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Multi-modal and Temporal Logics with Universal Formula—Reduction of Admissibility to Validity and Unification*

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Abstract

The article studies multi-modal (in particular temporal, tense, logics) possessing universal formulas. We prove (Theorems 11 and 12) that the admissibility problem for inference rules (inference rules with parameters) is decidable for all decidable multi-modal (in particular, temporal) logics possessing an universal formula and decidable w.r.t. unification (unification with parameters). These theorems use characterizations of admissible rules in terms of valid rules and unifiable formulas. Results are applied to wide range of multi-modal logics, including all linear transitive temporal logics, all temporal logics with bounded zigzag, all multi-modal logics with explicit universal modality. As consequence, we show that the admissibility problem for all such logics is decidable (e.g. for all logics of the series $S4_nU$, $n \in N$).

Keywords: Decidability, algorithms, logical consecutions, inference rules, temporal logic, linear temporal logic, admissible rules.

1 Introduction

Admissible rules were (perhaps, first time explicitly) introduced into consideration by Lorenzen [19]. Initially there were only observations on existence of interesting examples of admissible but not derivable rules [14, 20]. In 1975, Friedman [4] set the problem: whether the intuitionistic logic IPC is decidable w.r.t. admissible inference rules. This problem (together with its counterpart for modal logic $S4$) was solved affirmatively in Rybakov [22, 26]. Algorithms deciding admissibility for some transitive modal logics and *IPC* based on projective formulas and unification were discovered by Ghilardi [8–10]. P. Roziere [21] found a solution of the Friedman problem for *IPC* within methods of proof theory. If a logic $\mathcal{L}$ itself is decidable and has a decidable basis for admissible rules, then, basing on recursive enumeration of not-admissible rules, a deciding algorithm for admissibility may be derived. Therefore, in particular, study of bases for admissible rules has been launched. It has been shown Rybakov [23] that *IPC* and $S4$ do not have finite bases, but later (cf. Iemhoff [15] and Jerábek [16], Rybakov [29]) for some modal logics (e.g. $S4, K4, Grz$) and IPC some infinite, explicit and decidable bases were constructed. Till 2006, only one—rather artificial but nice one—example of a decidable modal logic with undecidable admissibility problem was known (cf. Chagrov [5]). Quite recently Wolter and Zakharyaschev [36] showed that modal logics with additional universal modality situated between K_u and $K4_u$ are undecidable w.r.t admissibility and even w.r.t. just unification. Ref. [36] sets up an open question whether logics $S4_u$ are admissible w.r.t. admissibility.

In Rybakov's works [27, 28], a refined technique for deciding admissibility, which is based on dropping up, rarefication and co-cover property (extension property), has been developed. This

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technique allows by general and uniform way to construct algorithms recognizing admissibility in solid infinite classes of transitive modal logics. It covers all previously proved cases of decidability by admissibility, and solves also the problem of unification (unification with parameters, problem of logical equations) in all these logics. But it turns out, that even this technique does not work perfectly for intransitive and temporal logics. Therefore, the research passed to the phase of study such particular logics (for collecting experience to work on admissibility in this area). Decidability of the admissibility problem was proved for the temporal logics of all integer numbers $\mathcal{L}(\mathcal{Z})$ and natural numbers $\mathcal{L}(\mathcal{N})$ (cf. [30]) and for the intransitive temporal logic of all integer intervals [31], for a variation of LTL with Until and Since on integers [32]. The question how to extend these results to other linear temporal logics and just arbitrary tense logics, as well as other multi-modal logics, was open interesting problem.

The modal-like logics with linear alternative accessibility relations, linear temporal logics, were studied in details (and deserve it as most natural interpretation of the time flow). Bull [2, 3] and Segerberg [33] found finite axiomatizations for such logics, proved fmp (finite model property, and hence decidability) for some these logics. Though ref. [2] shows that the temporal logic $\mathcal{L}_Z$ of integers with strict order does not have fmp. The decidability of $\mathcal{L}_Z$, first, it seems, was proved by D. Gabbay basing at Rabin's results on monadic second-order theories of successor functions (perhaps the decidability also followed from earlier research of D. Scott using automata). Later a series of investigations concerning various properties of linear temporal logics was published in contemporary issues (cf., for instance, Gabbay, Hodkinson and Reynolds [6, 7], Goldblatt [11, 12], van Benthem [34] and Wolter [35]). Next interest to our research came from the easy observation that linear temporal logics possess the definable universal modality. The universal modality (it seems first in a regular way) was investigated in Goranko and Passy [13]. In study of hybrid logics, universal modality nowadays is one of standard constructors, cf. Areces, Blackburn, Marx [1]. Concerning the admissibility problem, the presence of definable universal modality, for certain, can give more instruments for work with inference rules.

We address our paper to all multi-modal logics (in particular, to all temporal logics), where the universal modality can be defined. We show, Theorems 11 and 12, that, for any decidable multi-modal logic $\mathcal{L}(\mathcal{K})$ (generated by a class of frames $\mathcal{K}$), where $\mathcal{K}$ admits an universal formula and were the unification (with parameters) in $\mathcal{L}(\mathcal{K})$ is decidable, the logic $\mathcal{L}(\mathcal{K})$ is also decidable by admissibility of inference rules (inference rules with parameters, meta-variables). We apply these results to various logics admitting universal formula: to all linear transitive temporal logics, to arbitrary temporal logics with bounded zigzag, to merely multi-modal logics with universal formula, and show the decidability of the admissibility problem.

2 Preliminaries, Definitions, Notation

The article uses standard notation and facts concerning multi-modal and temporal logics, primarily connected with Kripke/Hintikka semantics (we briefly recall some definitions and notation below). The language of multi-modal logics consists of the language of Boolean logic extended by new unary logical modal operations $\Box_i$. Basically only finite amount of modal operations, – n – of $\Box_i$, is used, then the introduced logics are called n-modal logics. Formation rules for formulas are standard.

A Kripke/Hintikka frame is a tuple $\mathcal{F} := \langle F, R_1, \ldots, R_n \rangle$, where F is the base of $\mathcal{F}$—a non-empty set, and all R_i are binary (accessibility) relations on F. $|\mathcal{F}| := F$, $a \in \mathcal{F}$ is denotation for $a \in |\mathcal{F}|$. If, for a set of propositional letters P, a valuation V of P in $|\mathcal{F}|$ is defined, i.e. $V : P \to 2^F$, $\forall p \in P(V(p) \subseteq F)$, we call $\mathcal{M} := \langle \mathcal{F}, V \rangle$ Kripke/Hintikka model (structure). The truth values of formulas are defined at elements of $\mathcal{F}$ by the following rules:

$$\forall p \in Prop,\ \forall a \in \mathcal{F},\ \ (\mathcal{F}, a) \Vdash_V p \Leftrightarrow a \in V(p);$$

$$(\mathcal{F}, a) \Vdash_V \varphi \wedge \psi \Leftrightarrow (\mathcal{F}, a) \Vdash_V \varphi \text{ and } (\mathcal{F}, a) \Vdash_V \psi;$$

$$(\mathcal{F}, a) \Vdash_V \varphi \vee \psi \Leftrightarrow (\mathcal{F}, a) \Vdash_V \varphi \text{ or } (\mathcal{F}, a) \Vdash_V \psi;$$

$$(\mathcal{F}, a) \Vdash \varphi \to \psi \Leftrightarrow \neg[(\mathcal{F}, a) \Vdash_V \varphi] \text{ or } (\mathcal{F}, a) \Vdash_V \psi;$$

$$(\mathcal{F}, a) \Vdash_V \neg\varphi \Leftrightarrow not[(\mathcal{F}, a) \Vdash_V \varphi];$$

$$(\mathcal{F}, a) \Vdash \Box_i \varphi \Leftrightarrow \forall a \in \mathcal{F}((aR_i b) \Rightarrow (\mathcal{F}, b) \Vdash \varphi)).$$

DEFINITION 1
For a Kripke–Hintikka structure $\mathcal{M} := \langle \mathcal{F}, V \rangle$ and a formula φ, φ is *true in* $\mathcal{M}$ (denotation—$\mathcal{M} \Vdash \varphi$) if $\forall a \in \mathcal{F}\ (\mathcal{F}, a) \Vdash_V \varphi$. $\mathcal{F} \Vdash_V \varphi \Leftrightarrow \forall w \in \mathcal{F}((\mathcal{F}, w) \Vdash_V \varphi)$.

DEFINITION 2
For a class $\mathcal{K}$ of frames, the logic $\mathcal{L}(\mathcal{K})$ generated by $\mathcal{K}$ is the set of all formulas which are true in all models based on frames from $\mathcal{K}$.

For instance, for the basic minimal normal modal logic K, $K = \mathcal{L}(\mathcal{K}_{fr})$, where $\mathcal{K}_{fr}$ is the set of all frames with single accessibility relation, for Lewis modal logic $S4$, $S4 := \mathcal{L}(\mathcal{K}_{rt})$, where $\mathcal{K}_{rt}$ is the set of all frames with a reflexive and transitive accessibility relation. We can take in the definitions above only finite frames, i.e. K and $S4$ have so called finite model property (fmp).

The definition for temporal logics can be given using special 2-modal logics. Consider a set $\mathcal{K}$ of all 2-frames of structure $\langle F, R, R^{-1} \rangle$, and the language with two modal operations $\Box^+$—for R, and $\Box^-$—for R^{-1}. The logic $\mathcal{L}(\mathcal{K})$ of $\mathcal{K}$ in this language is the temporal logic of $\mathcal{K}$. Operations $\Diamond^+$ and $\Diamond^-$ are derivatives of $\Box^+$ and $\Box^-$, as usual in modal logic. All Kripke-complete temporal logics can be generated this way. For any logic $\mathcal{L}(\mathcal{K})$, a formula φ is a theorem of $\mathcal{L}(\mathcal{K})$ if $\varphi \in \mathcal{L}(\mathcal{K})$, φ is satisfiable in $\mathcal{K}$ if, for some valuation V in some frame $\mathcal{F} \in \mathcal{K}$, φ is true w.r.t. V at a world of $\mathcal{F}$. A *consecution*, (or, synonymously,—a *rule, inference rule*) **c** is an expression

$$\mathbf{c} := \frac{\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)}{\psi(x_1, \ldots, x_n)},$$

where $\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are some formulas constructed out of letters $x_1, \ldots, x_n$. Letters $x_1, \ldots, x_n$ are called variables of **c**, for any consecution **c**, $Var(\mathbf{c}) := \{x_1, \ldots, x_n\}$. The formula $\psi(x_1, \ldots, x_n)$ is the conclusion of **c**, formulas $\varphi_j(x_1, \ldots, x_n)$ are the premises of **c**.

DEFINITION 3
A consecution **c** is said to be *valid* in a Kripke structure $\langle \mathcal{F}, V \rangle$ (we will use notation $\langle \mathcal{F}, V \rangle \Vdash \mathbf{c}$, or $\mathcal{F} \Vdash_V \mathbf{c}$) if $(\mathcal{F} \Vdash_V \bigwedge_{1 \le i \le m} \varphi_i) \Rightarrow (\mathcal{F} \Vdash_V \psi)$. Otherwise we say **c** is *refuted* in $\mathcal{F}$, or *refuted in $\mathcal{F}$ by V*, and write $\mathcal{F} \nVdash_V \mathbf{c}$. A consecution **c** is *valid* in a frame $\mathcal{F}$ (notation $\mathcal{F} \Vdash \mathbf{c}$) if, for any valuation V of $Var(\mathbf{c})$, $\mathcal{F} \Vdash_V \mathbf{c}$.

Problems which we will study in this article are connected with so-called admissible inference rules (proposed by Lorenzen [19]). The definition of these rules is as follows. Given a logic $\mathcal{L}$ ($Form_{\mathcal{L}}$ is the set of all formulas in the language of $\mathcal{L}$) and a consecution (rule) $\mathbf{c} := \varphi_1(x_1,\ldots,x_n),\ldots,\varphi_m(x_1,\ldots,x_n)/\psi(x_1,\ldots,x_n)$.

Definition 4
Rule $\mathbf{c}$ is said to be *admissible* for $\mathcal{L}$ if, $\forall \alpha_1 \in Form_{\mathcal{L}},\ldots,\forall \alpha_n \in Form_{\mathcal{L}}$, $[\bigwedge_{1\leq i\leq m}[\varphi_i(\alpha_1,\ldots,\alpha_n)\in \mathcal{L}]] \Longrightarrow [\psi(\alpha_1,\ldots,\alpha_n)\in\mathcal{L}]$.

That is $\mathbf{c}$ is admissible if, for every substitution s, $s(\varphi_1)\in\mathcal{L},\ldots,s(\varphi_n)\in\mathcal{L}$ implies $s(\psi)\in\mathcal{L}$. We list below few examples of admissible rules. The Harrop's rule (1960, [14]) $\neg x\to y\vee z/(\neg x\to y)\vee(\neg x\to z)$ is admissible in the intuitionistic logic IPC but not derivable in Heyting axiomatic system for IPC. The same properties hold for the rule $(x\to y)\to x\vee z/((x\to y)\to x)\vee((x\to y)\to z)$ found by G. Mints [20]. The Lemmon–Scott rule (cf. [27])

$$\Box(\Box(\Box\Diamond\Box p\to\Box p)\to(\Box p\vee\Box\neg\Box p))\ /\ \Box\Diamond\Box p\vee\Box\neg\Box p$$

is admissible (but non-derivable in standard Hilbert-style axiomatic systems) for modal logics $S4$, $S4.1$ and Grz. The *admissibility problem* for inference rules in $\mathcal{L}$ is to determine, given by arbitrary inference rule $\mathbf{c}$, whether $\mathbf{c}$ is admissible for $\mathcal{L}$. If there is an algorithm solving this problem for any rule, $\mathcal{L}$ is said to be decidable by admissibility.

3 Logics with universal formula

For any frame $\mathcal{F} := \langle F, R_1,\ldots,R_n\rangle$ and any $w\in F$, the frame $\mathcal{F}(w)^s$ strongly generated by w in $\mathcal{F}$ is the set of all $a\in\mathcal{F}$, were $a = w$ or

$$wQ_{i1}w_1, w_1Q_{i2}w_2,\ldots w_{k-1}Q_kw_k, w_k = a$$

for some $Q_j\in\{R_1,\ldots R_n, R_1^{-1},\ldots R_n^{-1}\}, w_i\in\mathcal{F}$.

Definition 5
We say a class of frames $\mathcal{K}$ admits an universal formula if there is a formula $\Box_u(x)$ (called universal formula) constructed from a single letter x and logical operations, such that the following holds. For any $\mathcal{F}\in\mathcal{K}$, any world $w\in\mathcal{F}$ and any valuation V, $(\mathcal{F},w)\Vdash_V \Box_u(x)$ if and only if $(\mathcal{F},a)\Vdash_V x$ for all $a\in\mathcal{F}(w)^s$.

Trivial example is any class of linear reflexive and transitive temporal frames (but, say, classes of just 1-modal frames have no universal formula).

Definition 6
For a logic $\mathcal{L}$, a formula φ is unifiable (in $\mathcal{L}$), if there is a substitution σ, where $\sigma\varphi\in\mathcal{L}$. Then σ is said to be a unifier for φ (in $\mathcal{L}$).

Unification problem for $\mathcal{L}$ is, given an arbitrary formula φ, to determine whether φ has a unifier in $\mathcal{L}$, and if yes to compute at least one. Unification in $\mathcal{L}$ is decidable if there is an algorithm solving this problem. Turning to consecutions, it is clear that in the sequel we can consider only consecutions with single premiss.

Lemma 7
A consecution φ/ψ is admissible in a logic $\mathcal{L}(\mathcal{K})$, where $\mathcal{K}$ admits an universal formula, if and only if φ/ψ is valid in all frames from $\mathcal{K}$, or φ is not unifiable in $\mathcal{L}(\mathcal{K})$.

PROOF. If φ is not unifiable, then evidently φ/ψ is admissible. If φ/ψ is valid in all frames from $\mathcal{K}$ then we immediately infer that φ/ψ is admissible.

For the converse, suppose that φ/ψ is not valid in some $\mathcal{F}_1 \in \mathcal{K}$, $\mathcal{F}_1 \not\Vdash \varphi/\psi$, but there is a substitution σ such that $\sigma\varphi \in \mathcal{L}(\mathcal{K})$. Take the universal formula $\Box_u(x)$ and the substitution

$$\varepsilon x_i := (\Box_u(\varphi) \land x_i) \lor (\neg\Box_u(\varphi) \land \sigma x_i)$$

for every variable letter x_i from φ/ψ.

Consider an $\mathcal{F} \in \mathcal{K}$, a valuation V of letters from $\varepsilon\varphi$ in $\mathcal{F}$ and an $a \in \mathcal{F}$. If $(\mathcal{F}, a) \Vdash_V \Box_u(\varphi)$ then, for any $b \in \mathcal{F}(a)^s$, $(\mathcal{F}, b) \Vdash_V \Box_u(\varphi)$ and $(\mathcal{F}, b) \Vdash_V \varphi$. Therefore, for any $b \in \mathcal{F}(a)^s$, $(\mathcal{F}, b) \Vdash_V \varepsilon x_i \equiv x_i$ and hence $(\mathcal{F}, a) \Vdash_V \varepsilon\varphi$.

If $(\mathcal{F}, a) \not\Vdash_V \Box_u(\varphi)$ then, for any $b \in \mathcal{F}(a)^s$, $(\mathcal{F}, b) \not\Vdash_V \Box_u(\varphi)$ and consequently $(\mathcal{F}, b) \Vdash_V \varepsilon x_i \equiv \sigma x_i$. Therefore from $\sigma\varphi \in \mathcal{L}(\mathcal{K})$ we conclude that, for any $b \in \mathcal{F}(a)^s$, $(\mathcal{F}, b) \Vdash_V \varepsilon\varphi$, in particular, $(\mathcal{F}, a) \Vdash_V \varepsilon\varphi$. Thus, we proved $\varepsilon\varphi \in \mathcal{L}(\mathcal{K})$.

From $\mathcal{F}_1 \not\Vdash \varphi/\psi$ it follows, for some valuation V, $\mathcal{F}_1 \Vdash_V \varphi$ and $(\mathcal{F}_1, a) \not\Vdash_V \psi$ for an $a \in \mathcal{F}_1$. Then $\mathcal{F}_1(a)^s \Vdash_V \varphi$, and, for any $b \in \mathcal{F}_1(a)^s$, $(\mathcal{F}_1, b) \Vdash_V \Box_u(\varphi)$, and $(\mathcal{F}, b) \Vdash_V \varepsilon x_i \equiv x_i$, so $(\mathcal{F}_1, a) \not\Vdash_V \varepsilon\psi$, and hence $\varepsilon\psi \notin \mathcal{L}(\mathcal{K})$. ■

Actually the proof of this lemma works for more general consecutions—consecutions (rules) with parameters. To recall, a consecution with parameters is an expression

$$\mathbf{c} := \frac{\varphi_1(x_1, \dots, x_n, p_1, \dots, p_m), \dots, \varphi_k(x_1, \dots, x_n, p_1, \dots, p_m)}{\psi(x_1, \dots, x_n, p_1, \dots, p_m)},$$

where formulas $\varphi_j(x_1, \dots, x_n, p_1, \dots, p_m)$ and $\psi(x_1, \dots, x_n, p_1, \dots, p_m)$ are constructed out of letters $x_1, \dots, x_n$ (called variables) and special letters $p_1, \dots, p_m$ called parameters (meta-variables). Consecution $\mathbf{c}$ is said to be admissible in a logic $\mathcal{L}$ if, for any substitution ε of all letters x_j to arbitrary formulas and of all parameters p_i to themselves, $\varepsilon(p_i) = p_i$ (so ε lets all p_i to be intact),

$$[\forall s \in [1, k]\ \varepsilon(\varphi_s) \in \mathcal{L}] \Rightarrow \varepsilon(\psi) \in \mathcal{L}.$$

For a logic $\mathcal{L}$, a formula $\varphi_1(x_1, \dots, x_n, p_1, \dots, p_m)$ with parameters $p_1, \dots, p_m$ is unifiable (in $\mathcal{L}$), if there is a substitution σ, which lets all parameters p_j intact, such that $\sigma\varphi \in \mathcal{L}$.

LEMMA 8
Let a class of frames $\mathcal{K}$ to admit an universal formula. A consecution φ/ψ with parameters is admissible in $\mathcal{L}(\mathcal{K})$, if and only if φ/ψ is valid in all frames from $\mathcal{K}$, or the formula φ with parameters is not unifiable in $\mathcal{L}(\mathcal{K})$.

PROOF. verbatim follows the proof of Lemma 7, only we have to use all necessary substitutions which let parameters intact. ■

In the light of Lemmas 7 and 8, dealing with logics with universal formulas, to work on admissibility, we only need to recognize valid consecutions and unifiable formulas. Below we refer to just unification if we consider formulas without parameters, otherwise we say unification with parameters (same is meant for consecutions).

LEMMA 9
Let $\mathcal{K}$ be a class of frames which admits an universal formula. A consecution φ/ψ, where φ is unifiable in $\mathcal{L}(\mathcal{K})$, is valid in any frame from $\mathcal{K}$ if and only if the formula $\Box_u(\varphi) \to \psi$ is true in any $\mathcal{F}$ from $\mathcal{K}$.

PROOF. Notice that $\Box_u(\varphi) \to \psi$ is true in any $\mathcal{F}$ from $\mathcal{K}$ iff it is true in $\mathcal{F}(a)^s$ for all $\mathcal{F} \in \mathcal{K}$ and $a \in \mathcal{F}$. If $\mathcal{F}(a)^s \Vdash \Box_u\varphi \to \psi$ for any $\mathcal{F} \in \mathcal{K}$ and any $a \in \mathcal{F}$, we immediately get φ/ψ is true in all $\mathcal{F}$ from $\mathcal{K}$. If $\mathcal{F}(a)^s \nVdash_V \Box_u\varphi \to \psi$, take the valuation V_1 of letters from $\Box_u\varphi \to \psi$ on whole $\mathcal{F}$ as follows. Let $\varepsilon\varphi \in \mathcal{L}(\mathcal{K})$, and $\varepsilon(x_i) := x_i$ for all letters x_i, where $x_i \in Var(\psi)$ and $x_i \notin Var(\varphi)$.

Let V_2 be the valuation of all letters y_i from $\varepsilon\varphi$ in $\mathcal{F} \setminus \mathcal{F}(a)^s$, where $V_2(y_i) = \emptyset$. For any letter x_i from φ/ψ we set $V_1(x_i) := V(x_i) \bigcup V_2(\varepsilon x_i)$. From $\varepsilon\varphi \in \mathcal{L}(\mathcal{K})$ we get $(\mathcal{F} \setminus \mathcal{F}(a)^s) \Vdash_{V_1} \varphi$, and $\mathcal{F}(a)^s \Vdash_V \Box_u\varphi$ yields $\mathcal{F}(a)^s \Vdash_{V_1} \varphi$. So, in sum, we get $\mathcal{F} \Vdash_{V_1} \varphi$ and $\mathcal{F} \nVdash_{V_1} \psi$, so φ/ψ is not valid in $\mathcal{F}$. ■

Again, as before, this result can be immediately extended to consecutions with parameters.

LEMMA 10
Let $\mathcal{K}$ be a class of frames which admits an universal formula. A consecution φ/ψ with parameters, where the formula φ with parameters is unifiable in $\mathcal{L}(\mathcal{K})$, is valid in any frame from $\mathcal{K}$ if and only if the formula $\Box_u(\varphi) \to \psi$ is true in any $\mathcal{F}$ from $\mathcal{K}$.

PROOF. Literally follows the proof of Lemma 10. ■

From Lemmas 7 and 9 we immediately infer the following.

THEOREM 11
Let $\mathcal{L}(\mathcal{K})$ be a logic, where $\mathcal{K}$ admits universal formula. If $\mathcal{L}(\mathcal{K})$ is decidable and unification for $\mathcal{L}(\mathcal{K})$ is decidable then $\mathcal{L}(\mathcal{K})$ is decidable w.r.t. admissibility of consecutions.

Lemmas 8 and 10 yield the following.

THEOREM 12
Let $\mathcal{L}(\mathcal{K})$ be a logic, where $\mathcal{K}$ admits an universal formula. If $\mathcal{L}(\mathcal{K})$ is decidable and unification with parameters for $\mathcal{L}(\mathcal{K})$ is decidable then $\mathcal{L}(\mathcal{K})$ is decidable w.r.t. admissibility of consecutions with parameters.

Decidability of just unification is often a trivial problem as we show below. We say a logic $\mathcal{L}$ has no *irreflexive terminal points* if $\neg\Box_i\bot \in \mathcal{L}$ holds for all operations $\Box_i$.

LEMMA 13
If a logic $\mathcal{L}$ has no irreflexive terminal points then unification in $\mathcal{L}$ is decidable in exponential time (NP-complete).

PROOF. This is trivial. Indeed, if $\varepsilon : x_i \to \psi_i$ is a unifier for φ in $\mathcal{L}$, then for the substitution $\varepsilon_1 : p_j \to \top$, the composition $\varepsilon_1\varepsilon$ is again a unifier for φ. Since $\neg\Box_i\bot \in \mathcal{L}$, $\varepsilon_1\psi_i \equiv \top \in \mathcal{L}$ or $\varepsilon_1\psi_i \equiv \bot \in \mathcal{L}$. So, it is sufficient to check as unifiers for φ only substitutions in $\{\top, \bot\}$. Computation of values of such substitutions is polynomial (because $\neg\Box_i\bot \in \mathcal{L}$). Thus, overall complexity is the same as satisfiability in PC, i.e the task is NP-complete. ■

However, in non-transitive modal logics just unification may be undecidable, cf. Wolter, Zakharyaschev [36], where it is shown that just unification in the modal logic K extended by universal modality is undecidable (for K itself the question is open). But for vast set of modal logics just unification is trivial (Lemma 13).

On the contrary, decidability of unification with parameters is quite non-ordinary task. In Russian logical school it goes back to problem of logical equations in intuitionistic logic (substitution problem) which was set by P. Novikov in early 60s. First, it seems, this problem—decidability problem for unification with parameters—was solved in 1986 by Rybakov [24] for intuitionistic logic IPC itself and for modal logic $S4$. Then, in Rybakov [25], this problem was solved for modal provability logics, and later, in Rybakov [26] it was solved for intuitionistic logic basing on saturation property. Finally, in the book by Rybakov [27], some generalized technique for decidability of the problem of unification with parameters in wide classes of transitive modal logics and superintuitionistic logics was found. Since 1999, a new approach to the problem of admissibility via projective formulas (which is based on existence of unifiers with particular properties) was found by S. Ghilardi [8–10]. This technique efficiently works for many logics, which emphasizes again importance of the research on unification problem. Regarding unification with parameters, in spite of it being a subtask of admissibility with parameters, this problem stays already very close to itself admissibility with parameters.

4 Applications

The above developed technique works for all transitive and linear temporal logics (with operations for past and future). Recall that a Kripke-complete temporal logic $\mathcal{L}(\mathcal{K})$ is linear iff any frame from $\mathcal{K}$ is linear. It is clear that any transitive and linear temporal logic admits the universal formula for the generating class: $x \wedge \Box^+ x \wedge \Box^- x$ is the one. Thus, from Theorems 11 and 12 we immediately infer the following.

Theorem 14
Let $\mathcal{L}(\mathcal{K})$ be a linear and transitive temporal logic. If $\mathcal{L}(\mathcal{K})$ is decidable and unification (with parameters) for $\mathcal{L}(\mathcal{K})$ is decidable then the admissibility problem for consecutions (with parameters) in $\mathcal{L}(\mathcal{K})$ is decidable.

Thus, to apply Theorem 14 we need linear, transitive and decidable temporal logics with decidable unification. Below we, first list such logics from Segerberg [33] and Bull [2]. The logic *LinT* is $\mathcal{L}(\langle Rat, \leq, \geq\rangle)$, $LinTK := \mathcal{L}(\langle Re, \leq, \geq\rangle)$ and $LinTDum := \mathcal{L}(\langle Z, \leq, \geq\rangle)$, where *Rat*, *Re* and *Z* are the sets of rational numbers, real numbers and integer numbers respectively. Evidently logics *LinT*, *LinTK* and *LinTDum* obey Lemma 13. It is known (say, follows from fmp and finite axiomatization, or follows explicitly from Litak and Wolter, cf. [18, 35]), that all these logics are decidable. Therefore, from Theorem 14 we get the following.

Proposition 15
Logics *LinT*, *LinTK* and *LinDum* are decidable by admissibility.

For irreflexive logics: $LinDA := \mathcal{L}(\langle Rat, <, >\rangle)$, $LinDAS := \mathcal{L}(\langle Re, <, >\rangle)$ and $LinZD := \mathcal{L}(\langle Z, <, >\rangle)$ introduced in [33], Lemma 13 holds again, and, cf. [18, 35], all these logics are decidable, so Theorem 14 yields the following.

Proposition 16
The admissibility problem is decidable for logics *LinDA*, *LinDAS* and *LinZD*.

But above-mentioned results follow from more general fact that may be derived based on Litak and Wolter [18], where it is shown that all finitely axiomatizable linear tense logics (without reference to fmp or even Kripke completeness) are decidable and coNP-complete. So, using decidability of all such finitely axiomatizable linear logics and Theorem 14, we infer:

Theorem 17
Let $\mathcal{L}(\mathcal{K})$ be a linear and transitive temporal logic. If $\mathcal{L}(\mathcal{K})$ is finitely axiomatizable and unification for $\mathcal{L}(\mathcal{K})$ is decidable then $\mathcal{L}(\mathcal{K})$ is decidable w.r.t. admissibility.

We can also apply our technique to more general case—to non-linear logics with bounded zigzag.

DEFINITION 18
A class of temporal frames $\mathcal{K}$ is said to have n-bounded zigzag if, for any $\mathcal{F}\in\mathcal{K}$, any $a\in\mathcal{F}$, and any $b\in\mathcal{F}(a)^s$ either $b=a$ or there are $a_1,\dots,a_k\in\mathcal{F}(a)^s$, were $a=a_1Q_1a_2Q_2\dots Q_{k-2}\ a_{k-1}Q_{k-1}a_k=b$, $Q_j\in\{R,R^{-1}\}$, and $k\leq n$.

A temporal logic $\mathcal{L}$ has n-bounded zigzag if $\mathcal{L}=\mathcal{L}(\mathcal{K})$ for some class $\mathcal{K}$ of frames with n-bounded zigzag. If $\mathcal{K}$ has n-bounded zigzag, $\mathcal{L}(\mathcal{K})$ has an universal formula:

$$x\wedge\bigwedge_{1\leq i\leq n,\ \circ_j\in\{+,-\}}\Box^{\circ_1}\Box^{\circ_2}\dots\Box^{\circ_i}x.$$

It is clear that $\mathcal{L}(\mathcal{K})$ has n-bounded zigzag iff

$$\varphi_n:=x\vee\bigvee_{1\leq i\leq n+1,\ \circ_j\in\{+,-\}}\Diamond^{\circ_1}\Diamond^{\circ_2}\dots\Diamond^{\circ_i}x\rightarrow$$

$$x\vee\bigvee_{1\leq i\leq n,\ \circ_j\in\{+,-\}}\Diamond^{\circ_1}\Diamond^{\circ_2}\dots\Diamond^{\circ_i}\in\mathcal{L}(\mathcal{K})$$

So, by Theorem 11 we get the following.

THEOREM 19
If a temporal logic $\mathcal{L}(\mathcal{K})$ has n-bounded zigzag, is decidable and is decidable w.r.t. unification, then $\mathcal{L}(\mathcal{K})$ is decidable w.r.t. admissibility.

As examples, let $\mathcal{TL}_{z(n)}$ be the temporal logic of all transitive and reflexive frames with n-bounded zigzag. Then any $\mathcal{TL}_{z(n)}$ is decidable, which is easy to show by standard filtration technique, and obeys Lemma 13, so Theorem 19 works for all logics $\mathcal{TL}_{z(n)}$. As one more application, consider just multi-modal logics extended with universal modality. For these logics universal modality is presented as new modal operation. To recall definition, a multi-modal logic $\mathcal{L}$ is said to have the universal modality if its language contains a modal operation $\Box_u$ (called the *universal modality*) such that the following formulas are theorems of $\mathcal{L}$ (for any i): $\Box_u p\rightarrow\Box_i p\wedge p$, $\Box_u p\rightarrow\Box_u\Box_u p\wedge p$, $\Diamond_u p\rightarrow\Box_u\Diamond_u p$. $\mathtt{S4_nU}$ denotes the smallest $(n+1)$-modal logic with the universal modality $\Box_u$ containing the formulas $\Box_i p\rightarrow p$, $\Box_i p\rightarrow\Box_i\Box_i p$ for all $i\leq n$. Let $\mathcal{K}$ be a class of $n+1$-frames $\mathcal{F}=\langle F,R_1,\dots,R_n,R_u\rangle$, where R_u is an equivalence relation and $R\supseteq R_i\cup R_i^{-1}$ for all i. R_u is said to be universal accessibility relation. The logic $\mathcal{L}(\mathcal{K})$ (with the modal operation $\Box_u$ corresponding to R_u) is evidently a modal logic with universal modality (with the $\Box_u$ one); we will denote it by $\mathcal{L}_u(\mathcal{K})$ to emphasize the especial role of $\Box_u$. For a logic $\mathcal{L}$, $Fr(\mathcal{L}):=\{\mathcal{F}\mid\mathcal{F}\Vdash\mathcal{L}\}$, denotes the class of all $\mathcal{L}$-frames.

For any given n-multi-modal logic $\mathcal{L}$, the logic with universal modality $\mathcal{L}_u$ obtained from $\mathcal{L}$ is constructed as follows. For any $\mathcal{F}\in Fr(\mathcal{L})$, the frame $\mathcal{F}_u$ is $\mathcal{F}$ expanded by the minimal universal accessibility relation R_u, where

$$\forall a\in\mathcal{F},\ \forall b\in\mathcal{F},\ aR_ub\Leftrightarrow b\in\mathcal{F}(a)^s;$$

$$\mathcal{L}_u:=\mathcal{L}(\{\mathcal{F}_u\mid\mathcal{F}\in Fr(\mathcal{L})\}).$$

Theorem 11 is evidently applicable to all logics $\mathcal{L}_u$, since $\Box_u x$ is the universal formula. To specify these logics more, we need a general definition of multi-modal logics admitting filtration.

DEFINITION 20
A logic $\mathcal{L}$ admits the *effective filtration* if the following holds. For any model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ based on an $\mathcal{L}$-frame $\mathcal{F}$, and any finite set of formulas $\mathcal{X}$ (in letters from the domain of V) closed w.r.t. subformulas, there is a model $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1 \rangle$ based on an $\mathcal{L}$-frame $\mathcal{F}_1$, and a mapping f from $|\mathcal{F}|$ onto $|\mathcal{F}_1|$ such that for any $a, b \in \mathcal{F}$, any $\Box_i$, any R_i, and for any formula φ,

(i) $\forall p_j \in Dom(V),\ (\mathcal{M}, a) \Vdash_V p_j \Leftrightarrow (\mathcal{M}_1, f(a)) \Vdash_{V_1} p_j$,
(ii) $aR_ib \Rightarrow f(a)R_if(b)$,
(iii) $f(a)R_if(b) \& \Box_i\varphi \in \mathcal{X} \& (\mathcal{M}, a) \Vdash_V \Box_i\varphi \Rightarrow (\mathcal{M}, b) \Vdash_V \varphi$, and
(iv) $||\mathcal{F}_1|| \leq g(||\mathcal{X}||)$, where g is a computable function.

This definition just formalizes and generalizes the well–known notion of standard filtration, therefore the mapping f has the following filtration property: $\forall \varphi \in \mathcal{X} \forall a \in \mathcal{M}\ (\mathcal{M}, a) \Vdash_V \varphi \Leftrightarrow (\mathcal{M}_1, f(a)) \Vdash_{V_1} \varphi$. If, for a multi-modal logic $\mathcal{L}$, $\mathcal{L}_u$ admits the effective filtration and finite $\mathcal{L}_u$-frames are effectively recognizable, then it is easy to see that $\mathcal{L}_u$ is decidable. Thus, using Theorem 11 it follows:

THEOREM 21
If a multi-modal logic $\mathcal{L}_u$ with universal modality (i) has effectively recognizable finite frames (e.g. finitely axiomatizable), (ii) admits effective filtration and (iii) is decidable w.r.t. unification (e.g. $\mathcal{L}_u \supseteq \mathtt{S4_nU}$), then the admissibility problem for $\mathcal{L}_u$ is decidable.

For instance, all logics $\mathtt{S4_nU}$ (= $S4_{n,u}$ in our notation) satisfy this theorem. Hence, the admissibility problem for $S4_u$ itself is decidable, which solves the problem set up in Wolter and Zakharyaschev [36]. Notice that (i) and (ii) holds also for $K4_u$, but admissibility problem for $K4_u$ is undecidable, exactly because (iii) fails (cf. [36]).

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