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Decompositions of Edge-Colored Complete Graphs

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We prove an asymptotic existence theorem for decompositions of edge-colored complete graphs into prespecified edge-colored subgraphs. Many combinatorial design problems fall within this framework. Applications of our main theorem require calculations involving the numbers of edges of each color and degrees of each color class of edges for the graphs allowed in the decomposition. We do these calculations to provide new proofs of the asymptotic existence of resolvable designs, near resolvable designs, group divisible designs, and grid designs. Two further applications are the asymptotic existence of skew Room d-cubes and the asymptotic existence of $(v, k, 1)$-BIBDs with any group of order $k-1$ as an automorphism group.

1. INTRODUCTION

A large family of combinatorial problems can be formulated or discussed in the following language of graph decomposition.

We consider finite edge-r-colored directed graphs. Here, *edge-r-colored* means that each edge has a color chosen from a set of r colors. We often require edge-r-colored digraphs to be *simple*, i.e. there are no loops and for each ordered pair (x, y) of distinct vertices, there is at most one edge directed from x to y. For such a simple edge-colored digraph, the *opposite* of an edge directed from x to y is the edge directed from y to x, if present, no matter what its color. Undirected and/or "mixed" graphs may be included: we can identify an undirected edge of some color with a pair of opposite directed edges of that color. The term graph will be used below to mean "edge-colored directed graph," though often we use all or part of the latter term for emphasis. We require that isomorphisms between edge-r-colored digraphs preserve the colors of edges.

Let $K_n^{(r)}$ be a complete digraph on n vertices with exactly one edge of color i joining any vertex x to any other vertex y for every color i in a set of r colors. The digraph $K_n^{(r)}$ has a total of $rn(n-1)$ edges and, of course, is not simple if $r > 1$.

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A family $\mathscr{F}$ of subgraphs of a graph K will be called a *decomposition* of K if every edge $e \in E(K)$ belongs to exactly one member of $\mathscr{F}$. Given a family $\mathscr{G}$ of edge-r-colored digraphs, a $\mathscr{G}$-*decomposition* of K is a decomposition $\mathscr{F}$ such that every graph $F \in \mathscr{F}$ is isomorphic to some graph $G \in \mathscr{G}$. Often $\mathscr{G} = \{G\}$ consists of a single digraph G, and we speak of a G-*decomposition*.

The existence of certain combinatorial structures can be seen to be equivalent to the existence of a $\mathscr{G}$-decomposition of $K_n^{(r)}$ for some $\mathscr{G}$ and r. Sometimes the correspondence is natural and immediate (e.g. Mendelsohn triple systems, whist tournaments, Steiner pentagon systems, balanced weighing designs); other instances require a translation (and may appear somewhat artificial). For example, both the existence of certain group divisible designs and the existence of resolvable designs can be expressed as such decompositions; see Sections 8 and 10.

We discuss some elementary examples in Section 2. Generalizations of some of these will be discussed in later sections.

The most interesting examples/applications of $\mathscr{G}$-decompositions at this time would seem to be those where the family $\mathscr{G}$ has the properties that (1) each simple graph $G \in \mathscr{G}$ is "complete" in the sense that there is an edge directed from either x to y or from y to x for every ordered pair x, y of vertices, and (2) the colors are paired in the sense that the opposites of the edges of a color i over the members of $\mathscr{G}$, if any are present, are exactly the edges of color j for some j, possibly $j = i$. (In the case that the opposites of the edges of color i are exactly the edges of color j, $j \neq i$, the edges of color j can be suppressed without loss of information.)

Given a family $\mathscr{G}$ of edge-r-colored digraphs, we may ask for which values of n the complete digraph $K_n^{(r)}$ admits a $\mathscr{G}$-decomposition. (Of course, we are assuming that the r colors used for $K_n^{(r)}$ are the same r colors that appear on edges of members of $\mathscr{G}$.) We state our existence result first in the special case when $\mathscr{G}$ consists of a single graph. The case $r = 1$ of Theorem 1.1 below was proved by R. M. Wilson in [36]. Since then, it has become apparent that there are applications for a more colorful version. The proof of the main theorem, Theorem 1.2, is similar to the proof given in [36], but it needs to be written out carefully as a number of complications arise.

We require a generalized concept of degree. For a vertex x of an edge-r-colored digraph G, the *degree-vector* of x is the $2r$-vector

$$\tau(x) = (\mathrm{in}_1(x), \mathrm{out}_1(x), \mathrm{in}_2(x), \mathrm{out}_2(x), ..., \mathrm{in}_r(x), \mathrm{out}_r(x))$$

where $\mathrm{in}_j(x)$ and $\mathrm{out}_j(x)$ denote, respectively, the indegree and outdegree of vertex x in the spanning subgraph of G determined by the edges of color

j, $1 \leqslant j \leqslant r$. (Occasionally, we use $\text{in}_j(G, x)$, $\text{out}_j(G, x)$, and $\tau_G(x)$, when necessary to avoid confusion.) We denote by $\alpha(G)$ the greatest common divisor of the integers t such that the $2r$-vector $(t, t, ..., t)$ is an integral linear combination of the vectors $\tau(x)$ as x ranges over the vertex set $V(G)$ of G. Equivalently, $\alpha(G)$ is the least positive integer t_0 so that $(t_0, t_0, ..., t_0)$ is an integral linear combination of the vectors $\tau(x)$.

Note that if a G-decomposition of $K_n^{(r)}$ exists, then the set of $2r(n-1)$ edges incident with some fixed vertex of $K_n^{(r)}$ are partitioned by the isomorphic copies of G so that the $2r$-vector $(n-1, n-1, ..., n-1)$ is a nonnegative integral combination of $\tau(x)$, $x \in V(G)$. Thus $\alpha(G)$ divides $n-1$ whenever a decomposition exists.

THEOREM 1.1. *Let G be a simple edge-r-colored digraph with m edges of each of r different colors. There exists a constant $n_0 = n_0(G)$ such that the complete edge-r-colored digraph $K_n^{(r)}$ admits a G-decomposition for all integers $n \geqslant n_0$ that satisfy the following conditions*:

$$n(n-1) \equiv 0 \pmod{m},$$

$$n-1 \equiv 0 \pmod{\alpha(G)}.$$

Now let $\mathscr{G}$ be a family of simple edge-r-colored digraphs. Let $\alpha(\mathscr{G})$ denote the greatest common divisor of the integers t such that the constant vector $(t, t, ..., t)$ is an integral linear combination of the degree-vectors $\tau(x)$ as x ranges over *all* vertices of *all* graphs in $\mathscr{G}$. For each G, let $\mu(G) = (m_1, m_2, ..., m_r)$ where m_i is the number of edges of color i in G. We denote by $\beta(\mathscr{G})$ the greatest common divisor of the integers m such that $(m, m, ..., m)$ is an integral linear combination of the vectors $\mu(G)$, $G \in \mathscr{G}$. Equivalently, $\beta(\mathscr{G})$, if not zero, is the least positive integer m_0 so that $(m_0, m_0, ..., m_0)$ is an integral linear combination of the vectors $\mu(G)$. So for a family $\mathscr{G}$ consisting of a single graph G, $\beta(\{G\})$ (or simply $\beta(G)$) is m if G has the same number m of edges of each color and is zero otherwise.

If $K_n^{(r)}$ admits a $\mathscr{G}$-decomposition, then the constant vector $n(n-1) \cdot (1, 1, ..., 1)$ is certainly a nonnegative integral linear combination of the vectors $\mu(G)$, $G \in \mathscr{G}$, so that $\beta(\mathscr{G})$ divides $n(n-1)$.

We remark that $\alpha(\mathscr{G})$ is always a divisor of $\beta(\mathscr{G})$. To see this, note that the sum of $\tau(x)$ over all vertices x of a graph G is $(m_1, m_1, m_2, m_2, ..., m_r, m_r)$, where $(m_1, m_2, ..., m_r) = \mu(G)$. Since the constant vector of length r with all entries $\beta(\mathscr{G})$ is an integral linear combination of the vectors $\mu(\mathscr{G})$, it is clear that the constant vector of length $2r$ with all entries $\beta(\mathscr{G})$ is an integral linear combination of the vectors $\tau(x)$; hence $\beta(\mathscr{G})$ is a multiple of $\alpha(\mathscr{G})$.

We say that a graph G_0 is *useless* in $\mathcal{G}$ when in any nonnegative rational linear relation

$$(1, 1, ..., 1) = \sum_{G \in \mathcal{G}} c_G \mu(G) \qquad \text{with all} \quad c_G \geqslant 0, \tag{1.1}$$

we have $c_{G_0} = 0$. Such graphs cannot occur in any $\mathcal{G}$-decomposition of a complete graph. For example, if $\mathcal{G} = \{G_1, G_2, G_3\}$, $\mu(G_1) = (5, 5)$, $\mu(G_2) = (2, 3)$, and $\mu(G_3) = (1, 2)$, then G_2 and G_3 are useless in $\mathcal{G}$. Finally, we say that $\mathcal{G}$ is *admissible* when there exists a nonnegative rational linear relation (1.1) and when no member of $\mathcal{G}$ is useless in $\mathcal{G}$.

THEOREM 1.2. *Let $\mathcal{G}$ be an admissible family of simple edge-r-colored digraphs. Then there exists a constant $n_0 = n_0(\mathcal{G})$ such that $\mathcal{G}$-decompositions of $K_n^{(r)}$ exist for all $n \geqslant n_0$ satisfying the congruences*

$$\begin{aligned} n(n-1) &\equiv 0 \pmod{\beta(\mathcal{G})}, \\ n-1 &\equiv 0 \pmod{\alpha(\mathcal{G})}. \end{aligned} \tag{1.2}$$

Since Theorem 1.1 is a special case of Theorem 1.2, we will only consider the latter in the sections below.

The question of whether a graph G_0 is useless in $\mathcal{G}$ is a linear programming problem, i.e. maximize c_{G_0} subject to (1.1) and see whether it can be made strictly positive. (By duality, G_0 is useless in $\mathcal{G}$ if and only if there is a vector $\mathbf{v}$ so that $\langle \mathbf{v}, (1, 1, ..., 1) \rangle \geqslant 0$, $\langle \mathbf{v}, \mu(G) \rangle \leqslant 0$ for all $G \in \mathcal{G}$, and $\langle \mathbf{v}, \mu(G_0) \rangle < 0$.) The evaluation of $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$ can be done efficiently by computations involving the Smith form of the matrices whose entries include the vectors $\tau(x)$ or the vectors $\mu(G)$.

For later use, we point out at this time if $\mathcal{G}$ is admissible, then there exists a positive rational linear relation

$$(1, 1, ..., 1) = \sum_{G \in \mathcal{G}} c_G \mu(G) \qquad \text{with all} \quad c_G > 0 \tag{1.3}$$

(the converse is obviously true also), since if no members of $\mathcal{G}$ are useless and we take a solution of (1.1) with $c_{G_0} > 0$ for each $G_0 \in \mathcal{G}$, add them, and divide by $|\mathcal{G}|$, we get a relation as in (1.3).

Finally, it is desirable to have a version of these theorems for "$\lambda > 1$." We will state and prove in Section 13 the corresponding asymptotic existence result concerning decompositions of the complete digraph $K_n^{[\lambda_1, \lambda_2, ..., \lambda_r]}$ on n vertices where there are exactly λ_i edges of color i joining x to y for any ordered pair (x, y) of vertices. It turns out that Theorem 1.2 is sufficiently general that Theorem 13.1 can be derived from it by elementary means.

2. SOME EXAMPLES

In this section, we give several examples of relations between certain known structures and certain decompositions of complete directed graphs using graphs with more than one color. There are a number of examples of decompositions of K_v using graphs with just one color. For example, a Steiner triple system on v points, a $(v, 3, 1)$-BIBD, is equivalent to a decomposition of K_v into triangles. Similarly, balanced incomplete block designs, $(v, k, 1)$-BIBDs, give decompositions of K_v into K_k. Decompositions of K_v into cycles and other small graphs have also been investigated, and surveys of these results can be found in [20], [21], and [7]. There have also been a few investigations of decompositions using small graphs with more than one color, see [13], [11], and [10].

Example 2.1: Nested Triple Systems. A Steiner triple system, a $(v, 3, 1)$-BIBD, is called *nested* when there is a $(v, 4, 2)$-BIBD on the same set of v points so that each block of the triple system is contained in a block of the latter design. The existence of a nested Steiner triple system on v points is equivalent to a G_1-decomposition of $K_v^{(2)}$ where G_1 is the edge-2-colored graph shown in Fig. 2.1. Each unordered pair of vertices of $K_v^{(2)}$ appears in the vertex set of exactly two subgraphs of a G_1-decomposition, in one as a solid edge and in the other as a dashed edge; the vertex sets of the subgraphs provide the blocks of a $(v, 4, 2)$-BIBD and the vertex sets of their solid edge triangle-subgraphs are the triples of a $(v, 3, 1)$-BIBD.

D. R. Stinson has shown that nested triple systems exist for all values of $v \equiv 1 \pmod 6$ in [30]. Theorem 1.1 with $m = 6$ would imply the existence for sufficiently large $v \equiv 1 \pmod 6$ after a simple calculation. If the solid edges of G_1 represent color 1 and the dashed edges color 2, we have

$$\tau(x) = \tau(y) = \tau(z) = (2, 2, 1, 1), \quad \tau(c) = (0, 0, 3, 3)$$

and, we find $\alpha(G_1) = 6$. (Recall that we think of an undirected edge as a pair of opposite directed edges.)

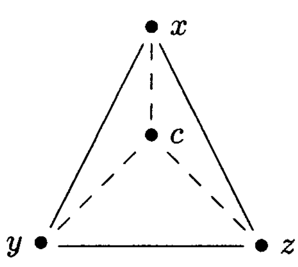


FIGURE 2.1

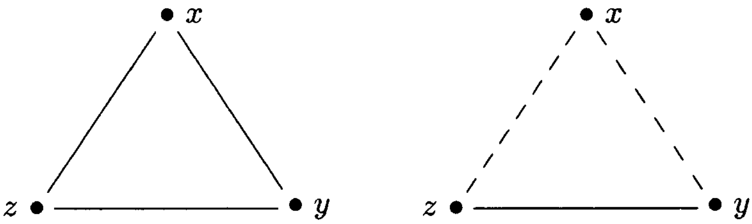


FIGURE 2.2

EXAMPLE 2.2: Reverse Triple Systems. A Steiner triple system is said to be a *reverse triple system* with respect to a particular point p when the permutation ϕ_p that fixes p but simultaneously interchanges all pairs of points x, y for which $\{p, x, y\}$ is a triple of the system is, in fact, an automorphism of the triple system. Necessary and sufficient conditions for the existence of reverse triple systems on v points are $v \equiv 1, 3, 9, 19 \pmod{24}$, [28], [32]. We claim that the existence of a reverse triple system on $2n+1$ points is equivalent to a $\mathscr{G}$-decomposition of $K_n^{(2)}$ where $\mathscr{G}$ consists of the two edge-2-colored graphs G_2, G_3 illustrated in Fig. 2.2.

We construct a reverse triple system from such a decomposition of $K_n^{(2)}$ as follows. Let X consist of a special point p and two points x_1, x_2 for every vertex x of $K_n^{(2)}$. The triples are to include $\{p, x_1, x_2\}$ for all x in the vertex set. For each graph F in the decomposition isomorphic to G_2, take two triples $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$. For each graph F in the decomposition isomorphic to G_3, take two triples $\{x_1, y_2, z_2\}$ and $\{x_2, y_1, z_1\}$. Conversely, a $\mathscr{G}$-decomposition of $K_n^{(2)}$ arises from a reverse triple system of order $2n+1$ when the points other than the special point p are partitioned into two n-sets that are interchanged by ϕ_p.

The values of $\tau(x)$ as x ranges over the vertices of the graphs in $\mathscr{G}$ are $(2, 2, 0, 0)$, $(0, 0, 2, 2)$, and $(1, 1, 1, 1)$, so $\alpha(\mathscr{G}) = 1$. The values of $\mu(G)$ for the two graphs G in $\mathscr{G}$ are $(6, 0)$ and $(2, 4)$, so $\beta(\mathscr{G}) = 12$. The condition $n(n-1) \equiv 0 \pmod{12}$ is equivalent to $2n+1 \equiv 1, 3, 9, 19 \pmod{24}$. In Section 12, we consider a generalization of reverse triple systems.

EXAMPLE 2.3: Complementing 3-Paths. Granville, Moisiadis, and Rees [15] considered decompositions of the complete graph K_v into paths of length 3 such that complementing each path (within the complete graph on its 4 vertices) yields another decomposition of K_v into paths of length 3. They prove that these exist if and only if $v \equiv 1 \pmod 3$. That they exist for all sufficiently large $v \equiv 1 \pmod 3$ is a consequence of Theorem 1.1 in the case $r = 2$ applied to the edge-2-colored graph G_4 shown in Fig. 2.3. Here $m = 6$, and there are two types of degree-vectors, $\tau(x) = (2, 2, 1, 1)$ and $\tau(y) = (1, 1, 2, 2)$. So $\alpha(G_4) = 3$.

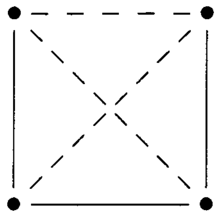


FIGURE 2.3

EXAMPLE 2.4: Skew Room Squares. A *Room square* of side n, $RS(n)$, defined on a set S of $n+1$ elements (symbols), is an n by n array R with the following properties:

(1) Every cell of R is either empty or contains an unordered pair of distinct elements from S.

(2) Every element of S occurs once in each row and once in each column of R.

(3) Every unordered pair of distinct elements from S occurs precisely once in R.

Note that n must be odd. A Room square of side n is in *standard form* (with respect to the element ∞) if cell (i, i) contains the pair $\{\infty, i\}$ for each i. A $RS(n)$ in standard form is called *skew* if for every pair of cells (i, j) and (j, i) with $i \neq j$ precisely one is filled. The existence of skew Room squares was established by D. R. Stinson in [31]: There exists a skew $RS(n)$ for $n \equiv 1 \pmod 2$, $n \neq 3, 5$.

Let $r=4$ and let G_5 denote the edge-4-colored digraph in Fig. 2.4. Here the colors are represented by solid, dashed, dotted, and wavy lines. Edges of the solid and wavy colors appear in opposite pairs while edges of the dotted and dashed colors have no opposites in G_5. We claim that G_5-decompositions of $K_n^{(4)}$ are equivalent to skew Room squares of side n; we will explain how to get the skew Room square from such a decomposition $\mathscr{F}$.

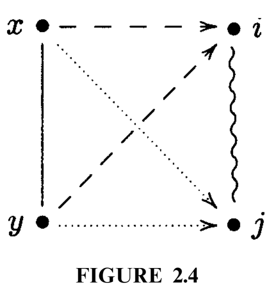


FIGURE 2.4

(The construction can be reversed to get the G_5-decomposition from skew Room squares.)

The rows of an n by n array R are indexed by the vertices of $K_n^{(4)}$. The symbols for the Room square are the vertices of $K_n^{(4)}$ together with a special symbol ∞. For every $x \in V(K_n^{(4)})$, we put the unordered pair $\{\infty, x\}$ in row x and column x. For every $F \in \mathscr{F}$ with vertices x, y, i, j as indicated in Fig. 2.4, put the unordered pair $\{x, y\}$ in the cell in row i and column j. We verify that R is a skew Room square of side n. Every unordered pair of symbols $\{x, y\}$ occurs in a (unique) cell because the solid edge joining x and y in $K_n^{(4)}$ is in exactly one G_5-subgraph of $\mathscr{F}$. A symbol x occurs exactly once in row (column) ℓ since the dashed (respectively, dotted) edge directed from x to ℓ in $K_n^{(4)}$ is in exactly one G_5-subgraph of $\mathscr{F}$, unless $x = \ell$ when x is in the diagonal cell. Given distinct ℓ, ℓ', the (undirected) wavy edge joining ℓ and ℓ' is in a unique copy of G_5; if the dashed edges are incident with ℓ, say, then the cell (ℓ, ℓ') contains an unordered pair while cell (ℓ', ℓ) is empty.

We have four degree-vectors:

$$\tau(x) = (1, 1, 0, 1, 0, 1, 0, 0), \qquad \tau(y) = (1, 1, 0, 1, 0, 1, 0, 0),$$

$$\tau(i) = (0, 0, 2, 0, 0, 0, 1, 1), \qquad \tau(j) = (0, 0, 0, 0, 2, 0, 1, 1).$$

So $\alpha(G_5) = 2$. Applying Theorem 1.1 with $m = 2$, we have the existence of skew Room squares for all sufficiently large $v \equiv 1 \pmod 2$. In Section 11, we generalize this example to skew Room d-cubes.

EXAMPLE 2.5: Self-Orthogonal Latin Squares. A Latin square L with row, column, and symbol set $\{1, 2, ..., n\}$, say, is *self-orthogonal* when L is orthogonal to its transpose. Brayton, Coppersmith, and Hoffman [8] have shown that self-orthogonal Latin squares exist for all positive integers n except for $n = 2, 3, 6$. The symbols $L(i, i)$ on the diagonal are necessarily distinct and we may assume $L(i, i) = i$ for all $i = 1, 2, ..., n$. If for each unordered pair $\{i, j\}$ of distinct symbols we form an edge-4-colored digraph G_6 with vertices i, j, $L(i, j)$, $L(j, i)$ as shown in Fig. 2.5, we obtain a G_6-decomposition of $K_n^{(4)}$.

Conversely, a G_6-decomposition of $K_n^{(4)}$ yields a self-orthogonal Latin square. The degree-vectors are

$$\tau(i) = \tau(j) = (1, 1, 0, 1, 0, 1, 0, 0)$$

and

$$\tau(x) = \tau(y) = (0, 0, 1, 0, 1, 0, 1, 1).$$

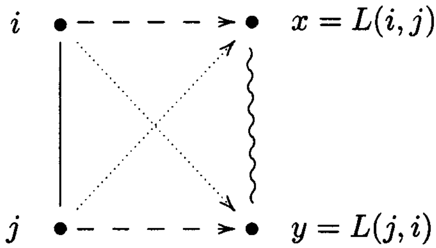


FIGURE 2.5

So $\alpha(G_6) = 1$ and $m = 2$, and applying Theorem 1.1 gives us the asymptotic existence of a pair of self orthogonal Latin squares.

Example 2.6: Orthogonal Idempotent Latin Squares. It is well known that there exists a pair of idempotent orthogonal Latin squares for all n except $n = 2, 3, 6$, [1]. The asymptotic existence follows from applying Theorem 1.1 with $m = 1$ and G_7 as described below. A pair L_1, L_2 of orthogonal Latin squares of order n, both of which are idempotent quasigroups (i.e. they have the same row, column, and symbol sets and $L_i(x, x) = x$ for each symbol x), corresponds to a G_7-decomposition of $K_n^{(6)}$ where G_7 is the digraph with four vertices x_1, x_2, x_3, x_4 and six edges of six different colors, one edge directed from x_i to x_j for $1 \leqslant i < j \leqslant 4$. (Or we could use 12 edges and 12 colors, if we also include edges directed from x_j to x_i.) It is easy to see that $\alpha(G_7) = 1$.

Example 2.7: Whist Tournaments. A problem that in part originally motivated this work is the Whist Tournament Problem of E. H. Moore [24]; see also [6].

A whist tournament $Wh(v)$ for $v = 4n$ (or $4n + 1$) players is a schedule of games each involving two players opposing two others, such that

(1) the games are arranged into $4n - 1$ (or $4n + 1$) rounds of n games each;

(2) each player plays in exactly one game in each of the rounds (or in each of $4n$ rounds and sits out in the remaining round);

(3) each player partners every other player exactly once;

(4) each player opposes every other player exactly twice.

$Wh(v)$ for $v = 4n$, $4n + 1$ exist for all $n \geqslant 1$, [2], [19]. If we relax the condition of partitioning the games into rounds and consider just Whist tables for v players, then the existence of Whist tables for v players is equivalent to a G_8-decomposition of $K_v^{[2,1]}$ where G_8 is the left graph in Fig. 2.6. It

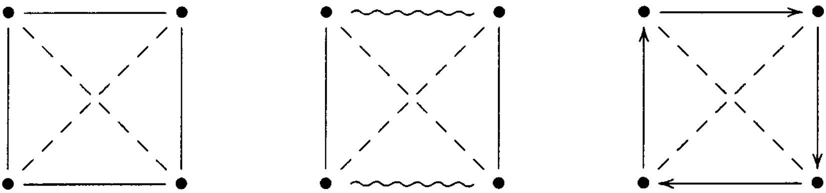


FIGURE 2.6

is easy to check that $\alpha(G_8; 2, 1) = 1$ and $\beta(G_8; 2, 1) = 4$ (with the notation defined in Section 13). So by Theorem 13.1, whist tables exist for all sufficiently large $v \equiv 0, 1 \pmod 4$.

Let G'_8 be the graph on 5 vertices and 3 colors constructed by adding a new vertex x to G_8 and an edge directed from each of the vertices of G_8 to x in the new color. Then the existence of a $Wh(4n+1)$ is equivalent to a G'_8-decomposition of $K_{4n+1}^{[2,1,1]}$. Since $\alpha(G'_8; 2, 1, 1) = \beta(G'_8; 2, 1, 1) = 4$, by Theorem 13.1, $Wh(v)$ exist for all sufficiently large $v \equiv 1 \pmod 4$.

A directed whist tournament $DWh(v)$ is a $Wh(v)$ with condition (4) replaced by: each player has every other player once as an opponent on the right and once on the left. A triple whist tournament $TWh(v)$ is a $Wh(v)$ with condition (4) replaced by: each player has every other player once as an opponent of the first kind and once as an opponent of the second kind. $DWh(v)$ and $TWh(v)$ are known to exist for $v = 4n, 4n+1$ with just a small number of possible exceptions; see [23], [3], and [4].

If we again relax the condition of the rounds, then triple whist tables and directed whist tables are equivalent respectively to G_9-decompositions of $K_v^{(2)}$ and G_{10}-decompositions of $K_v^{(3)}$, where G_9 is the graph in the middle of Fig. 2.6 and G_{10} is the graph on the right of Fig. 2.6. In both cases, $\alpha = 1$ and $\beta = m = 4$. Using Theorem 1.1, we get the existence of directed whist tables and triple whist tables for sufficiently large $v \equiv 0, 1 \pmod 4$.

If we add a new color and a fifth vertex as described above for whist tournaments, then the existence of $TWh(4n+1)$ and the existence of $DWh(4n+1)$ are equivalent respectively to a G'_9-decomposition of $K_{4n+1}^{(3)}$ and a G'_{10}-decomposition of $K_{4n+1}^{(4)}$. In each case, $\alpha = m = 4$ and using Theorem 1.1, we have that $DWh(v)$ and $TWh(v)$ exist for sufficiently large $v \equiv 1 \pmod 4$.

Example 2.8: Steiner Pentagon Systems. Let $r = 2$ and let G_{11} denote the edge-colored digraph in Fig. 2.7. A G_{11}-decomposition of $K_n^{(2)}$ is called a *Steiner pentagon system* in [22], where it is shown that such systems exist for all positive integers $n \equiv 1$ or $5 \pmod{10}$ except $n = 15$. Theorem 1.1 would prove this for sufficiently large $n \equiv 1$ or $5 \pmod{10}$, since $\alpha = 2$ and $m = 10$.

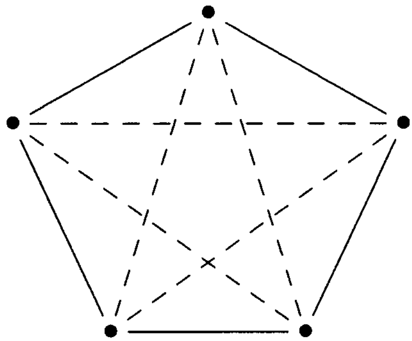


FIGURE 2.7

EXAMPLE 2.9: Kirkman Designs. A Kirkman design of order v is a resolvable $(v, 3, 1)$-BIBD. In [26], Ray-Chaudhuri and Wilson prove that a resolvable $(v, 3, 1)$-BIBD exists if and only if $v \equiv 3 \pmod 6$. The existence of Kirkman designs can be established asymptotically using Theorem 1.2. Let $v = 2n + 1$. The existence of a Kirkman design on $2n + 1$ points is equivalent to a $\mathcal{G}$-decomposition of $K_n^{(5)}$ where $\mathcal{G}$ is the family of four edge-5-colored digraphs displayed in Fig. 2.8. The more general case of resolvable $(v, k, 1)$-BIBDs is described in detail in Section 10. We just note here that, in this case, $\alpha(\mathcal{G}) = 3$ and $\beta(\mathcal{G}) = 6$.

EXAMPLE 2.10: Resolvable Reverse Triple Systems. If there is a Steiner triple system on v points that is both resolvable and reverse, then $v \equiv 3$ or $9 \pmod{24}$; cf. Examples 2.2. and 2.9. Such triple systems with $v = 2n + 1$ with the property that ϕ_p preserves each parallel class of a resolution can be seen to correspond to $\mathcal{G}$-decompositions of $K_n^{(3)}$ where $\mathcal{G}$ consists of the graphs G_{12} and G_{13} in Fig. 2.9. We have $\alpha(\mathcal{G}) = 3$ and $\beta(\mathcal{G}) = 12$.

The construction of a triple system from a decomposition is similar to that described in Example 2.2 and we use the notation introduced there. The parallel classes of triples will be called $\mathcal{A}_w$, $w \in V(K_n^{(3)})$. For each vertex w, $\mathcal{A}_w$ is to contain $\{p, w_1, w_2\}$. For each graph F in the decomposition isomorphic to G_{12}, take two triples $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ and put

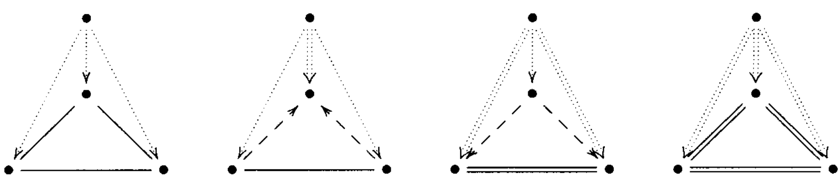


FIGURE 2.8

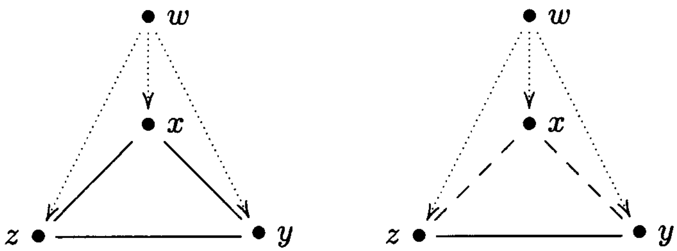


FIGURE 2.9

them in the parallel class $\mathscr{A}_w$. For each graph F in the decomposition isomorphic to G_{13}, take two triples $\{x_1, y_2, z_2\}$ and $\{x_2, y_1, z_1\}$ and put them in the parallel class $\mathscr{A}_w$.

Since there is a $\{G_{12}, G_{13}\}$-decomposition of $K_4^{(3)}$, with one graph isomorphic to G_{12} and three graphs isomorphic to G_{13}, and since there exist $(n, 4, 1)$-BIBD's for all $n \equiv 1$ or 4 (mod 12), we can obtain $\{G_{12}, G_{13}\}$-decompositions of $K_n^{(3)}$ for these values of n (see the beginning of Section 3), and thus resolvable reverse triple systems exist for all $v \equiv 3$ or 9 (mod 24). This would follow only for sufficiently large v of this form by Theorem 1.2.

3. SUMMARY OF THE PROOF OF THE MAIN THEOREM

Fix r and an admissible family $\mathscr{G}$ of edge-r-colored digraphs.

A *pairwise balanced design* (PBD) is a decomposition of a complete graph K_v into complete subgraphs; equivalently, a PBD consists of a set X and a family $\mathscr{A}$ of subsets (blocks) of X with the property that every 2-element subset of X is contained in a unique block $A \in \mathscr{A}$. By taking the union of $\mathscr{G}$-decompositions of complete edge-r-colored graphs on each block of a PBD, $(X, \mathscr{A})$, it is readily seen that if $K_{|A|}^{(r)}$ admits a $\mathscr{G}$-decomposition for every $A \in \mathscr{A}$, then $K_{|X|}^{(r)}$ admits a $\mathscr{G}$-decomposition. That is, in the terminology of [33], the set of integers

$$S(\mathscr{G}) = \{n : K_n^{(r)} \text{ admits a } \mathscr{G}\text{-decomposition}\}$$

is PBD-closed. The main result of [33] asserts that a PBD-closed set S that contains integers greater than 1 is eventually periodic with some positive period $\beta(S)$. This means that

$$n \in S \Rightarrow n + t\beta(S) \in S \text{ for all sufficiently large } t.$$

Now the hypothesis that $\mathscr{G}$ is admissible implies that there exists a positive integer m such that the constant vector $(m, m, ..., m)$ of length r is a nonnegative integral linear combination of the $\mu(G)$'s for $G \in \mathscr{G}$. This in turn means that we can form an edge-r-colored graph G_0 that is the disjoint union (or any edge-disjoint union) of graphs isomorphic to members of $\mathcal{G}$ and such that G_0 has exactly m edges of each color. We can further assume that m is even, by taking the disjoint union of two copies if necessary. By Theorem 4.2, there are (infinitely many) values of n for which $K_n^{(r)}$ admits a G_0-decomposition, and hence a $\mathcal{G}$-decomposition. Thus we have the existence of an eventual period $\beta_0 \neq 0$ for $S(\mathcal{G})$.

A multiple of an eventual period is also an eventual period of $S(\mathcal{G})$, so we may assume β_0 is divisible by $\beta(\mathcal{G})$. To complete the proof of Theorem 1.2, it will suffice to show for any one of the finitely many residue classes n modulo β_0 that satisfy

$$n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})} \qquad \text{and}$$

$$n-1 \equiv 0 \pmod{\alpha(\mathcal{G})},$$

that there exists an element n_0 of $S(\mathcal{G})$ such that $n_0 \equiv n \pmod{\beta_0}$. This is the content of Theorem 6.2.

In order to prove Theorem 6.2, we first show that conditions (1.2) are sufficient for the existence of a solution in integers to a certain system of linear equations. This is done in Section 5. In summary, the proof of Theorem 1.2 will be completed by the material in the next three sections.

4. EXAMPLES FROM CYCLOTOMY IN FINITE FIELDS

Let q be a prime power, $q \equiv 1 \pmod{m}$. The cyclic multiplicative subgroup $GF(q)^\times$ of nonzero elements in the field of q elements has a unique subgroup C_0 of index m. The multiplicative cosets $C_0, C_1, ..., C_{m-1}$ of C_0 are called the cyclotomic classes of index m and may be indexed so that $a \in C_i$ and $b \in C_j$ imply $ab \in C_{i+j}$ where the subscripts are read modulo m; if ω is a primitive element for $GF(q)$, we may take $C_i = \{\omega^t : t \equiv i \pmod{m}\}$.

The following lemma is proved in [35] (also see the comments in [37], [9], [18] for other proofs).

LEMMA 4.1. *Let m and k be given. There exists a constant $q_0 = q_0(m, k)$ such that for all prime powers $q \equiv 1 \pmod{m}$ with $q \geqslant q_0$, and for all choices of $k(k-1)/2$ cyclotomic classes $C(s, t)$, $1 \leqslant s < t \leqslant k$ of index m, there exists a k-tuple $(a_1, a_2, ..., a_k)$ of elements of $GF(q)$ such that $a_t - a_s \in C(s, t)$ for all s, t, $1 \leqslant s < t \leqslant k$.*

THEOREM 4.2. *Let G_0 be an edge-r-colored digraph with m edges of each of r colors. Further assume that m is even. Then $K_q^{(r)}$ admits a G_0-decomposition* *for every prime power* $q \equiv m+1 \pmod{2m}$ *with* $q \geqslant q_0(m, k)$, *where k is the number of vertices of* G_0.

Proof. Let Γ denote the group of $q(q-1)/m$ permutations

$$\{x \mapsto ax+b : a \in C_0, b \in GF(q)\}$$

of $GF(q)$. For each $i = 0, 1, ..., m-1$, Γ is sharply transitive on each orbit

$$\{(x, y) : y-x \in C_i\}$$

of Γ on the set of ordered pairs of distinct field elements. Suppose that there is an injective mapping ϕ: $V(G_0) \to GF(q)$ such that for each color i, the m field elements

$$\phi(\text{head of } e) - \phi(\text{tail of } e), \qquad \text{as } e \text{ ranges over the edges of color } i \text{ in } G_0,$$

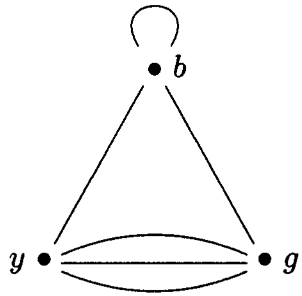
form a system of representatives for the cyclotomic classes $C_0, C_1, ..., C_{m-1}$ of index m. We will call this *Condition R*. When we apply the permutations in Γ to the vertices of the image of G_0, we obtain a decomposition of $K_q^{(r)}$ into $q(q-1)/m$ subgraphs isomorphic to G_0.

Lemma 4.1 asserts that, provided q is sufficiently large, we can map vertices of G_0 to field elements so that the difference $\phi(x) - \phi(y)$ $(x, y \in V(G_0))$ in one direction is in whichever cyclotomic class C_i we may wish, but then the difference $\phi(y) - \phi(x)$ in the other direction will belong to the cyclotomic class $C_{i+\ell}$ where ℓ is such that $-1 \in C_\ell$. If q is a prime power with $q \equiv m+1 \pmod{2m}$ and m even, $-1 \in C_{m/2}$. (One way to see this is to note $-1 = \omega^{(q-1)/2}$ and $(q-1)/2 \equiv m/2 \pmod{m}$. Thus if $a \in C_i$, then $-a \in C_{i+m/2}$.

It is clear, from Lemma 4.1, that there exists an injection ϕ satisfying Condition R if the edges of G_0 have no opposites, and it is also easy to see that Condition R can be satisfied if the colors are "paired" in the sense of Section 1. To handle the general situation, we proceed as follows.

We partition the mr edges of G_0 into $h = mr/2$ pairs of edges $A_1, A_2, ..., A_h$ in such a way that every pair $\{e_1, e_2\}$ of opposite edges is included as one of the sets A_i. Edges of G_0 that have no opposite mates may be paired arbitrarily.

We now introduce an (undirected, non-edge-colored, not necessarily simple) graph R whose vertices are the r colors and whose edges are the h pairs $A_1, A_2, ..., A_h$. As an edge of R, the pair $A_i = \{e_1, e_2\}$ is to join the vertices of R corresponding to the colors of e_1 and e_2 in G_0. Since G_0 has m edges of each color, R is regular of degree m. For example, if G_0 has 4 edges of colors blue, green, yellow and the 6 pairs of edges are



$\mathcal{F}_1$	$\mathcal{F}_2$
(b_1, b_2)	(b_3, g_1)
(g_2, y_2)	(y_1, b_4)
(y_3, g_3)	(g_4, y_4)

FIGURE 4.1

$$A_1 = \{b_1, b_2\}, \qquad A_2 = \{b_3, g_1\}, \qquad A_3 = \{b_4, y_1\},$$

$$A_4 = \{g_2, y_2\}, \qquad A_5 = \{g_3, y_3\}, \qquad A_6 = \{g_4, y_4\}$$

where the b_i's are blue, etc., then R is as shown on the left of Fig. 4.1.

Since m is even, Petersen's Theorem [25] asserts that R has a 2-factorization: the set of pairs $\{A_1, A_2, ..., A_h\}$ may be partitioned into 2-factors $\mathscr{F}_0, \mathscr{F}_1, ..., \mathscr{F}_{(m/2)-1}$, i.e. each $\mathscr{F}_i$ is the edge set of a union of disjoint polygons in R that cover all vertices of R. The pairs $A_{i_1}, ..., A_{i_r}$ in each 2-factor $\mathscr{F}_i$ can then be directed so that every vertex of R has indegree and outdegree 1 in $\mathscr{F}_i$. We do this for each $i = 0, 1, ..., (m/2) - 1$.

In other words, the h pairs A_j can be *ordered*, each pair A_j having a *primary* edge and a *secondary* edge, say, so that each color appears exactly once on a primary edge and exactly once on a secondary edge of the pairs in each 2-factor $\mathscr{F}_i$. For our example in Fig. 4.1, a 2-factorization is shown on the right with an appropriate ordering of the pairs.

For q sufficiently large, we can choose an injection ϕ so that

$$\phi(\text{head of } e) - \phi(\text{tail of } e) \in C_i$$

for the primary edge e of each pair in $\mathscr{F}_i$, $i = 0, 1, 2, ..., (m/2) - 1$. We can also have

$$\phi(\text{head of } e') - \phi(\text{tail of } e') \in C_{i+m/2}$$

for the secondary edge e' of each pair in $\mathscr{F}_i$, either because e and e' are opposites and we have no choice, or because neither has an opposite and we have complete freedom about which cyclotomic class we want to contain this latter difference. With such a choice of ϕ, Condition R holds and we obtain the required G_0-decomposition of $K_q^{(r)}$. ▮

5. INTEGRAL SOLUTIONS FOR A CERTAIN LINEAR SYSTEM

We show that the conditions (1.2) are sufficient for the existence of a solution in integers to a certain system of linear equations. Here and in later sections, we use the following well known lemma; see e.g. [29].

LEMMA 5.1. *Let M be a rational s by t matrix and* $\mathbf{c}$ *a rational column vector of length s. The equation $M\mathbf{x}=\mathbf{c}$ has an integral solution* $\mathbf{x}$, *a column vector of length t, if and only if*

$$\mathbf{y}M \text{ integral implies } \mathbf{yc} \text{ is an integer}$$

for all rational row vectors $\mathbf{y}$ *of length s.*

We also require two simple lemmas about digraphs.

LEMMA 5.2. *Let X be a finite set, let $\Gamma \subseteq X \times X$, and let F be a mapping from Γ into an abelian group $\mathscr{A}$. There exist mappings $g, h\colon X \to \mathscr{A}$ so that*

$$F(x, y) = g(x) + h(y) \qquad \text{for all} \quad (x, y) \in \Gamma \tag{5.1}$$

if and only if F satisfies the following condition:

$$\begin{aligned} &F(x_0, y_0) + F(x_1, y_1) + \cdots + F(x_{k-2}, y_{k-2}) + F(x_{k-1}, y_{k-1}) \\ &\quad = F(x_0, y_1) + F(x_1, y_2) + \cdots + F(x_{k-2}, y_{k-1}) + F(x_{k-1}, y_0) \end{aligned} \tag{5.2}$$

whenever $x_0, x_1, ..., x_{k-1}$ and $y_0, y_1, ..., y_{k-1}$ are elements of X, not necessarily distinct, so that all indicated pairs (x_i, y_i) and (x_i, y_{i+1}) (subscripts modulo k) are in Γ.

Proof. That (5.1) implies (5.2) is clear.

A mapping f from the edge set of a simple digraph to $\mathscr{A}$ satisfies *Kirchhoff's voltage law* when the "signed sum" of the values $\pm f(e)$ over the edges of any closed path p in the underlying undirected graph is zero; here "signed sum" means we take the term $f(e)$ with a "+" sign when the edge e is traversed by p according to its orientation, and a "−" sign if e is traversed the "wrong" way in p. If f satisfies Kirchhoff's voltage law, it is well known that there is then a function ρ from the vertices of the digraph to $\mathscr{A}$ so that

$$f(e) = \rho(\text{head of } e) - \rho(\text{tail of } e) \tag{5.3}$$

for all edges e.

From X, Γ, and F, we construct a simple bipartite digraph B and a mapping $f\colon E(B) \to \mathscr{A}$ as follows. For each element $z \in X$, take two vertices $\bar{z}$ and $\hat{z}$. So $|V(B)| = 2\,|X|$. For every $(x, y) \in \Gamma$, put an edge e directed from $\bar{x}$ to $\hat{y}$ into B, and define $f(e) = F(x, y)$ for this edge. Condition (5.2) implies Kirchhoff's voltage law: A closed path p of length $2k$ in the underlying undirected graph of B has vertex terms, say,

$$\hat{y}_0, \bar{x}_0, \hat{y}_1, \bar{x}_1, ..., \hat{y}_{k-1}, \bar{x}_{k-1}, \hat{y}_0$$

where

$$(x_0, y_0), (x_0, y_1), (x_1, y_1), (x_1, y_2), ... \in \Gamma.$$

The path p traverses the edges $(\bar{x}_i, \hat{y}_{i+1})$ according to their direction and traverses the edges $(\bar{x}_i, \hat{y}_i)$ "backwards." The equation (5.2) is equivalent to the vanishing of the signed sum of the values of f on the edges of p.

So we may conclude that (5.3) holds for some ρ. When we define $h(z) = \rho(\hat{z})$ and $g(z) = -\rho(\bar{z})$, (5.3) can be written $F(x, y) = g(x) + h(y)$. ∎

LEMMA 5.3. *Let X be a finite set and let F be a mapping from the set of all ordered pairs (x, y) of distinct elements of X to an abelian group $\mathscr{A}$. There exist mappings $g, h\colon X \to \mathscr{A}$ so that*

$$F(x, y) = g(x) + h(y) \qquad \textit{for all distinct} \quad x, y \in X \tag{5.4}$$

if and only if F satisfies the following condition:

$$F(x, y) + F(u, v) = F(x, v) + F(u, y) \tag{5.5}$$

whenever x, y, u, v are distinct elements of X. Moreover (5.5) determines g and h up to constant functions: if $g(x) + h(y) = g_1(x) + h_1(y)$ for all distinct $x, y \in X$, then for some constant ℓ, $g_1(z) = g(z) + \ell$ for all z and $h_1(z) = h(z) - \ell$ for all z.

Proof. We remark that it is possible to derive Lemma 5.3 from Lemma 5.2 by taking Γ to be $(X \times X) \backslash \{(x, x)\colon x \in X\}$ and showing that (5.5) implies (5.2) for this choice of Γ. But it is quicker to prove it directly. (Special cases of Lemma 5.3 were used in [36] and [9].)

It is clear that (5.4) implies (5.5).

Assume (5.5) and let p and q be two distinct elements. Choose rationals $g(p)$ and $h(q)$ so that $F(p, q) = g(p) + h(q)$. Then we have no further choice: For any element $x \neq q$, define $g(x) = F(x, q) - h(q)$, and for any element $y \neq p$, define $h(y) = F(p, y) - g(p)$. Then (5.5) gives

$$\begin{aligned} F(x, y) &= F(p, y) + F(x, q) - F(p, q) \\ &= g(p) + h(y) + g(x) + h(q) - g(p) - h(q) \\ &= g(x) + h(y) \end{aligned}$$

whenever x, y, p, q are distinct. Finally, define $g(q) = F(q, z) - h(z)$ and $h(p) = F(z, p) - g(z)$, for $z \neq p, q$. Then (5.5) ensures that $g(q)$ and $h(p)$ are well defined, and it is straightforward, with these definitions, to verify (5.4) for all choices of x, y with $x \neq y$. (In the case that $|X| < 4$, which we are not particularly interested in, our conclusion can be checked to hold even though (5.5) is vacuous.)

The value of $g(p)$ determines g and h completely so that (5.4) holds; hence the uniqueness up to constant functions. ∎

THEOREM 5.4. *Let $\mathscr{G}$ be an admissible family of simple edge-r-colored digraphs and let $\mathscr{H}$ denote the set of all subgraphs H of $K_n^{(r)}$ that are isomorphic to some member of $\mathscr{G}$. In addition, assume $n \geqslant 2 + |V(G)|$ for all G in $\mathscr{G}$. Then there exists a family $\{a_H : H \in \mathscr{H}\}$ of integers such that*

$$\sum_{H : e \in E(H)} a_H = 1 \qquad \textit{for every edge} \quad e \in E(K_n^{(r)})$$

if and only if

$$n(n-1) \equiv 0 \pmod{\beta(\mathscr{G})},$$
$$n - 1 \equiv 0 \pmod{\alpha(\mathscr{G})}.$$

Proof. Let M be the matrix whose rows are indexed by the edges of $K_n^{(r)}$ and whose columns are indexed by the members of $\mathscr{H}$, and where the entry in row e and column H of M is 1 if $e \in E(H)$ and 0 otherwise. The vector $\mathbf{c}$ is a vector of length $rn(n-1)$, indexed by the edge set of $K_n^{(r)}$, of all 1's. The coordinates of an integral vector $\mathbf{x}$ satisfying $M\mathbf{x} = \mathbf{c}$, if it exists, provide the family $\{a_H : H \in \mathscr{H}\}$ as required.

Lemma 5.1 asserts that the existence of $\mathbf{x}$ will follow if we show that whenever rationals $b(e)$ are assigned to the edges of $K_n^{(r)}$ in such a way that

$$b(H) = \sum_{e \in E(H)} b(e)$$

is an integer for every $H \in \mathscr{H}$, then

$$b(K_n^{(r)}) = \sum_{e \in E(K_n^{(r)})} b(e)$$

is also an integer.

So let rationals $b(e)$ be given such that $b(H)$ is integral for all $H \in \mathscr{H}$. For notational purposes, let $b_i(x, y)$ denote the value $b(e)$ where e is the edge of color i from a vertex x to a distinct vertex y in the graph $K_n^{(r)}$.

For rational numbers a, b, we write $a \equiv b$ to mean that the difference $b - a$ is an integer.

We say that colors c and d, not necessarily distinct, are *linked* when there exists a graph in $\mathscr{G}$ that has vertices x, y such that there is an edge of color c from x to y, and an edge of color d from y to x. We say that a color c *occurs unpaired* when there exists a graph in $\mathscr{G}$ that has vertices x, y such that there is an edge of color c from x to y, but no edge from y to x.

We require the existence of $2r$ rational valued functions γ_c, ξ_c, $c = 1, 2, ..., r$, on $V(K_n^{(r)})$ so that

$$b_c(x, y) \equiv \gamma_c(x) + \xi_c(y) \qquad \text{for all distinct } x, y \tag{5.6}$$

whenever color c occurs unpaired and also

$$b_c(x, y) + b_d(y, x) \equiv \gamma_c(x) + \xi_c(y) + \gamma_d(y) + \xi_d(x) \qquad \text{for all distinct } x, y \tag{5.7}$$

whenever colors c and d are linked.

First, suppose that a color c occurs unpaired and let $G \in \mathscr{G}$ have a pair of vertices p, q so that there is an edge of color c directed from p to q and no edge from q to p. Let x, y, u, v be any four vertices of $K_n^{(r)}$ and let H_1 be an isomorphic copy of G in $K_n^{(r)}$ so that H_1 contains the edge in $K_n^{(r)}$ of color c from x to y and no edge from y to x, such that both $u, v \notin V(H_1)$. Let H_2, H_3, and H_4 be, respectively, the images of H_1 under the permutations (xu), (yv), and $(xu)(yv)$. Now since $b(H_i)$ is integral for $i = 1, 2, 3, 4$, certainly we have

$$b(H_1) + b(H_4) \equiv b(H_2) + b(H_3). \tag{5.8}$$

Each side of this congruence is a sum of many terms $b(e)$ but there is a substantial amount of cancellation. For example, if an edge e joins $z \neq x, y$ in H_1 to x, then $b(e)$ contributes to and only to the sums $b(H_1)$ and $b(H_3)$ and may be cancelled. But if, as another example, e is the edge of color c

in $K_n^{(r)}$ from u to v, then $b(e)=b_c(u, v)$ contributes to and only to the sum $B(H_4)$. A consideration of cases shows that (5.8) reduces to

$$b_c(x, y)+b_c(u, v)\equiv b_c(x, v)+b_c(u, y). \tag{5.9}$$

Let $\mathscr{C}_1$ denote the set of colors c for which (5.9) holds for all choices of distinct vertices x, y, u, v of $K_n^{(r)}$. All colors that occur unpaired are in $\mathscr{C}_1$. If $c\in\mathscr{C}_1$, then $F(x, y)=b_c(x, y)$ satisfies (5.5) and by Lemma 5.3, there exist functions γ_c and ξ_c, which we now choose and fix, so that (5.6) holds. If $c, d\in\mathscr{C}_1$ and c and d also happen to be linked, (5.7) will clearly hold. Let $\mathscr{C}_2$ be the set of colors not in $\mathscr{C}_1$; it remains to define γ_c and ξ_c for $c\in\mathscr{C}_2$.

Suppose colors c and d are linked and that $G\in\mathscr{G}$ has a pair of vertices p, q so that there is an edge of color c directed from p to q and an edge of color d directed from q to p. Let x, y, u, v be any four vertices of $K_n^{(r)}$ and let H_1 be an isomorphic copy of G in $K_n^{(r)}$ so that H_1 contains the edge of color c from x to y and the edge of color d from y to x, and such that both $u, v\notin V(H_1)$. Let H_2, H_3, and H_4 be, respectively, the images of H_1 under the permutations (xu), (yv), and $(xu)(yv)$. Again, we have $b(H_1)+b(H_4)\equiv b(H_2)+b(H_3)$. Again, each side of this congruence is a sum of many terms $b(e)$ but there is a substantial amount of cancellation, and a consideration of cases shows that it reduces to

$$\begin{aligned} &b_c(x, y)+b_c(u, v)+b_d(y, x)+b_d(v, u)\\ &\qquad\equiv b_c(x, v)+b_c(u, y)+b_d(v, x)+b_d(y, u). \end{aligned}\tag{5.10}$$

Now suppose a color d is linked to $c\in\mathscr{C}_1$ (that may or may not occur unpaired). When we subtract (5.9) from (5.10), we discover that d is itself in $\mathscr{C}_1$. Thus no color in $\mathscr{C}_2$ is linked to a color in $\mathscr{C}_1$.

Let Γ be the set of ordered pairs (c, d) of linked colors in $\mathscr{C}_2$. Fix a vertex x_0 of $K_n^{(r)}$ Congruence (5.10) shows that for each $(c, d)\in\Gamma$, the function $F(x, y)=b_c(x, y)+b_d(y, x)$ satisfies (5.5), so Lemma 5.3 asserts there exist functions S_{cd} and T_{cd} from $V(K_n^{(r)})$ to Q/Z so that

$$b_c(x, y)+b_d(y, x)\equiv S_{cd}(x)+T_{cd}(y) \qquad \text{for all distinct } x, y. \tag{5.11}$$

We uniquely determine these functions by requiring that $S_{cd}(x_0)=0$ for all $(c, d)\in\Gamma$.

If $(c, d)\in\Gamma$, then $(d, c)\in\Gamma$. We have

$$S_{cd}(u)+T_{cd}(v)\equiv b_c(u, v)+b_d(v, u)=b_d(v, u)+b_c(u, v)\equiv S_{dc}(v)+T_{dc}(u).$$

By the uniqueness part of Lemma 5.3, the functions T_{cd} and S_{dc} differ by a constant function (values in Q/Z). If $T_{cd}(z) \equiv S_{dc}(z) + C$, we find $C = T_{cd}(x_0)$ by replacing z by x_0; that is,

$$T_{cd}(z) \equiv S_{dc}(z) + T_{cd}(x_0). \tag{5.12}$$

We remark that replacing u by x_0 above gives $T_{cd}(v) = S_{dc}(v) + T_{dc}(x_0)$ for $v \neq x_0$, so we evidently have $C = T_{dc}(x_0)$ also.

Now we will apply Lemma 5.2 where $X = N$, Γ is as above, $\mathscr{A}$ is the set of mappings from the vertex set of $K_n^{(r)}$ into Q/Z, and the mapping from Γ to $\mathscr{A}$ given by $\Phi(c, d) = S_{cd}$. We check the condition (5.2): Consider colors $c_0, ..., c_{k-1}$ and $d_0, ..., d_{k-1}$ so that c_i is linked to d_i and d_{i+1} (subscripts modulo k). We have, clearly,

$$\sum_{i=0}^{k-1} (b_{c_i}(x, y) + b_{d_i}(x, y)) \equiv \sum_{i=0}^{k-1} (b_{c_i}(x, y) + b_{d_{i+1}}(x, y)).$$

By (5.11),

$$\sum_{i=0}^{k-1} (S_{c_i d_i}(x) + T_{c_i d_i}(y)) \equiv \sum_{i=0}^{k-1} (S_{c_i d_{i+1}}(x) + T_{c_i d_{i+1}}(y)).$$

The uniqueness part of Lemma 5.3 asserts that

$$\sum_{i=0}^{k-1} S_{c_i d_i} - \sum_{i=0}^{k-1} S_{c_i d_{i+1}}$$

is a constant function. Since the value of the above at x_0 is 0, this is the 0-function, and (5.2) holds. We note for later reference that, also,

$$\sum_{i=0}^{k-1} T_{c_i d_i} - \sum_{i=0}^{k-1} T_{c_i d_{i+1}}. \tag{5.13}$$

is the 0-function.

Now Lemma 5.2 asserts there exist functions $g, h: X \to \mathscr{A}$ so that (5.1) holds: $\Phi(c, d) = g(c) + h(d)$ for all linked pairs (c, d) of colors. If we write $\hat{\gamma}_c$ for $g(c)$ and $\hat{\xi}_c$ for $h(c)$, then

$$S_{cd}(z) = \hat{\gamma}_c(z) + \hat{\xi}_d(z) \qquad \text{for all } z.$$

Then from (5.12),

$$T_{cd}(z) = \hat{\gamma}_d(z) + \hat{\xi}_c(z) + T_{cd}(x_0) \qquad \text{for all } z.$$

From (5.11),

$$b_c(x, y) + b_d(y, x) \equiv \hat{\gamma}_c(x) + \hat{\xi}_d(y) + \hat{\gamma}_d(y) + \hat{\xi}_c(x) + T_{cd}(x_0) \quad (5.14)$$

for all distinct x, y.

Finally, we claim that there exist rationals g_c and h_c, $c \in N$ so that $T_{cd}(x_0) = g_c + h_d$ for all $(c, d) \in \Gamma$. This follows from Lemma 5.2 and (5.13). We take $\gamma_c(z) = \hat{\gamma}_c(z) - g_c/2$ and $\xi_c(z) = \hat{\xi}_c(z) - h_c/2$, and (5.14) gives the desired claim (5.7).

Let z be a vertex of a graph $G \in \mathcal{G}$. Given vertices x, y of $K_n^{(r)}$, choose an isomorphic copy $H \in \mathcal{H}$ of G so that $x \in V(H)$, $y \notin V(H)$, and such that x corresponds to z under the isomorphism. Let H' be the image of H under the permutation (xy). We have $\tau_H(x) = \tau_{H'}(y) = \tau_G(z)$.

Of course, $b(H) \equiv b(H')$, as both have been assumed to be integers. After cancelling terms $b(e)$ that appear on both sides, we have

$$\sum (b(e) : e \in E(H) \text{ incident with } x)$$
$$\equiv \sum (b(e) : e \in E(H') \text{ incident with } y). \quad (5.15)$$

By construction, the term $b_i(x, a)$ or $b_j(a, x)$ occurs in the sum on the left if and only if the term $b_i(y, a)$ or $b_j(a, y)$ occurs in the sum on the right.

Let A_i denote the set of vertices a of H for which the edge of color i from x to a in $K_n^{(r)}$ is in H, but no edge from a to x is in H (or, equivalently, such that the edge of color i from y to a is in H' but no edge from y to a is in H'). Let B_i denote the set of vertices a of H for which the edge of color i from a to x is in H, but no edge from x to a is in H (or, equivalently, such that the edge of color i from a to y is in H' but no edge from a to y is in H'). Let C_{ij} denote the set of vertices a of H for which the edge of color i from x to a and the edge of color j from a to x are in H (or, equivalently, such that the edge of color i from y to a and the edge of color j from a to y are in H').

The left-hand side of (5.15) is

$$\sum_{i=1}^{r} \sum_{a \in A_i} b_i(x, a) + \sum_{i=1}^{r} \sum_{a \in B_i} b_i(a, x) + \sum_{i,j=1}^{r} \sum_{a \in C_{i,j}} (b_i(x, a) + b_j(a, x)), \quad (5.16)$$

and the right-hand side of (5.15) is the same expression with x replaced by y. For each i, the total number of terms $b_i(x, a)$, for various a, that appear on the left-hand side of (5.15) is $\mathrm{out}_i(z)$, and the total number of terms $b_i(a, x)$, for various a, that appear on either side of (5.15) is $\mathrm{in}_i(z)$; a similar statement hold for y and the right-hand side.

Choose and fix a vertex p of $K_n^{(r)}$ distinct from x, y. By (5.9) and (5.10), we have

$$b_i(x, a) - b_i(x, p) \equiv b_i(y, a) - b_i(y, p), \qquad \text{or}$$

$$b_i(a, x) - b_i(p, x) \equiv b_i(a, y) - b_i(p, y), \qquad \text{or}$$

$$(b_i(x, a) + b_j(a, x)) - (b_i(x, p) + b_j(p, x))$$
$$\equiv (b_i(y, a) + b_j(a, y)) - (b_i(y, p) + b_j(p, y)),$$

in the cases that $a \in A_i$, $a \in B_i$, or $a \in C_{ij}$. The point is that we may replace a value of a in terms $b_i(x, a)$ and $b_j(a, x)$ on the left of (5.15), and simultaneously in terms $b_i(y, a)$ and $b_j(a, y)$ on the right, by the fixed vertex p and preserve the congruence. That is, with the notation of (5.16), the expression on the left of (5.15) is, modulo an integer,

$$\sum_{i=1}^{r} |A_i|\, b_i(x, p) + \sum_{i=1}^{r} |B_i|\, b_i(p, x) + \sum_{i, j=1}^{r} |C_{i, j}|\, (b_i(x, p) + b_j(p, x)),$$

and the expression on the left of (5.15) is the same with x replaced by y. From (5.6) and (5.7), the expression on the right of (5.15) is, modulo an integer,

$$\sum_{i=1}^{r} |A_i|\, (\gamma_i(x) + \xi_i(p)) + \sum_{i=1}^{r} |B_i|\, (\gamma_i(p) + \xi_i(x))$$
$$+ \sum_{i, j=1}^{r} |C_{i, j}|\, (\gamma_i(x) + \xi_i(p) + \gamma_j(p) + \xi_j(x)),$$

and the expression on the right of (5.15) is the same with x replaced by y. The congruence (5.15), after cancelling terms involving p on both sides, reduces to

$$\sum_{i=1}^{r} (\text{out}_i(z)\, \gamma_i(x) + \text{in}_i(z)\, \xi_i(x)) \equiv \sum_{i=1}^{r} (\text{out}_i(z)\, \gamma_i(y) + \text{in}_i(z)\, \xi_i(y)). \qquad (5.17)$$

(We reiterate that H, H', and G are isomorphic graphs with x, y, and z corresponding to one another, so that the color indegrees and outdegrees of these vertices are the same.)

Congruence (5.17) holds for all vertices x, y of $K_n^{(r)}$ and vertices z of any member of $\mathcal{G}$. It can be written

$$\langle \tau(z), \mathbf{u}_x \rangle \equiv \langle \tau(z), \mathbf{u}_y \rangle \qquad (5.18)$$

where the angle brackets denote dot product of vectors and where

$$\mathbf{u}_x = (\xi_1(x), \gamma_1(x), \xi_2(x), \gamma_2(x), ..., \xi_r(x), \gamma_r(x))$$

and $\mathbf{u}_y$ is similarly defined. Since the $2r$-vector $\alpha(\mathscr{G})(1, 1, ..., 1)$ is an integral linear combination of the vectors $\tau(z)$, (5.18) implies

$$\alpha(\mathscr{G})\langle (1, 1, ..., 1), \mathbf{u}_x \rangle \equiv \alpha(\mathscr{G})\langle (1, 1, ..., 1), \mathbf{u}_y \rangle,$$

or

$$\alpha(\mathscr{G}) \sum_{k=1}^{r} (\gamma_i(x) + \xi_i(x)) \equiv \alpha(\mathscr{G}) \sum_{k=1}^{r} (\gamma_i(y) + \xi_i(y)). \tag{5.19}$$

This holds for any two vertices x, y of $K_n^{(r)}$.

Given a graph $G \in \mathscr{G}$, choose a graph $H \in \mathscr{H}$ isomorphic to G. Let p be a vertex of $K_n^{(r)}$. Let $E_i(H)$ denote the set of (x, y) so that there is an edge of color i in H directed from x to y. By hypothesis, $b(H)$ is an integer, and we have

$$b(H) = \sum_{e \in E(H)} b(e) = \sum_{i \in \mathscr{C}_1} \sum_{(x, y) \in E_i(H)} b_i(x, y) + \sum_{i \in \mathscr{C}_2} \sum_{(x, y) \in E_i(H)} b_i(x, y).$$

Every term $b_i(x, y)$ in the second double sum is naturally paired with a term $b_j(y, x)$ for a unique j. We apply (5.6) to the terms in the first double sum and (5.7) to the pairing of terms in the second double sum, and use (5.17) for the second congruence, to find that

$$\begin{aligned} b(H) &\equiv \sum_{v \in V(H)} \sum_{i=1}^{r} (\mathrm{out}_i(v)\, \gamma_i(v) + \mathrm{in}_i(v)\, \xi_i(v)) \\ &\equiv \sum_{v \in V(H)} \sum_{i=1}^{r} (\mathrm{out}_i(v)\, \gamma_i(p) + \mathrm{in}_i(v)\, \xi_i(p)) \\ &\equiv \sum_{i=1}^{r} \left(\left(\sum_{v \in V(H)} \mathrm{out}_i(v) \right) \gamma_i(p) + \left(\sum_{v \in V(H)} \mathrm{in}_i(v) \right) \xi_i(p) \right) \\ &\equiv \sum_{i=1}^{r} m_i(\gamma_i(p) + \xi_i(p)), \end{aligned}$$

where m_i is the number of edges of color i in H (or G).

This can be written as

$$0 \equiv b(H) \equiv \langle \mu(H), \mathbf{v}_p \rangle = \langle \mu(G), \mathbf{v}_p \rangle,$$

where

$$\mathbf{v}_p = (\gamma_1(p) + \xi_1(p), \gamma_2(p) + \xi_2(p), ..., \gamma_r(p) + \xi_r(p)).$$

Since $\langle \mu(G), \mathbf{v}_p \rangle$ is an integer for all G and since $\beta(\mathcal{G})(1, 1, ..., 1)$ is an integral linear combination of the vectors $\mu(G)$, it follows that the dot product of $\beta(\mathcal{G})(1, 1, ..., 1)$ with $\mathbf{v}_p$ is an integer; that is, for any vertex p,

$$\beta(\mathcal{G}) \sum_{i=1}^{r} (\gamma_i(p) + \xi_i(p)) \equiv 0. \tag{5.20}$$

Finally, we will show that $b(K_n^{(r)}) = \sum_e b(e)$ is an integer. We require a permutation π of the r colors so that for each color c, (5.7) holds with $d = \pi(c)$. Our hypothesis that $\mathcal{G}$ is admissible implies the existence of this permutation: If G_0 is a graph as described in Section 3 and a graph R, whose vertices are the r colors, is constructed from G_0 as described in the proof of Theorem 4.2, then any single 2-factor $\mathcal{F}_1$ of R provides such a permutation when oriented and taken as the cycle decomposition of π, since edges in R join two colors that are linked or else two colors that occur unpaired.

(Here is an alternate proof of the existence of such a permutation. Define $\pi(c) = c$ for $c \in \mathcal{C}_1$. For $G \in \mathcal{G}$, let $W(G)$ be the symmetric matrix with rows and columns indexed by $i, j \in \mathcal{C}_2$ and where the (i, j) entry is the number of ordered pairs (u, v) of vertices of G so that there is an edge of color i from u to v and an edge of color j from v to u. Then $W(G)$ times the column vector of all 1's is the column vector $\mu'(G)$ listing the number of edges of colors $i \in \mathcal{C}_2$. If $\sum_{G \in \mathcal{G}} w_G \mu(G)$ is the vector of all 1's, with the w_G's nonnegative, then $W_0 = \sum_{G \in \mathcal{G}} w_G W(G)$ is doubly stochastic and so has term rank $|\mathcal{C}_2|$. This means there is a permutation π of $\mathcal{C}_2$ so that for each $c \in \mathcal{C}_2$ the $(c, \pi(c))$ entry of W_0 is nonzero, which means that the colors c and $\pi(c)$ are linked in some graph $G \in \mathcal{G}$.)

We continue. The first inner sum below is to be extended over the $n(n-1)$ ordered pairs (x, y) of distinct vertices of $K_n^{(r)}$. We think of the vertices as being linearly ordered below only for notational convenience.

$$\begin{aligned} b(K_n^{(r)}) &= \sum_{i=1}^{r} \sum_{(x, y)} b_i(x, y) = \sum_{i=1}^{r} \sum_{x<y} (b_i(x, y) + b_i(y, x)) \\ &= \sum_{i=1}^{r} \sum_{x<y} (b_i(x, y) + b_{\pi(i)}(y, x)) \\ &\equiv \sum_{i=0}^{r} \sum_{x<y} (\gamma_i(x) + \xi_i(y) + \gamma_{\pi(i)}(y) + \xi_{\pi(i)}(x)) \\ &= \sum_{i=0}^{r} \sum_{(x, y)} (\gamma_i(x) + \xi_i(y)) = (n-1) \sum_{i=0}^{r} \sum_{x} (\gamma_i(x) + \xi_i(x)). \end{aligned}$$

Pick a vertex p. Since $n-1\equiv 0 \pmod{\alpha(\mathscr{G})}$, (5.19) gives

$$b(K_n^{(r)})\equiv(n-1)\sum_{i=0}^{r}\sum_{x}(\gamma_i(x)+\xi_i(x))\equiv n(n-1)\sum_{i=0}^{r}(\gamma_i(p)+\xi_i(p)).$$

Since $n(n-1)\equiv 0$ (mod $\beta(\mathscr{G})$, (5.20) immediately gives $b(K_n^{(r)})\equiv 0$.

This concludes the proof of the existence of the family $\{a_H: H\in\mathscr{H}\}$ of integers with the required property.

The conditions of Theorem 5.4 can be seen to be necessary as follows. Assume the existence of the family $\{a_H: H\in\mathscr{H}\}$ of integers. For each color i,

$$n(n-1)=\sum_{e \text{ of color } i} 1=\sum_{e \text{ of color } i}\left(\sum_{H:\, e\in E(H)} a_H\right)=\sum_H a_H m_i(H),$$

where the first sum is extended over all edges of $K_n^{(r)}$ of color i, and $m_i(H)$ is the number of edges of color i in H. Then

$$n(n-1)(1, 1, ..., 1)=\sum_H a_H \mu(H),$$

and so $n(n-1)\equiv 0 \pmod{\beta(\mathscr{G})}$.

If $E_i^{\pm}(x)$ is the set of edges of color i that leave (superscript $-$) or enter (superscript $+$) a vertex x of $K_n^{(r)}$,

$$n-1=\sum_{e\in E_i^{\pm}(x)} 1=\sum_{e\in E_i^{\pm}(x)}\left(\sum_{H:\, e\in E(H)} a_H\right)$$

$$=\sum_H a_H \operatorname{out}_i(H, x) \quad \text{or} \quad \sum_H a_H \operatorname{in}_i(H, x).$$

It follows that $(n-1)(1, 1, ..., 1)$, of length $2r$, is an integral linear combination of the vectors $\tau(x)$ as x ranges over vertices of graphs in $\mathscr{G}$; whence $n-1\equiv 0 \pmod{\alpha(\mathscr{G})}$. ∎

6. A LINEAR ALGEBRAIC CONSTRUCTION

We use Theorem 5.4 and techniques introduced in [34] to prove that examples of $\mathscr{G}$-decompositions exist representing all feasible congruence classes modulo β_0. We first note that the remarks in Section 3 on PBD-closure together with Theorem 4.2 prove the following lemma.

LEMMA 6.1. *Let $\mathscr{G}$ be an admissible family of edge-r-colored digraphs. There exists a positive integer β_0 which is divisible by $\beta(\mathscr{G})$ with the property*:

If $K_{v_0}^{(r)}$ *admits a* $\mathcal{G}$*-decomposition for some positive integer* v_0*, then* $K_v^{(r)}$ *can be* $\mathcal{G}$*-decomposed for all sufficiently large integers* $v \equiv v_0 \pmod{\beta_0}$.

THEOREM 6.2. *Let* $\mathcal{G}$ *be an admissible family of edge-r-colored digraphs. Let n be a positive integer satisfying*

$$n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})},$$

$$n-1 \equiv 0 \pmod{\alpha(\mathcal{G})}.$$

Then there exists an integer v_0 *so that* $v_0 \equiv n \pmod{\beta_0}$ *and such that* $K_{v_0}^{(r)}$ *admits a* $\mathcal{G}$*-decomposition.*

Proof. Equation (1.3) holds for some positive rationals c_G, $G \in \mathcal{G}$. Given $G \in \mathcal{G}$, the number of $H \in \mathcal{H}$ with $H \cong G$ that contain an edge e of $K_n^{(r)}$ depends only on its color. More precisely, there is a constant M_G so that if $\mu(G) = (m_1, ..., m_r)$, then the number of $H \in \mathcal{H}$ with $H \cong G$ that contain an edge e of color i is $m_i M_G$. Let $d_H = c_G / M_G$ for $H \cong G$. Then

$$\sum_{H : e \in E(H)} d_H = 1 \qquad \text{for every edge } e \text{ of } K_n^{(r)}.$$

Define $z_H = M d_H$, where M is a positive integer chosen so that all z_H are (positive) integers. Then

$$\sum_{H : e \in E(H)} z_H = M \qquad \text{for every edge } e \text{ of } K_n^{(r)}.$$

Let n be a positive integer satisfying $n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})}$ and $n-1 \equiv 0 \pmod{\alpha(\mathcal{G})}$. We may assume that $n \geqslant 2 + |V(G)|$ for all G in $\mathcal{G}$, by increasing n by a multiple of β_0 if necessary. Let $\{a_H : H \in \mathcal{H}\}$ be as in Theorem 5.4. If we let $a'_H = a_H + t z_H$ for each $H \in \mathcal{H}$, then

$$\sum_{H : e \in E(H)} a'_H = 1 + tM \qquad \text{for every edge } e \text{ of } K_n^{(r)}.$$

We choose and fix t so that

(1) $a'_H = a_H + t z_H \geqslant 0$ for each $H \in \mathcal{H}$, and

(2) $q = 1 + tM$ is a prime or a power of a prime congruent to 1 modulo β_0.

Take each subgraph $H \in \mathcal{H}$ with multiplicity a'_H to get a multiset $G_1, G_2, ..., G_N$ of subgraphs in $\mathcal{H}$ such that each edge (i, j) of color c of $K_n^{(r)}$ appears in exactly q of these subgraphs, $c = 1, 2, ..., r$. Next, choose an integer $d \geqslant n^2$ which is large enough so that by Lemma 6.1 $K_{q^d}^{(r)}$ has a

$\mathcal{G}$-decomposition. Let $v_0 = nq^d$; then $v_0 \equiv n \pmod{\beta_0}$. We show that $K_{v_0}^{(r)}$ can be $\mathcal{G}$-decomposed.

Let W be a d-dimensional vector space over $GF(q)$, and let ℓ: $W \to GF(q)$ be any nonzero linear functional. If $d \geqslant n^2$, there exist linear transformations $T_1, T_2, ..., T_n$ of W to itself with the following properties (see [36], [34]): $S_{ij} = (T_i - T_j)^{-1}$ exists whenever $i \neq j$ and for any choice of $n(n-1)/2$ scalars ρ_{ij}, $1 \leqslant i < j \leqslant n$, there exist vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ such that for $1 \leqslant i < j \leqslant n$,

$$\ell(S_{ij}(\mathbf{x}_j - \mathbf{x}_i)) = \rho_{ij}.$$

(Note that $S_{ji} = -S_{ij}$.)

Let $\{1, 2, ..., n\}$ be the vertex set of $K_n^{(r)}$, and let $V = W \times \{1, 2, ..., n\}$ be the vertex set of $K_v^{(r)}$. By our choice of d, $K_{q^d}^{(r)}$ defined on the vertex set $W \times \{i\}$ can be $\mathcal{G}$-decomposed for each i. So we need to construct a $\mathcal{G}$-decomposition for the complete multipartite edge-colored digraph $\mathcal{M}$ with vertices V and whose edges do not join any two vertices in any set $W \times \{i\}$.

For each subgraph G_h, $h = 1, 2, ..., N$, we want to assign scalars $\rho_h(i, j) \in GF(q)$ to all ordered pairs (i, j) of vertices of G_h with $i < j$ and for which i and j are adjacent (being joined in either or both directions) so that: for every pair (i, j) with $1 \leqslant i < j \leqslant n$ and every color c, $1 \leqslant c \leqslant r$,

(A) the scalars $\rho_h(i, j)$, as h ranges over the q subscripts for which there is an edge of color c from i to j in G_h, comprise all elements of $GF(q)$, each appearing exactly once, and

(B) the scalars $\rho_h(i, j)$, as h ranges over the q subscripts for which there is an edge of color c from j to i in G_h, comprise all elements of $GF(q)$, each appearing exactly once.

We postpone the demonstration that such scalars exist and first describe the rest of the construction.

For each G_h, associate vectors $\mathbf{x}_h(k) \in W$ to the vertices k of G_h whenever $i < j$ and i and j are adjacent vertices of G_h so that

$$\ell(S_{ij}(\mathbf{x}_h(j) - \mathbf{x}_h(i))) = \rho_h(i, j). \tag{6.1}$$

(We don't care what value the left-hand side in (6.1) has if i and j are not adjacent.) Let $\mathcal{G}_h$ be the family of subgraphs of $\mathcal{M}$ obtained as images of G_h under the $q^{d(d-1)}$ mappings (where colors are preserved under the mappings)

$$\phi_{\mathbf{y}, \mathbf{z}}^h : i \mapsto (\mathbf{x}_h(i) + T_i(\mathbf{y}) + \mathbf{z}, i)$$

where $\mathbf{y} \in \operatorname{kernel}(\ell)$, $\mathbf{z} \in W$, and $i = 1, 2, ..., n$. We claim that the subgraphs $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_N$ give a $\mathcal{G}$-decomposition of $\mathcal{M}$.

To verify this claim, consider the edge e of color c from $(\mathbf{w}, i)$ to $(\mathbf{w}', j)$, where $i \neq j$. First assume $i < j$. To find the subgraph containing this edge, find the unique index h so that G_h contains the edge of $K_n^{(r)}$ of color c from i to j and for which also

$$\ell(S_{ij}(\mathbf{x}_h(j) - \mathbf{x}_h(i))) = \ell(S_{ij}(\mathbf{w} - \mathbf{w}'));$$

this h exists and is unique by property (A) above. Then choose the unique $\mathbf{y} \in \text{kernel}(\ell)$ so that

$$S_{ij}(\mathbf{x}_h(j) - \mathbf{x}_h(i)) + \mathbf{y} = S_{ij}(\mathbf{w} - \mathbf{w}'),$$

which is equivalent to

$$(\mathbf{x}_h(j) - \mathbf{x}_h(i)) + (T_i(\mathbf{y}) - T_j(\mathbf{y})) = (\mathbf{w} - \mathbf{w}').$$

Then there is a unique $\mathbf{z} \in W$ so that

$$\begin{cases} \mathbf{x}_h(i) + T_i(\mathbf{y}) + \mathbf{z} = \mathbf{w} \\ \mathbf{x}_h(j) + T_j(\mathbf{y}) + \mathbf{z} = \mathbf{w}'. \end{cases}$$

This means that the edge e is contained in the image of G_h under the mapping $\phi^h_{\mathbf{y}, \mathbf{z}}$. A careful look at this argument shows that no other member of $\mathscr{G}_1 \cup \mathscr{G}_2 \cup \cdots \cup \mathscr{G}_N$ contains the edge e.

The claim is verified for $i > j$ by a similar argument.

It remains to show that scalars $\rho_h(i, j)$ can be chosen to satisfy the conditions (A) and (B). This is to be done separately for each pair (i, j) with $i < j$.

We regard the subgraphs $G_1, G_2, ..., G_N$ as "formally disjoint" (even though, to be precise, each edge of $K_n^{(r)}$ appears in q of these subgraphs). We pair the qr edges from i to j that appear in $G_1, G_2, ..., G_N$ with the qr edges from j to i in such a way that if there are edges both from i to j and from j to i in the same graph G_h, then those edges are paired; edges of a graph G_h without opposites in G_h can be paired arbitrarily with an edge in the other direction in some other graph $G_{h'}$. We then construct an (undirected) bipartite graph B with $2r$ vertices, two vertices $\hat{c}, \bar{c}$ for each color $c = 1, 2, ..., r$; each of the qr pairs $\{e, e'\}$ (with e from i to j and e' from j to i) may be thought of as an edge of B, joining $\hat{c}$ and $\bar{c}'$, where c is the color of e and c' the color of e'. The bipartite graph B is regular of degree q, and so admits a 1-factorization, i.e. the qr pairs of edges may be partitioned into families $\mathscr{F}_z$ indexed by $z \in GF(q)$, so that among the r pairs in each $\mathscr{F}_z$, there is one edge from i to j of each of the r colors, and one edge from j to i of each of the r colors. Each of the q edges of color c from i to j occurs in $\mathscr{F}_z$ for one and only one $z \in GF(q)$; each of the q edges of color c from j to i occurs in $\mathscr{F}_z$ for one and only one $z \in GF(q)$.

When an edge e from i to j belongs to G_h and the pair containing e belongs to $\mathscr{F}_z$, we define $\rho_h(i, j) = z$. When an edge e from j to i belongs to G_h and the pair containing e belongs to $\mathscr{F}_z$, we define $\rho_h(i, j) = z$. Of course, if edges exist both from i to j and from j to i in the same graph G_h, they are paired, so that $\rho_h(i, j)$ is well defined. The comments of the preceding paragraph ensure that properties (A) and (B) are valid. ▮

7. NOTE ON APPLICATIONS

In this and the following sections we give proofs of some known results and several new theorems concerning the asymptotic existence of combinatorial designs. In each case, we first observe that the problem can be stated as, or is equivalent to, a decomposition problem for some r and family $\mathscr{G}$ of edge-r-colored digraphs, so that Theorem 1.2 applies. This is usually the easy part. Then, it would appear, we must compute $\alpha(\mathscr{G})$ and $\beta(\mathscr{G})$ for a family $\mathscr{G}$ of edge-colored digraphs, and this can be tedious. And we must show that $\mathscr{G}$ is admissible.

Often $\alpha(\mathscr{G})$ and $\beta(\mathscr{G})$ are not, or need not, be computed explicitly. We will need to know that an integer n satisfying the hypotheses (congruences) stated in a theorem has the properties that $n(n-1) \equiv 0 \pmod{\beta(\mathscr{G})}$ and $n-1 \equiv 0 \pmod{\alpha(\mathscr{G})}$. But, by definition, these latter congruences are equivalent to showing that the vector $n(n-1)(1, 1, ..., 1)$ (of length r) is an integral linear combination of the vectors $\mu(G)$, $G \in \mathscr{G}$, and that the vector $(n-1)(1, 1, ..., 1)$ (of length $2r$) is an integral linear combination of the vectors $\tau(x)$, as x ranges over vertices of digraphs $G \in \mathscr{G}$.

Sometimes it is easy or convenient to explicitly exhibit such linear combinations. But at least as often, it is necessary or desirable to use Lemma 5.1 to establish the existence of such integral linear combinations.

When Lemma 5.1 is used for the purpose of showing that $n(n-1)(1, 1, ..., 1)$ is an integral linear combination of the vectors $\mu(G)$, we take, in the statement of that lemma, the vector $\mathbf{c}$ to be the column vector of height r of all $n(n-1)$'s, and the matrix M to have its rows indexed by the r colors and its columns indexed by the graphs $G \in \mathscr{G}$, the column labeled G containing the vector $\mu(G)$. The required integral linear combination exists if we show that whenever an assignment of rational numbers y_i to the r colors is such that $\sum_i m_i y_i$ is an integer for every $\mu(G) = (m_1, ..., m_r)$, then $n(n-1) \sum_i y_i$ is an integer. Similarly, the vector $(n-1)(1, 1, ..., 1)$ (of length $2r$) will be an integral linear combination of the vectors $\tau(x)$, as x ranges over vertices of digraphs $G \in \mathscr{G}$ if we show that whenever an assignment of a pair of rational numbers y_i, y'_i to each of the r colors is such that $\sum_i (\mathrm{out}_i(x)\, y_i + \mathrm{in}_i(x)\, y'_i)$ is an integer for every vertex x of any member of $\mathscr{G}$, then $(n-1)\sum_i (y_i+y'_i)$ is an integer.

In evaluating $\alpha(\mathscr{G})$ and $\beta(\mathscr{G})$, directly or indirectly, we sometimes do not use or consider *all* members of $\mathscr{G}$; it may be sufficient for our purposes (e.g. to evaluate α and β) to use a proper subfamily $\mathscr{G}'$. Then we may sometimes save a certain amount of effort by showing that $\mathscr{G}'$ is admissible, and not bothering to show that $\mathscr{G}$ is admissible, since the $\mathscr{G}'$-decompositions provided by Theorem 1.2 are also $\mathscr{G}$-decompositions.

In the following sections, we continue to use the notation $a \equiv b$ to mean that the difference $b-a$ is an integer.

8. APPLICATION TO GDDS

In his thesis [12], K. Chang proved an asymptotic existence result for group divisible designs. A *group divisible design* of index λ is a triple $(X, \mathscr{P}, \mathscr{A})$ where X is a set, $\mathscr{P}$ is a partition of X into nonempty subsets, and $\mathscr{A}$ is a family of subsets of X (each of cardinality at least 2) such that $|A \cap B| \leqslant 1$ for $A \in \mathscr{A}$ and $B \in \mathscr{P}$, and where any two distinct $x, y \in X$ that belong to different groups (members of $\mathscr{P}$) are together contained in exactly λ blocks (members of $\mathscr{A}$). Chang's thesis [12] and his original proof were never published. We give a proof here that is substantially shorter than the original one, because most of the work has been done in proving Theorem 1.2.

THEOREM 8.1 (K. Chang). *Let integers g, k be given with $g \geqslant 2$, $k \geqslant 2$. There exists a constant $n_0 = n_0(g, k)$ such that group divisible designs with n groups of size g, blocks of size k, and index $\lambda = 1$ exist for all integers $n \geqslant n_0$ that satisfy*

$$g^2 n(n-1) \equiv 0 \pmod{k(k-1)},$$

$$g(n-1) \equiv 0 \pmod{k-1}.$$

Proof. It is well known and easy to see that the above conditions are necessary for the existence of such a group divisible design.

We claim that the existence of such a GDD is equivalent to the existence of a $\mathscr{G}$-decomposition of $K_n^{(r)}$ where $r = g^2$ and $\mathscr{G}$ is the family of edge-colored graphs described below:

As colors, we use the ordered pairs from $\{1, 2, ..., g\}$. Let $\mathscr{T}(g, k)$ denote the set of g-sequences $\mathbf{t} = (t_1, t_2, ..., t_g)$ of nonnegative integers summing to k, let $G(\mathbf{t})$ be the digraph with vertices $V(G(\mathbf{t})) = T_1 \cup T_2 \cup \cdots \cup T_g$ where $|T_i| = t_i$ and where for all distinct $x, y \in V(G(\mathbf{t}))$, there is an edge from x to

y of color (i, j) where i and j are such that $x \in T_i$ and $y \in T_j$. Let $\mathscr{G}$ be the collection of all such $G(\mathbf{t})$, $\mathbf{t} \in \mathscr{T}(g, k)$.

It is simple to obtain a GDD from a $\mathscr{G}$-decomposition of $K_n^{(r)}$, if it exists. (The converse is also easily seen to be true.) Let $V = V(K_n^{(r)})$ and let $X = V \times \{1, 2, ..., g\}$. Let $\mathscr{P} = \{\{x\} \times \{1, 2, ..., g\}: x \in V\}$. For each $F \in \mathscr{F}$, there will be a unique partition $V(F) = S_1 \cup S_2 \cup \cdots \cup S_g$ so that the edge from x to y in F has color (i, j) if and only if $x \in S_i$ and $y \in S_j$; let

$$A_F = \bigcup_{i=1}^{g} S_i \times \{i\}$$

and let $\mathscr{A} = \{A_F: F \in \mathscr{F}\}$. It is not difficult to check that $(X, \mathscr{P}, \mathscr{A})$ is the required GDD; the block containing two points (x, i) and (y, j), $x \neq y$, is A_F where F is the subgraph in $\mathscr{F}$ that contains the edge of color (i, j) from x to y and the edge of color (j, i) from y to x.

Next we claim that $g^2 n(n-1) \equiv 0 \pmod{k(k-1)}$ and $g(n-1) \equiv 0 \pmod{k-1}$ together imply $n(n-1) \equiv 0 \pmod{\beta(\mathscr{G})}$. To this end we will use Lemma 5.1. Assume the first two congruences. We want to show that $n(n-1) \cdot (1, 1, ..., 1)$ is an integral linear combination of the vectors $\mu(G(\mathbf{t}))$, $\mathbf{t} \in \mathscr{T}(g, k)$. The vector $\mu(G(\mathbf{t}))$ has g^2 coordinates indexed by the ordered pairs (i, j) from $\{1, 2, ..., g\}$; the coordinate (i, i) is $t_i(t_i - 1)$ and for $i \neq j$, the coordinate (i, j) is $t_i t_j$. To establish the implication, it will suffice to show: Whenever g^2 rationals x_{ij} are given, $1 \leqslant i, j \leqslant g$, in such a way that

$$\sum_{i \neq j} t_i t_j x_{ij} + \sum_i t_i(t_i - 1)\, x_{ii} \equiv 0 \quad \text{for all } t \in \mathscr{T}(g, k), \tag{8.1}$$

then

$$n(n-1) \sum_{i, j} x_{ij} \equiv 0.$$

Assume (8.1) holds, fix i and j, and consider the three choices for $\mathbf{t} = (t_1, ..., t_g)$ where $t_i = k$, where $t_i = k-1$, $t_j = 1$, and where $t_i = k-2$, $t_j = 2$ (all other coordinates being zero). The implications of (8.1) arising from these three choices of t are:

$$\begin{aligned} k(k-1)\, x_{ii} &\equiv 0, \\ (k-1)(k-2)\, x_{ii} + (k-1)\, x_{ij} + (k-1)\, x_{ji} &\equiv 0, \quad \text{and} \\ (k-2)(k-3)\, x_{ii} + 2(k-2)\, x_{ij} + 2(k-2)\, x_{ji} + 2x_{jj} &\equiv 0. \end{aligned} \tag{8.2}$$

If we add the first and the third of these and subtract twice the second, we find that

$$2x_{ij}+2x_{ji}\equiv 2x_{ii}+2x_{jj} \tag{8.3}$$

for any i, j, $i\neq j$. Then surely

$$n(n-1)\,x_{ij}+n(n-1)\,x_{ji}\equiv n(n-1)\,x_{ii}+n(n-1)\,x_{jj}$$

and thus

$$n(n-1)\sum_{i,\,j} x_{ij}\equiv n(n-1)\,g\sum_{i} x_{ii}. \tag{8.4}$$

If we subtract the second relation of (8.2) from the first, we get

$$2(k-1)\,x_{ii}\equiv (k-1)(x_{ij}+x_{ji}), \tag{8.5}$$

and since this holds when j is replaced by i,

$$2(k-1)\,x_{ii}\equiv 2(k-1)\,x_{jj}$$

for all i and j. If k is even and $g(n-1)\equiv 0 \pmod{k-1}$, then $2(k-1)$ divides $gn(n-1)$ so

$$gn(n-1)\,x_{ii}\equiv gn(n-1)\,x_{jj}. \tag{8.6}$$

If k is odd, then multiply (8.3) by $(k-1)/2$ and combine it with (8.5) to obtain $(k-1)\,x_{ii}\equiv(k-1)\,x_{jj}$, and we again have (8.6). Then from (8.4), (8.6), and the first relation of (8.2), respectively, we find

$$n(n-1)\sum_{i,\,j} x_{ij}\equiv n(n-1)\,g\sum_{i} x_{ii}\equiv n(n-1)\,g^2x_{11}\equiv 0.$$

Now we want to show that $n-1\equiv 0 \pmod{\alpha(\mathscr{G})}$, assuming that $g(n-1)\equiv 0 \pmod{k-1}$. We use Lemma 5.1. We must show that $(n-1)\cdot(1, 1, ..., 1)$, of length $2g^2$, is an integral linear combination of the vectors $\tau(x)$ as x ranges over vertices of $G(\mathbf{t})$, $\mathbf{t}\in\mathscr{T}(g, k)$.

A vector $\tau(x)$ for x a vertex of $G(\mathbf{t})$ has $2g^2$ coordinates, corresponding to the color (i, j) indegrees and the color (i, j) outdegrees. If $\mathbf{t}=(t_1, ..., t_g)$ and x is a vertex in the vertex set T_ℓ, then the color (i, ℓ) indegree and the color (ℓ, i) outdegree is t_i for $i\neq\ell$ and $t_\ell-1$ for $i=\ell$; all other color (i, j) indegrees and color (i, j) outdegrees are zero.

To establish the implication, it will suffice to show: Whenever $2g^2$ rationals x_{ij}, y_{ij} are given, $1 \leqslant i, j \leqslant g$, in such a way that

$$(t_\ell - 1)(x_{\ell\ell} + y_{\ell\ell}) + \sum_{i \neq \ell} t_i(x_{i\ell} + y_{\ell i}) \equiv 0$$

$$\text{for all} \quad t \in \mathscr{T}(g, k) \quad \text{and} \quad \ell = 1, ..., g, \tag{8.7}$$

then

$$(n-1) \sum_{i,j} (x_{ij} + y_{ij}) \equiv 0.$$

Assume (8.7) holds. We will write z_{ij} for $x_{ij} + y_{ji}$. Consider the choices for $\mathbf{t} = (t_1, ..., t_g)$ and ℓ where $t_\ell = k$ and where $t_\ell = k-1$, $t_i = 1$. The implications of (8.7) arising from these choices are:

$$\begin{aligned} (k-1)\, z_{\ell\ell} &\equiv 0, \\ (k-2)\, z_{\ell\ell} + z_{i\ell} &\equiv 0 \end{aligned} \tag{8.8}$$

If we subtract the first from the second, we find $z_{i\ell} \equiv z_{\ell\ell}$ for all $i \neq \ell$. Then

$$(n-1) \sum_{i,j} z_{ij} \equiv g(n-1) \sum_{\ell} z_{\ell\ell} \equiv 0, \tag{8.9}$$

the last equivalence from the first relation of (8.8).

It remains only to show that some positive rational linear combination of the vectors $\mu(G(\mathbf{t}))$, $\mathbf{t} \in \mathscr{T}(g, k)$, is a positive scalar multiple of the all-ones vector. Let $\mathbf{m}_1$ denote the sum of $\mu(G(\mathbf{t}))$ as $\mathbf{t}$ ranges over the set of all integral vectors of length g that sum to k and have coordinates "as equal as possible," that is, when we write $k = gq + p$ with $0 \leqslant p < g$, where $\mathbf{t}$ has $g - p$ coordinates equal to q and p coordinates equal to $q+1$. It is easily checked that $\mathbf{m}_1$ has coordinates t_{ij} that for some A, B with $A < B$, $t_{ii} = A$ for all i and $t_{ij} = B$ for $i \neq j$. Let $\mathbf{m}_2$ denote the sum of $\mu(G(\mathbf{t}))$ as $\mathbf{t}$ ranges over the set of all integral vectors of length g with k in one coordinate and zeros elsewhere. Then $\mathbf{m}_1$ has coordinates s_{ij} where $s_{ii} = k(k-1)$ for all i and $s_{ij} = 0$ for $i \neq j$. Let $\mathbf{m}_3$ denote the sum of $\mu(G(\mathbf{t}))$ as $\mathbf{t}$ ranges over all integral vectors of length g that sum to k. Then $\mathbf{m}_3$ has coordinates u_{ij} that for some C, D, $u_{ii} = C$ for all i and $u_{ij} = D$ for $i \neq j$. Then $\mathbf{m}_3$ can be adjusted by adding a nonnegative scalar multiple of either $\mathbf{m}_1$ or $\mathbf{m}_2$ to produce a constant vector. ▮

9. APPLICATION TO GRID DESIGNS

An $n \times g$ *grid design* of index λ with block size k consists of a set X of ng points partitioned in two ways

$$X = G_1 \cup G_2 \cup \cdots \cup G_n = N_1 \cup N_2 \cup \cdots \cup N_g,$$

so that each "horizontal group" G_i has $|G_i| = g$, each "vertical group" N_j has $|N_j| = n$, and $|G_i \cap N_j| = 1$ for $i = 1, ..., n$, $j = 1, ..., g$, together with a set $\mathscr{A}$ of *blocks* of size k with $|A \cap G_i| \leqslant 1$, $|A \cap N_j| \leqslant 1$ for each i and j, so that two points $x, y \in X$ which do not belong to the same horizontal group or vertical group are contained in exactly λ blocks.

We are proposing use of the term "grid design." In K. Chang's thesis [12], these are called "lattice designs." They are called "modified GDDs" in [5]. These designs are mentioned much earlier in E. H. Moore's *Tactical Memoranda* [24]. We note that $k \times n$ grid designs with $\lambda = 1$ and block size k are equivalent to certain "transitive" orthogonal arrays, which were used by Bose, Parker, and Shrikhande in their disproof of Euler's conjecture. Chang's original proof of the following theorem was not published.

THEOREM 9.1 (K. Chang). *Let integers g, k be given with $g \geqslant k \geqslant 2$. There exists a constant $n_0 = n_0(g, k)$ such that $n \times g$ grid designs of index $\lambda = 1$ with block size k exist for all integers $n \geqslant n_0$ that satisfy*

$$\begin{aligned} g(g-1)\, n(n-1) &\equiv 0 \pmod{k(k-1)}, \\ (g-1)(n-1) &\equiv 0 \pmod{k-1}. \end{aligned} \tag{9.1}$$

Proof. It is easy to see that the above conditions are necessary for the existence of such a grid design.

We claim that the existence of such a grid design is equivalent to the existence of a $\mathscr{G}$-decomposition of $K_n^{(r)}$ where $r = g(g-1)$ and $\mathscr{G}$ is the family of edge-colored graphs described below:

As colors, we use the ordered pairs from $\{1, 2, ..., g\}$ with distinct coordinates. For any k-subset S of $\{1, 2, ..., g\}$, let $G(S)$ be the digraph with vertices $V(G(S)) = S$ and where for all distinct $x, y \in S$, there is an edge from x to y of color (x, y). Let $\mathscr{G}$ be the collection of all such $G(S)$.

It is simple to obtain a grid design from a $\mathscr{G}$-decomposition $\mathscr{F}$ of $K_n^{(r)}$, if it exists. (The converse is also easily seen to be true.) Let $V = V(K_n^{(r)})$ and let $X = V \times \{1, 2, ..., g\}$. Let $G_x = \{x\} \times \{1, 2, ..., g\}$, for $x \in V$, and $N_j = V \times \{j\}$ for $j = 1, 2, ..., g$. For each digraph $F \in \mathscr{F}$ with vertices $x_1, x_2, ..., x_k$, say, there is a subset $\{\ell_1, \ell_2, ..., \ell_k\}$ of $\{1, ..., g\}$ so that the edge from x_i to x_j has color (ℓ_i, ℓ_j); we take as a block the subset $A_F = \{(x_i, \ell_i): 1 \leqslant i \leqslant k\}$ of X. It is not difficult to check that we have the required grid design: the block containing two points (x, s) and (y, t), $x \neq y$, $s \neq t$, is A_F where F is the subgraph in $\mathscr{F}$ that contains the edge of color (s, t) from x to y.

If we sum $\mu(G(S))$ over all k-subsets S of $\{1, 2, ..., g\}$, we clearly get a positive constant vector. Thus $\mathscr{G}$ is admissible.

It remains to show that the congruences (9.1) imply that $n(n-1)$ $(1, 1, ..., 1)$, of length $g(g-1)$, is an integral linear combination of the vectors $\mu(G(S))$ and that $(n-1)(1, 1, ..., 1)$, of length $2g(g-1)$, is an integral linear combination of the degree-vectors $\tau(x)$ that arise from vertices x of graphs in $\mathscr{G}$.

It is shown in [38] and [16] that one can assign integers c_S to the k-subsets S of $\{1, 2, ..., g\}$ so that for every $i, j \in \{1, 2, ..., g\}$, $i \neq j$, the sum $\sum c_S$ over those k-subsets that contain i and j is a constant m, provided that $mg(g-1) \equiv 0 \pmod{k(k-1)}$ and $m(g-1) \equiv 0 \pmod{k-1}$. Clearly, this would imply that the linear combination $\sum c_S \mu(G(S))$ is $(m, m, ..., m)$. We may take $m = n(n-1)$ if the congruences (9.1) hold. (Remark: The result of [38] and [16] is related to Theorem 5.2 (replace the n there by our g) in the case that $\mathscr{G} = \{K_k\}$, $r = 1$. Assignments of integers to the k-subsets as above are called "integral designs" in [16].)

For $\ell \in S$, S a k-subset of $\{1, 2, ..., g\}$, let $t(\ell, S)$ denote the vector $\tau(\ell)$ for ℓ considered as a vertex of the graph $G(S)$. The coordinates of $t(\ell, S)$ are 1's in positions corresponding to color (ℓ, i) outdegree and color (i, ℓ) indegree for $i \in S \backslash \{\ell\}$, and 0's in all other positions. We can choose a family $\mathscr{T}_\ell$ of $(k-1)$-subsets of the $(g-1)$-set $\{1, 2, ..., g\} \backslash \{\ell\}$ so that every element $i \neq \ell$ occurs in exactly $c = (k-1)/\gcd(k-1, g-1)$ members of $\mathscr{T}_\ell$. (This is a 1-design on $g-1$ points.) If we sum $t(\ell, T \cup \{\ell\})$ over $T \in \mathscr{T}_\ell$, we get a vector $\mathbf{c}_\ell$ with c's in coordinates corresponding to color (ℓ, i) outdegree and color (i, ℓ) indegree for all $i \neq \ell$, and 0's in all other positions. If we sum the vectors $\mathbf{c}_\ell$ over $\ell = 1, 2, ..., g$, we get the constant vector $\mathbf{c} = (c, c, ..., c)$. If $(g-1)(n-1) \equiv 0 \pmod{k-1}$, then $n-1$ is divisible by c, and so the vector $(n-1)(1, 1, ..., 1)$ is an integral linear combination of vectors $t(\ell, S)$. ▮

10. APPLICATION TO RESOLVABLE DESIGNS

A (v, k, λ)-BIBD D is said to be *resolvable* (and denoted by (v, k, λ)-RBIBD) if the blocks of D can be partitioned into classes $R_1, R_2, ..., R_r$ (*resolution classes*) where $r = \lambda(v-1)/(k-1)$ such that each element of D is contained in precisely one block of each class. The classes $R_1, R_2, ..., R_r$ form a *resolution* of D. A necessary condition for the existence of a $(v, k, 1)$-RBIBD is $v \equiv k \pmod{k(k-1)}$. In this section we give, as another application of Theorem 2.1, a proof of the following result from [27].

THEOREM 10.1. *Given $k \geq 3$, there exists a constant $v_0 = v_0(k)$ such that $(v, k, 1)$-RBIBD exist for all $v \geq v_0$ with $v \equiv k \pmod{k(k-1)}$.*

Proof. We claim that the existence of such a $(v, k, 1)$-RBIBD is equivalent to the existence of a $\mathcal{G}$-decomposition of $K_n^{(r)}$ where $r = k^2 - k$ and $\mathcal{G}$ is the family of graphs described below:

As colors, we use the $(k-1)(k-1)$ ordered pairs from $\{1, 2, ..., k-1\}$ and the $k-1$ singletons (i), $i = 1, 2, ..., k-1$. For each $(k-1)$-tuple $\mathbf{t} = (t_1, t_2, ..., t_{k-1})$ of nonnegative integers summing to k, let $G(\mathbf{t})$ be the digraph with $k+1$ vertices

$$V(G(\mathbf{t})) = \{\omega\} \,\dot\cup\, T_1 \,\dot\cup\, T_2 \,\dot\cup \cdots \dot\cup\, T_{k-1} \tag{10.1}$$

where $|T_i| = t_i$ and ω is a $(k+1)$-st vertex. Here, for all distinct $x, y \in V(G(\mathbf{t}))$, there is an edge from x to y of color (i, j) where i and j are such that $x \in T_i$ and $y \in T_j$, and an edge of color (i) from the special vertex ω to each x in T_i. Let $\mathcal{G}$ be the collection of all such $G(\mathbf{t})$.

It is simple to obtain a $((k-1)\,n+1, k, 1)$-RBIBD from a $\mathcal{G}$-decomposition $\mathcal{F}$ of $K_n^{(r)}$, if it exists. (The converse is also easily seen to be true, but isn't necessary for the proof of the theorem.) Let V be the vertex set of $K_n^{(r)}$ and let $X = \{\infty\} \cup (V \times \{1, 2, ..., k-1\})$. Let

$$B_x = \{\infty\} \cup (\{x\} \times \{1, 2, ..., k-1\}), \qquad \mathcal{B} = \{B_x : x \in V\}.$$

The elements V will be used to index the parallel classes, which we will denote $\mathcal{C}_x$, $x \in V$; B_x will be in $\mathcal{C}_x$. For each $F \in \mathcal{F}$, there will be a unique partition of the $k+1$ vertices $V(F) \subseteq V$ as

$$V(F) = \{w\} \,\dot\cup\, S_1 \,\dot\cup\, S_2 \,\dot\cup \cdots \dot\cup\, S_{k-1}$$

as in (10.1). Let

$$A_F = \bigcup_{i=1}^{k-1} S_i \times \{i\};$$

this block is to be in the parallel class $\mathcal{C}_w$. Let $\mathcal{A} = \{A_F : F \in \mathcal{F}\}$. It is easy to check that $(X, \mathcal{A} \cup B)$ is a $((k-1)\,n+1, k, 1)$-BIBD, and that each $\mathcal{C}_w$ is a parallel class. For example, the unique block in $\mathcal{C}_w$ that contains a point (y, i), $y \neq w$, is A_F where F is the graph in $\mathcal{F}$ that contains the edge of color (i) from w to y.

If we show that $\mathcal{G}$-decompositions of $K_n^{(r)}$ exist for all large integers $n \equiv 1 \pmod{k}$, then Theorem 10.1 will follow.

To use Theorem 1.2, we want first to show that $n \equiv 1 \pmod{k}$ implies that $n(n-1)(1, 1, ..., 1)$ is an integral linear combination of the vectors $\mu(G(\mathbf{t}))$.

The vectors $\mu(G(\mathbf{t}))$ have coordinates indexed by the r colors. To establish the implication, it will suffice by Lemma 5.1 to show: Whenever k^2-k rationals x_{ij}, $1\leqslant i, j\leqslant k-1$, and x_i, $1\leqslant i\leqslant k-1$, are given in such a way that

$$\sum_{i\neq j} t_i t_j x_{ij}+\sum_i (t_i x_i+t_i(t_i-1)\, x_{ii})\equiv 0 \tag{10.2}$$

for all $\mathbf{t}=(t_1, ..., t_{k-1})$, then

$$n(n-1)\left(\sum_i x_i+\sum_{i, j} x_{ij}\right)\equiv 0.$$

Consider the three choices for $\mathbf{t}=(t_1, ..., t_{k-1})$ where $t_i=k$, where $t_i=k-1$, $t_j=1$, and where $t_i=k-2$, $t_j=2$ (all other coordinates being zero). The implications of (10.2) arising from these three choices of t are:

$$kx_i+k(k-1)\, x_{ii}\equiv 0,$$

$$(k-1)\, x_i+x_j+(k-1)(k-2)\, x_{ii}+(k-1)\, x_{ij}+(k-1)\, x_{ji}\equiv 0, \qquad \text{and}$$

$$\begin{aligned}(k-2)\, x_i+2x_j+(k-2)(k-3)\, x_{ii}&\\ +2(k-2)\, x_{ij}+2(k-2)\, x_{ji}+2x_{jj}&\equiv 0.\end{aligned} \tag{10.3}$$

If we add the first and the third of these and subtract twice the second, we find that

$$2x_{ij}+2x_{ji}\equiv 2x_{ii}+2x_{jj} \tag{10.4}$$

for any i, j, $i\neq j$. Then surely

$$n(n-1)\, x_{ij}+n(n-1)\, x_{ji}\equiv n(n-1)\, x_{ii}+n(n-1)\, x_{jj}. \tag{10.5}$$

If we subtract the second relation of (10.3) from the first, we get

$$x_i-x_j+2(k-1)\, x_{ii}\equiv (k-1)(x_{ij}+x_{ji}),$$

and since this holds when j is replaced by i,

$$2x_i+2(k-1)\, x_{ii}\equiv 2x_j+2(k-1)\, x_{jj} \tag{10.6}$$

for all i and j. From (10.5), (10.6), the first relation of (10.3), and $n \equiv 1 \pmod{k}$, we have

$$n(n-1)\left(\sum_i x_i + \sum_{i,j} x_{ij}\right) \equiv n(n-1)(k-1)\sum_i x_{ii} + n(n-1)\sum_i x_i$$
$$\equiv n(n-1)(k-1)((k-1)\,x_{11} + x_1) \equiv 0.$$

Next we want to show that $n \equiv 1 \pmod{k}$ implies that $(n-1)(1, 1, ..., 1)$ is an integral linear combination of the vectors $\tau(x)$ as x ranges over vertices of the graphs $G(\mathbf{t})$. Let $\mathbf{t}$ be a permutation of (2, 1, 1..., 1), say with the 2 in coordinate ℓ, and choose x to be one of the two vertices in the set T_ℓ as in (10.1). Then $\tau(x)$ has a 1 in coordinates corresponding to color (ℓ) indegree and color (j, ℓ) indegree and color (ℓ, j) outdegree for all $j = 1, 2, ..., k-1$; all other coordinates are 0. And $\tau(\omega)$ has a 2 in the coordinate corresponding to color (ℓ) outdegree and a 1 in the coordinates corresponding to color (i) outdegree for $i \neq \ell$. The sum $\mathbf{a}$ of $\tau(x)$ over the $k-1$ choices of ℓ has 1's in all coordinates, except in those corresponding to color (i) outdegrees which are 0. The sum $\mathbf{b}$ of $\tau(\omega)$ over the $k-1$ choices of ℓ has k's in all coordinates corresponding to color (i) outdegrees and 0's in all other coordinates. Then $k\mathbf{a} + \mathbf{b}$ is a constant vector of all k's.

It remains only to show that some positive rational linear combination of the vectors $\mu(G(\mathbf{t}))$ is a positive constant vector. Let $\mathbf{m}_1$ denote the sum of $\mu(G(\mathbf{t}))$ as $\mathbf{t}$ ranges over permutations of $(k, 0, 0, ..., 0)$; let $\mathbf{m}_2$ denote the sum of $\mu(G(\mathbf{t}))$ as $\mathbf{t}$ ranges over all permutations of (2, 1, 1, ..., 1). Let $\mathbf{m}_3$ denote the sum of $\mu(G(\mathbf{t}))$ as $\mathbf{t}$ ranges over all $(k-1)$-tuples of nonnegative integers summing to k. The entries of these vectors in coordinates corresponding to colors (i), (i, i) and (i, j), $i \neq j$, are

	(i)	(i, i)	(i, j)
$\mathbf{m}_1$	k	$k(k-1)$	0
$\mathbf{m}_2$	k	2	$k+1$
$\mathbf{m}_3$	A	B	C

where A, B, C are positive integers with $(k-1)\,A = B + (k-2)\,C$. To see that this linear relation holds, write $m_{(i)}(\mathbf{t})$ for the number of edges of color (i) and $m_{(ij)}(\mathbf{t})$ for the number of edges of color (i, j) of $G(\mathbf{t})$, and note that $\sum_i m_{(i)}(\mathbf{t}) = k$ and $\sum_{i,j} m_{(ij)}(\mathbf{t}) = k(k-1)$, where $i = 1, 2, ..., k-1$ and $i, j = 1, 2, ..., k-1$. So

$$(k-1)\sum_i m_{(i)}(\mathbf{t}) = \sum_i m_{(ii)}(\mathbf{t}) + \sum_{i \neq j} m_{(ij)}(\mathbf{t}).$$

Summing over all choices of $\mathbf{t}$, we find $(k-1)\sum_i A = \sum_i B + \sum_{i \neq j} C$, from which the relation follows.

If $B \geqslant C$, then $\mathbf{m}_3$ plus a nonnegative scalar multiple of $\mathbf{m}_2$ will be a positive constant vector; if $B \leqslant C$, then $\mathbf{m}_3$ plus a nonnegative scalar multiple of $\mathbf{m}_1$ will be a positive constant vector. ▮

11. APPLICATION TO SKEW ROOM *d*-CUBES

A *Room d-cube R* of side v defined on an $(n+1)$-set V is a d-dimensional array with the following properties:

(1) every unordered pair of distinct elements from $V \cup \{\infty\}$ occurs precisely once in the array, and

(2) each 2-dimensional projection of R is a Room square of side v.

A Room d-cube is *skew* if each 2-dimensional projection of R is a skew Room square of side v.

The asymptotic existence of Room d-cubes was established in [17]. In this section, we generalize Example 2.4 and prove the asymptotic existence of skew Room d-cubes.

THEOREM 11.1. *Let d be a positive integer. Then there is an integer v_0 such that for all odd $v \geqslant v_0$, there exists a skew Room d-cube of side v.*

Proof. We claim that the existence of a skew Room d-cube of side v is equivalent to the existence of a G-decomposition of $K_v^{(r)}$ where $r = 1 + d + \binom{d}{2}$ and G is the graph described below.

Let $\{a, b, x_1, x_2, ..., x_d\}$ be the $d+2$ vertices of G. The r colors are $\{i\}$, $i = 0, 1, ..., d$, and the unordered pairs $\{i, j\}$, $i \neq j$ and $1 \leqslant i, j \leqslant d$. The directed edges of G are:

(i) $(a, b), (b, a)$, color $\{0\}$,

(ii) $(a, x_i), (b, x_i)$, color $\{i\}$, $i = 1, 2, ..., d$, and

(iii) $(x_i, x_j), (x_j, x_i)$, color $\{i, j\}$, $i \neq j$, $1 \leqslant i, j \leqslant d$.

So G contains two edges of each of the r colors.

We first show that if there exists a skew Room d-cube of side v, then there is a G-decomposition of $K_v^{(r)}$. Let R be a skew Room d-cube of side

v defined on the element set $V = \{1, 2, ..., v\} \cup \{\infty\}$. Suppose that R is in standard form so that $\{\infty, i\}$ is in cell $(i, i, ..., i)$ for $i = 1, 2, ..., v$. Let $\{a, b\}$ be a pair in cell $(x_1, x_2, ..., x_d)$ of R where $a, b \in \{1, 2, ..., v\}$. Then we construct the graph G described above. We verify that this gives a G-decomposition of $K_v^{(r)}$.

Every unordered pair $\{a, b\}$ of distinct elements from $\{1, 2, ..., v\}$ occurs precisely once in R. So each edge (a, b), (b, a) of color $\{0\}$ occurs once in the decomposition. Let R_{ij} be the projection of R in the i-th and j-th coordinates, $i \neq j$. R_{ij} is a skew Room square. Every element of V occurs precisely once in each row (and column) of R_{ij}. Consider row k of R_{ij}. The pairs in row k are $\{s_\ell, t_\ell\}$, $\ell = 1, 2, ..., (v-1)/2$, where $\bigcup_\ell \{s_\ell, t_\ell\} = V - \{\infty, k\}$. The graphs constructed from these pairs contain the edges (s_ℓ, k), (t_ℓ, k) of color $\{i\}$, $\ell = 1, 2, ..., (v-1)/2$. So every edge of $K_v^{(r)}$ of color $\{i\}$ occurs precisely once in the G-decomposition for $i = 1, 2, ..., d$. Finally, we consider the edges of color $\{i, j\}$. Since R_{ij} is a skew Room square, one of the cells (u, w) and (w, u) is nonempty for $1 \leqslant u, w \leqslant v$, $u \neq v$. So the edges (u, w) and (w, u) of color $\{i, j\}$ occur in precisely one graph of the decomposition. Thus we have a G-decomposition of $K_v^{(r)}$.

Next we show that this construction can be reversed; we use a G-decomposition of $K_v^{(r)}$ to construct a skew Room d-cube of side v. Let R be a d-dimensional array indexed by the vertices of $K_v^{(r)}$, say $\{1, 2, ..., v\}$. We place $\{\infty, i\}$ in cell $(i, i, ..., i)$ of R for $i = 1, 2, ..., v$. For each graph G, we place $\{a, b\}$ in cell $(x_1, x_2, ..., x_d)$ of R.

The decomposition for color $\{0\}$ insures that each unordered pair of distinct elements $\{a, b\}$, $a, b \in \{1, 2, ..., v\}$ occurs precisely once in R. Thus, every unordered pair of distinct elements of $V = \{1, 2, ..., v\} \cup \{\infty\}$ occurs precisely once in R. Let R_{ij} be the projection of R in the ith and jth coordinates, $i \neq j$. The rows in R_{ij} are constructed from the directed edges of color $\{i\}$. For fixed k, the $v-1$ directed edges (y, k) of color $\{i\}$ come from the $(v-1)/2$ pairs in the graphs: (s_ℓ, k), (t_ℓ, k), $\ell = 1, 2, ..., (v-1)/2$. Row k of R_{ij} contains the pairs $\{s_\ell, t_\ell\}$, $\ell = 1, 2, ..., (v-1)/2$. So every element of V occurs once in row k, $k = 1, 2, ..., v$. Similarly, every element of V occurs once in column k of R_{ij} for $k = 1, 2, ..., v$. So R_{ij} is a Room square. Since the edges (u, w) and (w, u), $u \neq w$, of color $\{i, j\}$ occur together in precisely one graph of the decomposition, only one of the cells (u, w) and (w, u) in R_{ij} is filled. Therefore R_{ij} is skew. This verifies that R is a skew Room d-cube of side v.

In order to apply Theorem 1.1, we need to find $\alpha(G)$. Recall that $m = \beta(G) = 2$. For vertex $y \in G$, $\tau(y) = (\text{in}_1(y), \text{out}_1(y), ..., \text{in}_r(y), \text{out}_r(y))$. We order the colors of τ as follows:

$$\{0\}, \{1\}, ..., \{d\}, \{1, 2\}, \{1, 3\}, ..., \{1, d\}, \{2, 3\}, ..., \{2, d\}, ..., \{d-1, d\}.$$

So we have:

$$\tau(a) = (1, 1, 0, 1, 0, 1, ..., 0, 1, 0, 0, 0, 0, ..., 0, 0, 0, 0, ..., 0, 0, 0, 0, ..., 0, 0)$$
$$\tau(b) = (1, 1, 0, 1, 0, 1, ..., 0, 1, 0, 0, 0, 0, ..., 0, 0, 0, 0, ..., 0, 0, 0, 0, ..., 0, 0)$$
$$\tau(x_1) = (0, 0, 2, 0, 0, 0, ..., 0, 0, 1, 1, 1, 1, ..., 1, 1, 0, 0, ..., 0, 0, 0, 0, ..., 0, 0)$$
$$\tau(x_2) = (0, 0, 0, 0, 2, 0, ..., 0, 0, 1, 1, 0, 0, ..., 0, 0, 1, 1, ..., 1, 1, 0, 0, ..., 0, 0)$$
$$\vdots$$

We would like to find the least positive integer t such that the $2r$-vector $(t, t, ..., t)$ is an integral linear combination of the vectors $\{\tau(x): x \in V(G)\}$. First note that $t \geqslant 2$ since there are positions (for example, the third) in the $2r$-vectors where the only nonzero entry is 2. Next we consider $(\sum_{i=1}^{d} \tau(x_i)) + \tau(a) + \tau(b)$. It is easy to see that this is the $2r$ vector $(2, 2, ..., 2)$. So we have $t = 2$ or $\alpha(G) = 2$. By Theorem 1.1, there exists a constant v_0 such that $K_v^{(r)}$ has a G-decomposition for all $v \geqslant v_0$ that satisfy $v(v-1) \equiv 0 \pmod{2}$ and $v - 1 \equiv 0 \pmod{2}$. ∎

12. APPLICATION TO DESIGNS WITH AUTOMORPHISMS

The theorem below produces reverse triple systems when $k = 3$.

THEOREM 12.1. *Let $\mathbb{G}$ be a group of order $k-1$. There exists a $(v, k, 1)$-BIBD that admits a group $\mathbb{G}^*$ of automorphisms isomorphic to $\mathbb{G}$ such that for some point x_0, $\mathbb{G}^*$ fixes x_0 but acts transitively on the other $k-1$ points of all blocks that contain x_0 for all but finitely many integers $v = n(k-1)+1$ for which*

$$n(n-1) \equiv 0 \begin{cases} \pmod{4k} & \text{if } k \equiv 3 \pmod{4}, \\ \pmod{k} & \text{if } k \equiv 0, 1, \text{ or } 2 \pmod{4}. \end{cases}$$

These conditions are necessary for all n.

Proof. We consider edge-$(k-1)$-colored graphs and take the elements of $\mathbb{G}$ as our colors. For each mapping $f: \mathbb{G} \to Z^+$ (the nonnegative integers) such that $\sum_{g \in \mathbb{G}} f(g) = k$, let $G(f)$ denote the graph with vertex set $V(G(f)) = \bigcup_{g \in \mathbb{G}} T_g$ where the T_g's are disjoint sets with $|T_g| = f(g)$ and where for all distinct $x, y \in V(G(f))$, there is an edge from x to y of color $a^{-1}b$ where a and b are such that $x \in T_a$ and $y \in T_b$. Let $\mathcal{G}$ be the collection of all such $G(f)$.

We claim that the existence of a $\mathcal{G}$-decomposition of $K_n^{(r)}$ implies the existence of a $(v, k, 1)$-BIBD with the required automorphism group.

Given a $\mathcal{G}$-decomposition of $K_n^{(r)}$, let $V = V(K_n^{(r)})$ and take as points $X = \{\infty\} \cup (\mathbb{G} \times V)$. For each graph $F \in \mathcal{F}$, we construct $k-1$ blocks $A_{F,h}$, $h \in \mathbb{G}$, as follows: Write $V(F) = \bigcup_{g \in \mathbb{G}} S_g$ in any way so that the edge from x to y has color $a^{-1}b$ when $x \in S_a$ and $y \in S_b$ and take

$$A_{F,h} = \bigcup_{g \in \mathbb{G}} (S_g \times \{gh\}).$$

For the blocks of the $((k-1)n+1, k, 1)$-BIBD, take

$$\{A_{F,h} : h \in \mathbb{G}, F \in \mathcal{F}\} \cup \{\{\infty\} \cup (\{x\} \times \mathbb{G}) : x \in V\}.$$

To find a block containing two points (x, a) and (y, b) with $x \neq y$, let F be the unique graph in $\mathcal{F}$ containing the edge of color $a^{-1}b$ from x to y.

Conversely, we claim that the existence of a $(v, k, 1)$-BIBD with the required automorphism group is equivalent to the existence of a $\mathcal{G}$-decomposition of $K_n^{(r)}$. Each orbit of blocks will give us one subgraph isomorphic to a member of $\mathcal{G}$. We omit the details.

To see that the congruences in Theorem 12.1 are necessary conditions for the decomposition, first notice that the sum of all coordinates of a vector $\mu(G(f))$ is $k(k-1)$, and this must divide the sum $(k-1)\,n(n-1)$ of the coordinates of $n(n-1)(1, 1, ..., 1)$ in the case that this vector is an integral linear combination of the former vectors. So $n(n-1) \equiv 0 \pmod{k}$ is necessary. Now suppose $k \equiv 3 \pmod 4$. We will show that $n(n-1) \equiv 0 \pmod 4$ is also necessary and, since k is odd, we must have $n(n-1) \equiv 0 \pmod{4k}$.

It is well known that in a group $\mathbb{G}$ of order $k-1 \equiv 2 \pmod 4$, the elements of odd order form a subgroup $\mathbb{H}$ of index 2. Now assume that $n(n-1)(1, 1, ..., 1)$ is an integral linear combination of vectors $\mu(G(f))$. The sum of the coordinates corresponding to elements not in $\mathbb{H}$ is $n(n-1)(k-1)/2$. But the sum of the coordinates corresponding to elements of $\mathbb{G} \backslash \mathbb{H}$ in any vector $\mu(G(f))$ is $\sum f(a)\, f(b)$ where the sum is extended over pairs (a, b) where one of a or b is in $\mathbb{H}$ and the other is not. This is $2m(k-m)$ where $m = \sum_{a \in \mathbb{H}} f(a)$, and this is $\equiv 0 \pmod 4$ since k is odd. It follows that $n(n-1)(k-1)/2 \equiv 0 \pmod 4$, which, since $(k-1)/2$ is odd, means that $n(n-1) \equiv 0 \pmod 4$ in this case.

We now consider $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$. If f_1 assigns 2 to the identity and 1 to each other group element, the graph $G(f_1)$ has a vertex x with $\tau(x) = (1, 1, ..., 1)$, namely when x is one of the two vertices in $T(id)$. Thus $\alpha(\mathcal{G}) = 1$.

The vector $\mu(G(f_1))$ is $\mathbf{m}_1 = (2, k+1, k+1, ..., k+1)$ of length $k-1$, where the first coordinate corresponds to the color *id*. Let f_2 assign k to the identity of $\mathbb{G}$ and 0 to all other group elements. The vector $\mu(G(f_2))$ is $\mathbf{m}_2 = (k(k-1), 0, 0, ..., 0)$. We have

$$\mathbf{m}_2 + k\mathbf{m}_1 = k(k+1)(1, 1, ..., 1)$$

and we may conclude that $\beta(\mathscr{G})$ divides $k(k+1)$.

For $g \in \mathbb{G}$, $g \neq id$, let f_g assign 1 to g, $k-1$ to id, and 0 to all other group elements. Then the vector $\mu(G(f_g))$ has $(k-1)(k-2)$ in the first coordinate and two $(k-1)$'s, or a single $2(k-1)$ in case g has order 2, in the other (nonzero) coordinate positions. The sum of the vectors $\mu(G(f_g))$ over the nonidentity group elements g is $\mathbf{m}_3 = (k-2)((k-1)(k-2), 2, 2, ..., 2)$. Then

$$\mathbf{m}_3 + (k^2 - 4k + 2)\, \mathbf{m}_1 = (k^3 - 3k^2)(1, 1, ..., 1)$$

and we may conclude that $\beta(\mathscr{G})$ divides $k^2(k-3)$.

Since $\beta(\mathscr{G})$ divides both $k(k-1)$ and $k^2(k-3)$, it divides their greatest common divisor, which is k if k is even, and $2k$ or $4k$ for odd k, depending on whether k is congruent to 1 or 3 modulo 4. In the case $k \equiv 1 \pmod 4$, $n(n-1) \equiv 0 \pmod k$ and $n(n-1) \equiv 0 \pmod{2k}$ are, of course, equivalent.

The family $\mathscr{G}'$ consisting of $G(f_1)$, $G(f_2)$, and all $G(f_g)$ is, by the above equations, admissible. Since only graphs in $\mathscr{G}'$ were used to calculate $\beta(\mathscr{G})$ and $\alpha(\mathscr{G})$, we have $\mathscr{G}'$-decompositions (and hence $\mathscr{G}$-decompositions) for all sufficiently large integers n satisfying the conditions (congruences) in the statement of the theorem. ∎

13. EXTENSION TO MULTIPLICITIES

We consider the problem of finding a family $\mathscr{F}$ of subgraphs of $K_n^{(r)}$, each of which is isomorphic to a member of $\mathscr{G}$, so that each edge of $K_n^{(r)}$ of color i occurs in exactly λ_i of the members of $\mathscr{F}$. We can think of this as a $\mathscr{G}$-decomposition of $K_n^{[\lambda_1, \lambda_2, ..., \lambda_r]}$, where this denotes the digraph on n vertices where there are exactly λ_i edges of color i joining x to y for any ordered pair (x, y) of distinct vertices.

With this notation, $K_n^{(r)} = K_n^{[1, 1, ..., 1]}$, where there are r ones.

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a vector of positive integers. Let $\alpha(\mathscr{G}; \boldsymbol{\lambda})$ denote the least positive integer t such that the constant vector $t\boldsymbol{\lambda}$ is an integral linear combination of $\tau(x)$ over all vertices x of graphs in $\mathscr{G}$. Let $\beta(\mathscr{G}; \boldsymbol{\lambda})$ denote the least positive integer m such that the constant vector $m\boldsymbol{\lambda}$ is an integral linear combination of $\mu(G)$, $G \in \mathscr{G}$. We say $\mathscr{G}$ is $\boldsymbol{\lambda}$-*admissible* when the vector $\boldsymbol{\lambda}$ is a positive rational linear combination of $\mu(G)$, $G \in \mathscr{G}$.

THEOREM 13.1. *Let $\mathcal{G}$ be a $\boldsymbol{\lambda}$-admissible family of simple edge-r-colored digraphs, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_r)$. Then there exists a constant $n_0 = n_0(\mathcal{G}; \boldsymbol{\lambda})$ such that $\mathcal{G}$-decompositions of $K_n^{[\lambda_1, \lambda_2, ..., \lambda_r]}$ exist for all $n \geqslant n_0$ satisfying the conditions*

$$n - 1 \equiv 0 \pmod{\alpha(\mathcal{G}; \boldsymbol{\lambda})}, \qquad n(n-1) \equiv 0 \pmod{\beta(\mathcal{G}; \boldsymbol{\lambda})}. \tag{13.1}$$

For the proof, we use the following simple lemma. Given a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_r)$ of positive integers, we define a $\boldsymbol{\lambda}$-*refinement* of a nonnegative integral vector $\mathbf{a} = (a_1, ..., a_r)$ to be any nonnegative integral vector $\mathbf{b} = (b_1, ..., b_p)$, where $p = \lambda_1 + \cdots + \lambda_r$, such that $b_1, b_2, ..., b_{\lambda_1}$ sum to a_1, $b_{\lambda_1+1}, b_{\lambda_1+2}, ..., b_{\lambda_1+\lambda_2}$ sum to a_2, the next λ_3 coordinates of $\mathbf{b}$ sum to a_3, etc. For example, $(6, 1, 0, 4, 5)$ is a $(3, 2)$-refinement of $(7, 9)$.

LEMMA 13.2. *The vector $\boldsymbol{\lambda}$ of length r is a positive rational linear combination of $\mathbf{a}_1, \mathbf{a}_2, ...$ if and only if the vector $(1, 1, ..., 1)$ of length p is a positive rational linear combination of $\boldsymbol{\lambda}$-refinements of $\mathbf{a}_1, \mathbf{a}_2, ...$. Given a positive integer k, $k\boldsymbol{\lambda}$ is an integral linear combination of $\mathbf{a}_1, \mathbf{a}_2, ...$ if and only if $k(1, 1, ..., 1)$ is an integral linear combination of $\boldsymbol{\lambda}$-refinements of $\mathbf{a}_1, \mathbf{a}_2, ...$.*

Proof. First note that if $k(1, 1, ..., 1)$ of length p is a linear combination $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots$ where each $\mathbf{b}_i$ is a $\boldsymbol{\lambda}$-refinement of a vector $\mathbf{a}_i$ of length r, then $k\boldsymbol{\lambda} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots$ with the same coefficients. This follows by summing the coordinates in each block of λ_i coordinates in the vectors of length p.

For any vector $\mathbf{a} = (a_1, ..., a_r)$, the sum $\Sigma(\mathbf{a})$ of all $\boldsymbol{\lambda}$-refinements of $\mathbf{a}$ has, by symmetry, constant coefficients within each block of length λ_i, and so if $N(\mathbf{a})$ is the total number of such $\boldsymbol{\lambda}$-refinements, then

$$\Sigma(\mathbf{a}) = N(\mathbf{a}) \left(\frac{a_1}{\lambda_1}, ..., \frac{a_1}{\lambda_1}, \frac{a_2}{\lambda_2}, ..., \frac{a_2}{\lambda_2}, ..., \frac{a_r}{\lambda_r}, ..., \frac{a_r}{\lambda_r}\right).$$

If

$$(\lambda_1, ..., \lambda_r) = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots$$

with nonnegative coefficients c_i, then

$$(1, 1, ..., 1) = \frac{c_1}{N(\mathbf{a}_1)}\, \Sigma(\mathbf{a}_1) + \frac{c_2}{N(\mathbf{a}_2)}\, \Sigma(\mathbf{a}_2) + \cdots,$$

and the first part of the lemma follows.

Suppose that for some integer k,

$$k\boldsymbol{\lambda} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots \tag{13.2}$$

with integer coefficients c_i. To show that $k(1, 1, 1, ..., 1)$ (length p) is an integral linear combination of vectors $\mathbf{b}$ that are refinements of the $\mathbf{a}_i$'s, it will suffice by Lemma 5.1 to show that if $y_1, ..., y_p$ are rational numbers so that the inner product $(y_1, ..., y_p) \cdot \mathbf{b}$ is integral for every refinement $\mathbf{b}$ of an $\mathbf{a}_i$, then $k(y_1 + \cdots + y_p)$ is an integer. Let $\mathbf{y} = (y_1, ..., y_p)$ be given with the former property.

Let $\mathbf{a} = (a_1, a_2, ..., a_r)$ be one of the $\mathbf{a}_j$'s. If $\lambda_1 > 1$, then among the $\boldsymbol{\lambda}$-refinements of $\mathbf{a}$ we find the vector $\mathbf{b}_1$ where the i-th coordinate is replaced by $(a_i, 0, ..., 0)$ of length λ_i for all $i = 1, 2, ..., r$, and the vector $\mathbf{b}_2$ where the first coordinate is replaced by $(a_1 - 1, 1, 0, ..., 0)$ of length λ_1 but which otherwise agrees with $\mathbf{b}_1$. Since $\mathbf{y} \cdot \mathbf{b}_1$ and $\mathbf{y} \cdot \mathbf{b}_2$ are integral, so is $\mathbf{y} \cdot (\mathbf{b}_1 - \mathbf{b}_2)$. That is, the difference $y_1 - y_2$ is integral. In a similar manner, the difference $y_i - y_j$ is integral for $1 \leqslant i, j \leqslant \lambda_1$. And, also similarly, $y_i - y_j$ will be integral for any i, j in one of the intervals of length λ_ℓ that partition $\{1, 2, ..., p\}$. Thus if y_{i_ℓ} is any coordinate of $\mathbf{y}$ in the ℓ-th interval, then $k(y_1 + \cdots + y_p)$ differs by an integer from

$$\begin{aligned} k(\lambda_1 y_{i_1} + \cdots + \lambda_r y_{i_r}) &= k(y_{i_1}, ..., y_{i_r}) \cdot (\lambda_1, ..., \lambda_r) \\ &= (y_{i_1}, ..., y_{i_r}) \cdot (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots), \end{aligned}$$

the latter equality from (13.2). The proof will be complete once we note that the above is an integer. This is the case because $(y_{i_1}, ..., y_{i_r}) \cdot \mathbf{a}_\ell$ is equal to the integer $(y_1, ..., y_p) \cdot \mathbf{b}_\ell$ where $\mathbf{b}_\ell$ is obtained from $\mathbf{a}_\ell$ by putting the jth coordinate of $\mathbf{a}_\ell$ in position i_j of $\mathbf{b}_\ell$ and filling all other coordinates with zeros. ▮

Remark. The proof of Lemma 13.2 is easily modified/generalized to show, given a $\boldsymbol{\lambda}$-refinement $\mathbf{s}$ of a vector $\mathbf{t}$, that $\mathbf{t}$ is an integral linear combination of vectors from a set S if and only if $\mathbf{s}$ is an integral linear combination of $\boldsymbol{\lambda}$-refinements of members of S.

Proof of Theorem 13.1. Let $p = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. Let $\mathscr{H}$ denote the family of edge-p-colored digraphs that are obtained from digraphs in $\mathscr{G}$ as follows. For each edge-r-colored digraph $G \in \mathscr{G}$, take all edge-p-colored digraphs that result when the edges of color 1 of G are recolored in all possible ways with colors $1, 2, ..., \lambda_1$, the edges of color 2 are recolored in all possible ways with colors $\lambda_1 + 1, \lambda_1 + 2, ..., \lambda_1 + \lambda_2$, the edges of color 3 are recolored in all possible ways with the next λ_3 colors, etc. If $K_n^{(p)}$ admits an $\mathscr{H}$-decomposition, then $K_n^{[\lambda_1, \lambda_2, ..., \lambda_r]}$ admits a $\mathscr{G}$-decomposition. (We might think of the original r colors as red, green, etc., and then of introducing λ_1 "shades" of red, λ_2 "shades" of green, etc.)

The vectors $\mu(H)$, $H \in \mathscr{H}$, are exactly the $(\lambda_1, \lambda_2, ..., \lambda_r)$-refinements of $\mu(G)$, $G \in \mathscr{G}$. Similarly, the vectors $\tau_H(x)$, $H \in \mathscr{H}$, $x \in V(H)$ are the $(\lambda_1, \lambda_1, ..., \lambda_r, \lambda_r)$-refinements of the vectors $\tau_G(x)$, $G \in \mathscr{G}$, $x \in V(G)$. The first part of Lemma 13.2 makes it clear that $\mathscr{G}$ is $\boldsymbol{\lambda}$-admissible if and only if $\mathscr{H}$ is admissible.

By Theorem 1.2, $K_n^{(p)}$ admits an $\mathscr{H}$-decomposition for all sufficiently large n satisfying

$$n(n-1) \equiv 0 \pmod{\beta(\mathscr{H})} \quad \text{and} \quad n-1 \equiv 0 \pmod{\alpha(\mathscr{H})}.$$

However, Lemma 13.2 makes it clear that $\alpha(\mathscr{H}) = \alpha(\mathscr{G}; \lambda_1, \lambda_2, ..., \lambda_r)$ and $\beta(\mathscr{H}) = \beta(\mathscr{G}; \lambda_1, \lambda_2, ..., \lambda_r)$. ∎

Corollary 13.3. *$K_n^{[\lambda, \lambda, ..., \lambda]}$ admits a $\mathscr{G}$-decomposition for all sufficiently large integers n satisfying*

$$\lambda(n-1) \equiv 0 \pmod{\alpha(\mathscr{G})}$$
$$\lambda n(n-1) \equiv 0 \pmod{\beta(\mathscr{G})}$$

provided that $\mathscr{G}$ is admissible.

Proof. The congruence $\lambda n(n-1) \equiv 0 \pmod{\beta(\mathscr{G})}$ means $\lambda n(n-1)$ $(1, 1, ..., 1)$ is an integral linear combination of the vectors $\mu(G)$, $G \in \mathscr{G}$. The congruence $n(n-1) \equiv 0 \pmod{\beta(\mathscr{G}; \lambda, \lambda, ..., \lambda)}$ means $n(n-1)(\lambda, \lambda, ..., \lambda)$ is an integral linear combination of $\mu(G)$, $G \in \mathscr{G}$. These are obviously equivalent. Similarly, $\lambda(n-1) \equiv 0 \pmod{\alpha(\mathscr{G})}$ is equivalent to $n-1 \equiv 0 \pmod{\alpha(\mathscr{G}; \lambda, \lambda, ..., \lambda)}$. ∎

14. GDDS AND GRID DESIGNS WITH INDEX >1

With Theorem 13.1 and its corollary, we can get versions of the theorems on GDDs and grid designs for index $\lambda > 1$ with only the simplest changes in the proofs. The following two theorems are due to K. Chang [12].

Theorem 14.1. *Let integers g, k, λ be given with $g, k \geq 2$ and $\lambda \geq 1$. There exists a constant $n_0 = n_0(g, k, \lambda)$ such that group divisible designs with n groups of size g, blocks of size k, and index λ exist for all integers $n \geq n_0$ that satisfy*

$$\lambda g^2 n(n-1) \equiv 0 \pmod{k(k-1)},$$
$$\lambda g(n-1) \equiv 0 \pmod{k-1}.$$

Proof. The existence of a GDD of index unity is equivalent to a $\mathcal{G}$-decomposition of $K_n^{(r)}$, where $\mathcal{G}$ is the family of k-vertex edge-r-colored graphs described in the proof of Theorem 8.1, where $r = g^2$. The existence of a GDD with index λ is easily seen to be equivalent to a $\mathcal{G}$-decomposition of $K_n^{[\lambda, \lambda, ..., \lambda]}$ (where the number of λ's is r).

In the proof of Theorem 8.1, it was shown that $g^2 n(n-1) \equiv 0 \pmod{k(k-1)}$ and $g(n-1) \equiv 0 \pmod{k-1}$ together imply $n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})}$ and $n-1 \equiv 0 \pmod{\alpha(\mathcal{G})}$. We now claim that $\lambda g^2 n(n-1) \equiv 0 \pmod{k(k-1)}$ and $\lambda g(n-1) \equiv 0 \pmod{k-1}$ together imply $\lambda n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})}$ and $\lambda(n-1) \equiv 0 \pmod{\alpha(\mathcal{G})}$. To this end, we want to show, assuming the congruences in the hypothesis of Theorem 14.1, that if g^2 rationals x_{ij} are given so that (8.1) holds, then

$$\lambda n(n-1) \sum_{i,j} x_{ij} \equiv 0,$$

and if $2g^2$ rationals x_{ij}, y_{ij} are given so that (8.7) holds, then

$$\lambda(n-1) \sum_{i,j} (x_{ij} + y_{ij}) \equiv 0.$$

We may follow the same argument as in the proof of Theorem 8.1 except that some of the equations ('$\equiv$'s), namely (8.4), (8.6), the equation following (8.6), and (8.9), will have an extra factor of λ present on both sides. ∎

THEOREM 14.2. *Let integers g, k, λ be given with $g \geqslant k \geqslant 2$ and $\lambda \geqslant 1$. There exists a constant $n_0 = n_0(g, k, \lambda)$ such that $n \times g$ grid designs of index λ with block size k exist for all integers $n \geqslant n_0$ that satisfy*

$$\lambda g(g-1)\, n(n-1) \equiv 0 \pmod{k(k-1)},$$

$$\lambda(g-1)(n-1) \equiv 0 \pmod{k-1}.$$

Proof. The existence of a grid design of index unity is equivalent to a $\mathcal{G}$-decomposition of $K_n^{(r)}$, where $\mathcal{G}$ is the family of k-vertex edge-r-colored graphs described in the proof of Theorem 9.1, where $r = g(g-1)$. The existence of a grid design with index λ is easily seen to be equivalent to a $\mathcal{G}$-decomposition of $K_n^{[\lambda, \lambda, ..., \lambda]}$ (where the number of λ's is r).

In the proof of Theorem 9.1, it was shown that $g(g-1)\, n(n-1) \equiv 0 \pmod{k(k-1)}$ and $(g-1)(n-1) \equiv 0 \pmod{k-1}$ together imply $n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})}$ and $n-1 \equiv 0 \pmod{\alpha(\mathcal{G})}$. We now claim that $\lambda g(g-1)\, n(n-1) \equiv 0 \pmod{k(k-1)}$ and $\lambda(g-1)(n-1) \equiv 0 \pmod{k-1}$ together imply $\lambda n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})}$ and $\lambda(n-1) \equiv 0 \pmod{\alpha(\mathcal{G})}$.

To show that the congruences in the hypothesis of Theorem 14.2 imply that $\lambda n(n-1)(1, 1, ..., 1)$, of length $g(g-1)$, is an integral linear combination of the vectors $\mu(G(S))$, just take $m=\lambda n(n-1)$, rather than $n(n-1)$, in the penultimate paragraph of the proof of Theorem 9.1. To show that the second congruence in the hypothesis of Theorem 14.2 implies that $\lambda(n-1)(1, 1, ..., 1)$, of length $2g(g-1)$, is an integral linear combination of the degree-vectors $\tau(x)$ that arise from vertices x of graphs in $\mathcal{G}$, take the family $\mathcal{T}_\ell$ in the last paragraph of the proof of Theorem 9.1 to be a collection of $(k-1)$-subsets of $\{1, 2, ..., g\}\backslash\{\ell\}$ such that every element $i \neq \ell$ occurs in exactly $c=(k-1)/\gcd(k-1, \lambda(g-1))$ members of $\mathcal{T}_\ell$. This is a 1-(v', k', λ') design where $v'=g-1$, $k'=k-1$, $\lambda'=c$, which exists because $\lambda' v' \equiv 0 \pmod{k'}$. ∎

15. APPLICATION TO NEAR RESOLVABLE BIBDS

A (v, k, λ)-BIBD D is said to be *near resolvable* if the blocks of D can be partitioned into classes (*resolution classes*) $R_1, R_2, ..., R_v$ such that for each element x of D there is precisely one class which does not contain x in any of its blocks and each class contains precisely $v-1$ distinct elements of the design. The classes $R_1, R_2, ..., R_v$ form a *resolution* of D and D is denoted by NR(v, k, λ)-BIBD. Two necessary conditions for the existence of a NR(v, k, λ)-BIBD are $v \equiv 1 \pmod{k}$ and $\lambda = k-1$. The asymptotic existence of NR$(v, k, k-1)$-BIBDs was recently established in [14]. In this section, we give a new proof using Theorem 13.1.

THEOREM 15.1. *There exists a constant v_0 such that for all $v \geq v_0$ and $v \equiv 1 \pmod{k}$, there exists a NR$(v, k, k-1)$-BIBD.*

Proof. We claim that the existence of a NR$(v, k, k-1)$-BIBD is equivalent to the existence of a G-decomposition of $K_v^{[k-1, 1]}$ where G is the following graph. G has $k+1$ vertices $x_1, x_2, ..., x_k, y$, and two colors, $\{1\}$ and $\{2\}$. The directed edges of G are:

(i) $(x_i, x_j), (x_j, x_i)$ in color $\{1\}$, $i \neq j$, $1 \leq i, j \leq k$

(ii) (x_i, y) in color $\{2\}$, $i=1, 2, ..., k$.

So G contains $k(k-1)$ edges of color $\{1\}$ and k edges of color $\{2\}$.

We first show that if there exists a NR$(v, k, k-1)$-BIBD, then there exists a G-decomposition of $K_v^{[k-1, 1]}$. Let D be a NR$(v, k, k-1)$-BIBD defined on the element set $\{1, 2, ..., v\}$. Each element of D is missing from precisely one resolution class. Let R_i denote the resolution class which does not contain the element i in any of its blocks. Suppose $B=\{x_1, x_2, ..., x_k\}$

is a block in R_i. Then we construct a graph on $\{x_1, x_2, ..., x_k, i\}$ as described above. We verify that this gives a G-decomposition of $K_v^{[k-1,1]}$. Each edge (i, j), $i \neq j$, in color $\{1\}$ occurs $k-1$ times since each distinct pair $\{i, j\}$ occurs in $k-1$ blocks of D. Each edge (ℓ, i), $\ell \neq i$, in color $\{2\}$ occurs precisely once since every element of D except i occurs once in R_i. So we have a G-decomposition of $K_v^{[k-1,1]}$.

It is easy to reverse the construction and use a G-decomposition of $K_v^{[k-1,1]}$ to construct a NR$(v, k, k-1)$-BIBD. Let the vertices of $K_v^{[k-1,1]}$ be $\{1, 2, ..., v\}$. For each graph G, we construct a block $\{x_1, x_2, ..., x_k\}$ and place it in the resolution class which is missing element y, R_y. The decomposition for color $\{1\}$ insures that each unordered pair of distinct elements $\{i, j\}$, $i, j \in \{1, 2, ..., v\}$, occurs precisely $k-1$ times in the design. The decomposition for color $\{2\}$ insures that each resolution class R_i contains every element except i in its blocks, $i = 1, 2, ..., v$. Thus, we have a NR$(v, k, k-1)$-BIBD.

In order to apply Theorem 13.1, we need to find α and β. We order the vectors by color $\{1\}$ and color $\{2\}$. Since $\mu(G) = (k(k-1), k) = k(\lambda_1, \lambda_2)$, $\beta(G; k-1, 1) = k$. There are two types of degree-vectors $\tau : \tau(x_i) = (k-1, k-1, 0, 1)$ and $\tau(y) = (0, 0, k, 0)$. We would like to find the least positive integer t such that (t, t, t, t) is an integral linear combination of the vectors $\tau(x_i)$ and $\tau(y)$. First note that $t \geqslant k$ since the only nonzero entry of position 3 is k. Next note that $k\tau(x_i) + \tau(y) = (k(k-1), k(k-1), k, k)$. So this gives us $\alpha(G; k-1, 1) = k$. By Theorem 13.1, there exists a constant v_0 such that $K_v^{[k-1,1]}$ has a G-decomposition for all $v \geqslant v_0$ and $v \equiv 1 \pmod{k}$. ∎

Papers containing further applications of Theorems 1.2 and 13.1 are in preparation.

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