THE SET OF 2-BY-3 MATRIX PENCILS — KRONECKER STRUCTURES AND THEIR TRANSITIONS UNDER PERTURBATIONS*

ERIK ELMROTH† AND BO KÅGSTRÖM†

Abstract. The set (or family) of 2-by-3 matrix pencils $A - \lambda B$ comprises 18 structurally different Kronecker structures (canonical forms). The algebraic and geometric characteristics of the generic and the 17 non-generic cases are examined in full detail. The complete closure hierarchy of the orbits of all different Kronecker structures is derived and presented in a closure graph that show how the structures relate to each other in the 12-dimensional space spanned by the set of 2-by-3 pencils. Necessary conditions on perturbations for transiting from the orbit of one Kronecker structure to another in the closure hierarchy are presented in a labeled closure graph. The node and arc labels show geometric characteristics of an orbit's Kronecker structure and the change of geometric characteristics when transiting to an adjacent node, respectively. Computable normwise bounds for the smallest perturbations $(\delta A, \delta B)$ of a generic 2-by-3 pencil $A-\lambda B$ such that $(A+\delta A)-\lambda(B+\delta B)$ has a specific non-generic Kronecker structure are presented. First explicit expressions for the perturbations that transfer $A - \lambda B$ to a specified non-generic form are derived. In this context tractable and intractable perturbations are defined. Secondly, a modified `GUPTRI` that computes a specified Kronecker structure of a generic pencil is used. Perturbations devised to impose a certain non-generic structure is computed in a way that guarantees to find a KCF on the closure of the orbit of the intended KCF. Both approaches are illustrated by computational experiments. Moreover, a study of the behaviour of the non-generic structures under random perturbations in finite precision arithmetic (using the `GUPTRI` software) show for which sizes of perturbations the structures are invariant and also that structure transitions occur in accordance with the closure hierarchy. Finally, some of the results are extended to the general m-by-$(m+1)$ case.

Key words. Matrix pencils (2-by-3), Kronecker canonical form, generalized Schur decomposition, orbit, codimension, Kronecker structure hierarchy, closest non-generic structure, controllability.

AMS subject classifications. 65F15, 15A21, 15A22.

1. Introduction.

Singular matrix pencils $A - \lambda B$, where A and B are m-by-n matrices with real or complex entries, appear in several applications. Examples include problems in control theory relating to a linear system $E\dot{x}(t) = Fx(t) + Gu(t)$, where E and F are p-by-p matrices, and G is p-by-k. Solvability issues of a singular system (i.e., $\det(E) = 0$), such as the existence of a solution, consistent initial values, and its explicit solution can be revealed from the Kronecker structure of $A - \lambda B \equiv F - \lambda E$ (e.g. see [9, 20]). The problems to find the controllable subspace, uncontrollable modes or an upper bound on the distance to uncontrollability for a controllable system $E\dot{x}(t) = Fx(t) + Gu(t)$ can all be formulated and solved in terms of certain reducing subspaces of the matrix pencil $A - \lambda B \equiv [G \quad F] - \lambda[0 \quad E]$ (e.g. see [15, 17, 18, 6]).

In most applications it is enough to transfer $A - \lambda B$ to a *generalized Schur form* (e.g. to GUPTRI form [7, 8])

$$P^H(A - \lambda B)Q = \begin{bmatrix} A_r - \lambda B_r & * & * \\ 0 & A_{reg} - \lambda B_{reg} & * \\ 0 & 0 & A_l - \lambda B_l \end{bmatrix}, \tag{1.1}$$

where P (m-by-m) and Q (n-by-n) are unitary and $*$ denotes arbitrary conforming submatrices. Here the square upper triangular block $A_{reg} - \lambda B_{reg}$ is regular and has

* Received by the editors ... (********** this text is now put here just in order to give the correct size of the footnotes **********).

† Department of Computing Science, Umeå University, S-901 87 Umeå, Sweden (`elmroth@cs.umu.se` and `bokg@cs.umu.se`).

the same regular structure as $A - \lambda B$ (i.e., contains all generalized eigenvalues (finite and infinite) of $A - \lambda B$). The rectangular blocks $A_r - \lambda B_r$ and $A_l - \lambda B_l$ contain the singular structure (right and left minimal indices) of the pencil and are block upper triangular. The *singular blocks of right* (column) and *left* (row) *indices of grade* j are

$$L_j \equiv \begin{bmatrix} -\lambda & 1 & & \\ & \cdot & \cdot & \\ & & -\lambda & 1 \end{bmatrix} \quad \text{and} \quad L_j^T \equiv \begin{bmatrix} -\lambda & & \\ 1 & \cdot & \\ & \cdot & -\lambda \\ & & 1 \end{bmatrix}, \tag{1.2}$$

of size j-by-$(j+1)$ and $(j+1)$-by-j, respectively. $A_r - \lambda B_r$ has only right minimal indices in its Kronecker canonical form (KCF), indeed the same L_j blocks as $A - \lambda B$. Similarly, $A_l - \lambda B_l$ has only left minimal indices in its KCF, the same L_j^T blocks as $A - \lambda B$. If $A - \lambda B$ is singular at least one of $A_r - \lambda B_r$ and $A_l - \lambda B_l$ will be present in (1.1). The explicit structure of the diagonal blocks in staircase form can be found in [8]. If $A - \lambda B$ is regular $A_r - \lambda B_r$ and $A_l - \lambda B_l$ are not present in (1.1) and the GUPTRI form reduces to the upper triangular block $A_{reg} - \lambda B_{reg}$. Staircase forms that reveal the Jordan structure of the zero and infinite eigenvalues are contained in $A_{reg} - \lambda B_{reg}$.

Given $A - \lambda B$ in GUPTRI form we also know different pairs of reducing subspaces [18, 7]. Suppose the eigenvalues on the diagonal of $A_{reg} - \lambda B_{reg}$ are ordered so that the first k, say, are in Λ_1 (a subset of the spectrum of $A_{reg} - \lambda B_{reg}$) and the remainder are outside Λ_1. Let $A_r - \lambda B_r$ be m_r-by-n_r. Then the left and right reducing subspaces associated with Λ_1 are spanned by the leading $m_r + k$ columns of P and the leading $n_r + k$ columns of Q, respectively. When Λ_1 is empty, the corresponding reducing subspaces are called *minimal*, and when Λ_1 contains the whole spectrum the reducing subspaces are called *maximal*.

If $A - \lambda B$ is m-by-n, where $m \neq n$, then for almost all A and B it will have the same KCF, depending only on m and n (the *generic case*). The generic Kronecker structure for $A - \lambda B$ with $d = n - m > 0$ is

$$\text{diag}(L_\alpha, \ldots, L_\alpha, L_{\alpha+1}, \ldots, L_{\alpha+1}), \tag{1.3}$$

where $\alpha = \lfloor m/d \rfloor$, the total number of blocks is d, and the number of $L_{\alpha+1}$ blocks is $m \bmod d$ (which is 0 when d divides m) [16, 3]. The same statement holds for $d = m - n > 0$ if we replace $L_\alpha, L_{\alpha+1}$ in (1.3) by $L_\alpha^T, L_{\alpha+1}^T$. Square pencils are generically regular, i.e., $\det(A - \lambda B) = 0$ if and only if λ is an eigenvalue. The generic singular pencils of size n-by-n have the Kronecker structures [19]:

$$\text{diag}(L_j, L_{n-j-1}^T), \quad j = 0, \ldots, n-1. \tag{1.4}$$

In summary, generic rectangular pencils have only trivial reducing subspaces and no generalized eigenvalues at all. Generic square singular pencils have the same minimal and maximal reducing subspaces. Only if $A - \lambda B$ satisfies a special condition (lies in a particular manifold) does it have nontrivial reducing subspaces and generalized eigenvalues (the *non-generic case*). Moreover, only if it is perturbed so as to move continuously within that manifold do its reducing subspaces and generalized eigenvalues also move continuously and satisfy interesting error bounds [5, 7]. These requirements are natural in many control and systems theoretic problems such as computing controllable subspaces and uncontrollable modes.

Several authors have proposed (staircase-type) algorithms for computing a generalized Schur form (e.g. see [1, 4, 14, 13, 11, 12, 16, 20]). They are numerically stable in the sense that they compute the exact Kronecker structure (generalized Schur form or something similar) of a nearby pencil $A' - \lambda B'$. Let $\|\cdot\|_E$ denote the Euclidean (Frobenius) matrix norm. Then $\delta \equiv \|(A - A', B - B')\|_E$ is an upper bound on the distance to the closest $(A + \delta A, B + \delta B)$ with the KCF of (A', B'). Recently, robust software with error bounds for computing the GUPTRI form of a singular $A - \lambda B$ has been published [7, 8]. Some computational experiments that use this software will be discussed later.

The existing algorithms do not guarantee that the computed generalized Schur form is the "most" non-generic Kronecker structure within distance δ. However, if δ is of the size $O(\|(A, B)\|_E \epsilon)$, where ϵ is the relative machine precision, we know that (A, B) is close to a matrix with the Kronecker structure that the algorithm reports. It would of course be desirable to have algorithms that could solve the following "nearness" problems:

- Compute the closest non-generic pencil of a generic $A - \lambda B$.
- Compute the closest matrix pencil with a specified Kronecker structure.
- Compute the most non-generic pencil within a given distance δ.

If the closest structure is not unique we are mainly interested in the most non-generic KCF. From the perturbation theory for singular pencils [5] we know that all these problems are ill-posed in the sense that the generalized eigenvalues and reducing subspaces for a non-generic $A - \lambda B$ can change discontinuously as a function of A and B. Therefore, to be able to solve these problems we need to regularize them by restricting the allowable perturbations as mentioned above. In this contribution we make a comprehensive study of the set of 2-by-3 pencils in order to get a greater understanding of (i) these "nearness" problems and how to solve them, and (ii) existing algorithms/software for computing the Kronecker structure of a singular pencil. The full implications of this "case study" to general m-by-n pencils are topics for further research.

In the following we give a summary of our contribution and the organization of the rest of the paper. Section 2 is devoted to algebraic and geometric characteristics of the set of 2-by-3 pencils. In Section 2.1 we disclose the structurally different Kronecker structures and show how all the non-generic structures can be generated by a staircase-type algorithm, starting from the generic canonical form. Some algebraic and geometric characteristics of the 18 different Kronecker structures are summarized in three tables. Section 2.2 introduces the concepts of orbits of matrix pencils and their (co)dimensions. The codimensions of the orbits of the 2-by-3 matrix pencils, which depend only on their Kronecker structures [3], are displayed in Table 2.3. They vary between zero (the generic case) and 12 ($= 2mn$) for the zero pencil (the most non-generic case). Indeed, all 2-by-3 pencils "live" in a 12-dimensional space spanned by the set of all generic pencils. In Section 2.3 we derive a graph describing the closure hierarchy of the orbits of all 18 different Kronecker structures for the set of 2-by-3 pencils. The closure graph is presented in Figure 2.1. By labeling the nodes in the closure graph with their geometric characteristics and the arcs with the change in geometric characteristics for transiting to an adjacent node, we get a labeled graph showing necessary conditions on perturbations for transiting from one Kronecker structure to another. The labeled closure graph is presented in Figure 2.2 in Section 2.4.

Section 3 is devoted to an experimental study of how the non-generic Kronecker structures behave under random perturbations in finite precision arithmetic, using the `GUPTRI` software [7, 8]. Assuming a fixed relative accuracy of the input data, structure invariances and transitions of each non-generic case are studied as a function of the size of the perturbations added. The results summarized in Table 3.1 are discussed in terms of tolerance parameters used in `GUPTRI` for determining the Kronecker structure. For large enough perturbations all non-generic pencils turn generic (as expected). Some non-generic cases transit between several non-generic structures before turning generic. These transitions always go from higher to lower codimensions, along the arcs in the closure graph.

In Section 4 we present computable normwise bounds for the smallest perturbations $(\delta A, \delta B)$ of a generic 2-by-3 pencil $A - \lambda B$ such that $(A + \delta A) - \lambda(B + \delta B)$ has a specific non-generic Kronecker structure. Two approaches to impose a non-generic structure are considered. First, explicit expressions for the perturbations that transfer $A - \lambda B$ to a specified non-generic form are derived in Section 4.1. In this context tractable and intractable perturbations are defined. We compute a perturbation $(\delta A, \delta B)$ such that $(A+\delta A)-\lambda(B+\delta B)$ is guaranteed to be in the closure of the manifold (orbit) of a certain KCF. If the KCF found is the intended KCF, then the perturbation is said to be tractable. If the KCF found is even more non-generic then the perturbation is intractable. An intractable perturbation finds any other structure within the closure of the manifold, i.e., a structure that can be found by traveling along the arcs from the intended KCF in the closure graph. A summary of these perturbations is presented in a perturbation graph (Figure 4.1), where the path to each KCF's node shows the tractable perturbation required to find that KCF starting from the generic KCF (an L_2 block). After illustrating intractable perturbations we derive some results regarding the closest non-generic Kronecker structure of a generic 2-by-3 (and 1-by-2) pencil. In the second approach, we use a modified `GUPTRI` for computing a specified Kronecker structure of a generic pencil (Section 4.2). Computational experiments on random 2-by-3 pencils for the two approaches are presented in Section 4.3. It is the intractable perturbations, which impose the most non-generic structure (with highest codimension) for a given size of the perturbations (e.g. the relative accuracy of the data), that are requested in applications (e.g. computing the uncontrollable subspace). Finally, in Section 5 we comment on the general case and extend our results for the closest non-generic pencil to a generic m-by-$(m+1)$ pencil.

2. Algebraic and Geometric Characteristics of the Set of 2-by-3 Matrix Pencils. In this section we disclose the structurally different Kronecker structures and show how all the non-generic structures can be generated by a staircase-type algorithm, starting from the generic canonical form. Moreover, we discuss the codimensions of associated orbits and derive a closure graph, showing the Kronecker structure hierarchy of the set of 2-by-3 pencils.

2.1. Structurally Different Kronecker Structures. The *generic* case corresponds to A and B of size 2-by-3 both having full row rank and non-intersecting column nullspaces. This implies that $A - \lambda B$ is strictly equivalent to an L_2 block:

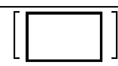
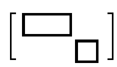
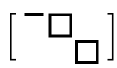
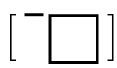
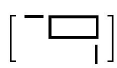
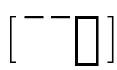
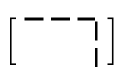
$$P^{-1}(A - \lambda B)Q = L_2 \equiv \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.1}$$

By inspection, we see that the A- and B-parts of L_2 have row rank 2 and non-intersecting 1-dimensional column nullspaces. The generic canonical form L_2 can be obtained by deleting the last row of $J_3(0)-\lambda I_3$, a 3-by-3 Jordan block corresponding to the zero eigenvalue. $J_3(0)$ is the generic canonical form of a 3-by-3 matrix with zero as a triple eigenvalue and the associated non-generic Jordan structures are $J_2(0) \oplus J_1(0)$

and $J_1(0) \oplus J_1(0) \oplus J_1(0)$ (i.e., a 3-by-3 zero matrix). Notice that a generic 3-by-3 matrix is diagonalizable with unspecified non-zero eigenvalues (i.e., all Jordan blocks of size 1-by-1).

In the following we disclose the structurally different non-generic singular cases of size 2×3. By structurally different we mean that all cases have different Kronecker structures (canonical forms). There exists 17 different non-generic singular cases. The simplest way to construct all non-generic canonical forms of size 2×3 is to generate all possible combinations of L_1, L_0, J_2, J_1, R_1, N_1, N_2, L_0^T, and L_1^T blocks as in Table 2.1. Algorithms for computing the Kronecker structure of a singular pencil reveal the right (or left) singular structure and the Jordan structure of the zero (or infinite) eigenvalue simultaneously. Therefore, we only distinguish the zero and infinite Jordan structures and put a non-zero and finite eigenvalue in R_1, a regular 1-by-1 block with an unspecified eigenvalue. We will use R_2 to denote a 2-by-2 block with non-zero finite eigenvalues, i.e., R_2 is used to denote any of the three structures $J_1(\alpha) \oplus J_1(\beta)$, $J_1(\alpha) \oplus J_1(\alpha)$, and $J_2(\alpha)$, where $\alpha,\ \beta \neq \{0, \infty\}$. Notice that if $R_2 = J_2(\alpha)$ then $A - \alpha B$ and B has $J_2(0)$ in its KCF. It is only for the case $L_0 \oplus R_2$ that we can have an $J_2(\alpha)$ block. If we treat these three cases separately we get 19 non-generic cases, but for our purposes it is sufficient to define R_2 as above.

TABLE 2.1
2×3 pencils built from different Kronecker and Jordan blocks.

Number of cases	Block structure	KCF
1		L_2
3		$L_1 \oplus \{J_1, R_1, N_1\}$
5		$L_0 \oplus \{J_1, R_1, N_1\} \oplus \{J_1, N_1\}$
3		$L_0 \oplus \{J_2, R_2, N_2\}$
1		$L_0 \oplus L_1 \oplus L_0^T$
1		$2L_0 \oplus L_1^T$
3		$2L_0 \oplus \{J_1, R_1, N_1\} \oplus L_0^T$
1		$3L_0 \oplus 2L_0^T$

In order to get more insight into the non-generic structures we would like to show how all the non-generic structures can be generated by a staircase-type algorithm. By dropping the row rank of the A-part and/or B-part of L_2 (2.1) and imposing different sizes of their "common column or row nullspace(s)" (see Table 2.3) we are able to generate all 17 non-generic cases starting from the generic canonical form (in the following denoted $A - \lambda B$). Algorithmically, we keep the rank of, for example, B constant and vary the row rank of A while imposing possible sizes of their "common nullspace(s)". A decrease of the row rank is done by deleting a non-zero element (= 1) in the first or second row of A and/or B and the dimension of the common column nullspace is imposed by permutations of the non-zero elements. After decreasing the row rank of B by one we repeat the procedure until the row rank of B equals zero. By doing so we can generate 12 structurally different non-generic pencils of size 2×3. These correspond to cases 2–13 in Table 2.2, where we display a case number i, the matrix pair (A_i, B_i), $\mathrm{r}(A_i), \mathrm{r}(B_i)$, the row ranks of A_i and B_i, respectively, $\mathrm{n}(A_i, B_i)$, the dimension of the common column nullspace of A_i and B_i. Finally, in the last column we display the generalized Schur forms (GUPTRI forms) which correspond to the Kronecker block structures displayed in Table 2.1.

Table 2.2

Summary of the 18 structurally different 2×3 pencils, numbered and presented in the order they are derived in Section 2.

i	A_i	B_i	$\mathrm{r}(A_i)$	$\mathrm{r}(B_i)$	$\mathrm{n}(A_i, B_i)$	GUPTRI form
1	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	2	2	0	$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1	2	0	$\begin{bmatrix} 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$
3	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	0	2	1	$\begin{bmatrix} 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$
4	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	2	1	$\begin{bmatrix} 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix}$
5	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	2	2	1	$\begin{bmatrix} 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$
6	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2	1	0	$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
7	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	1	1	1	$\begin{bmatrix} 0 & -\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$
8	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	0	1	2	$\begin{bmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{bmatrix}$
9	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	1	1	2	$\begin{bmatrix} 0 & 0 & 1-\lambda \\ 0 & 0 & 0 \end{bmatrix}$
10	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	2	1	1	$\begin{bmatrix} 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{bmatrix}$
11	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2	0	1	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
12	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	1	0	2	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
13	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	0	0	3	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$1'$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	2	2	0	$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$
$10'$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	2	1	1	$\begin{bmatrix} 0 & 1-\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$4'$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	2	1	$\begin{bmatrix} 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$
$7'$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	1	1	1	$\begin{bmatrix} 0 & -\lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}$
$9'$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	1	1	2	$\begin{bmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 1 \end{bmatrix}$

Case 1 in Table 2.2 corresponds to the generic structure. Cases 2–5 are obtained by keeping $\mathrm{r}(B_i) = 2$ and varying $\mathrm{r}(A_i)(2, 1, 0)$ and $\mathrm{n}(A_i, B_i)(0, 1)$. In cases 6–10 we keep $\mathrm{r}(B_i) = 1$ and vary $\mathrm{r}(A_i)$ (as before) and $\mathrm{n}(A_i, B_i)(0, 1, 2)$. Finally, in cases 11–13 $\mathrm{r}(B_i) = 0$, $\mathrm{r}(A_i)$ and $\mathrm{n}(A_i, B_i)$ are varied ((0, 1, 2) and (1, 2, 3), respectively). In cases 8, 9, 12 and 13, the matrix pairs have a common row nullspace as well, corresponding to L_0^T blocks in their KCF. The number of L_0^T blocks equals the dimension of the common row nullspace (1 for cases 8, 9 and 12 and 2 for case 13). Notice that $\mathrm{n}(A_i, B_i) = 2$ for three of these four cases and $\mathrm{n}(A_i, B_i) = 3$ for case 13. However, $\mathrm{n}(A_i, B_i) = 2$ is neither a necessary or sufficient condition for a 2-by-3 matrix pair to have a common row nullspace (see cases 7' and 9' below). If we exchange the roles of A and B in the derivation of the non-generic forms 2–13 they will appear in a different order with the N_k blocks and $J_k(0)$ blocks exchanged.

We have five more cases to retrieve, denoted 1', 10', 4', 7' and 9' in Table 2.2. Case x' denotes a case that has the same row-ranks and column-nullities as case x, and is obtained from case x by permuting rows or columns.

Case 1': By swapping columns 2 and 3 in B_1 we still have a matrix pair with $\mathrm{r}(A_i) = \mathrm{r}(B_i) = 2$ and $\mathrm{n}(A_i, B_i) = 0$. We denote this pencil case 1'. As can be seen in Table 2.2, GUPTRI delivers the KCF $L_1 \oplus R_1$ for $A_{1'} - \lambda B_{1'}$. After the first step of deflation in GUPTRI (which identifies that $A_i, i = 1, 1'$ has a 1-dimensional column nullspace ($n(A_i) = 1$) and that $\mathrm{n}(A_i, B_i) = 0, i = 1, 1'$) we are left with the pencils:

$$A_1^{(1)} - \lambda B_1^{(1)} = \begin{bmatrix} 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{1'}^{(1)} - \lambda B_{1'}^{(1)} = \begin{bmatrix} 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{2.2}$$

The difference is that $\mathrm{n}(A_1^{(1)}, B_1^{(1)}) = 0$ while $\mathrm{n}(A_{1'}^{(1)}, B_{1'}^{(1)}) = 1$. Is there any algebraic explanation? We find the answer in the classical characterization of a singular pencil with a right (column) index [9].

Let the matrix $R[A, B, i]$ of size $(i + 2)m \times (i + 1)n$ be defined by

$$R[A, B, i] = \begin{bmatrix} A & 0 & \cdots & 0 \\ B & A & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & B & A \\ 0 & \cdots & 0 & B \end{bmatrix}, \tag{2.3}$$

where A and B are $m \times n$ matrices. When it is clear from context we use the abbreviated notation $R[i]$ for $R[A, B, i]$. With the notation above we can state the following theorem.

THEOREM 2.1. *[9] The following statements are equivalent.*

- *$A - \lambda B$ is singular with a right (column) minimal index of lowest degree $k \geq 0$, i.e., $A - \lambda B$ has no right minimal indices of degree $< k$.*
- *$A - \lambda B$ is equivalent to the pencil*

$$\begin{bmatrix} L_k & 0 \\ 0 & A' - \lambda B' \end{bmatrix}, \tag{2.4}$$

  where L_k is a $k \times (k + 1)$ Kronecker block. $A' - \lambda B'$ may have indices of higher degree.
- *$R[i]$ has full column rank $\mathrm{r}(R[i]) = (i + 1)n$ for $i = 0, 1, \ldots, k - 1$, while $\mathrm{r}(R[k]) < (k + 1)n$, or equivalently, the column nullity $\mathrm{n}(R[i]) = 0$ for $i = 0, 1, \ldots, k - 1$ and $\mathrm{n}(R[k]) > 0$.*

By applying Theorem 2.1 to cases 1 and 1' we see that $\mathrm{n}(R[1]) = 0, \mathrm{n}(R[2]) = 1$ for case 1 while $\mathrm{n}(R[1]) = 1, \mathrm{n}(R[2]) = 2$ for case 1', which justify that case 1 has an L_2 block as its KCF and case 1' has an L_1 block in its KCF. After the second deflation of case 1', GUPTRI is left with the pencil $[1] - \lambda[1]$ which corresponds to R_1, a regular block of size 1×1.

Case 10': By swapping columns 2 and 3 of B_{10} we still get a matrix pair with $\mathrm{r}(A_i) = 2, \mathrm{r}(B_i) = 1$ and $\mathrm{n}(A_i, B_i) \equiv \mathrm{n}(R[0]) = 1$. We denote this pencil case 10'. This swapping does not change the singular structure. However, the N_2 block in case 10 is now split into two regular 1×1 blocks N_1 and R_1, i.e., one infinite eigenvalue is turned non-zero.

To get the remaining three cases we will swap rows 1 and 2 in A_i for $i = 4, 7$ and 9.

Case 4': If we swap rows 1 and 2 in A_4 we still get a matrix pair with $\mathrm{r}(A_i) = 1, \mathrm{r}(B_i) = 2$ and $\mathrm{n}(A_i, B_i) = 1$. We denote this pencil case 4'. The only difference is that the $J_2(0)$ block in case 4 is now split into two regular 1×1 blocks $J_1(0)$ and R_1, i.e., one zero eigenvalue is turned non-zero.

A dual form of Theorem 2.1 can be stated for a left (row) minimal index of lowest degree $k \geq 0$. Then L_k^T takes the place of L_k and $L[A, B, i]$ of size $(i+1)m \times (i+2)n$ replaces $R[A, B, i]$, where

$$L[A,B,i] = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ 0 & A & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A & B \end{bmatrix}, \tag{2.5}$$

and we are considering row ranks (or row nullities) of $L[A, B, i]$. (When it is clear from context we also here use the abbreviated notation $L[i]$ for $L[A, B, i]$.) We use this dual form to characterize the last two cases. Notice that $\mathrm{n}(R[A, B, 0])$ is equivalent to the dimension of the common column nullspace for A and B and that $\mathrm{n}(L[A, B, 0])$ is equivalent to the dimension of the common row nullspace for the two matrices.

Case 7': By swapping rows 1 and 2 in A_7 we still get a matrix pair with $\mathrm{r}(A_i) = 1, \mathrm{r}(B_i) = 1$ and $\mathrm{n}(A_i, B_i) = 1$. We denote this pencil case 7'. However, this swap imposes a common row nullspace of $A_{7'}$ and $B_{7'}$ as well, and will therefore change the singular structure completely. The regular part $(J_1(0) \oplus N_1)$ disappears and is replaced by $L_1 \oplus L_0^T$, i.e., the generic singular structure of a 2-by-2 pencil [19]. $\mathrm{n}(A_i, B_i) \equiv \mathrm{n}(R[0]) = 1$ for $i = 7$ and 7'. For case 7, $\mathrm{n}(R[1]) = 2, \mathrm{n}(L[0]) = 0$ while $\mathrm{n}(R[1]) = 3, \mathrm{n}(L[0]) = 1$ for case 7'.

Case 9': By swapping rows 1 and 2 in A_9 we still get a matrix pair with $\mathrm{r}(A_i) = 1, \mathrm{r}(B_i) = 1$ and $\mathrm{n}(A_i, B_i) = 2$. We denote this pencil case 9'. However, $A_{9'}$ and $B_{9'}$ do not have a common row nullspace. Also here the regular part disappears and $R_1 \oplus L_0^T$ turns into L_1^T, i.e., a generic 2-by-1 pencil. $\mathrm{n}(L[0]) = 1$ for case 9, while $\mathrm{n}(L[0]) = 0, \mathrm{n}(L[0]) = 1$ for case 9'.

In Table 2.3 we display ranks of A_i, B_i and nullities of $R[k]$ and $L[k]$ for some values of k together with our structurally different singular structures of the set of 2-by-3 pencils. The ordering of the cases is explained in Section 2.2.

2.2. Orbits and Their Codimensions. Each of the 18 singular canonical forms (A_i, B_i) in Table 2.3 defines a manifold of *strictly equivalent* pencils in $2mn(= 12)$ dimensional space:

$$\mathrm{orbit}(A_i - \lambda B_i) = \{P_i^{-1}(A_i - \lambda B_i)Q_i : \det(P_i)\det(Q_i) \neq 0\}.$$

TABLE 2.3
Geometric characteristics of the 18 structurally different 2×3 pencils.

Case	r(A_i)	r(B_i)	n(A_i, B_i)	n(R[1])	n(R[2])	n(L[0])	n(L[1])	KCF	Cod($A_i - \lambda B_i$)
1	2	2	0	0	1	0	0	L_2	0
1'	2	2	0	1	2	0	0	$L_1 \oplus R_1$	1
2	1	2	0	1	2	0	0	$L_1 \oplus J_1$	2
6	2	1	0	1	2	0	0	$L_1 \oplus N_1$	2
5	2	2	1	2	3	0	0	$L_0 \oplus R_2$	2
4'	1	2	1	2	3	0	0	$L_0 \oplus J_1 \oplus R_1$	3
10'	2	1	1	2	3	0	0	$L_0 \oplus N_1 \oplus R_1$	3
4	1	2	1	2	3	0	0	$L_0 \oplus J_2$	4
10	2	1	1	2	3	0	0	$L_0 \oplus N_2$	4
7	1	1	1	2	3	0	0	$L_0 \oplus J_1 \oplus N_1$	4
7'	1	1	1	3	5	1	2	$L_0 \oplus L_1 \oplus L_0^T$	5
3	0	2	1	2	3	0	0	$L_0 \oplus 2J_1$	6
11	2	0	1	2	3	0	0	$L_0 \oplus 2N_1$	6
9'	1	1	2	4	6	0	1	$2L_0 \oplus L_1^T$	6
9	1	1	2	4	6	1	2	$2L_0 \oplus R_1 \oplus L_0^T$	7
8	0	1	2	4	6	1	2	$2L_0 \oplus J_1 \oplus L_0^T$	8
12	1	0	2	4	6	1	2	$2L_0 \oplus N_1 \oplus L_0^T$	8
13	0	0	3	6	9	2	4	$3L_0 \oplus 2L_0^T$	12

The dimension of orbit$(A - \lambda B)$ is equal to the dimension of the tangent space, $\tan(A - \lambda B)$, to the orbit of $A - \lambda B$. The tangent space is defined as

$$f(X, Y) = X(A - \lambda B) - (A - \lambda B)Y, \tag{2.6}$$

where X is an $m \times m$ matrix and Y is an $n \times n$ matrix [3]. Since (2.6) maps a space of dimension $m^2 + n^2$ linearly to a space of dimension $2mn$, the dimension of the tangent space is $m^2 + n^2 - d$, where d is the number of (linearly) independent solutions of $f(X, Y) = 0$.

The codimension is the dimension of the space complementary to the tangent space, i.e.,

$$\operatorname{cod}(A - \lambda B) = 2mn - \dim(\tan(A - \lambda B)) = d - (m - n)^2.$$

The codimensions of the orbits depend only on their Kronecker structures. Demmel and Edelman [3] show that the codimension of the orbit of an $m \times n$ pencil $A - \lambda B$ can be computed as the sum of separate codimensions:

$$\operatorname{cod}(A - \lambda B) = c_{\text{Jor}} + c_{\text{Right}} + c_{\text{Left}} + c_{\text{Jor,Sing}} + c_{\text{Sing}},$$

where the different components are defined as follows.

The codimension of the Jordan structure is

$$c_{\text{Jor}} = \sum_{\lambda \neq 0, \infty} (q_1(\lambda) + 3q_2(\lambda) + 5q_3(\lambda) + \ldots - 1) + \sum_{\lambda = 0, \infty} (q_1(\lambda) + 3q_2(\lambda) + 5q_3(\lambda) + \ldots),$$

where the summation is over all eigenvalues and $q_1(\lambda) \geq q_2(\lambda) \geq q_3(\lambda) \ldots$, denote the sizes of the Jordan blocks corresponding to the eigenvalue λ. The first part of c_{Jor} corresponds to unspecified eigenvalues different from zero and infinity, which explains the term -1 in the codimension count.

The codimension of the right and left singular blocks are

$$c_{\text{Right}} = \sum_{j > k} (j - k - 1) \quad \text{and} \quad c_{\text{Left}} = \sum_{j > k} (j - k - 1),$$

respectively, where the summation for c_{Right} is over all pairs of blocks L_j and L_k, for which $j > k$, and the summation for c_{Left} is over all pairs of blocks L_j^T and L_k^T for which $j > k$.

The codimension due to interaction between the Jordan structure and the singular blocks is

$$c_{\text{Jor,Sing}} = (\text{size of complete regular part}) \cdot (\text{number of singular blocks}).$$

The codimension due to interaction between right and left singular blocks is

$$c_{\text{Sing}} = \sum_{j,k} (j + k + 2),$$

where the summation is over all pairs of blocks L_j and L_k^T.
The codimensions of our 18 different canonical forms are displayed in the last column of Table 2.3. We have ordered the cases by increasing codimension. In general, we see that by making A and B more rank deficient and increasing their "common nullspace(s)" ($\text{n}(R[k])$ and $\text{n}(L[k])$ for $k \geq 0$) we generate non-generic pencils with higher codimension. The generic pencil has codimension 0 while the matrix pair $(A, B) = (0_{2\times 3}, 0_{2\times 3})$ has codimension 12 ($= 2mn$), i.e., defines a "point" in 12-dimensional space.

2.3. The Closure Graph for Different Kronecker Structures. Since orbit(L_2) spans the complete 12-dimensional space, it is obvious that all other structures are in the closure of the orbit of L_2, and it is just as obvious that $3L_0 \oplus 2L_0^T$ (the zero pencil) is in the closure of the orbit of any other KCF. Since all other closure relations are not that obvious, we derive a complete closure graph for the set of 2-by-3 matrix pencils.

Throughout the paper we display graphs such that orbits (nodes) with the same codimension are displayed on the same horizontal level.

THEOREM 2.2. *For the set of* 2*-by-*3 *pencils, the directed graph in Figure* 2.1 *shows all closure relations as follows. One KCF is in the closure of the orbit of another KCF if and only if it exists a path to its node from the node of the KCF defining the closure* (*downwards in the graph*).

Proof. First we prove that each arc in the graph correspond to a closure relation, and then we prove that these are all arcs that can exist. We prove that one KCF is in the closure of the orbit of another KCF by showing that the one in the closure is just a special case of the one defining the closure. We show proofs for each arc starting from the zero pencil. Since the proof is rather space demanding, we here limit ourselves to prove one of the arcs and refer to appendix A for the complete proof.

Starting at the zero pencil, the first arc with non-trivial proof corresponds to that $2L_0 \oplus J_1 \oplus L_0^T$ is in the closure of orbit$(2L_0 \oplus R_1 \oplus L_0^T)$. This follows from the fact that $2L_0 \oplus J_1 \oplus L_0^T$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix},$$

which is equivalent to $2L_0 \oplus R_1 \oplus L_0^T$ for all other α (assuming that β is non-zero).

The proofs for all other arcs are done similarly. For some of them, an equivalence transformation is needed for transformation to KCF. $\square$

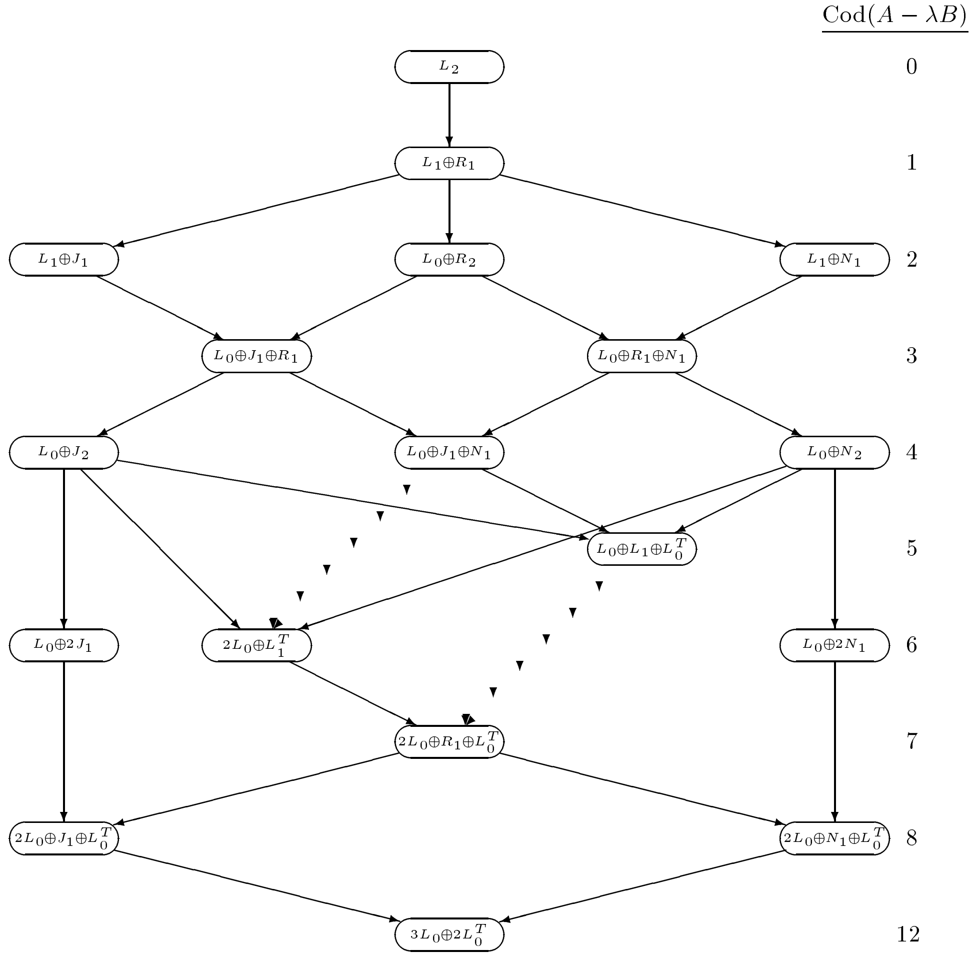


FIG. 2.1. *A graph displaying the closure hierarchy of the orbits of all* 18 *different KCF for the set of* 2*-by-*3 *matrix pencils.*

2.4. Labeled Closure Graph Showing Necessary Conditions on Perturbations for Transiting from One Structure to Another. One way to interpret a relation in the closure hierarchy is that a KCF that is in the closure of the orbit of another KCF "lives" in the space defined by that orbit. That is, if we consider the closure of the orbit of a non-generic KCF with certain rank-defects in Table 2.3, then to be in that closure a KCF must preserve or increase these defects. For example, since $L_1 \oplus J_1$ has $\operatorname{rank}(A) = 1$, no KCF with $\operatorname{rank}(A) > 1$ can be in its closure. A necessary condition for a KCF to be in the closure of $\operatorname{orbit}(L_1 \oplus J_1)$ is that the geometric characteristics $\mathrm{r}(A) \leq 1, \mathrm{r}(B) \leq 2, \mathrm{n}(A,B) \geq 0, \mathrm{n}(R[1]) \geq 1, \mathrm{n}(R[2]) \geq 2, \mathrm{n}([L[0]) \geq 0$ and $\mathrm{n}([L[1]) \geq 0$ are satisfied (see Table 2.3). Moreover, the change in geometric characteristics from, for example, $L_1 \oplus J_1$ whose orbit spans a 10-dimensional space (codimension is 2), to $L_0 \oplus J_1 \oplus R_1$ whose orbit spans a 9-dimensional space (codimension is 3), is nothing but a 1-dimensional restriction of the 10-dimensional space. We also note that $L_0 \oplus J_1 \oplus R_1$ is in the closure of $\operatorname{orbit}(L_0 \oplus R_2)$, which also spans a 10-dimensional space. Indeed, $L_0 \oplus J_1 \oplus R_1$ spans a 9-dimensional space in the intersection of the two 10-dimensional spaces spanned by the closures of orbit$(L_1 \oplus J_1)$ and orbit$(L_0 \oplus R_2)$.

When looking for perturbations corresponding to the arcs in the graph, a necessary condition for these perturbations is to fulfill the change in geometric characteristics. Indeed, by combining the geometric characteristics in Table 2.3 and the closure graph we get necessary conditions on perturbations $(\delta A, \delta B)$ for transiting from one structure to another.

We introduce the following labels. Let

$$[\mathrm{n}_r(A), \mathrm{n}_r(B), \mathrm{n}(A, B), \mathrm{n}(R[1]), \mathrm{n}(R[2]), \mathrm{n}([L[0]), \mathrm{n}([L[1])]$$

label the geometric characteristics for one node in the graph, where $n_r(A)$ and $n_r(B)$ denote the dimension of the row-nullspace in A and B, respectively, and all other characteristics are as in Table 2.3. Moreover, we label the change in geometric characteristics for transiting from one structure to an adjacent node by

$$< \mathrm{n}_r(A), \mathrm{n}_r(B), \mathrm{n}(A, B), \mathrm{n}(R[1]), \mathrm{n}(R[2]), \mathrm{n}([L[0]), \mathrm{n}([L[1]) > .$$

In Figure 2.2 a labeled closure graph is presented, with the geometric characteristics shown for each KCF and the change in geometric characteristics shown for each arc.

When transiting from one KCF to another, the geometric characteristics of the source node and the geometric characteristics on the arc are added to give the characteristics of the destination KCF. Since a KCF in the closure of another ones orbit cannot have a smaller dimensional nullspace for any of the matrices of the labels, the values on the arcs must all be non-negative.

Notice that the arc from $L_0 \oplus J_1 \oplus R_1$ to $L_0 \oplus J_2$ and the arc from $L_0 \oplus R_1 \oplus N_1$ to $L_0 \oplus N_2$ both have no change in the geometric characteristics. For these transitions the non-zero finite eigenvalue is turned to a zero eigenvalue and to an infinite eigenvalue, respectively. This does not affect any of the nullspaces displayed in the labels.

To transit several levels in the closure graph we just add the labels of changes in geometric characteristics for the arcs that are traveled during the transition. Each label of changes in geometric characteristics define necessary conditions on the perturbations $(\delta A, \delta B)$ to perform the transit. Later, we will derive perturbations required to transit from L_2 to any of the non-generic structures. In our derivation, however, we for most cases transit directly to the intended structure. There are only a few cases that require compound perturbations that transit via another KCF.

3. Structure Invariances and Transitions of Non-Generic Pencils under Perturbations.

Since computing the Kronecker structure of a singular pencil is a potentially ill-posed problem [5], it is interesting to see how the non-generic cases behave under perturbations in finite precision arithmetic. We add (uniformly distributed) random perturbations of different sizes $\epsilon_n (= 10^{-10}, 10^{-9}, \ldots, 10^{-2})$ to all A_i and B_i, corresponding to the generic and 17 non-generic cases, and compute their generalized Schur forms using `GUPTRI` [7, 8] assuming a fixed relative accuracy $\epsilon_u (= 10^{-8})$ of the input data. We repeat this procedure 100 times and study the structure invariances and transitions of each non-generic case as a function of the size of the perturbations added.

`GUPTRI` has two input parameters `EPSU` (ϵ_u above) and `GAP` which are used to make rank decisions in order to determine the Kronecker structure of an input pencil $A - \lambda B$. Inside `GUPTRI` the absolute tolerances $\mathtt{EPSUA} = \|A\|_E \cdot \mathtt{EPSU}$ and $\mathtt{EPSUB} = \|B\|_E \cdot \mathtt{EPSU}$

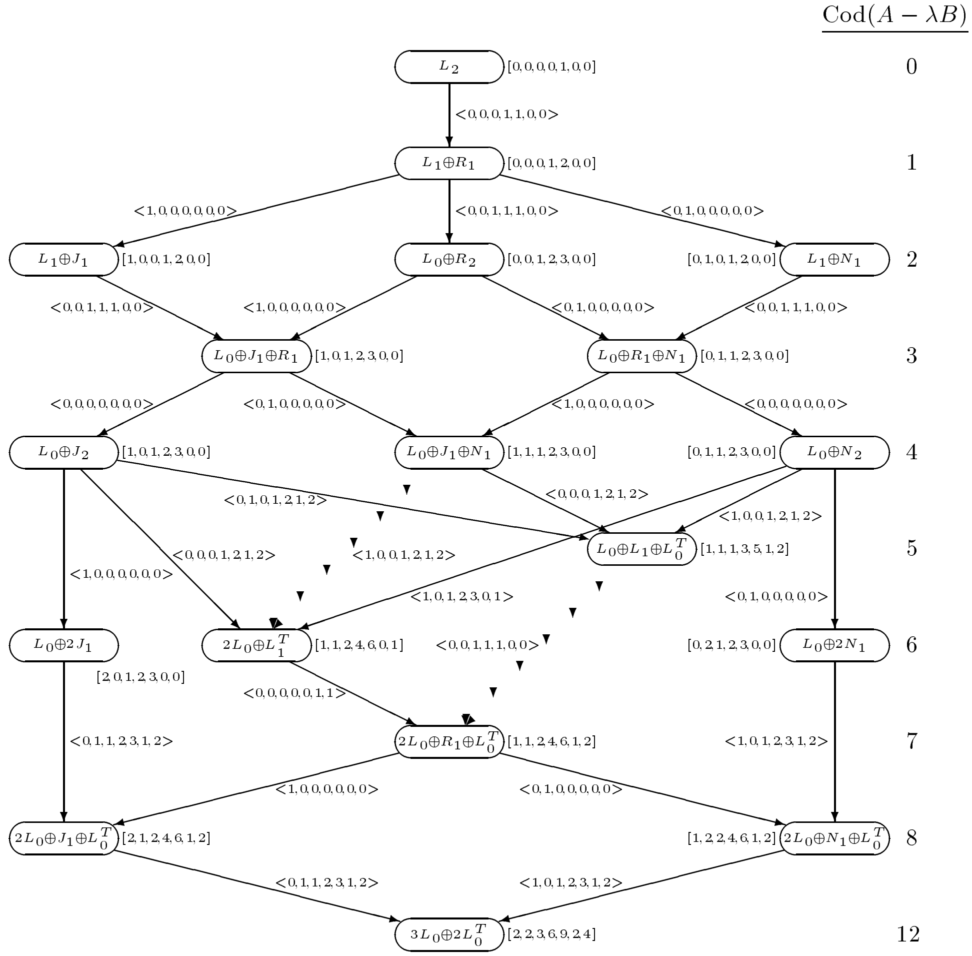


FIG. 2.2. *The labeled closure graph for all* 18 *different KCF for the set of* 2*-by-*3 *matrix pencils.*

are used in all rank decisions, where the matrices A and B, respectively, are involved. Suppose the singular values of A are computed in increasing order, i.e., $0 \le \sigma_1 \le \sigma_2 \le \ldots \le \sigma_k \le \sigma_{k+1} \le \ldots$; then all singular values $\sigma_k <$ `EPSUA` are interpreted as zeros. The rank decision is made more robust in practice: if $\sigma_k <$ `EPSUA` but $\sigma_{k+1} \ge$ `EPSUA`, `GUPTRI` insists on a gap between the two singular values such that $\sigma_{k+1}/\sigma_k \ge$ `GAP`. If $\sigma_{k+1}/\sigma_k <$ `GAP`, σ_{k+1} is also treated as zero. This process is repeated until an appreciable gap between the zero and non-zero singular values is obtained. In all of our tests we have used `EPSU` $= 10^{-8}$ and `GAP` $= 1000.0$. All computations (in sections 3 and 4) are performed on a SUN SPARC workstation in double precision complex arithmetic with unit roundoff $= O(10^{-17})$.

In Table 3.1 we display the computed Kronecker structures of the 17 perturbed non-generic pencils for 100 random perturbations for each ϵ_n. For each case all structure invariances and transitions are shown from left to right. The symbol $\stackrel{10^{-x}}{\longrightarrow}$ indicates that the Kronecker structure is invariant under perturbations smaller than $\epsilon_n = 10^{-x}$, and that the structure changes (at least for some of the 100 tests) for

TABLE 3.1

Computed Kronecker structures and transitions of 100 perturbed non-generic 2×3 pencils. The size ϵ_n of each perturbation is shown above the corresponding arrow.

$$1':\ L_1\oplus R_1 \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (81) \\ L_1\oplus R_1 & (19) \end{array}\right\} \xrightarrow{10^{-3}} \left\{\begin{array}{ll} L_2 & (98) \\ L_1\oplus R_1 & (2) \end{array}\right\} \xrightarrow{10^{-2}} L_2$$

$$2:\ L_1\oplus J_1 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (18) \\ L_1\oplus J_1 & (82) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (98) \\ L_1\oplus J_1 & (2) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$6:\ L_1\oplus N_1 \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (92) \\ L_1\oplus R_1 & (6) \\ L_1\oplus N_1 & (2) \end{array}\right\} \xrightarrow{10^{-3}} \left\{\begin{array}{ll} L_2 & (99) \\ L_1\oplus R_1 & (1) \end{array}\right\} \xrightarrow{10^{-2}} L_2$$

$$5:\ L_0\oplus R_2 \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (63) \\ L_1\oplus R_1 & (28) \\ L_0\oplus R_2 & (9) \end{array}\right\} \xrightarrow{10^{-3}} \left\{\begin{array}{ll} L_2 & (95) \\ L_1\oplus R_1 & (4) \\ L_0\oplus R_2 & (1) \end{array}\right\} \xrightarrow{10^{-2}} \left\{\begin{array}{ll} L_2 & (99) \\ L_1\oplus R_1 & (1) \end{array}\right\} \xrightarrow{10^{-1}} L_2$$

$$4':\ L_0\oplus J_1\oplus R_1 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (1) \\ L_1\oplus R_1 & (17) \\ L_0\oplus J_1\oplus R_1 & (82) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (82) \\ L_1\oplus R_1 & (16) \\ L_1\oplus J_1 & (1) \\ L_0\oplus J_1\oplus R_1 & (1) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$10':\ L_0\oplus N_1\oplus R_1 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (90) \\ L_0\oplus N_1\oplus R_1 & (10) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (98) \\ L_0\oplus R_2 & (2) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$4:\ L_0\oplus J_2 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (22) \\ L_0\oplus J_1\oplus R_1 & (17) \\ L_0\oplus J_2 & (61) \end{array}\right\} \xrightarrow{10^{-4}} L_2$$

$$10:\ L_0\oplus N_2 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (10) \\ L_0\oplus N_1\oplus R_1 & (15) \\ L_0\oplus N_2 & (75) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (99) \\ L_0\oplus N_2 & (1) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$7:\ L_0\oplus J_1\oplus N_1 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (5) \\ L_1\oplus N_1 & (13) \\ L_0\oplus J_1\oplus N_1 & (82) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (92) \\ L_1\oplus R_1 & (6) \\ L_1\oplus J_1 & (2) \end{array}\right\} \xrightarrow{10^{-3}} \left\{\begin{array}{ll} L_2 & (97) \\ L_1\oplus R_1 & (3) \end{array}\right\} \xrightarrow{10^{-2}} L_2$$

$$7':\ L_0\oplus L_1\oplus L_0^T \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (3) \\ L_1\oplus R_1 & (7) \\ L_1\oplus N_1 & (12) \\ L_0\oplus L_1\oplus L_0^T & (78) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (93) \\ L_1\oplus R_1 & (7) \end{array}\right\} \xrightarrow{10^{-3}} \left\{\begin{array}{ll} L_2 & (98) \\ L_1\oplus R_1 & (2) \end{array}\right\} \xrightarrow{10^{-2}} L_2$$

$$3:\ L_0\oplus 2J_1 \xrightarrow{10^{-10}} L_2$$

$$11:\ L_0\oplus 2N_1 \xrightarrow{10^{-10}} L_2$$

$$9':\ 2L_0\oplus L_1^T \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_0\oplus R_2 & (3) \\ L_0\oplus J_1\oplus R_1 & (34) \\ L_0\oplus N_2 & (15) \\ 2L_0\oplus L_1^T & (48) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (86) \\ L_1\oplus R_1 & (8) \\ L_0\oplus R_2 & (4) \\ L_1\oplus J_1 & (2) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$9:\ 2L_0\oplus R_1\oplus L_0^T \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_0\oplus R_2 & (2) \\ L_0\oplus J_1\oplus R_1 & (33) \\ L_0\oplus N_1\oplus R_1 & (16) \\ 2L_0\oplus R_1\oplus L_0^T & (49) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (80) \\ L_1\oplus R_1 & (17) \\ L_0\oplus R_2 & (1) \\ L_1\oplus J_1 & (1) \\ L_0\oplus J_1\oplus R_1 & (1) \end{array}\right\} \xrightarrow{10^{-3}} \left\{\begin{array}{ll} L_2 & (96) \\ L_1\oplus R_1 & (4) \end{array}\right\} \xrightarrow{10^{-2}} L_2$$

$$8:\ 2L_0\oplus J_1\oplus L_0^T \xrightarrow{10^{-10}} L_1\oplus N_1 \xrightarrow{10^{-6}} \left\{\begin{array}{ll} L_2 & (2) \\ L_1\oplus N_1 & (98) \end{array}\right\} \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (24) \\ L_1\oplus R_1 & (2) \\ L_1\oplus N_1 & (54) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (86) \\ L_1\oplus R_1 & (14) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$12:\ 2L_0\oplus N_1\oplus L_0^T \xrightarrow{10^{-10}} L_1\oplus J_1 \xrightarrow{10^{-5}} \left\{\begin{array}{ll} L_2 & (18) \\ L_1\oplus J_1 & (82) \end{array}\right\} \xrightarrow{10^{-4}} \left\{\begin{array}{ll} L_2 & (98) \\ L_1\oplus J_1 & (2) \end{array}\right\} \xrightarrow{10^{-3}} L_2$$

$$13:\ 3L_0\oplus 2L_0^T \xrightarrow{10^{-10}} L_2$$

perturbations of size 10^{-x}. For a size of the perturbations that has not given the same structure for all 100 tests, all KCF:s found are placed within curly brackets with a number within parentheses after each KCF showing the number of that particular KCF that has been found. As before, the cases are displayed in increasing codimension order and the transit KCF forms within curly brackets are ordered similarly.

From Table 3.1 we see that for large enough perturbations all non-generic structures turn generic (as expected). GUPTRI finds the same non-generic structure as long as $\epsilon_n < tol \equiv \min(\texttt{EPSUA}, \texttt{EPSUB}) \cdot \texttt{GAP}$. This behaviour is in agreement with the perturbation theory for singular pencils [5, 7]. Only if $A - \lambda B$ lies in a particular manifold does it have a non-generic Kronecker structure with non-trivial reducing subspaces and possibly eigenvalues. Moreover, only if it is perturbed so as to move continuously within that manifold does its original Kronecker structure remain. Actually, by choosing a $tol > 0$, we have thickened the manifolds so that they are no longer a set of measure zero.

All transitions from the initial case to the final generic case is clearly from cases with higher codimension to cases with lower. By a closer look we can also see that all the transitions are performed upwards (or backwards) along the arcs in the closure graph (Figure 2.1). This means that the perturbations cure the rank deficiencies in the non-generic pencil without contributing with any new singularities. GUPTRI increases the rank in A and B and decreases the size of their "common nullspace(s)", i.e., the "inverse" operations compared to what we did in Section 2.1. In other words, when a pencil $A - \lambda B$ with a given non-generic KCF is perturbed, by $\delta A - \lambda \delta B$ then $A - \lambda B$ is in the closure of $\text{orbit}((A + \delta A) - \lambda(B + \delta B))$.

Even if we see that all of the cases transit via some other non-generic structures before all 100 tests turn generic, we can also see that if we for each case and each size of the perturbation only consider the KCF that has been found in most tests, then it is only for cases 8 and 12 a transit KCF is found. Notice that all tests for cases 8 and 12 find the same other non-generic KCF for the smallest perturbation. In other words, when the perturbation is big enough to change the KCF for most tests of a case, then the generic KCF is the most likely to find, except for cases 8 and 12.

How can we explain the behaviour in cases 8 and 12? For these two cases one matrix is the zero matrix. This means that $tol \equiv \min(\texttt{EPSUA}, \texttt{EPSUB}) \cdot \texttt{GAP} = 0$ implying that $\epsilon_n > tol$ already for the smallest perturbation, which in turn explains why case transitions occur already for the smallest perturbation. Since either EPSUA or EPSUB is zero, all singular values in the perturbed zero matrix will be interpreted as non-zero, explaining why A or B are interpreted as a full rank matrix already for the smallest perturbations. Notice also the "jumps" these transitions correspond to in the closure graph. The argumentation here also explains why the zero pencil turns generic for the smallest perturbation.

We end this section by briefly discussing how the case invariances and transitions are affected by the choice of the fixed relative accuracy of the input data (EPSU). If we choose $\texttt{EPSU} = \epsilon_n$ then GUPTRI will retrieve the non-generic structure we started from for each ϵ_n considered. Notice that the distance from the input pencil to the computed Kronecker structure will normally be of size $O(\texttt{EPSU} \cdot \|(A, B)\|_E)$ [8]. Increasing EPSU means that the case invariances will remain longer before any case transition take place. Decreasing EPSU will impose the generic structure sooner. For example, with EPSU equal to the relative machine precision and $\epsilon_n > tol$, GUPTRI will always extract the generic structure. This corresponds to the fact that in infinite precision arithmetic any non-generic $A - \lambda B$ can be made generic with arbitrary small perturbations.

Moreover, travelling upwards in the closure hierarchy can always be effected with arbitrary small perturbations, while travelling downwards may require much larger perturbations.

4. Imposing Non-Generic Structures by Perturbing a Generic Pencil. In this section we study computable normwise bounds for the smallest perturbations $(\delta A, \delta B)$ of a generic 2-by-3 pencil $A - \lambda B$ such that $(A + \delta A) - \lambda(B + \delta B)$ has a specific non-generic Kronecker structure chosen from the 17 non-generic cases discussed earlier. Our goal is to find the closest non-generic pencil and the closest pencil with a specified non-generic Kronecker structure of a 2-by-3 generic pencil. We consider two approaches to impose a non-generic structure. First we derive explicit expressions for the perturbations that transfer $A - \lambda B$ to a specified non-generic form. Secondly, we have modified `GUPTRI` to be able to compute a specified Kronecker structure.

4.1. Explicit Perturbations to Impose Non-Generic Structures. We have seen in Section 2 that by making A and B more rank deficient and increasing their "common nullspace(s)" we can generate non-generic pencils with higher codimension. Here we elaborate on this fact and derive explicit expressions for the perturbations required to turn an arbitrary generic pencil into each of the 17 non-generic cases. The norms of these explicit expressions (measured as $\|(\delta A, \delta B)\|_E$) are upper bounds for the smallest perturbations required. Indeed, for 11 of the structures, the norms are the exact sizes of the smallest perturbations required.

We need the following notation. The size of the smallest perturbations $(\delta A, \delta B)$ such that $R[A + \delta A, B + \delta B, i]$ (2.3) of size $(i + 2)m \times (i + 1)n$ has a k-dimensional column nullspace is defined as

$$d_k(R[A, B, i]) = \min_{(\delta A, \delta B)} \{\|(\delta A, \delta B)\|_E : \mathrm{n}(R[A + \delta A, B + \delta B, i]) = k\}, \tag{4.1}$$

where δA and δB vary over all m-by-n matrices with complex (or real) entries. Similarly, we define $d_k(L[A, B, i])$ as the size of the smallest perturbations that impose a k-dimensional row nullspace on $L[i]$ (2.5). When it is clear from context we use the abbreviated notation $d_k(R[i])$ and $d_k(L[i])$. Also, let $d_k(A)$ denote the size of the smallest perturbations such that $\mathrm{rank}(A + \delta A) = \min(m, n) - k$.

In general, to find $d_k(R[i])$ (or $d_k(L[i])$) is a type of a structured singular value problem. For $i \geq 1$ it is an open problem to find explicit expressions for $d_k(R[i])$ and $d_k(L[i])$. The following theorem summarizes some of their properties for the case $m = 2, n = 3$ and $k = 1$:

THEOREM 4.1. *For a generic* 2*-by-*3 *pencil* (A, B) *the following inequalities hold:*

$$0 \equiv d_1(R[2]) < d_1(R[1]) \leq d_1(R[0]), \tag{4.2}$$

$$d_1(R[1]) \leq d_1(A), \quad d_1(R[1]) \leq d_1(B), \tag{4.3}$$

$$d_1(R[1]) < d_1(L[1]) \leq d_1(L[0]), \tag{4.4}$$

$$d_1(A) < d_2(A), \quad d_1(B) < d_2(B), \quad d_1(R[0]) < d_2(R[0]). \tag{4.5}$$

Proof. From Theorem 2.1 it follows that $d(R[2]) = 0$ for all 2-by-3 pencils (generic or non-generic). Decreasing the rank of the 4-by-3 $R[A, B, 0]$ by one gives that $R[A + \delta A, B + \delta B, 0]$ has only two linearly independent columns. The same perturbations make the 6-by-6 matrix $R[A + \delta A, B + \delta B, 1]$ rank deficient (a rank drop from six to four) showing that (4.2) holds. Similarly, decreasing the rank of A (or B) by one means that $A + \delta A$ (or $B + \delta B$) only has one linearly independent row. For the same perturbations $R[A + \delta A, B + \delta B, 1]$ is rank deficient with only one of the two first (or two last) rows linearly independent, resulting in the inequalities (4.3).

$L[1]$ is row rank deficient if and only if there exists at least one L_0^T or L_1^T block in the KCF. Since all KCFs with at least one L_0^T block or one L_1^T block have both A and B rank deficient (see Table 2.3), there will always exist a strictly smaller perturbation of size $d_1(A)$ that only lowers the rank in A. (The similar is of course true for B.) Now applying inequality (4.3) proves the first part of (4.4). The last part follows from similar arguments as proving $d_1(R[1]) \leq d_1(R[0])$ above. The inequalities (4.5) follow from the definition of $d_k(\cdot)$. $\square$

Theorem 4.1 will be used to identify the closest non-generic Kronecker structure of a generic 2-by-3 pencil. Notice that in general we cannot say anything about the relationship between $d_1(R[0])$ and $d_1(A)$ or $d_1(B)$ (see explicit expressions below). By varying α and β in

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

(i.e., a generic $A - \lambda B$ for non-zero α and β) we show that any of them can be the smallest quantity (see Table 4.1).

TABLE 4.1
The quantities $d_1(A)$, $d_1(B)$ and $d_1(R[0])$ for three examples.

Parameters	$d_1(A)$	$d_1(B)$	$d_1(R[0])$
$\alpha = \beta = 1$	1.000	1.000	**0.765**
$\alpha = 0.1,\ \beta = 1$	**0.100**	1.000	0.451
$\alpha = 1,\ \beta = 0.1$	1.000	**0.100**	0.451

The following explicit expressions, derived from the Eckart–Young and Mirsky theorem for finding the closest matrix of a given rank (e.g. see[10]), appear in our explicit bounds discussed next:

$$d_1(A) = \sigma_{\min}(A), \quad d_1(B) = \sigma_{\min}(B),$$

$$d_2(A) = \|A\|_E, \quad d_2(B) = \|B\|_E,$$

$$d_1(R[A, B, 0]) = \sigma_{\min}(R[0]), \quad d_1(L[A, B, 0]) = \sigma_{\min}(L[0]),$$

$$d_2(R[A, B, 0]) = (\sigma_{\min-1}^2(R[0]) + \sigma_{\min}^2(R[0]))^{1/2}.$$

Here, $\sigma_{\min}(X)$ and $\sigma_{\min-1}(X)$ (with $\sigma_{\min}(X) \leq \sigma_{\min-1}(X)$) denote the two smallest non-zero singular values of (a full rank) matrix X.

4.1.1. Tractable Perturbations. In order to make the problem more tractable we (first) put restrictions on allowable perturbations. We can compute a perturbation $\delta A - \lambda \delta B$ such that $(A + \delta A) - \lambda(B + \delta B)$ is guaranteed to fall on the closure of the manifold (orbit) of a certain KCF. (Necessary conditions on the required perturbations are given in the labeled closure graph in Figure 2.2.) If the KCF found is the intended KCF, then the perturbation is said to be tractable. If the KCF found is even more non-generic (i.e., its orbit has higher codimension but belongs to the closure of the intended manifold), then the perturbation is defined intractable. In other words, a tractable perturbation finds the generic KCF (i.e., the least non-generic KCF) in the closure of the manifold of the intended KCF. An intractable perturbation finds any other structure in the closure of the same manifold, i.e., any structure that can be found by traveling along the arcs (downwards) from the intended KCF in the closure graph in Figure 2.1.

When computing perturbations such that $(A + \delta A) - \lambda(B + \delta B)$ is given a non-generic KCF, we compute δA and δB such that one or more of the geometric characteristics presented in Table 2.3 for $(A+\delta A)-\lambda(B+\delta B)$ differ from the characteristics of the generic (A, B). In other words, we put restrictions on the size of the perturbed pencil's nullspaces so that at least one of them is larger than for the generic case. The space given by this restriction may contain several non-generic matrix pencils. For example, if we restrict the set of pencils to those who have a rank deficiency in the A-part, this space contains all pencils that fulfill the condition $\operatorname{rank}(A) < 2$. However, if we compute a perturbation such that $\operatorname{rank}(A + \delta A) < 2$, the perturbed pencil will most likely be the generic (least non-generic) KCF with a rank-deficient A-part, i.e., $L_1 \oplus J_1$. This corresponds to the KCF with rank-deficient A-part whose orbit has the smallest codimension and the corresponding perturbation $(\delta A, \delta B)$ is tractable. The perturbation is intractable if $(A + \delta A) - \lambda(B + \delta B)$ has any KCF (with $\operatorname{rank}(A) < 2$) that is more non-generic than $L_1 \oplus J_1$. The set of possible structures are the ones that are in the closure of $\operatorname{orbit}(L_1 \oplus J_1)$.

Eleven of the 17 non-generic structures (2, 6, 5, 7, 7', 3, 11, 9', 8, 12, and 13) are imposed by (minimal) tractable perturbations that effectuate one of the following rank-decreasing operations:

- Rank drop in A and/or B by one or two.
- Rank drop in $R[A, B, 0]$ by one or two, i.e., imposing a common one or two dimensional column nullspace.
- Rank drop in $L[A, B, 0]$ by one, i.e., imposing a common row nullspace.

In Table 4.2 the size of the perturbations required to impose each of the eleven singular structures are displayed. When both $d_i(A)$ and $d_j(B)$ are involved, the size of the total perturbations is $(d_i^2(A)+d_j^2(B))^{1/2}$. The singular cases are reported in increasing codimension order (see Table 2.3). Since all these perturbations are made as the smallest possible to impose the required ranks on $A, B, R[0]$ or $L[0]$, these bounds are attained for each non-generic form, i.e., the strongest possible, which is equivalent to that the bounds in Table 4.2 also are lower bounds. That these perturbations really give the forms shown in the table follows from the fact that we here only are considering tractable perturbations and these are the least non-generic forms that have the imposed rank-deficiencies (see Table 2.3). For example, by imposing a one dimensional rank drop in A we have restricted the 12-dimensional space to a space that contains a subset of all non-generic pencils. Since the perturbation is supposed to be tractable, the KCF found is the least non-generic in that space, i.e., $L_1 \oplus J_1$.

The rank decreasing operations performed in Table 4.2 "affect the codimension(s)"

TABLE 4.2

Minimal perturbations of a generic pencil to impose 11 *of the* 17 *non-generic structures.*

Case	KCF	Cod(·)	$d_1(A)$	$d_1(B)$	$d_1(R[0])$	$d_1(L[0])$	$d_2(A)$	$d_2(B)$	$d_2(R[0])$
2	$L_1 \oplus J_1$	2	×						
6	$L_1 \oplus N_1$	2		×					
5	$L_0 \oplus R_2$	2			×				
7	$L_0 \oplus J_1 \oplus N_1$	4	×	×					
7′	$L_0 \oplus L_1 \oplus L_0^T$	5				×			
3	$L_0 \oplus 2J_1$	6					×		
11	$L_0 \oplus 2N_1$	6						×	
9′	$2L_0 \oplus L_1^T$	6							×
8	$2L_0 \oplus J_1 \oplus L_0^T$	8		×			×		
12	$2L_0 \oplus N_1 \oplus L_0^T$	8	×					×	
13	$3L_0 \oplus 2L_0^T$	12					×	×	

in the following way: a rank drop by one in A, B or $R[0]$ increases the codimension by two, a rank drop by one in $L[0]$ increases the codimension by five, and a rank drop by two in A, B or $R[0]$ increases the codimension by six.

Two of the remaining six non-generic forms (4' and 10') are imposed by transiting via a non-generic form as shown in Table 4.3. For example, to derive perturbations of the generic $A - \lambda B$ that turn $(A + \delta A, B + \delta B)$ non-generic with KCF $L_0 \oplus J_1 \oplus R_1$ we have $(\delta A, \delta B) = (\delta A_1, \delta B_1) + (\delta A_2, \delta B_2)$, where $(\delta A_1, \delta B_1)$ is the smallest perturbation that lowers the rank of A (i.e., $\|(\delta A_1, \delta B_1)\|_E = d_1(A)$, $\delta B_1 = 0_{2\times 3}$) and $(\delta A_2, \delta B_2)$ is the smallest perturbation that imposes a common column nullspace on $(A + \delta A_1, B + \delta B_1)$ (i.e., $\|(\delta A_2, \delta B_2)\|_E = d_1(R[A + \delta A_1, B + \delta B_1, 0])$). In Table 4.3 we show how these forms are constructed. The size of the compound (total) perturbations $(\delta A, \delta B)$ for the two cases are obtained by adding the perturbations in Table 4.2 and Table 4.3. $\tilde{A} = A + \delta A_1$ and $\tilde{B} = B + \delta B_1$ in Table 4.3 represent the "transit" non-generic pencil. A rank drop by one in $R[\tilde{A}, \tilde{B}, 0]$ in Table 4.3 increases the codimension by one.

TABLE 4.3

Compound perturbations: Non-generic structures imposed by transiting via a non-generic form.

Case	KCF	Cod(·)	Transit KCF	$d_1(R[\tilde{A}, \tilde{B}, 0])$
4′	$L_0 \oplus J_1 \oplus R_1$	3	$L_1 \oplus J_1$	×
10′	$L_0 \oplus N_1 \oplus R_1$	3	$L_1 \oplus N_1$	×

The last four non-generic structures (1', 4, 10 and 9) require perturbations to parts of the `GUPTRI` form of a transiting pencil $\tilde{A} - \lambda \tilde{B}$:

$$P^H(\tilde{A} - \lambda \tilde{B})Q = \tilde{S} - \lambda \tilde{T} \equiv \begin{bmatrix} \tilde{s}_{11} & \tilde{s}_{12} & \tilde{s}_{13} \\ 0 & \tilde{s}_{22} & \tilde{s}_{23} \end{bmatrix} - \lambda \begin{bmatrix} \tilde{t}_{11} & \tilde{t}_{12} & \tilde{t}_{13} \\ 0 & \tilde{t}_{22} & \tilde{t}_{23} \end{bmatrix}, \tag{4.6}$$

where some $\tilde{s}_{ij}, \tilde{t}_{ij}$ may be zero. The size of the perturbations $(\delta \tilde{S}, \delta \tilde{T})$ imposed on $\tilde{S}$ and/or $\tilde{T}$ are displayed in Table 4.4. Case 1', which transits via the `GUPTRI` form of L_2, is retrieved by imposing a common column nullspace of the A- and B-parts of the deflated 1-by-2 pencil $[\tilde{s}_{22} \quad \tilde{s}_{23}] - \lambda[\tilde{t}_{22} \quad \tilde{t}_{23}]$. For cases 4 and 10 we retrieve the requested structures by setting elements $\tilde{s}_{12} = 0$ and $\tilde{t}_{12} = 0$, respectively, in the `GUPTRI` forms of $\tilde{A} - \lambda \tilde{B}$ (4.6). For case 4 we impose a zero multiple eigenvalue in $\tilde{A} - \lambda \tilde{B}$. Similarly, a multiple eigenvalue is imposed at infinity for case 10. In other

TABLE 4.4

Compound perturbations: Non-generic structures imposed by perturbing the GUPTRI *form (denoted Transit form) of the generic or some non-generic pencils.*

Case	KCF	Cod(·)	Transit form	$d_1(\begin{bmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{t}_{22} & \tilde{t}_{23} \end{bmatrix})$	$d_1(\begin{bmatrix} \tilde{s}_{12} & \tilde{s}_{13} \\ \tilde{t}_{12} & \tilde{t}_{13} \end{bmatrix})$	$\tilde{s}_{12}$	$\tilde{t}_{12}$
$1'$	$L_1 \oplus R_1$	1	L_2	×			
4	$L_0 \oplus J_2$	4	$L_0 \oplus J_1 \oplus R_1$			×	
10	$L_0 \oplus N_2$	4	$L_0 \oplus N_1 \oplus R_1$				×
9	$2L_0 \oplus R_1 \oplus L_0^T$	7	$L_0 \oplus L_1 \oplus L_0^T$		×		

words, $J_1 \oplus R_1$ and $N_1 \oplus R_1$ in $\tilde{A} - \lambda\tilde{B}$ are turned J_2 and N_2, respectively. Case 9 is obtained by giving the A- and B-parts of the L_1 block in $\tilde{A} - \lambda\tilde{B}$ a common column nullspace, which turns L_1 into $L_0 \oplus R_1$. Since P and Q in (4.6) are unitary the perturbations imposed on $\tilde{A}$ and $\tilde{B}$ are of the same size as $\delta\tilde{S}$ and $\delta\tilde{T}$. The size of the compound (total) perturbations $(\delta A, \delta B)$ for the four cases are obtained by adding the appropriate perturbations in tables 4.2, 4.3 and 4.4. The perturbations explicitly imposed for the four cases in Table 4.4 increase the codimensions by one, except for case 9 where the rank drop by one increases the codimension by two.

The compound perturbations discussed above are all supposed to be tractable, but are not necessarily optimal. A summary of the explicit perturbations in tables 4.2 - 4.4 is displayed in a perturbation graph in Figure 4.1, where the nodes are placed at the same positions as in the closure graph (Figure 2.1). The paths to a node show different ways to generate the tractable perturbation required to find the KCF of the node, starting from a generic $A - \lambda B$. Notice that some arcs are marked with a bullet and the corresponding paths from a generic pencil to a destination KCF generate perturbations which are not necessarily optimal (compound perturbations from Table 4.3 and Table 4.4). All other paths correspond to optimal perturbations from Table 4.2. We clarify the notation in Figure 4.1 with two examples. Let $(\delta A_1, \delta B_1)$ denote the optimal perturbation of size $d_1(A)$ that for a generic $A - \lambda B$ gives $\tilde{A} - \lambda\tilde{B} = (A + \delta A_1) - \lambda(B + \delta B_1)$ the Kronecker structure $L_1 \oplus J_1$. Similarly, let $(\delta\tilde{A}_2, \delta\tilde{B}_2)$ denote the optimal perturbation of size $d_1(R[\tilde{A}, \tilde{B}, 0])$ that moves $\tilde{A} - \lambda\tilde{B}$ to a pencil with Kronecker structure $L_0 \oplus J_1 \oplus R_1$. Then $(\delta A_1 + \delta\tilde{A}_2, \delta B_1 + \delta\tilde{B}_2)$ is not necessarily the optimal perturbation for moving a generic pencil to orbit$(L_0 \oplus J_1 \oplus R_1)$. Therefore the arc to orbit$(L_0 \oplus J_1 \oplus R_1)$ is marked with a bullet. On the other hand, adding the perturbations going from orbit(L_2) to orbit$(L_0 \oplus 2J_1)$ via orbit$(L_1 \oplus J_1)$ give us the optimal perturbation, which is already shown in Table 4.2.

In order to relate our explicit perturbations to the (labeled) closure graph we consider 2-dimensional rank drops in Table 4.2 as results of two 1-dimensional rank drops. In practice, these 2-dimensional rank drops are computed directly. Some of the perturbations in Table 4.2 do not give a unique path in the graph, since the generic $A - \lambda B$ in some cases is perturbed in A and B simultaneously. For these cases all alternative paths are shown in the graph, e.g. there are three different paths to $2L_0 \oplus J_1 \oplus L_0^T$ and all of them correspond to the same perturbation (in infinite arithmetic) of size $(d_2^2(A) + d_1^2(B))^{1/2}$. From the construction of the explicit perturbations it follows that each arc in the perturbation graph connects a KCF with another KCF within its orbit's closure. Therefore, for each arc in the perturbation graph it exists a corresponding path in the closure graph. It is of course possible to find other paths in the (labeled) closure graph that give tractable perturbations.

The sizes of the perturbations are shown on the corresponding arcs in the graph,

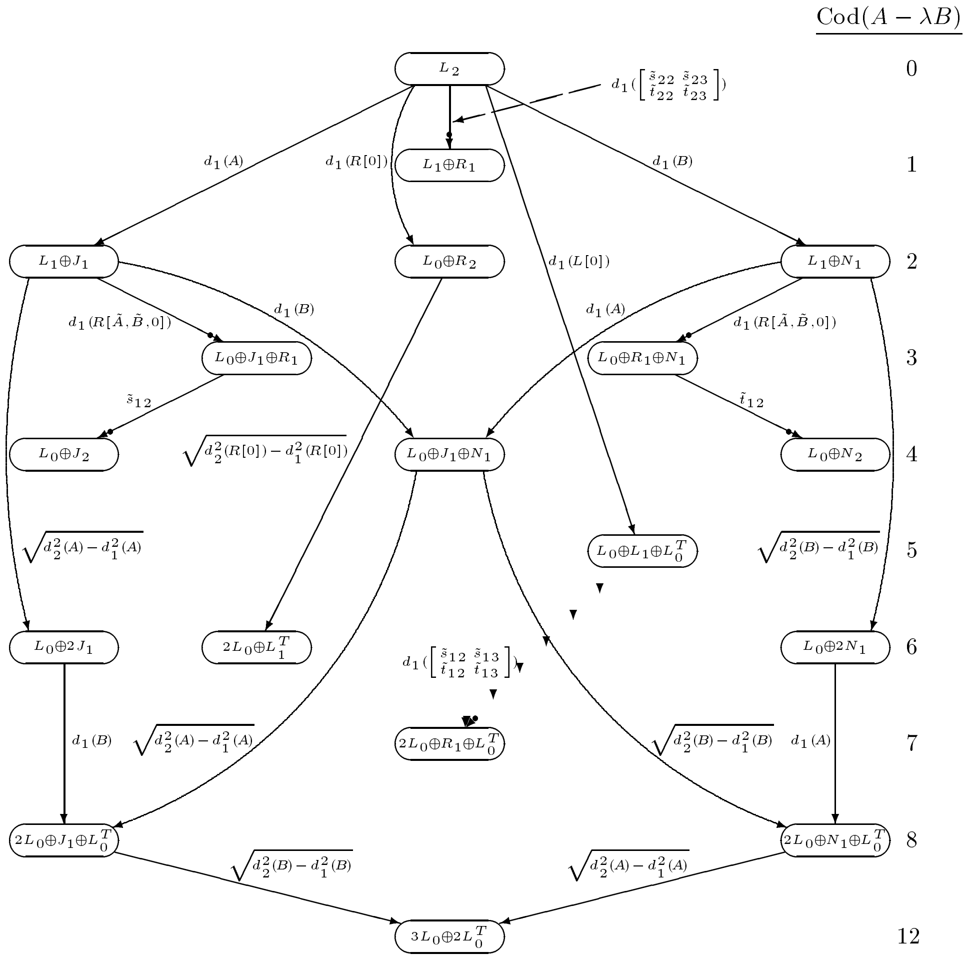


FIG. 4.1. *A graph displaying the tractable perturbations in tables* 4.2 – 4.4 *of a generic* 2*-by-*3 *pencil.*

with notation as before. The reason for perturbation sizes such as $\sqrt{d_2^2(A) - d_1^2(A)}$ is that the total perturbation needed for this 2-dimensional rank-drop in A is $d_2(A)$ (as shown in Table 4.2), but it is here shown as a further perturbation of a case where a perturbation of size $d_1(A)$ already has imposed a 1-dimensional rank-drop in A.

For each case in Table 4.2 that in Figure 4.1 is shown as a compound perturbation, even though it can be computed directly, the size of the total perturbation is the square root of the sum of the squares of the sizes of the components of the perturbation. For example, the case $2L_0 \oplus J_1 \oplus L_0^T$ is found by a compound perturbation $(\delta A, \delta B) = (\delta A_1, \delta B_1) + (\delta A_2, \delta B_2) + (\delta A_3, \delta B_3)$, where $\|(\delta A_1, \delta B_1)\|_E = d_1(A)$, $\|(\delta A_2, \delta B_2)\|_E = \sqrt{d_2^2(A) - d_1^2(A)}$, and $\|(\delta A_3, \delta B_3)\|_E = d_1(B)$. The size of the total perturbation is $\|(\delta A, \delta B)\|_E = (d_1^2(A) + (d_2^2(A) - d_1^2(A)) + d_1^2(B))^{1/2} = (d_2^2(A) + d_1^2(B))^{1/2}$. Notably, since the perturbation $d_1(A) = \sigma_{\min}(A)$ and $d_2(A) = \|A\|_E = (\sigma_{\min-1}^2(A) + \sigma_{\min}^2(A))^{1/2}$ the size $\sqrt{d_2^2(A) - d_1^2(A)}$ is equal to $\sigma_{\min-1}(A)$.

For each compound perturbation in tables 4.3 and 4.4, the size of the total perturbation is found by adding the components of the perturbation and then computing the norm of the resulting perturbation. However, an upper bound on the size of the compound perturbation can be achieved by adding the sizes of the components of the perturbation. For example, $L_0 \oplus J_1 \oplus R_1$ is found by the compound perturbation $(\delta A, \delta B) = (\delta A_1, \delta B_1) + (\delta A_2, \delta B_2)$, where $\|(\delta A_1, \delta B_1)\|_E = d_1(A)$ and $\|(\delta A_2, \delta B_2)\|_E = d_1(R[\tilde{A}, \tilde{B}, 0])$, and an upper bound on $\|(\delta A, \delta B)\|_E$ is $d_1(A) + d_1(R[\tilde{A}, \tilde{B}, 0])$.

4.1.2. Intractable Perturbations and the Closest Non-Generic Structure. The following example shows a situation where the perturbations incidentally create extra non-generic characteristics that raise the codimension of the perturbed pencil further than devised.

$$A = \begin{bmatrix} 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}, \quad B = \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \end{bmatrix}, \quad \epsilon_2 = \min_i \epsilon_i > 0. \tag{4.7}$$

Suppose we are looking for the minimal perturbations that impose the structure $L_1 \oplus J_1$ (case 2). They are of size $d_1(A)$ with

$$\delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon_2 \end{bmatrix}, \quad \delta B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Incidentally, δA and δB also lower the rank of $R[0]$. (For this example, δA and δB are the minimal perturbations that cause the rank drop, i.e., $d_1(A) = d_1(R[0])$ and the minima are attained for the same perturbations.) This fact implies that the perturbations aimed to impose the non-generic structure $L_1 \oplus J_1$ (with codimension two) result in a perturbed pencil with two zero eigenvalues corresponding to the structure $L_0 \oplus J_2$ with codimension four (case 4). One possible remedy is to further perturb the undesired non-generic pencil. To obtain $L_1 \oplus J_1$ we add, for example, the perturbations

$$\delta A' = \begin{bmatrix} \delta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta B' = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \end{bmatrix},$$

to $(A + \delta A, B + \delta B)$, where $\delta > 0$ is an arbitrary small number. These perturbations remove the common column nullspace $(\delta B')$ and the multiple eigenvalue at zero $(\delta A')$, making the compound perturbations tractable. If we start to look for the smallest perturbations of (A, B) that impose a common column nullspace that normally would generate the structure $L_0 \oplus R_2$ (case 5) we also get intractable perturbations and (in this case) the same structure $L_0 \oplus J_2$. We can also see from the closure graph in Figure 2.1 that $L_0 \oplus J_2$ is in the closure of each of the two orbits defined by $L_1 \oplus J_1$ and $L_0 \oplus R_2$.

Now, we turn to the problem of finding the closest non-generic Kronecker structure of a generic 2-by-3 pencil. Assume all inequalities relating to $d_1(R[1])$ in Theorem 4.1 are strict. Then the corresponding $R[A + \delta A, B + \delta B, 1]$ is rank deficient, and for all perturbations of size $\leq d_1(R[1])$, the (perturbed) matrices $A + \delta A, B + \delta B, R[A + \delta A, B + \delta B, 0]$ and $L[A + \delta A, B + \delta B, 0]$ must be of full rank, which correspond to the case $L_1 \oplus R_1$. Since all other non-generic cases require rank-deficiency in at least on of the matrices $A + \delta A, B + \delta B, R[A + \delta A, B + \delta B, 0]$ or $L[A + \delta A, B + \delta B, 0]$ (see necessary conditions in the labeled closure graph in Figure 2.2 or Table 2.3), we can formulate the following corollary.

COROLLARY 4.2. *If the inequalities* (4.2) *and* (4.3) *in Theorem* 4.1 *are strict,* $L_1 \oplus R_1$ *with codimension one* (*case* 1') *is the closest* (*unique*) *non-generic structure on distance* $d_1(R[1])$.

The presumptions of Corollary 4.2 are sufficient (but not necessary) to identify tractable perturbations that lowers the rank of $R[1]$. If equality holds in any of the inequalities of Theorem 4.1 (for the same perturbations $(\delta A, \delta B)$) we are faced with intractable perturbations which will result in non-generic structures with higher codimensions. We collect the different cases in the following corollary, where we list the closest Kronecker structure and the corresponding equality conditions. Notice that strict inequalities are assumed otherwise.

COROLLARY 4.3. *Assume strict inequalities hold in Theorem* 4.1 *when nothing else is stated. Then, if*

1. $d_1(R[1]) = d_1(R[0])$, $L_0 \oplus R_2$ (*case* 5) *is the closest non-generic form.*
2. $d_1(R[1]) = d_1(A)$, $L_1 \oplus J_1$ (*case* 2) *is the closest non-generic form.*
3. $d_1(R[1]) = d_1(B)$, $L_1 \oplus N_1$ (*case* 6) *is the closest non-generic form.*

All forms in Corollary 4.3 have codimension two. Notice that if it also exists some perturbations on distance $d_1(R[1])$ which do not lower the rank of $R[0], A$ and B, respectively, $L_1 \oplus R_1$ is also at the same distance as $L_0 \oplus R_2$, $L_1 \oplus J_1$ and $L_1 \oplus N_1$ for the three cases considered.

Assume that we can have equality in different combinations of the inequalities of Theorem 4.1. As before, we collect the possible cases in a corollary.

COROLLARY 4.4. *Assume two inequalities in Theorem* 4.1 *are satisfied with equality for the same perturbations* $(\delta A, \delta B)$. *Then, if*

1. $d_1(R[1]) = d_1(R[0]) = d_1(A)$, $L_0 \oplus J_1 \oplus R_1$ (*case* 4' *with codimension* 3) *or* $L_0 \oplus J_2$ (*case* 4 *with codimension* 4) *is the closest non-generic structure.*
2. $d_1(R[1]) = d_1(R[0]) = d_1(B)$, $L_0 \oplus R_1 \oplus N_1$ (*case* 10' *with codimension* 3) *or* $L_0 \oplus N_2$ (*case* 10 *with codimension* 4) *is the closest non-generic Kronecker structure.*

Notice that cases 4 and 10 have higher codimensions than cases 4' and 10', respectively, but the same algebraic characteristics in terms of the rank of $R[k]$ and $L[k]$ matrices as (see Table 2.3). The reason is that the 2-by-2 regular parts of cases 4 and 10 have one Jordan block with both eigenvalues specified, which increase the codimension by one compared to cases 4' and 10' (both with one eigenvalue unspecified).

The remark following Corollary 4.3, regarding a non-unique closest Kronecker structure can also be extended to apply to Corollary 4.4.

In applications (e.g. computing the uncontrollable subspace) we are interested to find the most non-generic structure (with highest codimension) for a given size of the perturbations. Is it possible to find intractable perturbations that result in a closest 2-by-3 non-generic structure with codimension > 4? The answer is no since all other cases require a rank drop of at least two in A, B or $R[0]$ or a simultaneous rank drop in A and B. It always exists strictly smaller perturbations that drop the rank by one (see (4.5)). Similar arguments also exclude $L_0 \oplus J_1 \oplus N_1$ with codimension 4 from being the closest non-generic pencil.

4.1.3. Closest Non-Generic Structures to a Generic 1-by-2 Pencil. Since we do not know any explicit expression for $d_1(R[1])$ it is hard to construct examples that illustrate different situations described in Section 4.1.2. By considering 1-by-2 pencils we overcome this problem. A generic 1-by-2 pencil has the Kronecker structure $L_1 = [-\lambda \quad 1] = [0 \quad 1] - \lambda[1 \quad 0] \equiv A - \lambda B$. The non-generic structures of size 1-by-2 are $L_0 \oplus R_1, L_0 \oplus J_1, L_0 \oplus N_1$ and $2L_0 \oplus L_0^T$ with codimensions 1, 2, 2 and 4, respectively.

Which form(s) can be the closest non-generic structure of a generic 1-by-2 pencil?

- $L_0 \oplus R_1$ if it exists perturbations of size $d_1(R[0])$ which do not simultaneously decrease the rank of A or B. This is, e.g. fulfilled if $d_1(R[0]) < \min(d_1(A), d_1(B))$.
- $L_0 \oplus J_1$ if $d_1(R[0]) = d_1(A)$.
- $L_0 \oplus N_1$ if $d_1(R[0]) = d_1(B)$.

Moreover, $2L_0 \oplus L_0^T$ can never be the closest non-generic structure. The size of the minimal perturbations that turn A and B to zero matrices is $(d_1^2(A) + d_1^2(B))^{1/2}$.

The following example illustrates a case where $d_1(R[0]) = d_1(A) = d_1(B)$ and there exist perturbations of size $d_1(R[0])$ that do not simultaneously decrease the rank of A or B. Consequently, $L_0 \oplus R_1, L_0 \oplus J_1$, and $L_0 \oplus N_1$ are all the closest non-generic Kronecker structure.

Let $A = [1 \quad 1]$ and $B = [-1 \quad 1]$. Then $R[0]$ has the singular value decomposition

$$R[0] \equiv \begin{bmatrix} A \\ B \end{bmatrix} = U\Sigma V^T \equiv \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$d_1(R[0])(= \sqrt{2})$ is attained for the (minimal) perturbations

$$\begin{bmatrix} \delta A \\ \delta B \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix},$$

while $A + \delta A$ and $B + \delta B$ remain full rank matrices, resulting in $L_0 \oplus R_1$ as the closest non-generic structure. The perturbations

$$\begin{bmatrix} \delta A_1 \\ \delta B_1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \delta A_2 \\ \delta B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

of the same minimal size make $R[0], A$ and $R[0], B$ drop rank, respectively. These perturbations generate the non-generic structures $L_0 \oplus J_1$ and $L_0 \oplus N_1$, respectively.

4.2. Using GUPTRI to Impose Non-Generic Structures. We have modified `GUPTRI` so that, for an $m \times n$ generic pencil $A - \lambda B$ as input, it is possible to impose a generalized Schur form with a specified Kronecker structure. (The modified `GUPTRI` also work for imposing a Kronecker structure of higher codimension on any non-generic pencil.) Given the block indices that define the specified Kronecker structure (n_i's and r_i's of the RZ-staircase and LI-staircase forms [8]), `GUPTRI` imposes the necessary rank deflations in order to compute the specified (non-generic) structure. The perturbations induced by these rank deflations are usually tractable. If the perturbations imposed by `GUPTRI` are intractable, `GUPTRI` computes the corresponding non-generic structure of higher codimension. The resulting generalized Schur decomposition can be expressed in finite arithmetic as

$$P^H((A + \delta A) - \lambda(B + \delta B))Q = \begin{bmatrix} A_r - \lambda B_r & * & * \\ 0 & A_{reg} - \lambda B_{reg} & * \\ 0 & 0 & A_l - \lambda B_l \end{bmatrix}, \tag{4.8}$$

where $*$ denotes arbitrary conforming submatrices. Let δ_σ^2 denote the sum of the squares of all deleted singular values (imposed as zeros) during the reduction to GUPTRI form. Then δ_σ is an accurate estimate of $\|(\delta A, \delta B)\|_E$ in (4.8). One interpretation is that `GUPTRI` computes an exact generalized Schur decomposition (with the specified Kronecker structure) for a pencil $A' - \lambda B'$ within distance δ_σ from the input pencil $A - \lambda B$. Moreover, δ_σ is an upper bound on the distance from $A - \lambda B$ to the nearest pencil with the Kronecker structure specified as input to GUPTRI.

Furthermore, this give us a method for computing an upper bound on the distance from a generic m-by-n pencil to the closest non-generic pencil:

- Compute the structure indices (n_i's and r_i's of the RZ-staircase and LI-staircase forms [8]) for all q structurally different non-generic GUPTRI forms of size $m \times n$. This is a finite integer matching problem.
- Use the modified version of GUPTRI to impose the q non-generic structures:

$$A_i - \lambda B_i = P_i^H((A + \delta A_i) - \lambda(B + \delta B_i))Q_i, \quad i = 1, \ldots, q. \tag{4.9}$$

- Compute the matrix pairs corresponding to the q non-generic structures:

$$\hat{A}_i = P_i A_i Q_i^H, \quad \hat{B}_i = P_i B_i Q_i^H, \qquad i = 1, \ldots, q. \tag{4.10}$$

- Compute

$$\delta = \min_{1 \le i \le q} \delta_i, \quad \delta_i = \|(A - \hat{A}_i, B - \hat{B}_i)\|_E. \tag{4.11}$$

Now, δ is an upper bound on the closest non-generic pencil to $A - \lambda B$ and the δ_i's are upper bounds on the closest non-generic pencils with the Kronecker structure of $A_i - \lambda B_i$ in (4.9).

The method described above is quite expensive already for moderate m and n (see Section 5) but is perfectly parallel. In a distributed memory environment it is possible to distribute the block indices for the different Kronecker structures evenly over the p ($\le q$) processors. Each processor also hold A and B and computes its local δ using the method above. Finally, a global minimum operation over all p processors gives us δ in (4.11).

4.3. Computational Experiments on Random 2-by-3 Pencils. We have performed computational experiments on 100 random 2-by-3 pencils $A - \lambda B$. The elements of A and B are chosen uniformly distributed in $(0, 1)$. For each random pencil we impose the 17 non-generic structures using the two approaches discussed in sections 4.1 and 4.2.

Table 4.5 displays the mean values of perturbations required to impose each of the 17 non-generic forms for 100 random examples. We measure the perturbations for each example and non-generic form as $\|(A - \tilde{A}, B - \tilde{B})\|_E$, where $\tilde{A} - \lambda\tilde{B}$ denotes a non-generic pencil. The matrices A and B are normalized such that $\|A\|_E = \|B\|_E$ and $\|(A, B)\|_E = 1$.

Columns 2 and 3 of Table 4.5 show the δ_i's in (4.11) computed by modified GUPTRI for the pencils $A - \lambda B$ and $B - \mu A$, respectively. Column 4 shows the explicit perturbations of tables 4.2, 4.3 and 4.4. The explicit perturbations that are proved to be the smallest possible are marked with the superscript *.

In Table 4.6 we display the smallest perturbations (measured as above) required to impose non-generic forms of each possible codimension for the same 100 random 2-by-3 examples. For example, we have three non-generic structures with codimension 2, so the smallest perturbations are in this case determined from 300 random examples. The singular structures (cases) that give the smallest perturbations are shown in columns directly following columns 2, 4 and 6 of Table 4.6.

Numbers in bold font in tables 4.5 and 4.6 indicate that the size of the perturbations (distances) computed by modified GUPTRI are the same as for the explicit

TABLE 4.5
Mean values of perturbations (measured as $\|(A-\tilde{A}, B-\tilde{B})\|_E$) required to impose each of the 17 non-generic forms for 100 random $A-\lambda B$ of size 2-by-3.

Case	$A-\lambda B$	$B-\mu A$	Explicit	Cod($A-\lambda B$)	Comment
1	0.000	0.000	0.000	0	
1′	0.160	0.154	0.127	1	
2	**0.181**	0.394	0.181*	2	
6	0.378	**0.190**	0.190*	2	
5	0.235	0.227	0.140*	2	
4′	0.218	0.268	0.211	3	
10′	0.287	0.227	0.220	3	
4	*0.456*	0.533	0.461	4	
10	0.538	*0.481*	0.524	4	
7	0.437	0.434	0.281*	4	
7′	0.589	0.602	0.326*	5	
3	**0.707**	**0.707**	0.707*	6	$A = 0_{2\times 3}$
11	**0.707**	**0.707**	0.707*	6	$B = 0_{2\times 3}$
9′	0.399	0.399	0.353*	6	
9	0.466	0.460	0.390	7	
8	**0.737**	**0.737**	0.737*	8	$A = 0_{2\times 3}$
12	**0.736**	**0.736**	0.736*	8	$B = 0_{2\times 3}$
13	**1.000**	**1.000**	1.000*	12	$A = B = 0_{2\times 3}$

TABLE 4.6
Minimum perturbations (measured as $\|(A-\tilde{A}, B-\tilde{B})\|_E$) required to impose non-generic forms of each possible codimension for 100 random $A-\lambda B$ of size 2-by-3.

Cod($A-\lambda B$)	$A-\lambda B$	Case	$B-\mu A$	Case	Explicit	Case
0	0.000	1	0.000	1	0.000	1
1	$2\cdot 10^{-4}$	1′	$3\cdot 10^{-4}$	1′	$1\cdot 10^{-4}$	1′
2	0.011	2	0.010	5	0.009	5
3	**0.036**	4′	0.037	4′	0.036	4′
4	0.111	4	**0.106**	10	0.106	7
5	0.192	7′	**0.119**	7′	0.119	7′
6	0.163	9′	0.163	9′	0.153	9′
7	0.233	9	0.224	9	0.184	9
8	**0.707**	12	**0.707**	12	0.707	12
12	**1.000**	13	**1.000**	13	1.000	13

perturbations, which for these cases also are shown to be the minimal perturbations. Numbers marked in italic font in Table 4.5 indicate that modified `GUPTRI` computed smaller upper bounds than corresponding bounds for the explicit perturbations.

All explicit perturbations of the 100 2-by-3 random pencils turned out to be tractable. The results show that the smallest distance from $A-\lambda B$ to a non-generic structure with fixed codimension k increases with increasing k, in accordance with the Kronecker structure hierarchy in Figure 2.1. Case 1' with KCF $L_1 \oplus R_1$ is the closest non-generic pencil. Our explicit bound for case 1' is not proved to be the smallest possible.

5. Some Comments on the General Case. The complexity and the intricacies of the problems considered are well-exposed in sections 2 – 4. In the following we discuss some extensions to general m-by-n pencils. The number of different KCFs grows rapidly with increasing m and n. Some cases are displayed in Table 5.1.

We have been able to generate 20098 structurally different KCFs for $m = 10, n = 20$. Notice that for a given m the number of different structures is fixed for $n \geq 2m$. For $m > n$ the number of KCFs are the same as for the transposed pencil. As an

TABLE 5.1
Number of structurally different Kronecker forms of size m-by-n ($m \leq n$).

m	n: 1	2	3	4	5	6	7	8	9	10
1	4	5	5	5	5	5	5	5	5	5
2		14	18	19	19	19	19	19	19	19
3			41	54	58	59	59	59	59	59
4				110	145	159	163	164	164	164
5					271	358	397	411	415	416

example we show all structurally different 3-by-4 Kronecker forms in Table 5.2, where we as before let R_2 denote a 2-by-2 regular block with any non-zero finite eigenvalues (see Section 2.1) and, similarly, we let R_3 denote a regular 3-by-3 block.

TABLE 5.2
All 54 structurally different 3-by-4 pencils.

KCF		
L_3	$L_0 \oplus R_2 \oplus N_1$	$2L_0 \oplus L_2^T$
$L_2 \oplus N_1$	$L_0 \oplus R_3$	$2L_0 \oplus N_2 \oplus L_0^T$
$L_2 \oplus R_1$	$L_0 \oplus J_3$	$2L_0 \oplus N_1 \oplus L_1^T$
$L_2 \oplus J_1$	$L_0 \oplus J_2 \oplus N_1$	$2L_0 \oplus 2N_1 \oplus L_0^T$
$L_1 \oplus N_2$	$L_0 \oplus J_2 \oplus R_1$	$2L_0 \oplus R_1 \oplus L_1^T$
$L_1 \oplus 2N_1$	$L_0 \oplus J_1 \oplus N_2$	$2L_0 \oplus R_1 \oplus N_1 \oplus L_0^T$
$L_1 \oplus R_1 \oplus N_1$	$L_0 \oplus J_1 \oplus 2N_1$	$2L_0 \oplus R_2 \oplus L_0^T$
$L_1 \oplus R_2$	$L_0 \oplus J_1 \oplus R_1 \oplus N_1$	$2L_0 \oplus J_2 \oplus L_0^T$
$L_1 \oplus J_2$	$L_0 \oplus J_1 \oplus R_2$	$2L_0 \oplus J_1 \oplus L_1^T$
$L_1 \oplus J_1 \oplus N_1$	$L_0 \oplus J_1 \oplus J_2$	$2L_0 \oplus J_1 \oplus N_1 \oplus L_0^T$
$L_1 \oplus J_1 \oplus R_1$	$L_0 \oplus 2J_1 \oplus N_1$	$2L_0 \oplus J_1 \oplus R_1 \oplus L_0^T$
$L_1 \oplus 2J_1$	$L_0 \oplus 2J_1 \oplus R_1$	$2L_0 \oplus 2J_1 \oplus L_0^T$
$2L_1 \oplus L_0^T$	$L_0 \oplus 3J_1$	$2L_0 \oplus L_1 \oplus 2L_0^T$
$L_0 \oplus N_3$	$L_0 \oplus L_2 \oplus L_0^T$	$3L_0 \oplus L_0^T \oplus L_1^T$
$L_0 \oplus N_1 \oplus N_2$	$L_0 \oplus L_1 \oplus L_1^T$	$3L_0 \oplus N_1 \oplus 2L_0^T$
$L_0 \oplus 3N_1$	$L_0 \oplus L_1 \oplus N_1 \oplus L_0^T$	$3L_0 \oplus R_1 \oplus 2L_0^T$
$L_0 \oplus R_1 \oplus N_2$	$L_0 \oplus L_1 \oplus R_1 \oplus L_0^T$	$3L_0 \oplus J_1 \oplus 2L_0^T$
$L_0 \oplus R_1 \oplus 2N_1$	$L_0 \oplus L_1 \oplus J_1 \oplus L_0^T$	$4L_0 \oplus 3L_0^T$

It is possible to extend Theorem 4.1 to general m-by-$(m+1)$ pencils.

THEOREM 5.1. *For a generic m-by-$(m+1)$ pencil (A, B) the following inequalities hold:*

$$0 \equiv d_1(R[m]) < d_1(R[m-1]) \leq \ldots \leq d_1(R[0]), \tag{5.1}$$

$$d_1(R[m-1]) \leq d_1(A), \quad d_1(R[m-1]) \leq d_1(B), \tag{5.2}$$

$$d_1(R[m-1]) \leq d_1(L[m-1]) \leq \ldots \leq d_1(L[0]), \tag{5.3}$$

$$\left.\begin{array}{l} d_k(A) < d_{k+1}(A) \\ d_k(B) < d_{k+1}(B) \\ d_k(R[0]) < d_{k+1}(R[0]) \end{array}\right\} \quad k = 1, \ldots, m-1. \tag{5.4}$$

Proof. From Theorem 2.1 it follows that $d_1(R[m]) = 0$ for all m-by-$(m+1)$ pencils (generic or non-generic). A perturbation that lowers the column rank in $R[k-1]$ will always lower the rank in $R[k]$, since a dependence between columns in $R[k-1]$ will make the corresponding columns in

$$R[k] = \begin{bmatrix} A & 0 \\ B & R[k-1] \\ 0 & \end{bmatrix}$$

linearly dependent, proving (5.1). A perturbation that reduces the rank in A (or B) will cause a linear dependence among the m first (or last) rows of

$$R[m-1] = \begin{bmatrix} A & 0 \\ & \dots \\ 0 & B \end{bmatrix}.$$

Since $R[m-1]$ is square $(m^2+m) \times (m^2+m)$, the row rank deficiency is equivalent to $R[m-1]$ being column rank deficient, which proves (5.2). The relations between $d_1(L[k]), k = 0, \dots, m-1$ in (5.3) can be proved similarly as the corresponding relations between the $R[k]$-matrices in (5.1). For the first inequality in (5.3) we recall the fact that a row rank deficient $L[m-1]$ is equivalent to at least one L_k^T block ($k = 0, \dots,$ or $m-1$) in the KCF. To match the dimensions of the pencil, the KCF must contain at least one L_i block ($i = 0, \dots,$ or $m-2$) which is equivalent to $R[i]$ being column rank deficient. Hence $L[m-1]$ row rank deficient is equivalent to $R[i]$ being column rank deficient for some $i = 0, \dots, m-2$. Now, the first inequality of (5.3) is obtained by applying (5.1) to the relation between $R[i]$ and $R[m-1]$. As in Theorem 4.1 the inequalities (5.4) follow from the definition of $d_k(\cdot)$. □

We can see that the closest non-generic structure to a generic m-by-$(m+1)$ pencil is on distance $d_1(R[m-1])$. Notably, when all inequalities relating to $d_1(R[m-1])$ in Theorem 5.1 are strict, the equation (5.1) excludes any L_k blocks for $k < m-1$ in the KCF of any pencil on distance $d_1(R[m-1])$ from the generic case. Similarly, the equation (5.2) excludes any J_i or N_i blocks, and (5.3) the existence of L_k^T blocks. Altogether, this extends Corollary 4.2 to m-by-$(m+1)$ pencils.

COROLLARY 5.2. *If all inequalities relating to* $d_1(R[m-1])$ *in Theorem* 5.1 *are strict, the closest non-generic structure to a generic* m*-by-*$(m+1)$ *pencil is* $L_{m-1} \oplus R_1$ (*with codimension* 1) *on distance* $d_1(R[m-1])$.

Corollary 5.2 can be used to characterize the distance to uncontrollability for a single input single output linear system $E\dot{x}(t) = Fx(t) + Gu(t)$, where E and F are p-by-p matrices, G is p-by-1, and E is assumed to be nonsingular. The linear system is completely controllable (i.e., the dimension of the controllable subspace equals p) if and only if $A - \lambda B \equiv [G|F - \lambda E]$ is generic. Under the assumptions in Corollary 5.2 the closest uncontrollable system is on distance $d_1(R[p-1])$ corresponding to the non-generic structure $L_{p-1} \oplus R_1$ (with the eigenvalue of R_1 finite and non-zero but otherwise unspecified).

Since B has full row rank $A - \lambda B \equiv [G|F - \lambda E]$ can have neither infinite eigenvalues nor L_j^T blocks in its KCF. Therefore, it can only have finite eigenvalues and L_j blocks in its KCF (and GUPTRI form) and the number of L_j blocks is equal to the number of columns of G. For $p = 2$ the possible uncontrollable systems correspond to cases 1', 2, 5, 4', 4 and 3 of Table 2.3.

Generalizations of corollaries 4.3 and 4.4 to m-by-$(m+1)$ pencils are straightforward, but there are several more cases to distinguish. The formulations and technicalities are omitted here.

Some results for general matrix pencils relating to problems studied here are presented in [2]. Eigenvalue perturbation bounds are used to develop computational bounds on the distance from a given pencil to one with a qualitatively different Kronecker structure.

Acknowledgements. We are grateful to Alan Edelman and the referees for constructive comments, which have improved both the contents and the organization of the paper.

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A. Proof of Theorem 2.2. *Proof.* First we prove that each arc in the graph correspond to a closure relation, and then we prove that these are all arcs that can exist. We prove that one KCF is in the closure of the orbit of another KCF by showing that the one in the closure is just a special case of the one defining the closure. We show proofs for each arc starting from the zero pencil.

Before looking at each arc we note that there is a symmetry regarding row ranks and column nullities between the Kronecker structures with J_i and N_i blocks replaced (see Table 2.3). From this follows that some of the proofs below that are shown for J_i blocks can be done similarly for the corresponding case with N_i blocks. Typically we have to work with specific elements in A instead of B or vice versa. For these cases we will just mention this similarity without repeating the computations.

In the following, α, β, γ, δ, and ϵ are supposed to be non-zero elements when nothing else is stated.

- $3L_0 \oplus 2L_0^T$ is in the closure of orbit$(2L_0 \oplus J_1 \oplus L_0^T)$, since $3L_0 \oplus 2L_0^T$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix},$$

  which is equivalent to $2L_0 \oplus J_1 \oplus L_0^T$ for all non-zero α.
- $3L_0 \oplus 2L_0^T$ is in the closure of orbit$(2L_0 \oplus N_1 \oplus L_0^T)$ follows from similar arguments based on the symmetry between J_i and N_i blocks.
- $2L_0 \oplus J_1 \oplus L_0^T$ is in the closure of orbit$(2L_0 \oplus R_1 \oplus L_0^T)$, since $2L_0 \oplus J_1 \oplus L_0^T$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix},$$

  which is equivalent to $2L_0 \oplus R_1 \oplus L_0^T$ for all non-zero α.
- $2L_0 \oplus N_1 \oplus L_0^T$ is in the closure of orbit$(2L_0 \oplus R_1 \oplus L_0^T)$ follows from similar arguments.
- $2L_0 \oplus J_1 \oplus L_0^T$ is in the closure of orbit$(L_0 \oplus 2J_1)$, since $2L_0 \oplus J_1 \oplus L_0^T$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \end{bmatrix},$$

  which multiplied by a permutation matrix can be shown to be equivalent to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

  and this pencil is equivalent to $L_0 \oplus 2J_1$ for all non-zero α.
- $2L_0 \oplus N_1 \oplus L_0^T$ is in the closure of orbit$(L_0 \oplus 2N_1)$ follows from similar arguments.

- $2L_0 \oplus R_1 \oplus L_0^T$ is in the closure of orbit$(2L_0 \oplus L_1^T)$, since $2L_0 \oplus R_1 \oplus L_0^T$ is the special case $\beta = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix},$$

which for non-zero β is shown to be equivalent to $2L_0 \oplus L_1^T$ by the following equivalence transformation

$$\begin{bmatrix} \frac{1}{\gamma} & \frac{-\alpha}{\beta\gamma} \\ 0 & \frac{1}{\beta} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- $2L_0 \oplus R_1 \oplus L_0^T$ is in the closure of orbit$(L_0 \oplus L_1 \oplus L_0^T)$, since $2L_0 \oplus R_1 \oplus L_0^T$ is the special case $\beta = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \beta & \gamma \\ 0 & 0 & 0 \end{bmatrix},$$

which for non-zero β is shown to be equivalent to $L_0 \oplus L_1 \oplus L_0^T$ by the following equivalence transformation

$$\begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \beta & \gamma \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha}{\beta} & -\frac{\gamma}{\beta} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- $2L_0 \oplus L_1^T$ is in the closure of orbit$(L_0 \oplus J_1 \oplus N_1)$, since $2L_0 \oplus L_1^T$ is the special case $\gamma = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & \delta \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.1}$$

This is shown by the following equivalence transformation:

$$\begin{bmatrix} \frac{1}{\delta} & \frac{-\alpha}{\beta\delta} \\ 0 & \frac{1}{\beta} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which is identical to $2L_0 \oplus L_1^T$. That the pencil (A.1) is equivalent to $L_0 \oplus J_1 \oplus N_1$ for all non-zero γ follows from the equivalence transformation:

$$\begin{bmatrix} 1 & \frac{-\alpha}{\beta} \\ 0 & \frac{1}{\beta} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & \delta \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\gamma} & \frac{-\delta}{\gamma} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.2}$$

- $2L_0 \oplus L_1^T$ is in the closure of orbit$(L_0 \oplus J_2)$, since $2L_0 \oplus L_1^T$ is a permutation of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

which is the special case $\gamma = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

and this pencil is equivalent to $L_0 \oplus J_2$ for all non-zero γ.

- $2L_0 \oplus L_1^T$ is in the closure of orbit$(L_0 \oplus N_2)$ follows from similar arguments.
- $L_0 \oplus L_1 \oplus L_0^T$ is in the closure of orbit$(L_0 \oplus J_2)$, since $2L_0 \oplus L_1^T$ is the special case $\beta = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

which is equivalent to $L_0 \oplus J_2$ for all non-zero β.
- $L_0 \oplus L_1 \oplus L_0^T$ is in the closure of orbit$(L_0 \oplus N_2)$ follows from similar arguments.
- $L_0 \oplus L_1 \oplus L_0^T$ is in the closure of orbit$(L_0 \oplus J_1 \oplus N_1)$, since $L_0 \oplus L_1 \oplus L_0^T$ is the special case $\beta = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & \delta \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.3}$$

This follows from the equivalence transformation

$$\begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & \delta \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha}{\gamma} & -\frac{\delta}{\gamma} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

That (A.3) is equivalent to $L_0 \oplus J_1 \oplus N_1$ for non-zero β is shown in (A.2).
- $L_0 \oplus 2J_1$ is in the closure of orbit$(L_0 \oplus J_2)$, since $L_0 \oplus 2J_1$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix},$$

which is equivalent to $L_0 \oplus J_2$ for all non-zero α.
- $L_0 \oplus 2N_1$ is in the closure of orbit$(L_0 \oplus N_2)$ follows from similar arguments.
- $L_0 \oplus J_2$ is in the closure of orbit$(L_0 \oplus J_1 \oplus R_1)$, since $L_0 \oplus J_2$ is the special case $\beta = 0$ of

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix},$$

which for non-zero β is shown to be equivalent to $L_0 \oplus J_1 \oplus R_1$ (with eigenvalue β/δ) by the following equivalence transformation

$$\begin{bmatrix} 1 & -\frac{\alpha}{\beta} \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\gamma} & \frac{\alpha}{\beta\gamma} \\ 0 & 0 & \frac{1}{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\beta}{\delta} \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- $L_0 \oplus N_2$ is in the closure of orbit$(L_0 \oplus N_1 \oplus R_1)$ follows from similar arguments.
- $L_0 \oplus J_1 \oplus N_1$ is in the closure of orbit$(L_0 \oplus J_1 \oplus R_1)$, since $L_0 \oplus J_1 \oplus N_1$ is the special case $\gamma = 0$ of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \lambda \begin{bmatrix} 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix},$$

which is equivalent to $L_0 \oplus J_1 \oplus R_1$ for all non-zero γ.
- $L_0 \oplus J_1 \oplus N_1$ is in the closure of orbit$(L_0 \oplus N_1 \oplus R_1)$ follows from similar arguments.

- $L_0 \oplus J_1 \oplus R_1$ is in the closure of orbit$(L_1 \oplus J_1)$, since $L_0 \oplus J_1 \oplus R_1$ is equivalent to $L_0 \oplus R_1 \oplus J_1$ which is the special case $\alpha = 0$ of

$$\begin{bmatrix} \alpha & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix},$$

which for non-zero α is shown to be equivalent to $L_1 \oplus J_1$ by the following equivalence transformation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \alpha & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix} \right) \begin{bmatrix} -\frac{\beta}{\alpha\gamma} & \frac{1}{\alpha} & 0 \\ \frac{1}{\gamma} & 0 & 0 \\ 0 & 0 & \frac{1}{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- $L_0 \oplus N_1 \oplus R_1$ is in the closure of orbit$(L_1 \oplus N_1)$ follows from similar arguments.
- $L_0 \oplus J_1 \oplus R_1$ is in the closure of orbit$(L_0 \oplus R_2)$, since $L_0 \oplus J_1 \oplus R_1$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix},$$

which is equivalent to $L_0 \oplus R_2$ for all non-zero α.
- $L_0 \oplus J_1 \oplus R_1$ is in the closure of orbit$(L_0 \oplus R_2)$ follows from similar arguments.
- $L_1 \oplus J_1$ is in the closure of orbit$(L_1 \oplus R_1)$, since $L_1 \oplus J_1$ is the special case $\beta = 0$ of

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} - \lambda \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix},$$

which is equivalent to $L_1 \oplus R_1$ for all non-zero β.
- $L_1 \oplus N_1$ is in the closure of orbit$(L_1 \oplus R_1)$ follows from similar arguments.
- $L_0 \oplus R_2$ is in the closure of orbit$(L_1 \oplus R_1)$, since $L_0 \oplus R_2$ is the special case $\alpha = 0$ of

$$\begin{bmatrix} \alpha & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} - \lambda \begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & \epsilon \end{bmatrix},$$

which for non-zero α is shown to be equivalent to $L_1 \oplus R_1$ (with eigenvalue γ/ϵ) by the following equivalence transformation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \alpha & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} - \lambda \begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\alpha} & -\frac{\beta}{\alpha\delta} & 0 \\ 0 & \frac{1}{\delta} & 0 \\ 0 & 0 & \frac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\gamma}{\epsilon} \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- $L_1 \oplus R_1$ is in the closure of orbit(L_2), since L_2 spans the complete 12-dimensional space.

Now we have shown that all arcs in the graph are valid. It remains to show that there are no arcs missing. This can be done by examining the KCF:s that cannot be in the closure of each other.

First we remark that one necessary condition for a KCF to be in the closure of the orbit of another is that it must have higher codimension than the one defining the closure.

Since $L_0 \oplus N_1 \oplus R_1$, $L_0 \oplus N_2$ and $L_0 \oplus 2N_1$ all require that A has full rank ($= 2$), none of them can be in the closure of orbit$(L_1 \oplus J_1)$, since that KCF requires A to have rank $= 1$. (Of course this also implies that none of these three KCF:s can be in the closure of the orbit of $L_0 \oplus J_1 \oplus R_1$, $L_0 \oplus 2J_1$ or any other KCF that is in the closure of orbit$(L_1 \oplus J_1)$.)

From the symmetry for J_i and N_i blocks, we have that none of $L_0 \oplus J_1 \oplus R_1$, $L_0 \oplus J_2$ and $L_0 \oplus 2J_1$ can be in the closure of orbit$(L_1 \oplus N_1)$, since they require B to have full rank and $L_1 \oplus N_1$ has rank$(B) = 1$.

Since $2L_0 \oplus J_1 \oplus L_0^T$ and $2L_0 \oplus R_1 \oplus L_0^T$ have a B of rank 1, none of them can be in the closure of orbit$(L_0 \oplus 2N_1)$ since that KCF requires a 2-dimensional rank deficiency in B. By similar arguments for the rank of A we see that $2L_0 \oplus N_1 \oplus L_0^T$ and $2L_0 \oplus R_1 \oplus L_0^T$ cannot be in the closure of orbit$(L_0 \oplus 2J_1)$. Since we have investigated all presumptive KCF:s the proof is complete. $\square$