon \to 0$, and $\operatorname{div} \mathbf{g}_\varepsilon = 0$ for each $\varepsilon > 0$. We define $\mathbf{g}_{\varepsilon,R} = \varphi_R \mathbf{g}_\varepsilon$. Note that $\operatorname{div}(\mathbf{g}_{\varepsilon,R}) = \nabla \varphi_R \cdot \mathbf{g}_\varepsilon$. If we choose sequences $\varepsilon(n) \to 0$ and $R(n) \to \infty$ as $n \to \infty$, then $\mathbf{g}_n = \varphi_{R(n)} \mathbf{g}_{\varepsilon(n)} \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$, $\mathbf{g}_n \to \mathbf{g}$ in L^2, and $\operatorname{div} \mathbf{g}_n \to 0$ in L^2 as $n \to \infty$.

Set $\mathbf{w}_n = \Gamma * \mathbf{g}_n$. These are well defined since the $\mathbf{g}_n$ have bounded support. Using (3.3), we see that

$$\operatorname{curl} \mathbf{w}_n = -\mathbf{u}(\mathbf{g}_n) \to -\mathbf{u}(\mathbf{g}) \text{ in } \check{H}^1 \text{ as } n \to \infty.$$

Consider

$$\operatorname{curl}\mathbf{u}(\mathbf{g}_n) = -\operatorname{curl}\operatorname{curl}\mathbf{w}_n = \Delta\mathbf{w}_n - \nabla(\operatorname{div}\mathbf{w}_n) = \mathbf{g}_n - \nabla(\operatorname{div}\mathbf{w}_n). \tag{3.4}$$

We know that $\nabla(\operatorname{div}\mathbf{w}_n) = \nabla\Gamma*(\operatorname{div}\mathbf{g}_n) \to \mathbf{0}$ in $\check{H}^1(\mathbb{R}^3)$ as $n\to\infty$ since $\operatorname{div}\mathbf{g}_n \to 0$ in L^2. Thus, using (3.1) we conclude that $\nabla(\operatorname{div}\mathbf{w}_n)\to\mathbf{0}$ in L^6 as $n\to\infty$. Furthermore, we have $\operatorname{curl}\mathbf{u}(\mathbf{g}_n)\to\operatorname{curl}\mathbf{u}(\mathbf{g})$ and $\mathbf{g}_n\to\mathbf{g}$ in L^2 as $n\to\infty$. As a consequence, $\operatorname{curl}\mathbf{u}(\mathbf{g}) = \mathbf{g}$ in $\mathbb{R}^3$.

Since $\mathbf{u}(\mathbf{g}_n) = -\operatorname{curl}\mathbf{w}_n$ we have $\operatorname{div}\mathbf{u}(\mathbf{g}_n) = 0$, which implies $\operatorname{div}\mathbf{u}(\mathbf{g}) = 0$. Finally, using (3.2) we see that $\mathbf{u}$ is unique in $\check{H}^1(\mathbb{R}^3)$. ∎

We can apply the preceding lemma to characterize weak solutions.

LEMMA 3.2. *Let* $(\zeta,\mathbf{B})$ *be a weak solution to* (1.2). *Then there is a gauge-equivalent solution* $(\psi,\mathbf{A})$ *such that* $\operatorname{div}\mathbf{A} = 0$ *and* $(\mathbf{A} - \frac{\mu_e h}{2}(-y,x,0)) \in \check{H}^1(\mathbb{R}^3)$. *Moreover, if* $(\tilde{\psi},\tilde{\mathbf{A}})$ *is another such solution, then* $\tilde{\psi} = a\psi$ *for some* $a\in\mathbb{C}, |a| = 1$, *and* $\tilde{\mathbf{A}} = \mathbf{A}$.

Proof. Set $\mathbf{g} = (\operatorname{curl}\mathbf{B} - \mu_e h\mathbf{e}_3) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. From the previous lemma there is a unique element $\mathbf{u}\in\check{H}^1(\mathbb{R}^3)$ such that $\operatorname{curl}\mathbf{u} = \mathbf{g}$ and $\operatorname{div}\mathbf{u} = 0$. Therefore, we find $\mathbf{A} = \mathbf{u} + \frac{\mu_e h}{2}(-y,x,0)$. ∎

We now characterize the normal state in three dimensions.

LEMMA 3.3. *There is a unique normal state satisfying* (1.3) *such that* $(\mathbf{a}_N - \frac{\mu_e}{2}(-y,x,0)) \in \check{H}^1(\mathbb{R}^3)$ *and* $\operatorname{div}\mathbf{a}_N = 0$.

Proof. Consider the strictly convex functional

$$E(\mathbf{b}) = G(0,\mathbf{b}) + \int_{\mathbb{R}^3}(\operatorname{div}\mathbf{b})^2 d\mathbf{x} = \int_{\boldsymbol{\mathcal{D}}}\frac{1}{2}\,d\mathbf{x} + \int_{\mathbb{R}^3}\left(\mu\left|\frac{1}{\mu}\operatorname{curl}\mathbf{b} - e_3\right|^2 + (\operatorname{div}\mathbf{b})^2\right)d\mathbf{x}$$

for the class $\boldsymbol{\mathcal{S}} = \{\mathbf{b} : (\mathbf{b} - \frac{\mu_e}{2}(-y,x,0)) \in \check{H}^1(\mathbb{R}^3)\}$. A unique equilibrium exists that also minimizes $E(\cdot)$. Let $\widetilde{\mathbf{b}}$ be this equilibrium. If $\operatorname{div}\widetilde{\mathbf{b}} \neq 0$, then by Lemma 3.2 we can find another vector field $\approx{\mathbf{b}} \in \boldsymbol{\mathcal{S}}$ such that $\operatorname{curl}\overset{\approx}{\mathbf{b}} = \operatorname{curl}\widetilde{\mathbf{b}}$ and $\operatorname{div}\overset{\approx}{\mathbf{b}} = 0$. This would imply $E(\overset{\approx}{\mathbf{b}}) < E(\widetilde{\mathbf{b}})$, which is impossible. Thus, $\operatorname{div}\widetilde{\mathbf{b}} = 0$. It follows that $\widetilde{\mathbf{b}}$ satisfies (1.3), and as a result a normal state $(0,\mathbf{a}_N)$ exists. Conversely, a normal state satisfying the hypothesis is an equilibrium for $E(\cdot)$, and so $\mathbf{a}_N$ is unique. ∎

Recall that the induction $\operatorname{curl}\mathbf{a}_N$, not $\mathbf{a}_N$, is the physically relevant quantity. Below we show that it is uniquely determined.

LEMMA 3.4. *Let* $(0,\mathbf{a}_N)$ *be a normal state. Then* $\operatorname{curl}\mathbf{a}_N$ *is uniquely determined,* $\operatorname{curl}\mathbf{a}_N$ *is harmonic in* $\mathbb{R}^3\backslash\partial\boldsymbol{\mathcal{D}}$, *and*

$$\operatorname{curl}\mathbf{a}_N \in C^{1,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{1,\alpha}(\boldsymbol{\mathcal{D}}^c).$$

Moreover, if $\operatorname{div}\mathbf{a}_N = 0$, then

$$\mathbf{a}_N \in C^{1,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{1,\alpha}(\boldsymbol{\mathcal{D}}^c).$$

Proof. Using Lemma 3.2 we see that any normal state is gauge equivalent to the normal state described in Lemma 3.3. Since a gauge transformation leaves the curl of a vector field invariant, we conclude that $\operatorname{curl}\mathbf{a}_N$ is uniquely determined for solutions to (1.3).

We can use the first equation in (1.3) to prove that there exists a function $p\in H^1_{\text{loc}}(\mathbb{R}^3)$ such that $\operatorname{curl}\mathbf{a}_N = \mu\nabla p$. Since

$$\operatorname{div}(\mu\nabla p) = 0 \text{ in } \mathbb{R}^3 \tag{3.5}$$

and μ is constant on the components of $\mathbb{R}^3 \setminus \partial\boldsymbol{\mathcal{D}}$, the function p (and thus $\operatorname{curl}\mathbf{a}_N$) is harmonic in each component. We apply the results from [13, Chap. 5, section 4] to the solution p for (3.5), to derive that $p \in C^{2,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{2,\alpha}(\boldsymbol{\mathcal{D}}^c)$, and as a consequence, $\operatorname{curl}\mathbf{a}_N \in C^{1,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{1,\alpha}(\boldsymbol{\mathcal{D}}^c)$.

Assume $\operatorname{div}\mathbf{a}_N = 0$. Let $\boldsymbol{\mathcal{U}}$ be an open neighborhood of $\overline{\boldsymbol{\mathcal{D}}}$ and consider $\mathbf{w} \in H^2(\boldsymbol{\mathcal{U}};\mathbb{R}^3)$ such that $\Delta\mathbf{w} = \operatorname{curl}\mathbf{a}_N$ in $\boldsymbol{\mathcal{U}}$. From [13, Chap. 5], we have $\mathbf{w} \in C^{2,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{2,\alpha}(\boldsymbol{\mathcal{U}} \setminus \boldsymbol{\mathcal{D}})$. The identity $\operatorname{curl}(\operatorname{curl}\mathbf{w}) + \Delta\mathbf{w} = \nabla(\operatorname{div}\mathbf{w})$ in $\boldsymbol{\mathcal{U}}$ yields

$$\operatorname{curl}(\operatorname{curl}\mathbf{w} + \mathbf{a}_N) = \nabla(\operatorname{div}\mathbf{w}),$$

from which we obtain

$$\operatorname{curl}\operatorname{curl}(\operatorname{curl}\mathbf{w} + \mathbf{a}_N) = \mathbf{0} \text{ in } \mathcal{D}'(\boldsymbol{\mathcal{U}}).$$

By hypothesis, $\operatorname{div}(\operatorname{curl}\mathbf{w} + \mathbf{a}_N) = 0$. Whence, from the identity above,

$$-\Delta(\operatorname{curl}\mathbf{w} + \mathbf{a}_N) = \operatorname{curl}\operatorname{curl}(\operatorname{curl}\mathbf{w} + \mathbf{a}_N) = \mathbf{0} \text{ in } \mathcal{D}'(\boldsymbol{\mathcal{U}}).$$

This implies $(\operatorname{curl}\mathbf{w} + \mathbf{a}_N) \in C^\infty(\boldsymbol{\mathcal{U}})$, and we conclude that

$$\mathbf{a}_N \in C^{1,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{1,\alpha}(\boldsymbol{\mathcal{D}}^c). \qquad \square$$

Consider the case $\mu \equiv 1$. Given $\mathbf{e}$, we can find a linear function $\mathbf{a}(\mathbf{x})$ such that $\operatorname{curl}\mathbf{a} \equiv \mathbf{e}$. Clearly $\mathbf{a}$ satisfies (1.3). It follows from the previous lemma, then, that $\operatorname{curl}\mathbf{a}_N \equiv \mathbf{e}$ when $\mu \equiv 1$.

We now derive a Sobolev estimate analogous to Lemma 2.3.

LEMMA 3.5. *Let* $(\zeta, \mathbf{B})$ *be a weak solution to* (1.2). *Let* $(\psi, \mathbf{A})$ *be the gauge-equivalent solution found in Lemma* 3.2 *and* $(0, h\mathbf{a}_N)$ *be the normal state found in Lemma* 3.3. *Then there is a constant* C_0 *depending only on* $\boldsymbol{\mathcal{D}}$ *such that*

$$\int_{\boldsymbol{\mathcal{D}}} |\mathbf{A} - h\mathbf{a}_N|^2 d\mathbf{x} \le C_0 \int_{\mathbb{R}^3} |\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N)|^2 d\mathbf{x}.$$

Proof. Using (3.1) and (3.2) we see

$$\|\mathbf{A} - h\mathbf{a}_N\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} \le \theta\|\nabla(\mathbf{A} - h\mathbf{a}_N)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} = \theta\|\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.$$

Since $\boldsymbol{\mathcal{D}}$ is bounded we have

$$\|\mathbf{A} - h\mathbf{a}_N\|_{L^2(\boldsymbol{\mathcal{D}};\mathbb{R}^3)} \le M(\boldsymbol{\mathcal{D}})\|\mathbf{A} - h\mathbf{a}_N\|_{L^6(\boldsymbol{\mathcal{D}};\mathbb{R}^3)}$$

and the lemma follows. $\square$

We proceed to derive the three-dimensional counterpart to Lemma 2.5.

LEMMA 3.6. *Let* $(\psi, \mathbf{A})$ *and* $(0, h\mathbf{a}_N)$ *be as in Lemma* 3.5. *Then there is a constant* $C_1 = C_1(\boldsymbol{\mathcal{D}}, \mu_e)$ *so that*

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + h\kappa\mathbf{a}_N)\psi|^2 d\mathbf{x} \le C_1\kappa^2 \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x}.$$

Proof. We proceed just as in Lemma 2.5 to obtain

$$\int_{\mathbb{R}^3} \frac{1}{\mu}[\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N) \cdot \operatorname{curl}\mathbf{B}]d\mathbf{x} \le \varepsilon^{-1} \int_{\boldsymbol{\mathcal{D}}} |\psi|^2 d\mathbf{x} + \varepsilon \int_{\boldsymbol{\mathcal{D}}} |\psi|^2|\mathbf{B}|^2 d\mathbf{x}$$

for any $\varepsilon > 0$ and $\mathbf{B} \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ with bounded support. However, since $\mathbf{A} - h\mathbf{a}_N \in \check{H}^1(\mathbb{R}^3)$, we can take $\mathbf{B} = \mathbf{B}_j \to \mathbf{A} - h\mathbf{a}_N$ in $\check{H}^1(\mathbb{R}^3)$ as $j \to \infty$. As a result, we have

$$\int_{\mathbb{R}^3} \frac{1}{\mu} |\operatorname{curl}(\mathbf{A} - h\mathbf{a}_N)|^2 d\mathbf{x} \le \varepsilon^{-1} \int_{\mathcal{D}} |\psi|^2 d\mathbf{x} + \varepsilon \int_{\mathcal{D}} |\psi|^2 |\mathbf{A} - h\mathbf{a}_N|^2 d\mathbf{x}.$$

The remainder of the proof is just as before. □

We next give a three-dimensional analogue for the eigenvalue estimate from [2].

Let $\mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $|\mathbf{v}| = 1$ and $\mathbf{x}_0 \in \mathbb{R}^3$. Let $T(\mathbf{x}_0, r, \mathbf{v})$ be a cylinder with central axis parallel to $\mathbf{v}$, height $2r$, and whose middle cross section is the disk of radius r with center $\mathbf{x}_0$.

LEMMA 3.7. *Let* $\mathbf{b} \in H^1(T(\mathbf{x}_0, r, \mathbf{v}); \mathbb{R}^3)$ *such that* $\operatorname{curl} \mathbf{b} = \mathbf{v}$. *Then*

$$\int_T |(i\nabla + \omega^2 \mathbf{b})\zeta|^2 d\mathbf{x} \ge \omega^2 \sigma(\omega r) \int_T |\zeta|^2 d\mathbf{x} \tag{3.6}$$

for all $\zeta \in \mathcal{H}^1(T)$, *where* $\sigma(\cdot)$ *is as in Proposition* 2.7.

Proof. We first transfer the problem to

$$T(\mathbf{0}, r, \mathbf{e}_3) = B_r(\mathbf{0}) \times (-r, r).$$

Let $Q \in SO(3)$ such that $\mathbf{e}_3 = Q\mathbf{v}$, and set $\mathbf{y}(\mathbf{x}) = Q(\mathbf{x} - \mathbf{x}_0)$. Then

$$\mathbf{y} : \mathbf{x} \in T(\mathbf{x}_0, r, \mathbf{v}) \to T(\mathbf{0}, r, \mathbf{e}_3).$$

Given $\zeta \in \mathcal{H}^1(T(\mathbf{x}_0, r, \mathbf{v}))$ we define $\xi(\mathbf{y}) = \zeta(\mathbf{x}(\mathbf{y}))$. Then

$$(i\nabla_{\mathbf{y}} + \omega^2 \mathbf{b} Q^t)\xi(\mathbf{y}) = (i\nabla_{\mathbf{x}} + \omega^2 \mathbf{b})\zeta(\mathbf{x}) Q^t.$$

By changing variables we see that (3.6) is equivalent to showing the following inequality:

$$\int_{T(\mathbf{0},r,\mathbf{e}_3)} |(i\nabla + \omega^2 \mathbf{b} Q^t)\xi|^2 d\mathbf{y} \ge \omega^2 \sigma(\omega r) \int_{T(\mathbf{0},r,\mathbf{e}_3)} |\xi|^2 d\mathbf{y}.$$

For any $\mathbf{w} \in \mathbb{R}^3$, we have

$$Q\mathbf{w} \cdot \operatorname{curl}_{\mathbf{y}}(\mathbf{b}Q^t) = \det \begin{bmatrix} \mathbf{w}^t Q^t \\ \nabla_{\mathbf{y}} \\ \mathbf{b} Q^t \end{bmatrix} = \det \begin{bmatrix} \mathbf{w}^t Q^t \\ \nabla_{\mathbf{x}} Q^t \\ \mathbf{b} Q^t \end{bmatrix} = \det \begin{bmatrix} \mathbf{w}^t \\ \nabla_{\mathbf{x}} \\ \mathbf{b} \end{bmatrix} = \mathbf{w} \cdot \operatorname{curl}_{\mathbf{x}} \mathbf{b}.$$

Therefore, $\operatorname{curl}_{\mathbf{y}}(\mathbf{b}Q^t) = Q(\operatorname{curl}_{\mathbf{x}} \mathbf{b}) = Q\mathbf{v} = \mathbf{e}_3$.

By changing the gauge if necessary, we can assume $\mathbf{b}Q^t = \frac{1}{2}(-y, x, 0)$. Then

$$\begin{aligned}
\int_{T(\mathbf{0},r,\mathbf{e}_3)} |(i\nabla + \omega^2 \mathbf{b} Q^t)\xi|^2 d\mathbf{y} &\ge \int_{T(\mathbf{0},r,\mathbf{e}_3)} |(i(D_x, D_y, 0) + \omega^2 \mathbf{b} Q^t)\xi|^2 d\mathbf{y} \\
&= \int_{-r}^{r} \int_{B_r} \left| \left(i(D_x, D_y, 0) + \frac{\omega^2}{2}(-y, x, 0) \right) \xi(x, y, z) \right|^2 dx\, dy\, dz \\
&\ge \int_{-r}^{r} \omega^2 \sigma(\omega r) \int_{B_r} |\xi(x, y, z)|^2 dx\, dy\, dz \\
&= \omega^2 \sigma(\omega r) \int_{T(\mathbf{0},r,\mathbf{e}_3)} |\xi|^2 d\mathbf{y},
\end{aligned}$$

where we have applied Proposition 2.7 for each $-r \le z \le r$. □

We go on to prove the three-dimensional counterpart to the eigenvalue estimate in Lemma 2.8.

LEMMA 3.8. *Let* $(0, \mathbf{a}_N)$ *be the normal state from Lemma* 3.3. *Assume* $\operatorname{curl} \mathbf{a}_N \neq \mathbf{0}$ *in* $\overline{\boldsymbol{\mathcal{D}}}$. *Then there exist constants* $m \geq 1$ *and* $0 < C_2 \leq 1$ *so that if* $\omega^2 \geq m$, *it holds that*

$$C_2 \omega^2 \int_{\boldsymbol{\mathcal{D}}} |\zeta|^2 d\mathbf{x} \leq \int_{\boldsymbol{\mathcal{D}}} |(i\nabla + \omega^2 \mathbf{a}_N)\zeta|^2 d\mathbf{x} \tag{3.7}$$

for all $\zeta \in \mathcal{H}^1(\boldsymbol{\mathcal{D}})$.

Proof. We argue as in Lemma 2.8. There exists a constant $d > 0$ so that, given $\xi \in \mathcal{H}^1(\boldsymbol{\mathcal{D}})$ and $\omega \geq 1$, either (3.7) is true with $C_2 = 1$ and $\xi = \zeta$ or

$$\frac{1}{2} \int_{\boldsymbol{\mathcal{D}}} |\xi|^2 d\mathbf{x} \leq \int_{\boldsymbol{\mathcal{D}} \setminus \boldsymbol{\mathcal{D}}_{\frac{d}{\omega}}} |\xi|^2 d\mathbf{x}. \tag{3.8}$$

Assume the latter. In this case we cover $\boldsymbol{\mathcal{D}} \setminus \boldsymbol{\mathcal{D}}_{\frac{d}{\omega}}$ by a family of cylinders $\{T_k : k = 1, \dots, N(\omega)\}$ such that $T_k = T(\mathbf{x}_k, \frac{d}{2\omega}, \operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)/|\operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)|)$ with $T_k \subset \boldsymbol{\mathcal{D}}$ for each k and $\sum_{k=1}^N \chi_{T_k} \leq K$, where K is independent of ω for $\omega \geq 1$. As a consequence,

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + \omega^2 \mathbf{a}_N)\xi|^2 d\mathbf{x} \geq K^{-1} \sum_{k=1}^{N} \int_{T_k} |(i\nabla + \omega^2 \mathbf{a}_N)\xi|^2 d\mathbf{x}. \tag{3.9}$$

In each T_k we write $\mathbf{a}_N(\mathbf{x}) = \ell_k(\mathbf{x}) + q_k(\mathbf{x})$, where $\ell_k(\mathbf{x}) = \mathbf{a}_N(\mathbf{x}_k) + \nabla \mathbf{a}_N(\mathbf{x}_k) \cdot (\mathbf{x} - \mathbf{x}_k)$. Note that $\operatorname{curl} \ell_k(\mathbf{x}) = \operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)$. Using (2.9), for each k we obtain

$$\int_{T_k} |(i\nabla + \omega^2 \mathbf{a}_N)\xi|^2 d\mathbf{x} \geq \frac{1}{2} \int_{T_k} |(i\nabla + \omega^2 \ell_k)\xi|^2 d\mathbf{x} - \omega^4 \int_{T_k} |q_k|^2 |\xi|^2 d\mathbf{x}.$$

From Lemma 3.7 we have

$$\begin{aligned} \int_{T_k} |(i\nabla + \omega^2 \ell_k)\xi|^2 d\mathbf{x} &\geq \omega^2 |\operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)| \sigma \left(|\operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)|^{\frac{1}{2}} \frac{d}{2} \right) \int_{T_k} |\xi|^2 d\mathbf{x} \\ &\geq \omega^2 M_0 \int_{T_k} |\xi|^2 d\mathbf{x}, \end{aligned}$$

where $M_0 > 0$ depends on $\inf_{\boldsymbol{\mathcal{D}}} |\operatorname{curl} \mathbf{a}_N| > 0$ and the structure of $\sigma(\cdot)$ (see Proposition 2.7).

Since $\mathbf{a}_N \in C^{1,\alpha}(\overline{\boldsymbol{\mathcal{D}}})$ we have

$$|q_k| \leq M_1 (\operatorname{diam} T_k)^{1+\alpha} \leq M_2 \omega^{-1-\alpha}, \quad \text{where } M_2 \text{ is independent of } k.$$

As a result, we see for each k that

$$\int_{T_k} |(i\nabla + \omega^2 \mathbf{a}_N)\xi|^2 d\mathbf{x} \geq \left(\frac{M_0}{2} \omega^2 - M_2^2 \omega^{2-2\alpha} \right) \int_{T_k} |\xi|^2 d\mathbf{x} \geq \frac{M_0}{4} \omega^2 \int_{T_k} |\xi|^2 d\mathbf{x}, \tag{3.10}$$

provided $\omega^2 \geq m = m(\boldsymbol{\mathcal{D}}, \mu_e)$ sufficiently large.

From (3.9) and (3.10), then,

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + \omega^2 \mathbf{a}_N)\xi|^2 d\mathbf{x} \geq M_3\omega^2 \int_{\cup_{k=1}^N T_k} |\xi|^2 d\mathbf{x}$$

for some $M_3 > 0$ independent of the cover. Using $\boldsymbol{\mathcal{D}} \setminus \boldsymbol{\mathcal{D}}_{\frac{d}{\omega}} \subset \cup_{k=1}^N T_k$ and (3.8), we derive

$$\int_{\cup_{k=1}^N T_k} |\xi|^2 d\mathbf{x} \geq \int_{\boldsymbol{\mathcal{D}}\setminus\boldsymbol{\mathcal{D}}_{\frac{d}{\omega}}} |\xi|^2 d\mathbf{x} \geq \frac{1}{2}\int_{\boldsymbol{\mathcal{D}}} |\xi|^2 d\mathbf{x}.$$

Setting $C_2 = \frac{M_3}{2}$, we have our lemma. □

The following theorem is proved in the same manner as Theorem 2.9. We establish the existence of $\overline{h}$ and derive an upper bound for it provided $\operatorname{curl}\mathbf{a}_N$ does not vanish on $\overline{\boldsymbol{\mathcal{D}}}$.

THEOREM 3.9. *Assume that* $\operatorname{curl}\mathbf{a}_N \neq \mathbf{0}$ *in* $\overline{\boldsymbol{\mathcal{D}}}$. *There are constants m and ϕ, depending on $\boldsymbol{\mathcal{D}}$ and μ_e, so that if $h > \max(\frac{m}{\kappa}, \phi\kappa)$, then any weak solution to* (1.2) *with $n = 3$ is normal.*

For the case $\mu \equiv 1$, we have $|\operatorname{curl}\mathbf{a}_N| \equiv 1$, and we can recover the following result.

COROLLARY 3.10. *If $\mu_e = 1$, then there exist constants m and ϕ depending on $\boldsymbol{\mathcal{D}}$ so that if $h > \max(\frac{m}{\kappa}, \phi\kappa)$, then any weak solution to* (1.2) *with $n = 3$ is normal.*

In general, one does not know whether or not $\operatorname{curl}\mathbf{a}_N$ vanishes somewhere in $\overline{\boldsymbol{\mathcal{D}}}$. Nevertheless, since $\operatorname{curl}\mathbf{a}_N$ is harmonic in $\boldsymbol{\mathcal{D}}$ it can vanish only on a small set.

LEMMA 3.11. *Let $(0, \mathbf{a}_N)$ be a normal state. Then* $\mathcal{L}^3(\{\mathbf{x} \in \boldsymbol{\mathcal{D}} : \operatorname{curl}\mathbf{a}_N(\mathbf{x}) = \mathbf{0}\}) = 0$.

Proof. Since $\operatorname{curl}\mathbf{a}_N$ is harmonic in $\boldsymbol{\mathcal{D}}$, either $\mathcal{L}^3(\{\mathbf{x} \in \boldsymbol{\mathcal{D}} : \operatorname{curl}\mathbf{a}_N(\mathbf{x}) = \mathbf{0}\}) = 0$ or $\operatorname{curl}\mathbf{a}_N \equiv \mathbf{0}$ in $\boldsymbol{\mathcal{D}}$. From Lemma 3.4, we know that there is a function $p \in C^{2,\alpha}(\overline{\boldsymbol{\mathcal{D}}}) \cap C^{2,\alpha}(\boldsymbol{\mathcal{D}}^c) \cap C(\mathbb{R}^3)$ such that

$$\operatorname{curl}\mathbf{a}_N = \mu\nabla p \text{ in } \mathbb{R}^3.$$

Hence, p satisfies

$$\begin{cases} \operatorname{div}(\mu\nabla p) = 0 & \text{in } \mathbb{R}^3, \\ (\nabla p - \mathbf{e}_3) \in L^2(\mathbb{R}^3). \end{cases} \tag{3.11}$$

Assume that $\operatorname{curl}\mathbf{a}_N \equiv \mathbf{0}$ in $\boldsymbol{\mathcal{D}}$. Then $p = \text{constant} = p_0$ in $\overline{\boldsymbol{\mathcal{D}}}$. It follows from the first equation in (3.11) that p solves

$$\Delta p = 0 \qquad \text{in} \quad \boldsymbol{\mathcal{D}}^c,$$

$$p = p_0, \frac{\partial p}{\partial n} = 0 \quad \text{on } \partial\boldsymbol{\mathcal{D}}^c,$$

where $\mathbf{n}$ is the exterior normal to $\partial\boldsymbol{\mathcal{D}}$. The Cauchy problem has the unique solution $p = p_0$. This would contradict the second equation in (3.11). □

To conclude, we show that there is a finite upper critical field for each κ.

THEOREM 3.12. *Let κ, μ_e, and γ be fixed. There is a constant $\overline{h} = \overline{h}(\kappa, \mu_e, \gamma, \boldsymbol{\mathcal{D}})$ so that if $h > \overline{h}$, then any weak solution to* (1.2) *with $n = 3$ is normal.*

Proof. Let $(0, \mathbf{a}_N)$ be as in Lemma 3.3. Assume that there exists a sequence $\{(\psi_j, \mathbf{A}_j)\}$, where for each j the pair is as in Lemma 3.5, solving (1.2) with $h = h_j$ for which $\lim_{j\to\infty} h_j = \infty$ and $\int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 dx > 0$.

Set $\varphi_j(x) = |\psi_j(x)|/\|\psi_j\|_{L^2(\boldsymbol{\mathcal{D}})}$. From (2.6) we have

$$\int_{\boldsymbol{\mathcal{D}}} |\nabla\varphi_j|^2 d\mathbf{x} \le \kappa^2,$$

and we can find a subsequence $\varphi_j \to \varphi_0$ in $L^2(\boldsymbol{\mathcal{D}})$ as $j \to \infty$ with $\|\varphi_0\|_{L^2(\boldsymbol{\mathcal{D}})} = 1$.

From Lemma 3.11 the set $\mathbf{Q} = \{\mathbf{x} \in \overline{\boldsymbol{\mathcal{D}}} : \operatorname{curl} \mathbf{a}_N(\mathbf{x}) = \mathbf{0}\}$ is a closed set of measure zero. It follows that there exists a ball $B_{2r} \subset \boldsymbol{\mathcal{D}} \setminus \mathbf{Q}$ such that $\int_{B_r} |\varphi_0|^2 d\mathbf{x} \equiv 2\delta > 0$ and $\inf_{B_{2r}} |\operatorname{curl} \mathbf{a}_N| > 0$. Note that for j sufficiently large we have

$$\int_{B_r} |\psi_j|^2 d\mathbf{x} \ge \delta \int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 d\mathbf{x}.$$

For each j we cover B_r by a finite family of cylinders,

$$\{T(\mathbf{x}_k, \omega_j^{-1}, \operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)/|\operatorname{curl} \mathbf{a}_N(\mathbf{x}_k)|)\} = \{T_{kj}, 1 \le k \le N(j)\},$$

such that $\omega_j^2 = h_j\kappa$, $\mathbf{x}_k \in B_r$, and such that the family has overlap of at most K uniformly in $\mathbf{x} \in \boldsymbol{\mathcal{D}}$ independent of j. For j sufficiently large, each $T_{kj} \subset B_{2r}$ and, as in Lemma 3.8, we derive

$$\int_{T_{kj}} |(i\nabla + \omega_j^2 \mathbf{a}_N)\psi_j|^2 d\mathbf{x} \ge \omega_j^2 M_0 \int_{T_{kj}} |\psi_j|^2 d\mathbf{x},$$

where $M_0 > 0$ depends on $\inf_{B_{2r}} |\operatorname{curl} \mathbf{a}_N|$ and $\|\mathbf{a}_N\|_{C^{1,\alpha}(\overline{\boldsymbol{\mathcal{D}}})}$ but not on j. Whence we can write

$$\begin{aligned}\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + h_j\kappa\mathbf{a}_N)\psi_j|^2 d\mathbf{x} &\ge K^{-1} \sum_{k=1}^{N(j)} \int_{T_{kj}} |(i\nabla + h_j\kappa\mathbf{a}_N)\psi_j|^2 d\mathbf{x} \\ &\ge M_1 h_j\kappa \int_{\cup_{k=1}^{N(j)} T_{kj}} |\psi_j|^2 d\mathbf{x} \ge M_1 h_j\kappa \int_{B_r} |\psi_j|^2 d\mathbf{x} \ge M_2 h_j\kappa \int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 d\mathbf{x},\end{aligned}$$

where M_1 and M_2 are positive and depend on $\inf_{B_{2r}} |\operatorname{curl} \mathbf{a}_N|$, $\mathbf{a}_N$, and δ. From Lemma 3.6 we know

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + h_j\kappa\mathbf{a}_N)\psi_j|^2 d\mathbf{x} \le C_1\kappa^2 \int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 d\mathbf{x},$$

which leads to

$$M_2 h_j\kappa \int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 d\mathbf{x} \le C_1\kappa^2 \int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 d\mathbf{x}$$

for all j sufficiently large. Since $h_j \to \infty$ as $j \to \infty$, we must have $\int_{\boldsymbol{\mathcal{D}}} |\psi_j|^2 d\mathbf{x} = 0$ for j large, and this is a contradiction. ◻

4. Estimates for Small κ. In this section we consider $\overline{h}$ for small κ in cases where $\operatorname{curl} \mathbf{a}_N \equiv \mathbf{e}$ in $\boldsymbol{\mathcal{D}}$.

THEOREM 4.1. *Let $n = 2$ with $\mu_e > 0$ or $n = 3$ with $\mu_e = 1$. Then $\overline{h} = O(1)$ as $\kappa \downarrow 0$.*

Proof. We consider the case $n = 3$. The argument for $n = 2$ is identical.

Let $\kappa \le 1$ and assume that (ψ, A) solves (1.2) with $\psi \not\equiv 0$. From (2.6) we have

$$\int_{\mathcal{D}} |\nabla|\psi|\,|^2 d\mathbf{x} \le \kappa^2 \int_{\mathcal{D}} |\psi|^2 d\mathbf{x} \le \int_{\mathcal{D}} |\psi|^2 d\mathbf{x}.$$

If we apply Lemma 2.6 with $\lambda = \lambda_0 = 1$, we find that there is a constant $d > 0$ so that

$$\frac{1}{2}\int_{\mathcal{D}} |\psi|^2 d\mathbf{x} \le \int_{\mathcal{D}\setminus\mathcal{D}_d} |\psi|^2 d\mathbf{x}.$$

We take r, $0 < r < d$, to be determined and cover $\mathcal{D} \setminus \mathcal{D}_d$ by a family of cylinders $\{T(\mathbf{x}_k, r, \mathbf{e})\}$ such that the cylinders have finite overlap independent of r and each is contained in $\mathcal{D}$. Then, arguing just as in Lemma 2.8, there exists a constant $C_2 > 0$ for which

$$C_2 h\kappa\ \sigma((h\kappa)^{\frac{1}{2}} r) \int_{\mathcal{D}} |\psi|^2 d\mathbf{x} \le \int_{\mathcal{D}} |(i\nabla + h\kappa \mathbf{a}_N)\psi|^2 d\mathbf{x}.$$

Applying Lemma 3.6 to the right-hand side and recalling that $\int_{\mathcal{D}} |\psi|^2 d\mathbf{x} > 0$, the following inequality holds:

$$C_2 h\kappa\ \sigma((h\kappa)^{\frac{1}{2}} r) \le C_1 \kappa^2. \tag{4.1}$$

From Corollary 3.10 we have $h\kappa \le M_1^2$ for some $M_1 < \infty$ for all $\kappa \le 1$. We now choose r so that $M_1 r \le 2^{\frac{1}{2}}$. The reason for this choice is that if $0 < t \le 2^{\frac{1}{2}}$, then $\sigma(t)$ is the principal eigenvalue for the Sturm–Liouville problem

$$g''(s) + \frac{g'(s)}{s} - \frac{s^2 g(s)}{4} = -\sigma(t) g(s) \qquad \text{for } 0 < s < t,$$
$$g'(t) = 0 \text{ and } g \text{ is bounded}$$

(see [2, section 6]). As such, from [7, Prop. 3.4], we have $\sigma(t) = \frac{t^2}{8} + o(t^2)$ as $t \to 0$. It follows that there is a constant $M_2 > 0$ so that

$$M_2 t^2 \le \sigma(t) \qquad \text{for } 0 \le t \le 2^{\frac{1}{2}}.$$

Combining this estimate with (4.1), we derive

$$h^2 \le \frac{C_1}{C_2 M_2 r^2} \qquad \text{for } 0 < \kappa \le 1.$$

Since this holds for all superconducting solutions, we conclude that

$$\overline{h}^2 \le \frac{C_1}{C_2 M_2 r^2} \qquad \text{for } 0 < \kappa \le 1. \qquad \square$$

The length scale for variations in superconducting solutions is $\frac{1}{\kappa}$. Because of this, it is of interest to consider domains with dimensions comparable to $\frac{1}{\kappa}$. To this end, let $\mathcal{D} \subseteq \mathbb{R}^n$ and define the dilated domain

$$\mathcal{D}(\kappa) = \{\mathbf{x} \in \mathbb{R}^n : \kappa\mathbf{x} \in \mathcal{D}\}.$$

Then $\operatorname{diam}(\boldsymbol{\mathcal{D}}(\kappa)) = \frac{1}{\kappa}\operatorname{diam}(\boldsymbol{\mathcal{D}})$. Let $\mu \equiv 1$ so that $\operatorname{curl}\mathbf{a}_N \equiv \mathbf{e}$ in $\boldsymbol{\mathcal{D}}(\kappa)$. Consider (ψ, A) satisfying (1.2) on $\boldsymbol{\mathcal{D}}(\kappa)$. Let

$$\widetilde{\psi}(\mathbf{x}) = \psi(\kappa^{-1}\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \boldsymbol{\mathcal{D}}$$

and

$$\widetilde{\mathbf{A}}(\mathbf{x}) = \mathbf{A}(\kappa^{-1}\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3.$$

Then $(\widetilde{\psi}, \widetilde{\mathbf{A}})$ satisfies

$$\text{(4.2)} \qquad \begin{cases} (i\nabla + \widetilde{\mathbf{A}})^2\widetilde{\psi} - \widetilde{\psi} + |\widetilde{\psi}|^2\widetilde{\psi} = 0 & \text{in } \boldsymbol{\mathcal{D}}, \\ \operatorname{curl}^2 \widetilde{\mathbf{A}} = -\kappa^{-2}\left(\frac{i}{2}(\widetilde{\psi}^*\nabla\widetilde{\psi} - \widetilde{\psi}\nabla\widetilde{\psi}^*) + \widetilde{\mathbf{A}}|\psi|^2\right)\chi_{\boldsymbol{\mathcal{D}}} & \text{in } \mathbb{R}^3, \\ n \cdot (i\nabla + \widetilde{\mathbf{A}})\widetilde{\psi} = -i\gamma\widetilde{\psi} & \text{on } \partial\boldsymbol{\mathcal{D}}, \\ (\operatorname{curl}\widetilde{\mathbf{A}} - h\kappa^{-1}\mathbf{e}) \in L^2(\mathbb{R}^3; \mathbb{R}^3). \end{cases}$$

Assume that $0 < \kappa \leq 1$. Arguing as in Lemma 2.5, taking into account the multiple κ^{-2} of the right-hand side of the second equation in (4.2), we obtain the analogue of (2.7),

$$\int_{\boldsymbol{\mathcal{D}}} |\widetilde{\mathbf{A}} - h\kappa^{-1}\mathbf{a}_N|^2 d\mathbf{x} \leq \kappa^{-4} M \int_{\boldsymbol{\mathcal{D}}} |\widetilde{\psi}|^2 d\mathbf{x},$$

where $M = M(\operatorname{diam}\boldsymbol{\mathcal{D}})$. Using $\kappa \leq 1$ we then obtain the analogue of (2.5),

$$\int_{\boldsymbol{\mathcal{D}}} |(i\nabla + h\kappa^{-1}\mathbf{a}_N)\widetilde{\psi}|^2 d\mathbf{x} \leq C_1\kappa^{-4} \int_{\boldsymbol{\mathcal{D}}} |\widetilde{\psi}|^2 d\mathbf{x}.$$

It follows as in Theorem 2.9, then, that there is a constant ϕ so that if $h\kappa^{-1} \geq \phi\kappa^{-4}$ then $\widetilde{\psi} \equiv 0$. Returning to $(\psi, \mathbf{A})$, we conclude that

$$\overline{h}(\kappa, 1, \gamma, \boldsymbol{\mathcal{D}}(\kappa)) = O(\kappa^{-3}) \qquad \text{as } \kappa \downarrow 0.$$

As we noted in the introduction, there are superconducting solutions for the case of the slab $-\infty < x < 0$, $-\infty < y, z < \infty$ for which h diverges as $\kappa \downarrow 0$ (see [5]). However, this occurs at the slower rate $h = O(\kappa^{-\frac{1}{2}})$.

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