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Some properties of windowed linear canonical transform and its logarithmic uncertainty principle

Mawardi Bahri
Department of Mathematics, Hasanuddin University
Makassar 90245, Indonesia
mawardibahri@gmail.com

Ryuichi Ashino
Division of Mathematical Sciences
Osaka Kyoiku University, Osaka 582-8582, Japan
ashino@cc.osaka-kyoiku.ac.jp

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Based on the relationship between the Fourier transform (FT) and linear canonical transform (LCT), a logarithmic uncertainty principle and Hausdorff–Young inequality in the LCT domains are derived. In order to construct the windowed linear canonical transform (WLCT), Gabor filters associated with the LCT is introduced. Using the basic connection between the classical windowed Fourier transform (WFT) and the WLCT, a new proof of inversion formula for the WLCT is provided. This relation allows us to derive Lieb's uncertainty principle associated with the WLCT. Some useful properties of the WLCT such as bounded, shift, modulation, switching, orthogonality relation, and characterization of range are also investigated in detail. By the Heisenberg uncertainty principle for the LCT and the orthogonality relation property for the WLCT, the Heisenberg uncertainty principle for the WLCT is established. This uncertainty principle gives information how a complex function and its WLCT relate. Lastly, the logarithmic uncertainty principle associated with the WLCT is obtained.

Keywords: Complex-valued function; windowed linear canonical transform; logarithmic uncertainty principle.

AMS Subject Classification: 11R52, 42C40, 30G35

1. Introduction

It is well-known that the classical windowed Fourier transform (WFT) provides simultaneously information in time and frequency domains. It is also called *the short-time Fourier transform* (FT) or *Gabor transform* when used with a Gaussian windowed. The WFT has been widely studied in communication theory, quantum mechanics, and many other fields.[1,11,13] Some attempts at constructing the windowed fractional Fourier transform (WFRFT) which is a generalization of the WFT in the fractional Fourier domain have been undertaken in Refs. 23, 16 and 26. In Ref. 17, the authors have proposed the WFT to the windowed linear canonical transform (WLCT). This generalized transform is constructed by replacing the FT kernel with the LCT kernel in the WFT definition. Some important properties of the WLCT are discussed. Those include covariance property, orthogonality property and reconstruction formula. Generalized Poisson summation formula, series expansions and sampling formulas were also studied in detail.

On the other hand, in the classical analysis, the uncertainty principle for the FT relates a function and its FT which cannot both be simultaneously sharply localized. One example of this fact is the Heisenberg uncertainty principle concerning position and momentum wave functions in quantum physics. In signal processing an uncertainty principle states that the product of the variances of the signal in the *time* and *frequency* domains has a lower bound. Until now, many generalizations of the uncertainty principles to various types of functions and integral transformations have already been proposed in the literature. The authors in Refs. 15, 16, 27 and 24 discussed the uncertainty principles associated with the LCT. In Ref. 18, the authors have established uncertainty principles for the WLCT. Theirs uncertainties are generalizations of Lieb's uncertainty principles[13] in the WLCT domains. The various versions of uncertainty principles for the Wigner–Ville distribution were introduced in Ref. 19. Recently, paper[8] studied the logarithmic uncertainty principles for the Wigner–Ville distribution. The principles were established using the relationship between the FT and Wigner–Ville distribution.

In this paper, we derive the inversion formula for the WLCT using the basic connection between the WFT and the WLCT. Based on this relation we obtain Lieb's uncertainty principle associated with the WLCT. We also investigate in detail some important properties of the WLCT such as bounded, shift, modulation, switching, orthogonality relation, and characterization of range. Further, we shall derive three uncertainty principles related to the LCT domains. First of all, we shall derive the logarithmic uncertainty principle and sharp Hausdorff–Young inequality in the LCT domains based on the relationship between the LCT and the FT. Second, we shall establish the Heisenberg uncertainty principle associated with the WLCT. This uncertainty principle describes that the spread of a complex-valued function and its WLCT are inversely proportional. Third, we shall apply the logarithmic uncertainty principle for the LCT to obtain logarithmic uncertainty principle associated with the WLCT.

The paper is organized as follows: Section 2 presents the LCT and investigate the relationship between the FT and the LCT. Using some properties of the LCT we establish logarithmic uncertainty principle associated with the LCT. The construction of the WLCT using Gabor filters are presented in Sec. 3. Some properties of the WLCT is also investigated in this section. Section 4 provides the uncertainty principles associated with the WLCT.

2. Logarithmic Uncertainty Principle and Sharp Hausdorff–Young Inequality Associated with LCT

Let us begin with the following definitions.

Definition 2.1. For $1 \leq p \leq \infty$, the Lebesgue space $L^p(\mathbb{R})$ is defined as the space of all measurable functions on $\mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p} < \infty. \tag{1}$$

Definition 2.2. Denote by $C^\infty(\mathbb{R})$, the set of smooth functions on $\mathbb{R}$. For every choice of α and β of non-negative integers, the Schwartz space is defined by

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} |x^\alpha D^\beta f(x)| < \infty \right\}. \tag{2}$$

The element of the dual space $\mathcal{S}'$ of $\mathcal{S}$ is called *tempered distribution.*

2.1. *Definition of LCT*

The concept of the linear canonical transform (LCT) is firstly proposed by Moshinsky and Collins[9,20] by generalizing the classical FT. Here, we briefly introduce the LCT definition.

Definition 2.3 (LCT). Let $A = (a, b, c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $\det(A) = ad - bc = 1$. The LCT of a signal $f \in L^2(\mathbb{R})$ is defined by

$$L_A\{f\}(\omega) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(x) K_A(\omega, x) dx, & b \neq 0 \\ \sqrt{d} e^{i\frac{cd}{2}\omega^2} f(d\omega), & b = 0, \end{cases} \tag{3}$$

where $K_A(x, \omega)$ is so-called kernel of the LCT given by

$$K_A(x,\omega) = \frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}.$$

The LCT kernel mentioned above has the following important property:

$$K_{A^{-1}}(x,\omega) = \overline{K_A(x,\omega)} = \frac{1}{\sqrt{2\pi b}} e^{-i\frac{1}{2}(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}.$$

Since the LCT of a signal is essentially a chirp multiplication when $b = 0$, we always assume $b \neq 0$ in this paper. As a special case, when $A = (a, b, c, d) = (0, 1, -1, 0)$, the LCT definition (3) reduces to the FT definition. The inverse transform of the LCT is given by

$$\begin{aligned} L_A^{-1}[L_A\{f\}](x) = f(x) &= \int_{\mathbb{R}} L_A\{f\}(\omega)\, K_A{-1}(\omega, x)\, d\omega \\ &= \int_{\mathbb{R}} L_A\{f\}(\omega)\, \frac{1}{\sqrt{2\pi b}} e^{-i\frac{1}{2}(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}\, d\omega. \end{aligned} \tag{4}$$

The LCT of $f \in L^2(\mathbb{R})$ can be computed via associated FT, namely,

$$L_A\{f\}(\omega) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{\frac{id}{2b}\omega^2} \mathcal{F}\{e^{\frac{ia}{2b}x^2} f(x)\} \left(\frac{\omega}{b}\right), \tag{5}$$

where $\mathcal{F}\{f\}(\omega) = \hat{f}(\omega)$ is the FT of $f \in L^1(\mathbb{R})$ defined by (see Ref. 7)

$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x} dx. \tag{6}$$

Note that Eq. (5) takes the form

$$e^{-\frac{id}{2b}\omega^2} L_A\{f\}(\omega) = \mathcal{F}\{h\} \left(\frac{\omega}{b}\right), \tag{7}$$

where $h(x)$ is given by

$$h(x) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{\frac{ia}{2b}x^2} f(x). \tag{8}$$

An important property of the LCT is Parseval's formula:

$$\int_{\mathbb{R}} f(x)\overline{g(x)} dx = (f, g)_{L^2(\mathbb{R})} = (L_A\{f\}, L_A\{g\})_{L^2(\mathbb{R})} = \int_{\mathbb{R}} L_A\{f\}(\omega)\overline{L_A\{g\}(\omega)}\, d\omega. \tag{9}$$

In particular, when $f = g$, we obtain Plancherel's formula for the LCT:

$$\|f\|^2_{L^2(\mathbb{R})} = \|L_A\{f\}\|^2_{L^2(\mathbb{R})}. \tag{10}$$

Some of the other properties of the LCT corresponding to the FT properties are summarized in the following theorem. As the proof of Theorem 2.1 is quite similar to that of Theorems 9 and 10 in Ref. 2, we omit it here.

Theorem 2.1. *Let $f \in L^1(\mathbb{R})$. Then the LCT satisfies*:

- $\lim_{|\omega|\to\infty} |L_A\{f\}(\omega| = 0$.
- $L_A\{f\}(\omega)$ *is uniformly continuous on* $\mathbb{R}$.

2.2. *Logarithmic uncertainty principle for LCT*

The authors in Refs. 5, 12 and 22 have proposed the logarithmic uncertainty principle associated with the FT as follows.

Theorem 2.2 (FT logarithmic uncertainty principle). *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$\int_{\mathbb{R}} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}} \ln|\omega||\mathcal{F}\{f\}(\omega)|^2\, d\omega \geq D \int_{\mathbb{R}} |f(x)|^2 dx, \tag{11}$$

where $D = \psi(\frac{1}{2}) - \ln\pi$, $\psi(x) = \frac{d}{dx}\ln[\Gamma(x)]$ and $\Gamma(x)$ is the Gamma function.

A generalization of (11) is logarithmic uncertainty principle associated with the LCT, which is given by the following inequality.

Theorem 2.3 (LCT logarithmic uncertainty principle). *Let $f \in \mathcal{S}(\mathbb{R})$. Then we have*

$$\int_{\mathbb{R}} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}} \ln|\omega||L_A\{f\}(\omega)|^2\, d\omega \geq (D + \ln|b|) \int_{\mathbb{R}} |f(x)|^2 dx. \tag{12}$$

Proof. As f, $L_A\{f\} \in \mathcal{S}(\mathbb{R})$, all the integrals in (12) are finite. Then, h, defined by (8), belongs to $\mathcal{S}(\mathbb{R})$. By replacing f with h in both sides of Eq. (11), we have

$$\int_{\mathbb{R}} \ln|x||h(x)|^2 dx + \int_{\mathbb{R}} \ln|\omega||\mathcal{F}\{h\}(\omega)|^2\, d\omega \geq D \int_{\mathbb{R}} |h(x)|^2 dx. \tag{13}$$

Now setting $\omega = \frac{\omega}{b}$, we further obtain

$$\begin{aligned} &\int_{\mathbb{R}} \ln|x| \left| \frac{e^{-i\frac{\pi}{4}}}{\sqrt{b}} e^{\frac{ia}{2b}x^2} f(x) \right|^2 dx + \int_{\mathbb{R}} \ln\left|\frac{\omega}{b}\right| \left| \mathcal{F}\{h\}\left(\frac{\omega}{b}\right) \right|^2 d\frac{\omega}{b} \\ &\quad \geq D \int_{\mathbb{R}} \left| \frac{e^{-i\frac{\pi}{4}}}{\sqrt{b}} e^{\frac{ia}{2b}x^2} f(x) \right|^2 dx. \end{aligned} \tag{14}$$

Subsequently,

$$\int_{\mathbb{R}} \frac{1}{|b|} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}} \frac{1}{|b|} (\ln|\omega| - \ln|b|) \left| \mathcal{F}\{h\}\left(\frac{\omega}{b}\right) \right|^2 d\omega \geq D \frac{1}{|b|} \int_{\mathbb{R}} |f(x)|^2 dx. \tag{15}$$

By Eq. (7), we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{|b|} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}} \frac{1}{|b|} (\ln|\omega| - \ln|b|) |e^{-\frac{id}{2b}\omega^2} L_A\{f\}(\omega)|^2\, d\omega \\ &\quad \geq D \frac{1}{|b|} \int_{\mathbb{R}} |f(x)|^2 dx. \end{aligned} \tag{16}$$

The above equation can be simplified to

$$\begin{aligned} &\int_{\mathbb{R}} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}} \ln|\omega||L_A\{f\}(\omega)|^2\, d\omega - \int_{\mathbb{R}} \ln|b||L_A\{f\}(\omega)|^2\, d\omega \\ &\quad \geq D \int_{\mathbb{R}} |f(x)|^2 dx. \end{aligned} \tag{17}$$

By applying Prancherel's formula (10) to the third term of (17), we have

$$\int_{\mathbb{R}} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}} \ln|\omega||L_A\{f\}(\omega)|^2\, d\omega \geq (D + \ln|b|) \int_{\mathbb{R}} |f(x)|^2 dx,$$

which completes the proof. □

The sharp Hausdorff–Young inequality is a basic tool in Fourier analysis. Many inequalities are derived from it.[12] In the following, we derive the sharp Hausdorff–Young inequality associated with the LCT (compare to Ref. 16).

Theorem 2.4 (Hausdorff–Young LCT). *Let* $1 \leq p \leq 2$ *and* q *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for all* $f \in L^p(\mathbb{R})$,

$$\|L_A\{f\}\|_{L^q(\mathbb{R})} \leq |b|^{-1/2+1/q} \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2} \|f\|_{L^p(\mathbb{R})}. \tag{18}$$

Proof. From the sharp Hausdorff–Young inequality for the FT, we have

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{f\}(\omega)|^q \, d\omega\right)^{1/q} \leq \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2} \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}. \tag{19}$$

Following the steps of the proof of the preceding theorem, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}} |\mathcal{F}\{h\}(\omega)|^q \, d\omega\right)^{1/q} &\leq \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2} \left(\int_{\mathbb{R}} |h(x)|^p dx\right)^{1/p}, \\
\left(\int_{\mathbb{R}} \frac{1}{|b|}\left|\mathcal{F}\{h\}\left(\frac{\omega}{b}\right)\right|^q d\omega\right)^{1/q} &\leq \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2}\left(\int_{\mathbb{R}}\left|\frac{e^{-i\frac{\pi}{4}}}{\sqrt{b}} e^{\frac{ia}{2b}x^2} f(x)\right|^p dx\right)^{1/p}, \qquad (20)\\
\frac{1}{|b|^{1/q}}\left(\int_{\mathbb{R}} \left|\mathcal{F}\{h\}\left(\frac{\omega}{b}\right)\right|^q d\omega\right)^{1/q} &\leq \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2} \frac{1}{|b|^{1/2}} \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}.
\end{aligned}$$

Thus

$$\left(\int_{\mathbb{R}} |e^{-\frac{id}{2b}\omega^2} L_A\{f\}(\omega)|^q \, d\omega\right)^{1/q} \leq \left(\frac{p^{1/p}}{q^{1/q}}\right)^{1/2} |b|^{-1/2+1/q} \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}, \tag{21}$$

which completes the proof. □

3. Windowed Linear Canonical Transform

Before constructing the WLCT using Gabor filters associated with the LCT, we will introduce the definition of the classical WFT. It is an important tool in the time-frequency analysis, which has been extensively applied in speech, acoustics, and many other signal processing domains (see Refs. 1, 3, 10, 14 and 26). A generalization of the WFT to quaternion algebra and the fractional FT were introduced in Refs. 3 and 21, respectively.

3.1. *Gabor filters associated with LCT*

Definition 3.1 (WFT). For a window function $\phi \in L^2(\mathbb{R})\backslash\{0\}$, its *window daughter function* or its *windowed Fourier kernel* $\phi_{\omega,u}$ is defined by

$$\phi_{\omega,u}(x) = \phi(x - u)e^{i\omega x}. \tag{22}$$

The WFT of $f \in L^2(\mathbb{R})$ with respect to the window function $\phi \in L^2(\mathbb{R})\backslash\{0\}$ is defined by

$$G_\phi f(\omega, u) = \int_{\mathbb{R}} f(x)\, \overline{\phi_{\omega,u}(x)} dx. \tag{23}$$

By the uncertainty principle, the optimal window for time-frequency localization is achieved by any Gaussian function:

$$g(x,\sigma) = e^{-\frac{x^2}{2\sigma^2}}, \tag{24}$$

where σ is the standard deviation of the Gaussian function and determines the width of the window. In particular, for fixed ω_0,

$$g_{\omega_0,0}(x,\sigma) = e^{i\omega_0 x} g(x,\sigma) \tag{25}$$

is called a *Gabor filter*. The extension of the Gabor filter to the LCT domain is given by the following definition.

Definition 3.2. For a window function $\phi \in L^2(\mathbb{R})\backslash\{0\}$, its window daughter function associated with LCT is defined by

$$\phi^A_{\omega,u}(x) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}\phi(x-u). \tag{26}$$

For a fixed $\omega = \omega_0$, the Gabor filter associated with the LCT is defined by

$$g^A_{\omega_0,u}(x) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega_0 + \frac{d}{b}\omega_0^2 - \frac{\pi}{4})} e^{-\frac{(x-u)^2}{2\sigma^2}}. \tag{27}$$

Lemma 3.1. *For $\phi^A_{\omega,u} \in L^2(\mathbb{R})$, we have*

$$\|\phi^A_{\omega,u}\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi|b|}\|\phi\|^2_{L^2(\mathbb{R})}. \tag{28}$$

3.2. *Definition of WLCT*

In this subsection, using Gabor filters associated with LCT, we present the WLCT (compare to Refs. 4 and 17). We also discuss the connection between the WLCT and LCT.

Definition 3.3 (WLCT). Let $\phi \in L^2(\mathbb{R})\backslash\{0\}$ be a window function. Denote by G^A_ϕ, the WLCT on $L^2(\mathbb{R})$. The WLCT of $f \in L^2(\mathbb{R})$ with respect to ϕ is defined by

$$\begin{aligned} G^A_\phi f(\omega,u) &= \int_{\mathbb{R}} f(x)\,\overline{\phi^A_{\omega,u}(x)}dx \\ &= \int_{\mathbb{R}} f(x)\overline{\frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}\phi(x-u)}dx \\ &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\frac{1}{\sqrt{2\pi b}} e^{\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}dx. \end{aligned} \tag{29}$$

We will discuss the followings:

- When $A = (a,b,c,d) = (0,1,-1,0)$, Definition 3.3 reduces to (23).
- Equation (29) shows that it is also generated using the inverse LCT kernel.
- If we take the Gaussian function as the window function in (29), then we get the Gabor linear canonical transform (GLCT).
- For a fixed u, we have

$$G^A_\phi f(\omega,u) = L_A\{f\,T_u\bar{\phi}\}(\omega), \tag{30}$$

where the *translation operator* is defined by

$$T_u f(x) = f(x-u). \tag{31}$$

Equation (30) implies that the WLCT can be regarded as the LCT of the product of a function f and a conjugated and translated window function.

- The linear canonical window daughter function

$$\phi^A_{\omega,u}(x) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}\phi(x-u) \tag{32}$$

is also called the *linear canonical windowed Fourier kernel.*

- Applying the inverse LCT to (29), we have

$$\begin{aligned} f(x)\overline{\phi(x-u)} &= L_A^{-1}\{G^A_\phi f(\omega,u)\} \\ &= \int_{\mathbb{R}} G^A_\phi f(\omega,u)\frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}\, d\omega. \end{aligned} \tag{33}$$

- The *energy density* of the WLCT is defined by

$$|G^A_\phi f(\omega,u)|^2 = \left|\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\frac{1}{\sqrt{2\pi b}} e^{\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})} dx\right|^2, \tag{34}$$

which measures the energy of a signal in the time-frequency plane in neighborhood of the point (ω,u).

The following result describes an inequality related to the WLCT.

Theorem 3.2. *Let $\phi \in L^p(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, then*

$$\|G^A_\phi f(\omega,\cdot)\|_{L^p(\mathbb{R})} \le \frac{1}{\sqrt{2\pi b}}\|\phi\|_{L^p(\mathbb{R})}\|f\|_{L^1(\mathbb{R})}. \tag{35}$$

Proof. Applying Minkowski's inequality, we get

$$\begin{aligned} \|G^A_\phi f(\omega,\cdot)\|_{L^p(\mathbb{R})} &= \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\frac{1}{\sqrt{2\pi b}} e^{\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})} dx\right|^p du\right)^{1/p} \\ &\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(x)\overline{\phi(x-u)}\frac{1}{\sqrt{2\pi b}} e^{\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}\right|^p du\right)^{1/p} dx \\ &= \frac{1}{\sqrt{2\pi b}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(x)\overline{\phi(x-u)}\right|^p du\right)^{1/p} dx. \end{aligned}$$

Substituting $x-u=y$ in the above equation, we have

$$\begin{aligned} \|G^A_\phi f(\omega,\cdot)\|_{L^p(\mathbb{R})} &= \frac{1}{\sqrt{2\pi b}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(x)\overline{\phi(y)}\right|^p dy\right)^{1/p} dx \\ &\le \frac{1}{\sqrt{2\pi b}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\phi(y)|^p\, dy\right)^{1/p} |f(x)| dx \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi b}} \left(\int_{\mathbb{R}} |\phi(y)|^p \, dy \right)^{1/p} \int_{\mathbb{R}} |f(x)| dx, \tag{36}$$

which yields (35). □

Similar to (35), we obtain the following result by applying Holder's inequality.

Lemma 3.3. *Let $\phi \in L^p(\mathbb{R})$, $f \in L^q(\mathbb{R})$, $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$|G_\phi^A f(\omega, u)| \leq \frac{1}{\sqrt{2\pi b}} \|\phi\|_{L^p(\mathbb{R})} \|f\|_{L^q(\mathbb{R})}. \tag{37}$$

Remark 3.1. Let $p = q = 2$. Then, (37) can be reduced to

$$|G_\phi^A f(\omega, u)| \leq \frac{1}{\sqrt{2\pi b}} \|\phi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}, \tag{38}$$

which shows that $G_\phi^A f(\omega, u)$ is bounded on $L^2(\mathbb{R})$.

The following Lemma 3.4 will be used to prove Lieb's inequality for the WLCT.

Lemma 3.4. *The WLCT of a function $f \in L^2(\mathbb{R})$ with matrix parameter $A = (a, b, c, d)$ can be reduced to the WFT, that is,*

$$e^{-\frac{id}{2b}\omega^2} G_\phi^A f(\omega, u) = G_\phi h\left(\frac{\omega}{b}, u\right). \tag{39}$$

Proof. A straightforward computation shows that

$$\begin{aligned} G_\phi^A f(\omega, u) &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)} \frac{1}{\sqrt{2\pi b}} e^{\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})} dx \\ &= e^{\frac{id}{2b}\omega^2} \int_{\mathbb{R}} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{\frac{ia}{2b}x^2} f(x)\overline{\phi(x-u)} e^{i\frac{x\omega}{b}} dx \\ &= e^{\frac{id}{2b}\omega^2} \int_{\mathbb{R}} h(x)\overline{\phi(x-u)} e^{i\frac{x\omega}{b}} dx \\ &= e^{\frac{id}{2b}\omega^2} G_\phi h\left(\frac{\omega}{b}, u\right), \end{aligned} \tag{40}$$

which completes the proof. □

3.3. *Properties of WLCT*

The following results describe several useful properties of the WLCT, which have not been established in Ref. 17. As we will see, all properties of the WFT can be established in the WLCT domain.

Theorem 3.5 (Linearity). *Let $\phi \in L^2(\mathbb{R})\backslash\{0\}$ be a window function. The WLCT is a linear operator, namely,*

$$[G_\phi^A(\lambda f + \mu g)](\omega, u) = \lambda G_\phi^A f(\omega, u) + \mu G_\phi^A g(\omega, u), \tag{41}$$

for arbitrary constants λ and μ.

Proof. This follows directly from the linearity of the product and the integration involved in Definition 3.3. □

Theorem 3.6 (Parity). *Let $\phi \in L^2(\mathbb{R})\backslash\{0\}$ be a window function. Then we have*

$$G^A_{P\phi}\{Pf\}(\omega,u) = G^A_\phi f(-\omega,-u), \tag{42}$$

where $P\phi(x) = \phi(-x)$ for every $\phi \in L^2(\mathbb{R})$.

Proof. A direct calculation gives, for every $f \in L^2(\mathbb{R})$,

$$\begin{aligned} G^A_{P\phi}\{Pf\}(\omega,u) &= \int_{\mathbb{R}} f(-x)\overline{\phi(-(x-u))}\,\frac{1}{\sqrt{2\pi b}}e^{\frac{1}{2}i(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}dx \\ &= \int_{\mathbb{R}} f(-x)\overline{\phi(-x-(-u))} \\ &\quad \times \frac{1}{\sqrt{2\pi b}}e^{\frac{1}{2}i(\frac{a}{b}(-x)^2-\frac{2}{b}(-x)(-\omega)+\frac{d}{b}(-\omega)^2-\frac{\pi}{4})}dx, \end{aligned} \tag{43}$$

which proves the theorem according to Definition 3.3. □

Theorem 3.7 (Shift). *Let $\phi \in L^2(\mathbb{R})\backslash\{0\}$ be a window function. Then, we have*

$$G^A_\phi(T_{x_0}f)(\omega,u) = e^{ix_0\omega c}e^{-i\frac{ax_0^2}{2}c}G^A_\phi f(\omega - x_0 a, u - x_0). \tag{44}$$

Proof. By Eq. (29), we have

$$G^A_\phi(T_{x_0}f)(\omega,u) = \int_{\mathbb{R}} f(x-x_0)\overline{\phi(x-u)}\,\frac{1}{\sqrt{2\pi b}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}dx. \tag{45}$$

By making the change of variable $t = x - x_0$ in the above expression, we obtain

$$\begin{aligned} &G^A_\phi(T_{x_0}f)(\omega,u) \\ &\quad = \int_{\mathbb{R}} f(t)\overline{\phi(t-(u-x_0))}\,\frac{1}{\sqrt{2\pi b}}e^{i\frac{1}{2}(\frac{a}{b}(t+x_0)^2-\frac{2}{b}(t+x_0)\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}\,dt \\ &\quad = \int_{\mathbb{R}} f(t)\overline{\phi(t-(u-x_0))} \\ &\qquad \times \frac{1}{\sqrt{2\pi b}}e^{i\left(\frac{1}{2}\frac{a}{b}t^2\right)}e^{i\left(\frac{1}{2}\frac{2a}{b}tx_0\right)}e^{i\left(-\frac{1}{2}\frac{2t\omega}{b}\right)}e^{i\left(\frac{1}{2}\frac{a}{b}x_0^2\right)}e^{i\left(-\frac{1}{2}\frac{2x_0\omega}{b}\right)}e^{i\left(\frac{1}{2}\frac{d}{b}\omega^2\right)}e^{-i\frac{\pi}{4}}\,dt \\ &\quad = \int_{\mathbb{R}} f(t)\overline{\phi(t-(u-x_0))} \\ &\qquad \times \frac{1}{\sqrt{2\pi b}}e^{i\frac{1}{2}\left(\frac{a}{b}t^2-\frac{2}{b}t(\omega-x_0a)+\frac{d}{b}\omega^2-\frac{\pi}{4}\right)}e^{i\left(\frac{1}{2}\frac{a}{b}x_0^2\right)}e^{i\left(-\frac{1}{2}\frac{2x_0\omega}{b}\right)}\,dt. \end{aligned} \tag{46}$$

Therefore, we get

$$
\begin{aligned}
&G_\phi^A(T_{x_0}f)(\omega,u)\\
&\quad= \int_{\mathbb{R}} f(t)\overline{\phi(t-(u-x_0))}\\
&\qquad\times \frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}\left(\frac{a}{b}t^2-\frac{2}{b}t(\omega-x_0a)+\frac{d}{b}(\omega-x_0a)^2-\frac{\pi}{4}\right)} e^{i\frac{1}{2}\frac{d}{b}(2(\omega-x_0a)x_0a+(x_0a)^2)}\\
&\qquad\times e^{i\left(\frac{1}{2}\frac{a}{b}x_0^2\right)} e^{i\left(-\frac{1}{2}\frac{2x_0\omega}{b}\right)} dy\\
&\quad= e^{i\frac{1}{2}\frac{d}{b}(2(\omega-x_0a)x_0a+(x_0a)^2)} e^{i\left(\frac{1}{2}\frac{a}{b}x_0^2\right)} e^{i\left(-\frac{1}{2}\frac{2x_0\omega}{b}\right)} G_\phi^A f(\omega-x_0a,u-x_0).
\end{aligned} \tag{47}
$$

Finally we arrive at

$$G_\phi^A(T_{x_0}f)(\omega,u) = e^{ix_0\omega c} e^{-i\frac{ax_0^2}{2}c} G_\phi^A f(\omega-x_0a,u-x_0),$$

which completes the proof. □

Theorem 3.8 (Modulation). *For every $f \in L^2(\mathbb{R})$, we have*

$$G_\phi^A(M_{\omega_0}f)(\omega,u) = e^{id\omega\omega_0} e^{-i\frac{1}{2}\,db\omega_0^2} G_\phi^A f(x,\omega-\omega_0 b). \tag{48}$$

Proof. Direct computations show that

$$
\begin{aligned}
&G_\phi^A(M_{\omega_0}f)(\omega,u)\\
&\quad= \int_{\mathbb{R}} e^{i\omega_0 x} f(x)\overline{\phi(x-u)}\,\frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}dx\\
&\quad= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2+2\omega_0x-\frac{\pi}{4})}dx\\
&\quad= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\,\frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x(\omega-\omega_0b)+\frac{d}{b}\omega^2-\frac{\pi}{4})}dx\\
&\quad= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\,\frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}\left(\frac{a}{b}x^2-\frac{2}{b}x(\omega-\omega_0b)+\frac{d}{b}((\omega-\omega_0b)+\omega_0b)^2-\frac{\pi}{4}\right)}dx\\
&\quad= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\\
&\qquad\times \frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}\left(\frac{a}{b}x^2-\frac{2}{b}x(\omega-\omega_0b)+\frac{d}{b}((\omega-\omega_0b)^2-\frac{\pi}{4}+2(\omega-\omega_0b)\omega_0b+\omega_0^2b^2)\right)}dx.
\end{aligned}
$$

Applying Definition 3.3 finishes the proof. □

Theorem 3.9 (Conjugation). *Let $f \in L^2(\mathbb{R})$. If ϕ is a real window function, then*

$$G_\phi^A\bar{f}(\omega,u) = \overline{G_\phi^{A^{-1}}f(\omega,u)}. \tag{49}$$

Proof. It follows from (29) that

$$\begin{aligned} G_\phi^A \bar{f}(\omega,u) &= \int_{\mathbb{R}} \overline{f(x)}\phi(x-u)\frac{1}{\sqrt{2\pi b}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}dx \\ &= \int_{\mathbb{R}} \overline{f(x)\overline{\phi(x-u)}\frac{1}{\sqrt{2\pi b}}e^{-i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}}dx \\ &= \overline{G_\phi^{A^{-1}} f(\omega,u)}, \end{aligned}$$

which completes the proof. □

Taking the matrix parameter $A=(a,b,c,d)=(0,1,-1,0)$, then Eq. (49) implies $G_\phi^A \bar{f}(\omega,u) = \overline{G_\phi^A f(\omega,-u)}$. This form resembles the complex conjugation property of the classical WFT.

Theorem 3.10 (Switching f with ϕ). *Let $f,\ \phi \in L^2(\mathbb{R})\backslash\{0\}$ be window functions. Then we obtain*

$$G_\phi^A f(\omega,u) = e^{i\left(\frac{1}{2}\frac{a}{b}u^2\right)}e^{-i\left(\frac{u\omega}{b}\right)}\overline{G_f^{A^{-1}}\phi(\omega-ua,-u)}. \tag{50}$$

Proof. By invoking the WLCT definition (29), we have

$$\begin{aligned} G_\phi^A f(\omega,u) &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\,\frac{1}{\sqrt{2\pi b}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}dx \\ &= \int_{\mathbb{R}} \overline{\phi(x-u)\overline{f(x)}\,\frac{1}{\sqrt{2\pi b}}e^{-i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}}dx. \end{aligned} \tag{51}$$

By means of the substitution $y=x-u$ in the above expression we obtain

$$\begin{aligned} &G_\phi^A f(\omega,u) \\ &= \int_{\mathbb{R}} \overline{\phi(y)\overline{f(y+u)}\,\frac{1}{\sqrt{2\pi b}}e^{-i\frac{1}{2}(\frac{a}{b}(y+u)^2-\frac{2}{b}(y+u)\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}}dy \\ &= \int_{\mathbb{R}} \overline{\phi(y)\overline{f(y+u)}\frac{1}{\sqrt{2\pi b}}e^{-i\frac{1}{2}(\frac{a}{b}(y+u)^2-\frac{2}{b}(y+u)\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}}dy \\ &= \int_{\mathbb{R}} \overline{\begin{matrix}\phi(y)\overline{f(y+u)}\frac{1}{\sqrt{2\pi b}}e^{-i(\frac{1}{2}\frac{a}{b}y^2)}e^{-i\frac{\pi}{4}}e^{-i(\frac{1}{2}\frac{2a}{b}yu)}e^{-i(-\frac{1}{2}\frac{2y\omega}{b})} \\ e^{-i(\frac{1}{2}\frac{a}{b}u^2)}e^{-i(-\frac{1}{2}\frac{2u\omega}{b})}e^{-i(\frac{1}{2}\frac{d}{b}\omega^2)}\end{matrix}}dy \\ &= \int_{\mathbb{R}} \overline{\phi(y)\overline{f(y+u)}\,\frac{1}{\sqrt{2\pi b}}e^{-i\frac{1}{2}(\frac{a}{b}y^2-\frac{2}{b}y(\omega-ua)+\frac{d}{b}\omega^2-\frac{\pi}{4})}e^{-i\left(\frac{1}{2}\frac{a}{b}u^2\right)}e^{-i\left(-\frac{1}{2}\frac{2u\omega}{b}\right)}}dy \\ &= e^{i\left(\frac{1}{2}\frac{a}{b}u^2\right)}e^{-i\left(\frac{u\omega}{b}\right)}\overline{G_f^{A^{-1}}\phi(\omega-ua,-u)}, \end{aligned} \tag{52}$$

which completes the proof. □

Theorem 3.11 (Orthogonality relation[17]). *Let* ϕ, $\psi \in L^2(\mathbb{R})\backslash\{0\}$ *be window functions and* f, $g \in L^2(\mathbb{R})$ *be arbitrary. Then, we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^A f(\omega, u)\,\overline{G_\psi^A g(\omega, u)}\, d\omega\, du = (\bar{\phi}, \bar{\psi})_{L^2(\mathbb{R})}(f, g)_{L^2(\mathbb{R})}. \tag{53}$$

From the above theorem, we obtain the following consequences.

(i) If $\phi = \psi$, then

$$\int_{\mathbb{R}} G_\phi^A f(\omega, u)\,\overline{G_\phi^A g(\omega, u)}\, du\, d\omega = \|\phi\|_{L^2(\mathbb{R})}(f, g)_{L^2(\mathbb{R})}. \tag{54}$$

(ii) If $f = g$ and $\phi = \psi$, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|G_\phi^A f(\omega, u)\right|^2\, du\, d\omega = \|f\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})}. \tag{55}$$

(iii) When $\|\phi\|_{L^2(\mathbb{R})} = 1$, (55) is reduced to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|G_\phi^A f(\omega, u)\right|^2 du\, d\omega = \|f\|^2_{L^2(\mathbb{R})}. \tag{56}$$

Remark 3.2. Suppose that $\|f\|^2_{L^2(\mathbb{R})} = 1$ and $\|g\|^2_{L^2(\mathbb{R})} = 1$. Then, (55) is reduced to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G_\phi^A f(\omega, u)|^2\, d\omega\, du = 1. \tag{57}$$

Equation (57) is known as the radar uncertainty principle in the WLCT domain. It is easily seen that the function $G_\phi^A f(\omega, u)$ cannot be concentrated arbitrarily close to the origin.

We will provide a new proof for the WLCT inversion formula using the connection between the WFT and WLCT. This theorem tells us that it is possible to restore the original signal f perfectly using the inverse WLCT as follows.

Theorem 3.12 (Inversion formula). *Let* $f \in L^2(\mathbb{R})$. *Then, the inversion formula of the WLCT can be derived from that of the WFT, namely,*

$$f(x) = \frac{1}{(\phi, \psi)}\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^A f(\omega, u)\overline{K_A(x, \omega)}\psi(x - u)\, d\omega\, du. \tag{58}$$

Under the same assumptions as in (54), *we obtain*

$$f(x) = \frac{1}{\|\phi\|^2_{L^2(\mathbb{R})}}\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^A f(\omega, u)\phi^A_{\omega,u}\, d\omega\, du. \tag{59}$$

Proof. Since $h \in L^2(\mathbb{R})$, then the inverse transform of the WFT (23) implies

$$\begin{aligned} h(x) &= \frac{1}{2\pi(\phi, \psi)}\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi h(\omega, u)e^{i\omega x}\psi(x - u)\, d\omega\, du \\ &= \frac{1}{2\pi(\phi, \psi)}\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi h\left(\frac{\omega}{b}, u\right) e^{i\frac{x\omega}{b}}\psi(x - u)\, d\frac{\omega}{b}\, du. \end{aligned} \tag{60}$$

Here, $h(x)$ is defined by (8). It means that we have

$$\begin{aligned}
\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{\frac{ia}{2b}x^2} f(x) &\overset{(39)}{=} \frac{1}{2\pi(\phi,\psi)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{id}{2b}\omega^2} G_\phi^A f(\omega,u) e^{i\frac{x\omega}{b}} \psi(x-u)\, d\frac{\omega}{b}\, du \\
f(x) &= \frac{1}{(\phi,\psi)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{-\frac{ia}{2b}x^2} e^{-\frac{id}{2b}\omega^2} G_\phi^A f(\omega,u) e^{i\frac{x\omega}{b}} \psi(x-u)\, d\omega\, du \\
&= \frac{1}{(\phi,\psi)} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(\omega,u) \\
&\quad \times \overline{\frac{1}{\sqrt{2\pi b}} e^{\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})}} \psi(x-u)\, d\omega\, du,
\end{aligned}$$

which proves (58). □

Theorem 3.13 (Characterization of range of G_ϕ^A). *Let $\phi \in L^2(\mathbb{R})\backslash\{0\}$ be a window function such that $|\phi\|_{L^2(\mathbb{R})} = 1$. Suppose that $h \in L^2(\mathbb{R})$. Then, $h \in G_\phi^A(L^2(\mathbb{R}))$ if and only if it satisfies*

$$h(\omega', u') = \int_{\mathbb{R}} \int_{\mathbb{R}} h(\omega,u)(\phi_{\omega,u}^A, \phi_{\omega',u'})_{L^2(\mathbb{R})}\, d\omega\, du. \tag{61}$$

Proof. If $h \in G_\phi^A(L^2(\mathbb{R}))$, then there exist $f \in L^2(\mathbb{R})$ such that $G_\phi^A f = h$. Therefore, we have from (29) that

$$\begin{aligned}
G_\phi^A f(\omega', u') &= \int_{\mathbb{R}} f(x)\, \overline{\phi_{\omega',u'}^A(x)} dx \\
&\overset{(59)}{=} \int_{\mathbb{R}} \left(\frac{1}{\|\phi\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(\omega,u)\, \phi_{\omega,u}^A(x) du\, d\omega \right) \overline{\phi_{\omega',u'}^A(x)} dx \\
&= \frac{1}{\|\phi\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(\omega,u) \left(\int_{\mathbb{R}} \phi_{\omega,u}^A(x) \overline{\phi_{\omega',u'}^A(x)} dx \right) du\, d\omega \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(\omega,u)(\phi_{\omega,u}^A, \phi_{\omega',u'}^A)_{L^2(\mathbb{R})}\, du\, d\omega \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} h(\omega,u)(\phi_{\omega,u}^A, \phi_{\omega',u'}^A)_{L^2(\mathbb{R})}\, du\, d\omega.
\end{aligned}$$

Conversely, let $h(\omega, u)$ be any square integrable function satisfying

$$h(\omega', u') = \int_{\mathbb{R}} \int_{\mathbb{R}} h(\omega,u)(\phi_{\omega,u}^A, \phi_{\omega',u'}^A)_{L^2(\mathbb{R})}\, d\omega\, du. \tag{62}$$

If

$$f(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(\omega,u) \frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}i(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{4})} \phi(x-u)\, d\omega\, du,$$

then $f \in L^2(\mathbb{R})$ and $G_\phi^A f = h$. Since

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} f(x)\,\overline{f(x)}dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega,u)\frac{1}{\sqrt{2\pi b}}e^{-\frac{1}{2}i(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}\phi(x-u) \\
&\quad \times \overline{\phi(x-u')}\frac{1}{\sqrt{2\pi b}}e^{\frac{1}{2}i(\frac{a}{b}x^2-\frac{2}{b}x\omega'+\frac{d}{b}\omega'^2-\frac{\pi}{4})}\,d\omega\,du\,d\omega'\,du'dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega,u)\phi^A_{\omega,u}\,\overline{\phi^A_{\omega',u'}}\,\overline{h(\omega',u')}\,d\omega\,du\,d\omega'\,du'dx,
\end{aligned}$$

it implies that

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega,u)(\phi^A_{\omega,u},\phi^A_{\omega',u'})_{L^2(\mathbb{R})}\overline{h(\omega',u')}\,d\omega\,du\,d\omega'\,du' \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega',u')\overline{h(\omega',u')}\,d\omega'\,du' \\
&= \|h\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

which means that $f \in L^2(\mathbb{R})$. For every $(\omega',u') \in \mathbb{R}\times\mathbb{R}$, Fubini's theorem implies

$$\begin{aligned}
&G_\phi^A f(\omega',u') \\
&\quad= \int_{\mathbb{R}} f(x)\,\overline{\phi^A_{\omega',u'}(x)}dx \\
&\quad= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega,u)\frac{1}{\sqrt{2\pi b}}e^{-\frac{1}{2}i(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{4})}\phi(x-u)\,d\omega\,du\right)\overline{\phi^A_{\omega',u'}(x)}dx
\end{aligned}$$

Table 1. Properties of the WLCT.

Property	Function	WLCT
Linearity	$\lambda f + \mu g$	$\lambda G_\phi^A f(\omega,u) + \mu G_\phi^A g(\omega,u)$
Parity	$P\phi(x)$	$G_\phi^A f(-\omega,-u)$
Conjugation	$G_\phi^A \bar{f}(\omega,u) =$	$\overline{G_\phi^{A^{-1}} f(\omega,u)}$
Shift	$f(x-x_0)$	$e^{ix_0\omega c}e^{-i\frac{ax_0^2}{2}c}G_\phi^A f(\omega - x_0 a, u - x_0)$
Modulation	$f(x)e^{i\omega_0 x}$	$e^{id\omega\omega_0}e^{-i\frac{1}{2}db\omega_0^2}G_\phi^A f(x,\omega-\omega_0 b)$
Formula		
Orthogonality	$\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^A f(\omega,u)\,\overline{G_\psi^A g(\omega,u)}\,d\omega\,du = (\bar{\phi},\bar{\psi})_{L^2(\mathbb{R})}(f,g)_{L^2(\mathbb{R})}$	
	$\int_{\mathbb{R}} G_\phi^A f(\omega,u)\,\overline{G_\phi^A g(\omega,u)}\,du\,d\omega = \|\phi\|_{L^2(\mathbb{R})}(f,g)_{L^2(\mathbb{R})}$	
	$\int_{\mathbb{R}}\int_{\mathbb{R}} \left\|G_\phi^A f(\omega,u)\right\|^2\,du\,d\omega = \|f\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})}$	
Reconstruction	$f(x) = \frac{1}{(\phi,\psi)}\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^A f(\omega,u)\overline{K_A(x,\omega)}\psi(x-u)\,d\omega\,du$	

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega,u)\left(\int_{\mathbb{R}} \phi^A_{\omega,u}(x)\overline{\phi^A_{\omega',u'}(x)}dx\right) du\, d\omega$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(\omega,u)(\phi^A_{\omega,u},\phi^A_{\omega',u'})_{L^2(\mathbb{R})}\, du\, d\omega$$
$$= h(\omega',u'),$$

which completes the proof. □

The above properties of the WLCT are summarized in Table 1.

4. Logarithmic Uncertainty Principle for the WLCT

Let us give a short and simple proof of the WLCT Lieb's uncertainty principle by considering the fundamental relationship between the WFT and WLCT. The proposed proof is quite different from the one presented in Ref. 18. Then, we will establish a generalization of the Heisenberg type uncertainty principle for the WLCT, which describes how a complex function relates to its WLCT. Finally, we will obtain a logarithmic uncertainty principle associated with the WLCT.

Theorem 4.1. *Let f, $\phi \in L^2(\mathbb{R})$ and $2 \le p < \infty$. Then,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G^A_\phi f(\omega,u)|^p\, d\omega\, dx \le \frac{2}{p}(E_A)^p\left(\|f\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\right)^p, \tag{63}$$

where $E_A = (2\pi)^{-1/2}|b|^{1/p-1/2}$.

Proof. Based on the Lieb inequality for the WFT (see Ref. 13), we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G_\phi f(\omega,u)|^p\, d\omega\, dx \le \frac{2}{p}\left(\|f\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\right)^p. \tag{64}$$

Since $f \in L^2(\mathbb{R})$, it implies that $h \in L^2(\mathbb{R})$. Replacing f in both sides of (64) with h given in (7), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G_\phi h(\omega,u)|^p\, d\omega\, dx \le \frac{2}{p}\left(\|h\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\right)^p$$
$$= \frac{2}{p}\left(\left(\int_{\mathbb{R}}\left|\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}}e^{\frac{ia}{2b}x^2}f(x)\right|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R})}\right)^p. \tag{65}$$

Setting $\omega = \frac{\omega}{b}$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|b|}\left|G_\phi h\left(\frac{\omega}{b},u\right)\right|^p d\omega\, dx \le \frac{2}{p}\frac{1}{(2\pi|b|)^{p/2}}\left(\left(\int_{\mathbb{R}}|f(x)|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R})}\right)^p. \tag{66}$$

Applying Lemma 3.4 to the left-hand side of (66), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|b|}|e^{-\frac{id}{2b}\omega^2}G^A_\phi f(\omega,u)|^p\, d\omega\, dx \le \frac{2}{p}\frac{1}{(2\pi|b|)^{p/2}}\left(\left(\int_{\mathbb{R}}|f(x)|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R})}\right)^p, \tag{67}$$

which can be simplified to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G_\phi^A f(\omega,u)|^p \, d\omega \, dx \le \frac{2}{p}\frac{|b|}{(2\pi|b|)^{p/2}} \left(\|f\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\right)^p$$
$$= ((2\pi)^{-1/2}|b|^{1/p-1/2})^p(\|f\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})})^p.$$

This completes the proof. □

Theorem 4.2 (LCT uncertainty principle [15, 27]). *Let* $f \in L^2(\mathbb{R})$ *and* $L_A\{f\} \in L^2(\mathbb{R})$. *Then, we have*

$$\int_{\mathbb{R}} x^2|f(x)|^2 \, dx \int_{\mathbb{R}} \omega^2|L_A\{f\}(\omega)|^2 \, d\omega \ge \frac{b^2}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2. \tag{68}$$

Equality holds if and only if f *is the Gaussian function, namely,*

$$f(x) = C_0 \, e^{-ax^2}, \tag{69}$$

where C_0 *is a complex constant and* a *is positive real constant.*

Substituting the inverse transform for the LCT (4) into the left-hand side of (68), we obtain

$$\int_{\mathbb{R}} x^2|L_A^{-1}[L_A\{f\}](x)|^2 \, dx \int_{\mathbb{R}} \omega^2|L_A\{f\}(\omega)|^2 \, d\omega \ge \frac{b^2}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2. \tag{70}$$

Further, applying Plancherel's theorem for the LCT (10) to the right-hand side of (68), we have

$$\int_{\mathbb{R}} x^2|L_A^{-1}[L_A\{f\}](x)|^2 \, dx \int_{\mathbb{R}} \omega^2|L_A\{f\}(\omega)|^2 \, d\omega \ge \left(\frac{b}{2}\int_{\mathbb{R}} |L_A\{f\}(\omega)|^2 \, d\omega\right)^2. \tag{71}$$

Now we arrive at the following important result.

Theorem 4.3 (WLCT uncertainty principle). *For a given window function* $\phi \in L^2(\mathbb{R})\backslash\{0\}$, *let* $G_\phi^A f \in L^2(\mathbb{R})$ *be the WLCT of* f. *Then, for every* $f \in L^2(\mathbb{R})$, *we have the following inequality*:

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \omega^2|G_\phi^A f(\omega,u)|^2 \, d\omega \, du\right)^{1/2}\left(\int_{\mathbb{R}} x^2|f(x)|^2 dx\right)^{1/2} \ge \frac{b}{2}\|f\|_{L^2(\mathbb{R})}^2\|\phi\|_{L^2(\mathbb{R})}. \tag{72}$$

In order to prove Theorem 4.3, we need the following lemma.

Lemma 4.4. *We have*

$$\|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} x^2|f(x)|^2 \, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} x^2|L_A^{-1}\{G_\phi^A f(\omega,u)\}(x)|^2 dx \, du. \tag{73}$$

Proof. Simple calculation shows that

$$\begin{aligned}\|\phi\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}} x^2|f(x)|^2dx &= \int_{\mathbb{R}} x^2|f(x)|^2dx\int_{\mathbb{R}}|\phi(x-u)|^2\,du\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} x^2|f(x)|^2|\phi(x-u)|^2dx\,du\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} x^2|f(x)\overline{\phi(x-u)}|^2dx\,du\\ &\overset{(33)}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} x^2|L_A^{-1}\{G_\phi^A f(\omega,u)\}(x)|^2dx\,du. \qquad (74)\end{aligned}$$

In the second equality, we have applied Fubini's theorem to reverse the integration order. This completes the proof. □

Let us begin with the proof of Theorem 4.3.

Proof of Theorem 4.3. Assume that $L_A\{f\}\in L^2(\mathbb{R})$. Since $G_\phi^A f\in L^2(\mathbb{R})$, we can replace the LCT of f by the WLCT of f on the both sides of (71). Then, we have

$$\int_{\mathbb{R}}\omega^2|G_\phi^A f(\omega,u)|^2\,d\omega\int_{\mathbb{R}} x^2|L_A^{-1}\{G_\phi^A f(\omega,u)\}(x)|^2dx \geq \left(\frac{b}{2}\int_{\mathbb{R}}|G_\phi^A f(\omega,u)|^2\,d\omega\right)^2. \qquad (75)$$

Taking the square root on both sides of (75) and integrating both sides with respect to u, we have

$$\begin{aligned}&\int_{\mathbb{R}}\left\{\left(\int_{\mathbb{R}}\omega^2|G_\phi^A f(\omega,u)|^2\,d\omega\right)^{1/2}\left(\int_{\mathbb{R}} x^2|L_A^{-1}\{G_\phi^A f(\omega,u)\}(x)|^2dx\right)^{1/2}\right\}du\\ &\qquad\geq \frac{b}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}|G_\phi^A f(\omega,u)|^2\,d\omega\,du. \qquad (76)\end{aligned}$$

Furthermore, applying the Cauchy–Schwarz inequality to the left-hand side of (76), we have

$$\begin{aligned}&\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\omega^2|G_\phi^A f(\omega,u)|^2d\omega\,du\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} x^2|L_A^{-1}\{G_\phi^A f(\omega,u)\}(x)|^2dx\,du\right)^{1/2}\\ &\qquad\geq \frac{b}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}|G_\phi^A f(\omega,u)|^2\,d\omega\,du. \qquad (77)\end{aligned}$$

Inserting Lemma 4.4 into the second term on the left-hand side of (77) and substituting (55) into the right-hand side of this inequality. Then, we have

$$\begin{aligned}&\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\omega^2|G_\phi^A f(\omega,u)|^2d\omega\,du\right)^{1/2}\left(\|\phi\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}} x^2|f(x)|^2\,dx\right)^{1/2}\\ &\qquad\geq \frac{b}{2}\|f\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})}. \qquad (78)\end{aligned}$$

Dividing both sides of (78) by $\|\phi\|_{L^2(\mathbb{R})}$, we obtain the desired result. □

Remark 4.1. When $A = (a, b, c, d) = (0, 1, -1, 0)$, the uncertainty principle (72) becomes the Heisenberg uncertainty principle for the Gabor transform presented in Refs. 25 and 6.

Applying Plancherel's theorem for the LCT (10) to the right-hand side of (12), we have the following corollary.

Corollary 4.5. *We have*

$$\int_{\mathbb{R}} \ln |x||L_A^{-1}[L_A\{f\}](x)|^2 dx + \int_{\mathbb{R}} \ln |\omega||L_A\{f\}(\omega)|^2\, d\omega \geq (D + \ln |b|) \int_{\mathbb{R}} |L_A\{f\}|^2 dx. \tag{79}$$

In a similar manner, we can obtain the following result.

Lemma 4.6. *We have*

$$\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \ln |x||f(x)|^2\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \ln |x||L_A^{-1}\{G_\phi^A f(\omega, u)\}(x)|^2 dx\, du. \tag{80}$$

We finally obtain the logarithmic uncertainty principle associated with the WLCT, which is described in the following theorem.

Theorem 4.7 (WLCT logarithmic uncertainty principle). *For $\phi \in \mathcal{S}(\mathbb{R})$, then for every $f \in \mathcal{S}(\mathbb{R})$ we have the following inequality*:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |\omega||G_\phi^A f(\omega, u)|^2 d\omega\, du + \|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \ln |x||f(x)|^2 dx \geq (D + \ln |b|)\|f\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})}. \tag{81}$$

Proof. Since f, $\phi \in \mathcal{S}(\mathbb{R})$, then $L_A\{f\}$ and $G_\phi^A f$ belong to $\mathcal{S}(\mathbb{R})$. Therefore, we can substitute the LCT of f with the WLCT of f on both sides of (79) in Corollary 4.5 and get

$$\int_{\mathbb{R}} \ln |\omega||G_\phi^A f(\omega, u)|^2\, d\omega + \int_{\mathbb{R}} \ln |x||L_A^{-1}\{G_\phi^A f(\omega, u)\}(x)|^2 dx \geq (D + \ln |b|) \int_{\mathbb{R}} |G_\phi^A f(\omega, u)|^2\, d\omega. \tag{82}$$

Integrating both sides with respect to du yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |\omega||G_\phi^A f(\omega, u)|^2\, d\omega\, du + \int_{\mathbb{R}} \int_{\mathbb{R}} \ln |x||L_A^{-1}\{G_\phi^A f(\omega, u)\}(x)|^2 dx\, du \geq (D + \ln |b|) \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\phi^A f(\omega, u)|^2 d\omega\, du. \tag{83}$$

Inserting Eq. (80) into the second term on the left-hand side of (83) and substituting (55) into the first term on the right-hand side of (83), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln|\omega||G_\phi^A f(\omega,u)|^2 d\omega\, du + \|\phi\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}} \ln|x||f(x)|^2 dx$$
$$\geq (D+\ln|b|)\|f\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})},$$

which gives the desired result. □

Remark 4.2. When the window function is normalized, namely, $\|\phi\|_{L^2(\mathbb{R})} = 1$, then (81) implies

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln|\omega||G_\phi f(\omega,b)|^2\, d\omega\, du + \int_{\mathbb{R}} \ln|x||f(x)|^2 dx \geq (D+\ln|b|)\|f\|^2_{L^2(\mathbb{R})}. \qquad (84)$$

References

1. A. Ahadi, A. Khoshnevis and M. Z. Saghir, Windowed Fourier transform as an essential digital interferometry tool to study coupled heat and mass transfer, *Opt. Laser Technol.* **57** (2014) 304–317.
2. M. Bahri and R. Ashino, A simplified proof of uncertainty principle for quaternion linear canonical transforms, *Abstract Appl. Anal.* (2016) Article ID: 5874930, 11 pp.
3. M. Bahri, E. Hitzer, R. Ashino and R. Vaillancourt, Windowed Fourier transform of two-dimensional quaternionic signals, *Appl. Math. Comput.* **28**(6) (2010) 2366–2379.
4. R.-F. Bai, B.-Z. Li and Q.-Y. Cheng, Wigner–Ville distribution associated with the linear canonical transform, *J. Appl. Math.* **2012** (2012), Article ID: 740161, 14 pp.
5. W. Beckner, Pitt's inequality and uncertainty principle, *Proc. Amer. Math. Soc.* **123**(6) (1995) 1897–1905.
6. A. Bonami, B. Demange and P. Jaming, Hermite functions and uncertainty principles for the Fourier transform and the windowed Fourier transform, *Rev. Mat. Iberoamericana* **19**(1) (2003) 23–55.
7. R. Bracewell, *The Fourier Transform and its Applications* (McGraw Hill Higher Education, Boston, 2000).
8. Y. J. Cao, B. Z. Li, Y. G. Li and Y. H. Chen, Logarithmic uncertainty relations for odd or even signals associate with Wigner–Ville distribution, *Circuits, Syst. Signal. Process.* (2015), DOI: 10.1007/s00034-015-0146-x.
9. S. A. Collins, Lens-system diffraction integral written in term of matrix optics, *J. Opt. Soc. Am.* **60**(9) (1970) 1168–1177.
10. L. Debnath, *Wavelet Transforms and Their Applications* (Birkhäuser, Boston, 2002).
11. L. Debnath, B. V. Shankara and N. Rao, On new two-dimensional Wigner–Ville nonlinear integral transforms and their basic properties, *Integral Transforms Spec. Funct.* **21**(3) (2010) 165–174.
12. G. B. Folland and A. Sitaram, The uncertainty principle: A mathematical survey, *J. Fourier Anal. Appl.* **3**(3) (1997) 207–238.
13. K. Gröchenig, *Foundation of Time-Frequency Analysis* (Birkhäuser, Boston, 2001).
14. K. Gröchenig and G. Zimmermann, Hardy's theorem and the short-time Fourier transform of Schwartz functions, *J. London Math. Soc.* **63**(1) (2001) 205–214.
15. X. Guanlei, W. Xiaotong and X. Xiaogang, New inequalities and uncertainty relations on linear canonical transform revisit, *EURASIP J. Adv. Signal Process.* **2009** (2009), Article ID: 563265, 7 pp.

16. X. Guanlei, W. Xiaotong and X. Xiaogang, Uncertainty inequalities for linear canonical transform, *IET Signal Process.* **3**(5) (2009) 392–402.
17. K. I. Kou and R. H. Xu, Windowed linear canonical transform and its applications, *Signal Process.* **92**(1) (2012) 179–188.
18. K. I. Kou, R. H. Xu and Y. H. Zhang, Paley–Wiener theorems and uncertainty principles for the windowed linear canonical transform, *Math. Methods Appl. Sci.* **35** (2012) 2122–2132.
19. Y.-G. Li, B.-Z Li and H.-F. Sun, Uncertainty principle for Wigner–Ville distribution associated with the linear canonical transform, *Abstract Appl. Anal.* **2014** (2014), Article ID: 470459, 9 pp.
20. M. Moshinsky and C. Quesnee, Linear canonical transform and their unitary representations, *J. Math. Phys.* **12**(8) (1971) 1772–1780.
21. H. M. Ozaktas, Z. Zalevsky and M. A. Kutay, *The Fractional Fourier Transform with Application in Optics and Signal Processing* (Willey, New York, 2001).
22. F. Sontani, Pitt's inequality and logarithmic uncertainty principle for the Dunkl transform on $\mathbb{R}$, *Acta Math. Hungar.* **143**(2) (2014) 480–490.
23. L. Stankovic, T. Alieva and M. I. Bastiaans, Time-frequency signal analysis based on the windowed fractional Fourier transform, *Signal Process.* **83** (2003) 2459–2468.
24. R. Tao, Y.-L. Li and Y. Wang, Uncertainty principles for linear canonical transforms, IEEE Trans. *Signal Process.* **57**(7) (2009) 2856–2858.
25. E. Wilczok, New uncertainty principles for the continuous Gabor transform and the continuous wavelet transform, *Doc. Math.* **5** (2000) 201–226.
26. G. I. Xu, X. T. Wang and X. G. Xu, The logarithmic, Heisenberg's and short-time uncertainty principles associated with fractional Fourier transforms, *Signal Process.* **89**(3) (2009) 339–343.
27. Y. Yang and K. I. Kou, On uncertainty principles for hypercomplex signals in the linear canonical transform domains, *Signal Process.* **95** (2014) 67–75.