A Combinatorial Proof of a Recursion For the q-Kostka Polynomials

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Abstract

The Kostka numbers $K_{\lambda\mu}$ are important in several areas of mathematics, including symmetric function theory, representation theory, combinatorics and invariant theory. The q-Kostka polynomials $K_{\lambda\mu}(q)$ are the q-analogues of the Kostka numbers. They generalize and extend the mathematical meaning of the Kostka numbers. Lascoux and Schützenberger proved one can attach a non-negative integer statistic called charge to a semistandard tableau of shape λ and content μ such that $K_{\lambda\mu}(q)$ is the generating function for charge on those semistandard tableaux. We will give two new descriptions of charge and prove several new properties of this statistic. In addition, the q-Kostka polynomials are known to satisfy a certain shape and content reducing recursion. We will give a combinatorial proof of a related recursion for the q-Kostka polynomials on words.

1 Introduction

The q-Kostka polynomials can be defined as the connection coefficients $K_{\lambda\mu}(q)$ in the expansion

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(q) P_\mu(x;q)$$

where $P_\mu(x;q)$ is the classical Hall-Littlewood polynomial and the s_λ are the Schur functions. [17]

If we specialize q=1 in this equation we obtain

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu} m_\mu(x)$$

where the $K_{\lambda\mu}$ are the classical Kostka numbers and the $m_\mu(x)$ is the monomial basis for the symmetric functions. The $K_{\lambda\mu}$ appear as multiplicities of the irreducible representations of S_n. Young's rule states that $K_{\lambda\mu}$ is equal to the number of column-strict tableaux of shape λ and type μ. In 1974, Foulkes conjectured that it should be possible to obtain the $K_{\lambda\mu}(q)$ by q-counting column-strict tableaux of shape λ and content μ [4].

The statistic charge was discovered by Lascoux and Schützenberger [13] for use in the proof that the Kostka polynomials have nonnegative coefficients. Charge is defined so that

$$K_{\lambda\mu}(q) = \sum_{\substack{T \\ sh T=\lambda \\ content T=\mu}} q^{ch(T)}.$$

Our interest is in properties of the charge statistic as it is applied to words of a given content, and is motivated by the following recursion stated by Garsia and Procesi [6]:

$$\sum_{\mathbf{w}\in W_\mu} q^{ch(\mathbf{w})} = \sum_i q^{r_{i,\mu}} \sum_{\mathbf{w}\in W_{\mu^{(i)}}} q^{ch(\mathbf{w})}, \tag{1}$$

where $r_{i,\mu} = |\{j > i : \mu_j = \mu_i\}|$. Our proof depends on the extension of properties of cyclage (defined by Lascoux and Schützenberger in [15]) to the cycling of 1's.

2 Words

We define a *word* $\mathbf{w} = w_n \cdots w_2 w_1$ to be any arrangement of a multiset of numbers and define the *content* $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of a word $\mathbf{w}$ so that μ_i is the number of i's in the word $\mathbf{w}$. We will often refer to these numbers as *letters*. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ then μ is a *partition*. We say that a word $\mathbf{w}$ has *partition content* if μ is a partition. The *length of a word* $\mathbf{w}$ with content μ is $n = \mu_1+\mu_2+\cdots+\mu_k$, i.e., the number of letters in $\mathbf{w}$. We define a *j-protopartition* to be $(a, \nu_1, \nu_2, \ldots, \nu_k)$ where $(\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{j-1} \geq a > \nu_j \geq \cdots \geq \nu_k)$ is a partition.

Let W_μ be the set of all words of content μ and $W_\mu^{(1)}$ be the set of all words of content μ with a 1 in the leftmost position, i.e., $w_n = 1$.

Define $\mu^{(i)} = (\mu_1, \ldots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \ldots, \mu_n)$.

For any partition μ, define $r_{i,\mu} = |\{j > i : \mu_j = \mu_i\}|$. For example, if $\mu = (6,5,5,4,3,3,3,3,1,1)$, then $r_{1,\mu} = 0, r_{2,\mu} = 1, r_{5,\mu} = 3, r_{6,\mu} = 4$, etc.

If μ is a partition and each part of μ is a multiple of d, then define $\mu/d = (\mu_1/d, \mu_2/d, \ldots, \mu_k/d)$. For example, if $\mu = (18,12,12,6)$, then $\mu/3 = (6,4,4,2)$ and $\mu/6 = (3,2,2,1)$.

Given any word $\mathbf{w}$ of content μ we can select any 1, any 2, any 3, ..., and any k, and write these letters down in the order in which they appear in $\mathbf{w}$. We will call this new word a *standard subword*.

For example, let

$$\mathbf{w} = 1365374212$$

then $w_9 w_8 w_7 w_5 w_4 w_2 w_1 = 3657412$ would be a standard subword of length 7 and $w_9 w_4 w_3 w_2 = 3421$ would be a standard subword of length 4, etc.

Given any word $\mathbf{w}$, define $ORB(\mathbf{w})$ to be the orbit of $\mathbf{w}$ under the action of the cyclic group Z_n. For $\mathbf{w} \in W_\mu$, define

$$ORB_1^+(\mathbf{w}) = ORB(\mathbf{w}) \cap W_\mu^{(1)}.$$

That is, $ORB_1^+(\mathbf{w})$ is the set of μ_1 elements of $ORB(\mathbf{w})$ which have a one in the leftmost position. Similarly, define $ORB_1^-(\mathbf{w})$ to be the set of μ_1 elements of $ORB(\mathbf{w})$ which have a one in the rightmost position.

For example, suppose

$$\mathbf{w} = \begin{matrix} 2 & 1 & 5 & 3 & 2 & 1 & 4 & 3 & 1 \end{matrix}.$$

Then $ORB(\mathbf{w}) =$

$$\begin{aligned} \{&121532143, \\ &312153214, \\ &431215321, \\ &143121532, \\ &214312153, \\ &321431215, \\ &532143121, \\ &153214312, \\ &215321431\} \end{aligned}$$

Also, $ORB_1^+(\mathbf{w}) = \{121532143, 143121532, 153214312\}$ and $ORB_1^-(\mathbf{w}) = \{215321431, 431215321, 532143121\}$.

Words with cyclic symmetry cause certain technical difficulties. Therefore, we will say a word $\mathbf{w}$ is n/d-cyclic if n/d is the smallest positive integer such that $cr^{n/d}(\mathbf{w}) = \mathbf{w}$. Thus, $n/d = \sharp ORB(\mathbf{w})$. Primitive words of length n are thus n-cyclic.

Define the *crank* of $\mathbf{w}, cr(\mathbf{w})$, to be the operation of cycling the rightmost letter of $\mathbf{w}$ to the leftmost position. Thus *crank* restricts to a bijection from $ORB_1^-(\mathbf{w})$ to $ORB_1^+(\mathbf{w})$.

Next we define an action of S_k on words using the letters $1, 2, \ldots, k$. We first define the action of an adjacent transposition σ_i, which will swap the multiplicities of the i's and the $i+1$'s, in the following way: look at only the subword of i's and $i+1$'s. Then below each i write a right parenthesis ")" and below each $i+1$ write a left parenthesis "(". Next pair the parentheses by matching left parentheses with right parentheses until we are left with a string of unmatched left parentheses to the right of a string of unmatched right parentheses. Suppose there are k unmatched right parentheses and m unmatched left parentheses. If $k > m$ switch the rightmost $k - m$ unmatched right parentheses to left parentheses. If $m > k$ switch the leftmost $m - k$ unmatched left parentheses to right parentheses. Leave all of the matched parentheses unaltered. Now write down an i for every right parenthesis and an $i+1$ for every left parenthesis. This gives us a word of with the multiplicities of the i's and $i+1$'s swapped.

For example, let $\mathbf{w} = 2131122123131222112331$ be a word of content $\mu = (985)$. To obtain $\sigma_2\mathbf{w}$ of content (958), first write down the subword of 2's and 3's and below it write a right parenthesis under every 2 and a left parenthesis under every 3. Below the parentheses we will label with the same number the left and right parentheses which pair:

$$\mathbf{w}_{23} = \begin{array}{ccccccccccccc} 2 & 3 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 3 & 3 \\ ) & (&) &) &) & (& (&) &) &) &) & (& (\\ & {\scriptstyle 1} & {\scriptstyle 1} & & & {\scriptstyle 3} & {\scriptstyle 2} & {\scriptstyle 2} & {\scriptstyle 3} & & & & \end{array}$$

The unpaired parentheses form the following sequence:

$$) \quad) \quad) \quad) \quad) \quad (\quad (\quad .$$

When we switch the multiplicities we have the sequence:

$$) \quad) \quad (\quad (\quad (\quad (\quad (\quad .$$

Thus our new word has the following parentheses and corresponding 23 subword.

$$\sigma_2\mathbf{w}_{23} = \begin{array}{ccccccccccccc}) & (&) &) & (& (& (&) &) & (& (& (& (\\ 2 & 3 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 3 & 3 & 3 & 3 \end{array}$$

This gives us the word $\sigma_2\mathbf{w} = 2131122133131223113331$ with content $\sigma_2\mu = (958)$.

Then S_k acts on words using letters $1, 2, \ldots, k$ since

$$\sigma_i^2\mathbf{w} = \mathbf{w},$$

$$\sigma_i\sigma_j\mathbf{w} = \sigma_j\sigma_i\mathbf{w}, |i-j| \geq 2,$$

and

$$\sigma_i\sigma_{i+1}\sigma_i\mathbf{w} = \sigma_{i+1}\sigma_i\sigma_{i+1}\mathbf{w}$$

We will define $\rho_j = (123\ldots j) = \sigma_1\sigma_2\ldots\sigma_{j-1} \in S_k$. We also have the following proposition:

Proposition 1. *The action of S_n on a word $\mathbf{w}$ commutes with the crank operation, i.e. $cr(\sigma\mathbf{w}) = \sigma cr(\mathbf{w})$.*

3 Charge

Let $\pi = \pi_n\pi_{n-1}\cdots\pi_1$ be a permutation of $[n]$ (i.e., the integers 1 to n each appear exactly once in π). For each i define the *charge value of i* to be $c_i(\pi) = |\{j : j \leq i \text{ and } j \text{ is to the right of } j-1\}|$. Then we define the *charge of π* as $ch(\pi) = \sum_i c_i(\pi)$.

For example, if

$$\pi = \begin{array}{cccccc} 5 & 6 & 2 & 4 & 1 & 3 \\ {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} \end{array},$$

then

$$ch(\pi) = 1 + 2 + 1 + 1 = 5$$

where $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $c_4 = 0$, $c_5 = 2$, $c_6 = 1$.

We now extend the definition of charge to words with content λ where λ is any partition. We will follow the description of charge given in Macdonald [17] and refer to this description of charge as the *standard* description.

If $\mathbf{w}$ is a word whose content is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then select standard subwords, $\mathbf{w}^1$, $\mathbf{w}^2$, $\ldots$, $\mathbf{w}^{\lambda_1}$, as described below. Then define the *charge* of $\mathbf{w}$ by

$$\mathrm{ch}(\mathbf{w}) = \mathrm{ch}(\mathbf{w}^1) + \mathrm{ch}(\mathbf{w}^2) + \cdots + \mathrm{ch}(\mathbf{w}^{\lambda_1}).$$

To select the subwords, read $\mathbf{w} = w_n w_{n-1} \cdots w_2 w_1$ from right to left, i.e. starting with w_1. Mark with a superscript ${\scriptstyle 1}$ the first 1 which occurs, then mark with a superscript ${\scriptstyle 1}$ the first 2 which occurs to the left of 1^1, then mark with a superscript ${\scriptstyle 1}$ the first 3 which occurs to the left of 2^1, and so on. If at any stage there is no $i+1$ to the left of i^1, then search for the first $i+1$ beginning at w_1, reading from right to left. Continue until the letter λ_1' is marked with a superscript ${\scriptstyle 1}$. The letters in $\mathbf{w}$ marked with a superscript ${\scriptstyle 1}$ form the subword $\mathbf{w}^1$.

For example, $\mathbf{w}= 215^1 4^1 253^1 12^1 47^1 6^1 31^1$, so $\mathbf{w}^1 = 5432761$.

Now select the subword $\mathbf{w}^2$ in the same manner, only mark with a superscript ${\scriptstyle 2}$ and ignore the letters already marked with a superscipt ${\scriptstyle 1}$.

As in the example before, $\mathbf{w} = 215^1 4^1 2^2 5^2 3^1 1^2 2^1 4^2 7^1 6^1 3^2 1^1$, so $\mathbf{w}^2 = 25143$.

Repeat for $\mathbf{w}^3, \ldots, \mathbf{w}^{\lambda_1}$. Then

$$ch(21542531247631) = ch(5432761) + ch(25143) + ch(21) = 2 + 3 + 0 = 5.$$

We define $c_i(\mathbf{w})$ to be the charge value of the letter in position i of the word $\mathbf{w}$, i.e., $c_i(\mathbf{w})$ is the charge value on w_i in its standard subword. For example, let $\mathbf{w} = 32514132$. Then $c_1 = c_2 = c_4 = c_6 = 1$ and $c_i = 0$ for all other i.

Now we define charge on words of non-partition type by choosing $\sigma \in S_k$ so that $\sigma\mathbf{w}$ has partition content, then defining $ch(\mathbf{w}) = ch(\sigma\mathbf{w})$.

For example, let $\mathbf{w} = 24132352431232$ be a word of content $\mu = (25421)$. In order to calculate the charge of $\mathbf{w}$, note that $\sigma\mathbf{w}$ will have content (54221) if $\sigma = \sigma_2\sigma_1$. We will have

$$\sigma\mathbf{w} = 24121352421131$$

and

$$\begin{aligned} ch(\sigma\mathbf{w}) &= ch(43521) + ch(2413) + ch(21) + ch(21) + ch(1) \\ &= 1 + 2 + 0 + 0 + 0 = 3 = ch(\mathbf{w}). \end{aligned}$$

It is easy to see that when $x \neq 1$ is cycled from the rightmost position in a word of partition content to the leftmost position in the word the charge value of x in the word decreases by one, which gives rise to the following lemma by Lascoux and Schützenberger [13].

Lemma 1. *Let* $\mathbf{v}x$ *be a word of partition content which does not have a 1 in the rightmost position, i.e.* $x \neq 1$. *Then*

$$ch(\mathbf{v}x) = ch(x\mathbf{v}) + 1.$$

In fact,

$$c_i(\mathbf{z}x) = c_{i-1}(x\mathbf{z}) \text{ for } 1 < i \leq n$$

and

$$c_1(\mathbf{z}x) = c_n(x\mathbf{z}) + 1.$$

The following theorem is an alternate characterization of charge due to Lascoux and Schützenberger [13].

Theorem 1. *Charge is the unique function from words to non-negative integers such that:*

1. $ch(\emptyset) = 0$.
2. $ch(\mathbf{w}) = ch(\sigma\mathbf{w})$, *for* $\sigma \in S_k$.
3. $ch(\mathbf{v}x) = ch(x\mathbf{v}) + 1$, *for* $x \neq 1$ *and* $\mathbf{v}x$ *a word of partition content.*
4. $ch(\mathbf{v}1^m) = ch(\mathbf{v})$ *for* $\mathbf{v}1^m$ *a word of partition content.*
5. *If* $\mathbf{w} \overset{K}{\cong} \tilde{\mathbf{w}}$ *then* $ch(\mathbf{w}) = ch(\tilde{\mathbf{w}})$.

4 New Descriptions of Charge

In this section we will give two new descriptions of charge which will be more useful for our purposes than the standard description of charge.

We will first give the description of this statistic, which we will call $\widehat{ch}$, for words with content λ, where λ is any partition, and then extend the definition to words of arbitrary content.

If $\mathbf{w}$ is a word of partition content $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then select standard subwords by reading $\mathbf{w} = w_n w_{n-1} \cdots w_1$ from left to right, i.e., starting with w_n as follows. Suppose k is the largest letter which appears in $\mathbf{w}$. Mark with a superscript $_1$ the first k which occurs (i.e. the leftmost k), then mark with a superscript $_1$ the first $k-1$ which occurs to the right of k^1, then mark with a superscript $_1$ the first $k-2$ which occurs to the right of $(k-1)^1$, and so on. If at any stage there is no $i-1$ to the right of i^1, then search for the first $i-1$ beginning at w_n, reading from left to right. Continue until a 1 is marked with a superscript $_1$. The letters in $\mathbf{w}$ marked with a superscript $_1$ form the standard subword $\mathbf{w}^1$. For example,

$$\mathbf{w} = 12^1 1^1 43521317^1 526^1 5^1 24^1 43^1, \tag{2}$$

so

$$\mathbf{w}^1 = 2176543.$$

Now select the subword $\mathbf{w}^2$ in the same manner, only mark with a superscript $_2$ and ignore the letters already marked with a superscript $_1$.

$$\mathbf{w} = 12^1 1^1 43^2 5^2 2^2 1^2 317^1 526^1 5^1 24^1 4^2 3^1,$$

so

$$\mathbf{w}^2 = 35214.$$

Repeat for $\mathbf{w}^3, \ldots, \mathbf{w}^{\lambda_1}$. Then we compute charge on these standard subwords as in the standard description of charge on permutations and define

$$\begin{aligned} \widehat{ch}(1214352131752652443) &= ch(2176543) + ch(35214) + ch(14352) + ch(12) \\ &= 5 + 2 + 5 + 1 = 13. \end{aligned}$$

As with charge, we can extend this statistic to words of non-partition content by choosing $\sigma \in S_k$ so that $\sigma\mathbf{w}$ has partition content, and then defining $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\sigma\mathbf{w})$.

Lemma 2. *Let* $1\mathbf{v}$ *be a word of partition content. Then*

$$\widehat{ch}(1\mathbf{v}) = \widehat{ch}(\mathbf{v}1) + l(\mathbf{w}^k) - 1$$

where the leftmost 1 *of* $1\mathbf{v}$ *is in the subword* $\mathbf{w}^k$*.*

For example, using $\mathbf{w} = 1214352131752652443$ as in the previous example, we have that

$$\widehat{ch}(1214352131752652443) = 13.$$

The leftmost 1 is in subword $\mathbf{w}^3$ which has length 5. Then we have

$$\begin{aligned} \widehat{ch}(1214352131752652443) &= \widehat{ch}(2143521317526524431) + l(\mathbf{w}^3) - 1 \\ &= 9 + 5 - 1 = 13. \end{aligned}$$

Proof. Because of the way that subwords are chosen under $\widehat{ch}$ we will choose the same subwords in $1\mathbf{v}$ as in $\mathbf{v}1$. All the subwords will be the same except for the kth subword which will have a 1 in the rightmost position instead of the leftmost position. From the calculation of charge on the subwords, this will decrease the charge value of each letter in the kth subword by 1 except for the charge value of the 1 which remains 0. Thus $\widehat{ch}(\mathbf{v}1)$ decreases by one less then the length of the kth subword. $\square$

Lemma 3. *Let $x\mathbf{v}$ be a word of partition content with $x \neq 1$. Then*

$$\widehat{ch}(x\mathbf{v}) = \widehat{ch}(\mathbf{v}x) - 1.$$

Proof. If x is not equal to the largest letter in some subword, then it is easy to see that the subword decomposition will remain the same and the $\widehat{ch}$ value of x will go up by one while all other $\widehat{ch}$ values will remain the same.

Suppose k is the largest letter in a word and that the multiplicity of k's in the word is λ_k. First we will show that the $k-1$'s which appear in the first λ_k subwords in $\mathbf{v}k$ are the same $k-1$'s that appear in the first λ_k subwords in $k\mathbf{v}$, though not necessarily in exactly the same subwords. Then we can reduce to the case of rectangular content, where all columns in the Ferrers diagram of the content have length k. We will show in this case that when we cycle a k, the $\widehat{ch}$ value on each letter cannot go down and that the $\widehat{ch}$ value goes up by one on some k in the word. Thus $\widehat{ch}$ must go up by exactly one when we cycle a k.

The $k-1$'s which appear in the first λ_k subwords are those $k-1$'s which pair with the k's under the plactic action of σ_{k-1}. Since cycling a k does not change the cyclic pairing of k's and $k-1$'s under the plactic action, the same $k-1$'s pair with the k's in both $k\mathbf{v}$ and $\mathbf{v}k$, though perhaps in a different order. Thus the $k-1$'s which appear in the first k subwords of $k\mathbf{v}$ are the same $k-1$'s which appear in the first k subwords of $\mathbf{v}k$, so it is enough to consider the case of rectangular content where all columns in the Ferrers diagram of the content have length k.

Now suppose $k\mathbf{v}$ has rectangular content. We will examine what will happen to the $\widehat{ch}$ value on each letter when we cycle a k.

To do so, we will label each letter in $k\mathbf{v}$ with its position in the word, read left to right. For example, for the following word of rectangular content we will place the position labels below each letter.

4	2	1	3	2	4	3	1	2	4	3	3	2	1	4	1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Let $A_1 = \{a_{11}, a_{12}, \ldots, a_{1\mu_1}\}$ be the set of labels on the largest letter, $A_2 = \{a_{21}, a_{22}, \ldots, a_{2\mu_1}\}$ the set of labels on the second largest letter, …, $A_k = \{a_{k1}, a_{k2}, \ldots, a_{k\mu_1}\}$ the set of labels on the 1's.

For the example, $A_1 = \{1, 6, 10, 15\}$, $A_2 = \{4, 7, 11, 12\}$, $A_3 = \{2, 5, 9, 13\}$ and $A_4 = \{3, 8, 14, 16\}$.

Now we will place these labels into a table with the set of labels in A_1 in the first row, the set of labels in A_2 in the second row, etc., until we have the set of labels in A_k in the last row. Now we describe the order of each row. Order the labels in A_1 in increasing order and let the resulting order be denoted by $B_1 = \{b_{11}, b_{12}, \ldots, b_{1\mu_1}\}$. Now let the first row in the table be

$$b_{11} \quad b_{12} \quad \ldots \quad b_{1\mu_1}$$

Now reorder the set A_2 to form $B_2 = \{b_{21}, b_{22}, \ldots, b_{2\mu_1}\}$. Let b_{21} be the smallest element of A_2 which is larger than b_{11}. If no such element of A_2 exists, let b_{21}

be the smallest element of A_2. Remove this element from A_2. Now let b_{22} be the smallest remaining element of A_2 which is smaller than b_{12}. Again, if no such element exists, let b_{22} be the smallest remaining element of A_2. Continue in this manner to form B_2 and then let the second row in the table be

$$b_{21} \quad b_{22} \quad \dots \quad b_{2\mu_1}$$

Continue in this manner for the remaining rows. From our previous example we will have the tableau

1	6	10	15
4	7	11	12
5	9	13	2
8	14	16	3

Now we will place dots in the tableau to denote descents in the columns.

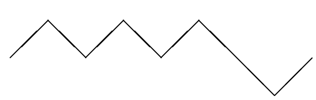
1	6	10	15
			•
4	7	11	12
			•
5	9	13	2
8	14	16	3

To find the value of $\widehat{ch}$ on our original word, we need only to find the total number of numbers above a dot. If a number is above more than one dot, we count it with the multiplicity of the dots it is above. For example, $\widehat{ch}$ on our original word $\mathbf{w} = 4213243124332141$ is 3.

This kind of tableau allows us to analyze the contribution to $\widehat{ch}$ given by the k's, for any letter k. To determine the number of dots between the kth row and the $(k+1)$st row, arrange the numbers in each of these rows in order from smallest to largest. Then draw a lattice path with one move for each number, starting with the smallest number and ending with the largest number. We will draw a "down" move if the number is in the kth row and an "up" move if the number is in the $(k+1)$st row. The height of this lattice path above the initial starting point will give the number of dots between the kth and $(k+1)$st rows. The height represents a number or numbers in the $(k+1)$st row for which there will be no smaller available number in the kth row. Thus this number in the $(k+1)$st row will be in the same column as a larger number in the kth row. This will give us a descent in this column. For example, arrange the labels in the 2nd and 3rd rows of the tableau above.

$$2 \quad 4 \quad 5 \quad 7 \quad 9 \quad 11 \quad 12 \quad 13$$

This corresponds to the lattice path



which has height one, corresponding to the one dot between the second and third rows of the tableau. Note that the number of descents between the kth and the $(k+1)$st row does not depend on the order of numbers in the kth row.

Now it is easy to see that cycling a largest letter from the leftmost position to the rightmost position will not affect the contribution to $\widehat{ch}$ from any of the rows except for between the first and second row. Thus the only possible place a change in $\widehat{ch}$ could occur would be on the largest letter in the word.

Now we will show that $\widehat{ch}$ must go up on a largest letter, i.e., when we move a largest letter from the leftmost position to the rightmost position a new dot will appear between the first and second rows. Or, in other words, the height of the lattice path given by the first and second rows will increase by one. Before cycling a largest letter, our tableau had a 1 in the first row corresponding to the largest letter in the leftmost position, so the lattice path started with a "down" move. When we cycle the largest letter to the rightmost position, we lose this 1 in the first row and instead have the largest label in the first row. The lattice path will remain exactly the same, except the first "down" move is removed and a "down" move is placed at the end of the path. This increases the height of the lattice path by one, thus a new dot appears in the first row and $\widehat{ch}$ on some largest letter is increased by one.

□

Theorem 2. *The statistic $\widehat{ch}$ is charge. That is, $\widehat{ch} = ch$.*

Proof. In order to show that $\widehat{ch} = ch$, we will show that it satisfies the five properties given in Theorem 1 which are uniquely satisfied by charge. Only property 5 requires detailed explanation.

We need to show that if $\mathbf{w} \overset{K}{\cong} \tilde{\mathbf{w}}$ then $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\tilde{\mathbf{w}})$.

Case 1: Suppose

$$\mathbf{w} = u_1 \cdots u_{i-1} xzy u_{i+1} \cdots u_n$$

$$\tilde{\mathbf{w}} = u_1 \cdots u_{i-1} zxy u_{i+1} \cdots u_n$$

with $x \leq y < z$. If $x < y < z$, then we are merely transposing the order of nonadjacent numbers, so we have the same subword decomposition and since $\widehat{ch}$ on the subwords depends only on the relative position of adjacent numbers, $\widehat{ch}$ on the subwords will not change. Thus $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\tilde{\mathbf{w}})$.

If $x = y < z$, then x is in some subword $\mathbf{w}^k$ and y is in some subword $\mathbf{w}^l$. If $z \neq y + 1$ then $k < l$ and the same subword choices are made for each subword, thus $\widehat{ch}$ on the subwords does not change. Suppose $z = y + 1$. If z is in subword $\mathbf{w}^m$ for $m > l$ and $m > k$, then again the subword decomposition does not change. Otherwise, z is in subword $\mathbf{w}^l$, the same subword as y. If $k < l$ then x is still in subword $\mathbf{w}^k$ in $\tilde{\mathbf{w}}$ and z and y are both still in subword $\mathbf{w}^l$. Thus the subword decomposition is the same in both $\mathbf{w}$ and $\tilde{\mathbf{w}}$. If $k > l$, then z and x are in subword $\mathbf{w}^l$ in $\tilde{\mathbf{w}}$ and y is in subword $\mathbf{w}^k$. However, the order of the letters in these subwords is the same, so $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\tilde{\mathbf{w}})$.

Case 2: Suppose

$$\mathbf{w} = u_1 \cdots u_{i-1} yzx u_{i+1} \cdots u_n,$$

$$\tilde{\mathbf{w}} = u_1 \cdots u_{i-1} yxz u_{i+1} \cdots u_n,$$

with $x < y \leq z$.

If $x < y < z$, then again we are merely transposing the order of nonadjacent numbers, so we have the same subword decomposition and since $\widehat{ch}$ on the subwords depends only on the relative position of adjacent numbers, $\widehat{ch}$ on the subwords will not change. Thus $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\tilde{\mathbf{w}})$.

If $x < y = z$, then in $\mathbf{w}$, y is in some subword $\mathbf{w}^k$ and z is in some subword $\mathbf{w}^l$ with $k < l$. If $x + 1 \neq y$, then the same subword choices are made in $\tilde{\mathbf{w}}$ since we are swapping nonadjacent numbers and thus the $\widehat{ch}$ of $\mathbf{w}$ equals the $\widehat{ch}$ of $\tilde{\mathbf{w}}$.

Now suppose $x + 1 = y = z$. If z is in subword $\mathbf{w}^k$ and y is in subword $\mathbf{w}^j$ and x is in subword $\mathbf{w}^i$ with $i < j < k$ then they will remain in the same subwords in $\tilde{\mathbf{w}}$, so $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\tilde{\mathbf{w}})$. Suppose z is in subword $\mathbf{w}^j$ and x and y are in subword $\mathbf{w}^i$, with $i < j$. In $\tilde{\mathbf{w}}$, z will still be in subword $\mathbf{w}^j$ and x and y will still be in subword $\mathbf{w}^i$ and will remain in the same order in that subword. Thus $\widehat{ch}(\mathbf{w}) = \widehat{ch}(\tilde{\mathbf{w}})$.

□

Our second description of charge is not for words of partition content but is for words of content (μ_1, λ) where $\mu_1 > 0$ and λ is a partition, i.e., for words of protopartition type. As before, we will first call this description $\widetilde{ch}$ and then we will prove that $\widetilde{ch} = ch$.

Define $\widetilde{ch}$ on a word of protopartition type by finding the smallest letter reading right to left, assigning it a charge of zero, and then building a subword from it as we do in the standard definition of charge. We then remove this subword from the word and repeat the process. We calculate charge on the subwords in the usual way, starting with the smallest letter (not necessarily a 1 in this description). The statistic $\widetilde{ch}$ will be the sum of the charge calculations on the subwords.

For example, let $\mathbf{w} = 154324322$. The first subword that we select and then remove is $\mathbf{w}^1 = 15432$ leaving 4322. The second subword is $\mathbf{w}^2 = 432$ and the third subword is $\mathbf{w}^3 = 2$. When we calculate charge on the subwords (starting with the smallest letter having a charge of zero) we find that $\mathbf{w}^1$ has a charge of 4, $\mathbf{w}^2$ has a charge of 0 and $\mathbf{w}^3$ has a charge of 0. Thus $\widetilde{ch}(\mathbf{w}) = 4$.

Again, we will define $\widetilde{ch}$ on j-protopartitions and then extend to all types through the plactic action, as we did with charge and with $\widehat{ch}$.

Theorem 3. *The statistic $\widetilde{ch}$ is charge, i.e. $\widetilde{ch} = ch$.*

Proof. As in Theorem 2, we will prove this theorem by showing that $\widetilde{ch}$ satisfies the five properties which uniquely define ch. To do this we will first define $\widetilde{ch}_k$ and show that for any k, $\widetilde{ch}_k = ch$. Then we will generalize to conclude that $\widetilde{ch} = ch$.

Define $\widetilde{ch}_k$ to be $\widetilde{ch}$ for words of content $\mu = (\mu_{n-k}, \mu_1, \ldots, \mu_{n-k-1}, \mu_{n-k+1}, \ldots, \mu_m)$ where $\mu_{n-k} \neq 0$ is the kth smallest part of μ and $\mu = (\mu_1, \ldots, \mu_m)$ is a partition. Then for any word $\mathbf{w}$ define $\widetilde{ch}_k(\mathbf{w}) = \widetilde{ch}_k(\sigma\mathbf{w})$ where $\sigma\mathbf{w}$ has content $(\mu_{n-k}, \mu_1, \ldots, \mu_{n-k-1}, \mu_{n-k+1}, \ldots, \mu_m)$.

By definition, $\widetilde{ch}_k(\emptyset) = 0$ and $\widetilde{ch}_k(\mathbf{w}) = \widetilde{ch}_k(\sigma\mathbf{w})$.

Now we must show that for words of partition content, $\widetilde{ch}_k(\mathbf{v}x) = \widetilde{ch}_k(x\mathbf{v}) + 1$, for $x \neq 1$. To compute $\widetilde{ch}_k$ for words of partition content, we first apply ρ_{n-k} to the word so that the word has content $(\mu_{n-k}, \mu_1, \ldots \mu_{n-k-1}, \mu_{n-k+1}, \ldots, \mu_m)$. Then in $\rho_{n-k}(\mathbf{v}x) = \tilde{\mathbf{v}}\tilde{x}$ we have $\tilde{x} \neq 1$ and $\tilde{x}$ not equal to a leading 2, i.e., a 2 which starts a word under the definition of $\widetilde{ch}$. Then in the word $\tilde{x}\tilde{\mathbf{v}}$, $\tilde{x}$ is still chosen as part of the same subword in the subword decomposition, but it now appears at the end of that subword instead of at the beginning. Thus the charge value on $\tilde{x}$ goes down by 1. Therefore, $\widetilde{ch}_k(\tilde{\mathbf{v}}\tilde{x}) = \widetilde{ch}_k(\tilde{x}\tilde{\mathbf{v}}) + 1$, and since $\widetilde{ch}_k(\tilde{\mathbf{v}}\tilde{x}) = \widetilde{ch}_k(\mathbf{v}x)$ by definition, and $\widetilde{ch}_k(\tilde{x}\tilde{\mathbf{v}}) = \widetilde{ch}_k(x\mathbf{v})$ by definition, then $\widetilde{ch}_k(\mathbf{v}x) = \widetilde{ch}_k(x\mathbf{v}) + 1$.

Next we must show that for $\mathbf{w} = \tilde{\mathbf{w}}1^{\mu_1}$ a word of partition content μ, $\widetilde{ch}_k(\mathbf{w}) = \widetilde{ch}_k(\tilde{\mathbf{w}})$. Again, to apply $\widetilde{ch}_k$ to $\mathbf{w}$ we must first apply ρ_{n-k} to $\mathbf{w}$. We can check that $\rho_{n-k}\mathbf{w} = \rho_{n-k}(\tilde{\mathbf{w}}1^{\mu_1}) = \rho_{n-k}(\tilde{\mathbf{w}})1^{\mu_{n-k}}2^{\mu_1-\mu_{n-k}}$. The 2's at the beginning of the word are the leading 2's in the subwords which do not contain a 1 under the definition of $\widetilde{ch}$, so they all have a charge value of zero. In addition, the 1's also all have a charge value of zero, so when we remove these particular 2's and the 1's, we do not change the charge on the word at all. Thus $\widetilde{ch}_k(\rho_{n-k}\mathbf{w}) = \widetilde{ch}_k(\rho_{n-k}\tilde{\mathbf{w}})$ and since $\widetilde{ch}_k(\rho_{n-k}\mathbf{w}) = \widetilde{ch}_k(\mathbf{w})$ and $\widetilde{ch}_k(\rho_{n-k}\tilde{\mathbf{w}}) = \widetilde{ch}_k(\tilde{\mathbf{w}})$ by definition, we have that $\widetilde{ch}_k(\mathbf{w}) = \widetilde{ch}_k(\tilde{\mathbf{w}})$.

Finally, we must show that for $\mathbf{w}$ and $\tilde{\mathbf{w}}$ words of partition type, if $\mathbf{w} \overset{K}{\cong} \tilde{\mathbf{w}}$ then $\widetilde{ch}_k(\mathbf{w}) = \widetilde{ch}_k(\tilde{\mathbf{w}})$. From properties of plactic swaps we know that if $\mathbf{w} \overset{K}{\cong} \tilde{\mathbf{w}}$ then $\sigma\mathbf{w} \overset{K}{\cong} \sigma\tilde{\mathbf{w}}$, so we only need to show that if $\mathbf{w}$ and $\tilde{\mathbf{w}}$ are words of type $(\mu_{n-k}, \mu_1, \ldots, \mu_{n-k-1}, \mu_{n-k+1}, \ldots, \mu_m)$ and $\mathbf{w} \overset{K}{\cong} \tilde{\mathbf{w}}$ then $\widetilde{ch}_k(\mathbf{w}) = \widetilde{ch}_k(\tilde{\mathbf{w}})$.

Case 1: Suppose

$$\mathbf{w} = u_1 \cdots u_{i-1} xzy u_{i+1} \cdots u_n$$
$$\tilde{\mathbf{w}} = u_1 \cdots u_{i-1} zxy u_{i+1} \cdots u_n$$

with $x \leq y < z$. If $x < y < z$, then we are merely transposing the order of nonadjacent numbers, so we have the same subword decomposition and since $\widetilde{ch}$ depends only on the relative position of adjacent numbers, the value of $\widetilde{ch}$ on the subwords will not change. Thus $\widetilde{ch}(\mathbf{w}) = \widetilde{ch}(\tilde{\mathbf{w}})$. If $x = y < z$, then x is in some subword $\mathbf{w}^l$ and y is in some subword $\mathbf{w}^k$ with $k < l$. If $z \neq y + 1$ then the same subword choices are made for each subword and thus the charge does not change. If $z = y + 1$, then z is either in $\mathbf{w}^k$ in both $\mathbf{w}$ and $\tilde{\mathbf{w}}$ or z is in

$\mathbf{w}^j$ in both $\mathbf{w}$ and $\tilde{\mathbf{w}}$ for $j < k$. In both cases, z is in the same relative position in both, and x and y remain in the same subwords in both $\mathbf{w}$ and $\tilde{\mathbf{w}}$, so the subwords don't change and thus $\widetilde{ch}$ doesn't change.

Case 2: Suppose

$$\mathbf{w} = u_1 \cdots u_{i-1} yzx u_{i+1} \cdots u_n,$$

$$\tilde{\mathbf{w}} = u_1 \cdots u_{i-1} yxz u_{i+1} \cdots u_n,$$

with $x < y \leq z$.

If $x < y < z$, then again we are merely transposing the order of nonadjacent numbers, so we have the same subword decomposition and since widetildech depends only on the relative position of adjacent numbers, the value of $\widetilde{ch}$ on the subwords will not change. Thus $\widetilde{ch}(\mathbf{w}) = \widetilde{ch}(\tilde{\mathbf{w}})$. If $x < y = z$, then in $\mathbf{w}$, y is in some subword $\mathbf{w}^1$ and z is in some subword $\mathbf{w}^k$ with $k < l$. If $x + 1 \neq y$, then the same subword choices are made in $\tilde{\mathbf{w}}$ since we are swapping nonadjacent numbers and thus $\widetilde{ch}$ of $\mathbf{w}$ equals $\widetilde{ch}$ of $\tilde{\mathbf{w}}$. Now suppose $x + 1 = y = z$. If z is in subword $\mathbf{w}^i$ and y is in subword $\mathbf{w}^j$ and x is in subword $\mathbf{w}^k$ with $i < j < k$ then they will remain in the same subwords in $\tilde{\mathbf{w}}$, so $\widetilde{ch}(\mathbf{w}) = \widetilde{ch}(\tilde{\mathbf{w}})$. Suppose z is in subword $\mathbf{w}^i$ and x and y are in subword $\mathbf{w}^j$, with $i < j$. In $\tilde{\mathbf{w}}$, z will still be in subword $\mathbf{w}^i$ and x and y will still be in subword $\mathbf{w}^j$ and will remain in the same order in that subword. Finally, suppose that x and z are in subword $\mathbf{w}^i$ and y is in subword $\mathbf{w}^j$ with $i < j$. Then in $\tilde{\mathbf{w}}$, x and y will be in subword $\mathbf{w}^i$ in the same order that x and z were in subword $\mathbf{w}^i$ in $\mathbf{w}$, and z will be in subword $\mathbf{w}^j$ in the same position as y was in the subword $\mathbf{w}^j$ of $\mathbf{w}$. Thus the order of the subwords remains the same, so the value of widetildech on the subwords doesn't change and thus $\widetilde{ch}(\mathbf{w}) = \widetilde{ch}(\tilde{\mathbf{w}})$.

□

With our two new descriptions of charge, we can discuss the relation between three multisets of numbers which are associated with a word $\mathbf{w}$ of partition content μ. For a fixed $\mathbf{w}$ of partition content μ and the accompanying orbit $ORB(\mathbf{w})$, we define the following multisets:

$$CC(\mathbf{w}) = \{j \mid j = \text{the change in charge when a 1 is cycled from the rightmost position in } \mathbf{w} \text{ to the leftmost position } \}.$$

$$CL(\mathbf{w}) = \{j \mid j = (\text{the length of a column of } \mu) - 1\}.$$

$$P(\mathbf{w}) = \{j \mid \ \rho_j \mathbf{w} \text{ has a 1 in the leftmost position and } \rho_{j+1}\mathbf{w} \text{ does not}\}.$$

Theorem 4. *For any word* $\mathbf{w}$ *of partition content,* $CC(\mathbf{w}) = CL(\mathbf{w}) = P(\mathbf{w})$.

Proof. The fact that $CC(\mathbf{w}) = P(\mathbf{w})$ (which obviously follows from the facts that $CC(\mathbf{w}) = CL(\mathbf{w})$ and $CL(\mathbf{w}) = P)\mathbf{w}))$ was proven directly by Mark Shimozono using an alternate description of charge given by Lascoux, LeClerc, and Thibon [12]. Details of his proof may be found in my thesis.

We can use Theorem 2 and Lemmas 2 and 3 to prove that $CC(\mathbf{w}) = CL(\mathbf{w})$. Let $\mathbf{w} = w_n \dots w_2 w_1$. Let k be the largest letter in $\mathbf{w}$ and choose an i so that $w_i = k$. Apply cr iteratively to $\mathbf{w}$. The only such iteration which results in an increase in the charge value of that k is when k cycles from the leftmost position to the rightmost position. According to Lemma 3 and Theorem 2, this will increase the charge value on this k by one. Since after cycling all the way through $ORB(\mathbf{w})$ this k must have its initial value, there must be exactly one word $1\mathbf{v} \in ORB(\mathbf{w})$ such that $cr(1\mathbf{v})$ decreases the charge value of that k. The proof of that $CC(\mathbf{w}) = CL(\mathbf{w})$ now follows from Lemma 2 and Theorem 2 and induction.

Before proving the second part of the theorem, $CL(\mathbf{w}) = P(\mathbf{w})$, we must prove the following lemma.

Lemma 4. *Let* $\mathbf{w}$ *be a word of partition content and let* $\mathbf{w}^i$ *be a subword of length* k *as determined by the* $\widehat{ch}$ *method of choosing subwords. Then for all* $j \leq k$, ρ_j *leaves each of the letters in* $\mathbf{w}^i$ *unchanged.*

We will say these letters are *pinned* by ρ_j, for $j \leq k$.

Proof. In the following proof, the ith subword will refer to the ith subword formed under the $\widehat{ch}$ decomposition into subwords. We will give the proof for ρ_k. The proof is identical for any ρ_j with $j < k$. To apply $\rho_k = \sigma_1\sigma_2 \dots \sigma_{k-1}$ to $\mathbf{w}$ we first apply σ_{k-1} to $\mathbf{w}$. Looking just at the k's and $k-1$'s in $\mathbf{w}$ we have the following picture:

$$\begin{array}{ccccccc} & & i & & i & & \\ \cdots & & k & \cdots & k-1 & & \cdots \\ & A & & B & & C & \end{array}$$

We know that the $k-1$ in the ith subword is cyclicly to the right of the k in the ith subword from the way that subwords are chosen under the $\widehat{ch}$ description of charge. In addition, if there are any $k-1$'s in region B, they must already be in previous subwords, for otherwise they would have been chosen for the ith subword. Thus they are paired with k's from region A. There may be k's in region B. Now when we apply σ_{k-1} to $\mathbf{w}$ we will be changing $k-1$'s to k's, since the multiplicity of the $k-1$'s is less than or equal to the multiplicity of the k's. The $k-1$ in the ith subword will be paired under σ_{k-1}, either with the k in the ith subword or with some k in region B. Thus it will not change to a k. We will say that this $k-1$ is *pinned.* After applying σ_{k-1} to $\mathbf{w}$ we will look at only the $k-1$'s and $k-2$'s in $\sigma_{k-1}\mathbf{w}$ and apply σ_{k-2}. Again we will have the picture

$$\begin{array}{ccccc} & \overset{i}{k-1} & & \overset{i}{k-2} & \\ \cdots & & \cdots & & \cdots \\ A & & B & & C \end{array}$$

and by the same argument as before, the $k-2$ in the ith subword is pinned by the $k-1$ in the ith subword. Continuing in this manner we find that at each step, the j in the ith subword is pinned by the $j+1$ in the ith subword. Thus we find that eventually the 1 in the ith subword is pinned by the 2 in the ith subword.

$\square$

Corollary 1. *Let* $1\mathbf{v}$ *be a word of partition content such that the leftmost one is in the ith subword* $\mathbf{v}^i$ *and suppose the length of* $\mathbf{v}^i = k$. *Then*

$$\rho_k 1\mathbf{v} = 1\tilde{\mathbf{v}}.$$

Using our example from the description of $\widehat{ch}$ we have

$$\begin{array}{ccccccccccccccccccccc} & & \scriptstyle 3 & \scriptstyle 1 & \scriptstyle 1 & \scriptstyle 3 & \scriptstyle 2 & \scriptstyle 2 & \scriptstyle 2 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 1 & \scriptstyle 3 & \scriptstyle 3 & \scriptstyle 1 & \scriptstyle 1 & \scriptstyle 4 & \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 1 \\ \mathbf{w} & = & 1 & 2 & 1 & 4 & 3 & 5 & 2 & 1 & 3 & 1 & 7 & 5 & 2 & 6 & 5 & 2 & 4 & 4 & 3 \end{array}$$

where the top row denotes which subword the letters appear in in the subword decomposition. The leftmost 1 is in the subword $\mathbf{w}^3$ which has length 5. Thus, from the lemma, $\rho_5\mathbf{w} = 1\tilde{\mathbf{w}}$. Indeed we find that

$$\rho_5\mathbf{w} \;=\; 1 \;\; 2 \;\; 1 \;\; 4 \;\; 3 \;\; 5 \;\; 2 \;\; 1 \;\; 3 \;\; 2 \;\; 7 \;\; 5 \;\; 2 \;\; 6 \;\; 5 \;\; 3 \;\; 4 \;\; 4 \;\; 3$$

which does have a one in the leftmost position.

Note that if we apply ρ_j to $\mathbf{w}$ for $j > k$, then when we apply σ_k to $\sigma_{k+1}\sigma_{k+2}\cdots\sigma_{j-1}\mathbf{w}$ the k in the ith subword is not pinned by any $k+1$ and thus it is changed to a $k+1$ under σ_k. Then the $k-1$ in the ith subword is no longer pinned by this k and thus it will change to a k under σ_{k-1}. Continuing in this manner we find that each of the letters in the ith subword is unpinned and eventually changed under ρ_j and thus the 1 in the ith subword is not preserved under ρ_j.

This completes our proof of that $CL(\mathbf{w}) = P(\mathbf{w})$, and thus that $CC(\mathbf{w}) = P(\mathbf{w})$. $\square$

We can now see easily that the charge change when we cycle the leftmost 1 in a word is given by one less than the length of the subword that this 1 belongs to. In addition we have shown that the plactic swaps which preserve this leftmost 1 are ρ_j for $j \leq k$ where the leftmost 1 is in a subword of length k.

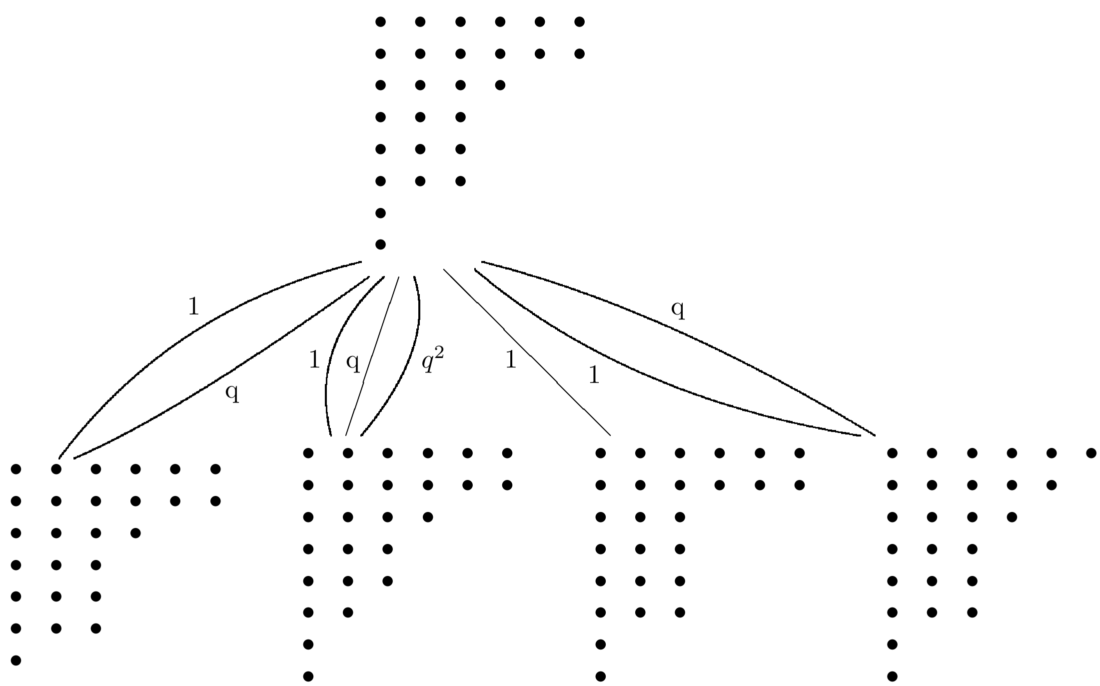
5 Bijection

As mentioned previously, Garsia and Procesi [6] prove the cocharge version of the following recurrence for the generating function for charge on words.

Theorem 5.

$$\sum_{\mathbf{w}\in W_\mu} q^{ch(\mathbf{w})} = \sum_i q^{r_{i,\mu}} \sum_{\mathbf{w}\in W_{\mu^{(i)}}} q^{ch(\mathbf{w})}$$

Since this recursion is a statement about charge on words of a certain content, we can draw the following diagram based on the relation between the content of these words. Recall that to compute the charge on a word $\mathbf{w} \in W_{\mu^{(i)}}$, i.e., on a word of content $\mu^{(i)}$, we rearrange the content $\mu^{(i)}$ to partition content. Thus we have the following diagram for words of content $\mu = (6, 6, 4, 3, 3, 3, 1, 1)$.



Reading from this diagram with $\mu = (66433311)$, we have

$$\sum_{\mathbf{w}\in W_\mu} q^{ch(\mathbf{w})} = (1+q)\sum_{\mathbf{w}\in W_{\mu(8)}} q^{ch(\mathbf{w})} + (1+q+q^2)\sum_{\mathbf{w}\in W_{\mu(6)}} q^{ch(\mathbf{w})} + \sum_{\mathbf{w}\in W_{\mu(3)}} q^{ch(\mathbf{w})} + (1+q)\sum_{\mathbf{w}\in W_{\mu(2)}} q^{ch(\mathbf{w})}$$

We will give a direct combinatorial proof.

5.1 Ferrers Diagram Theorem

Lemmas 2 and 3 allow us to partition $ORB(\mathbf{w})$ in a very precise way.

Theorem 6. *Let μ be a partition of n and let $\mathbf{w} \in W_\mu$ be an n/d-cyclic word. There is a bijection ϕ between the elements of $ORB(\mathbf{w})$ and the cells of the Ferrers diagram of μ/d satisfying*

1. *If $\phi(\mathbf{z}_j) = (1, j)$, then $\mathbf{z}_j \in ORB_1^+(\mathbf{w})$ and $ch(\mathbf{z}_j) = ch(cr(\mathbf{z_j})) + {\mu_j}' - 1$.*
2. *If $\mathbf{u}, \mathbf{v} \in ORB(\mathbf{w})$ with $\phi(\mathbf{u}) = (i-1, j)$ and $\phi(\mathbf{v}) = (i, j)$, $i \neq 1$, then $ch(\mathbf{v}) = ch(\mathbf{u}) - 1$.*

Thus part 1 says that elements of $ORB_1^+(\mathbf{w})$ appear at the top of the columns of μ/d, while part 2 says that charge values decrease by one down columns. Before giving the proof of this theorem we will introduce some additional notation and prove a technical lemma which will allow us to describe ϕ.

Define a *restricted decrease sequence* to be a cyclic sequence $(x_0, x_1, x_2, \ldots, x_{n-1})$ of integers such that either

$$x_i = x_{i-1} - 1,$$

or

$$x_i \geq x_{i-1}.$$

By a cyclic sequence of integers we mean that if $x_i = x_0$ then $x_{i-1} = x_n$. Now let $r_1, r_2, \ldots, r_k$ be the indices such that

$$x_{r_j} \geq x_{r_j - 1}.$$

Then we have the following lemma:

Lemma 5. *Let $(x_0, x_1, \ldots, x_{n-1}) = X$ be a restricted decrease sequence with increases at $(r_1, \ldots, r_k)$. Then*

$$\{x_0, \ldots, x_{n-1}\} = \{a_{11}, a_{12}, \ldots, a_{1t_1}\} \sqcup \cdots \sqcup \{a_{k1}, a_{k2}, \ldots, a_{kt_k}\}$$

where

1. $a_{ij} = a_{ij-1} - 1$

2. $a_{i1} = x_{r_i}$

3. $t_i = x_{r_i} - x_{r_i - 1} + 1$

and $\sqcup$ *means disjoint union.*

For example, suppose we have the restricted decrease sequence

$$\begin{array}{cccccccccc} 6 & 5 & 4 & 5 & 4 & 3 & 3 & 2 & 4 & 3. \\ r_1 & & & r_2 & & & r_3 & & r_4 & \end{array}$$

Then this sequence is equal to

$$a_1 \sqcup a_2 \sqcup a_3 \sqcup a_4$$

where

$$\begin{aligned} a_1 &= \{6, 5, 4, 3\} \\ a_2 &= \{5, 4\} \\ a_3 &= \{3\} \\ a_4 &= \{4, 3, 2\}. \end{aligned}$$

Proof. The proof will be a proof by induction on k. If $k = 1$ the statement of the lemma is trivially true. Suppose then that the properties of the lemma hold for $k - 1$. Let $t_i = x_{r_i} - x_{r_i - 1} + 1$ as before. Then it is easy to observe that there exists an r_i such that

$$\begin{aligned} x_{r_i+1} &= x_{r_i} - 1 \\ x_{r_i+2} &= x_{r_i+1} - 1 \\ &\vdots \\ x_{r_i+t_i-1} &= x_{r_i+t_i-2} - 1 \end{aligned}$$

Now let

$$\tilde{X} = \{x_0, x_1, \ldots, x_{r_i-1}, x_{r_i+t_i}, \ldots, x_{n-1}\}.$$

Note that:

$$\begin{aligned} x_{r_i+t_i} &\geq x_{r_i+t_i-1} - 1 \\ &= x_{r_i} - t_i \\ &= x_{r_{i-1}} - 1 \end{aligned}$$

Thus an increase at r_i in $\tilde{X}$ will occur if and only if there was an increase at r_i in X and t_i in $\tilde{X}$ is thus equal to t_i in X. Therefore, $\tilde{X}$ is a restricted decrease sequence and so by induction, $\tilde{X}$ decomposes as in the theorem. Then X decomposes as a disjoint union of $\tilde{X}$ and $\{x_{r_i}, x_{r_i+1}, \ldots, x_{r_i+t_i-1}\}$. □

Now we observe that the set of charge values on the words in $ORB(\mathbf{w})$ is a restricted decrease sequence, thus we know we can partition this set of charge values into disjoint sequences of charge values where the number of disjoint sequences corresponds to the number of columns in the Ferrers diagram of μ and the lengths of the sequences correspond to the lengths of the columns of μ. Now we will describe ϕ which tells us the manner in which to partition $ORB(\mathbf{w})$ into the columns of the Ferrers diagram.

Let $\mathbf{w} \in W_\mu$ be an n/d-cyclic word as in the statement of the theorem. Then we know that $\#\{ORB_1^+(\mathbf{w})\} = \mu_1/d$. We define the words in $ORB_1^+(\mathbf{w})$ to be *column headers*. We will define a *column* to be a subset of $ORB(\mathbf{w})$ such that exactly one word in the subset is in $ORB_1^+(\mathbf{w})$ and such that the number of words in the subset is equal to the length of one of the columns in the Ferrers diagram of μ. In addition, if the charge on the word which ends in a one is r and there are k words in the column, then exactly one word in the column has a charge of $r-1$, exactly one word has a charge of $r-2$, ..., and exactly one word has a charge of $r-k+1$. We now show how to partition the $ORB(\mathbf{w})$ for any word $\mathbf{w}$ into λ_1/d columns.

Let $1\mathbf{z}_1$, $1\mathbf{z}_2$, ..., $1\mathbf{z}_{\lambda_1/d}$ be the elements of $ORB_1^+(\mathbf{w})$ for some cyclic orbit $ORB(\mathbf{w})$, i.e., the column headers. Suppose the change in charge between $1\mathbf{z_i}$ and $\mathbf{z_i}1$ is d_i. Then $1\mathbf{z_i}$ will head a column of length d_i+1. Throughout this description we will consider $\mathbf{w}$ to be the first word in $ORB(\mathbf{w})$ and for any word $\mathbf{v}$ in $ORB(\mathbf{w})$ we will consider the "next" word in $ORB(\mathbf{w})$ after $\mathbf{v}$ to be $cr(\mathbf{v})$.

Label the first column header in $ORB(\mathbf{w})$ x_1. (Throughout this description, the label x_i does not necessarily correspond to the ith column in the Ferrers diagram.) Let d_i+1 be the length of the ith column and r_i be the charge of the ith column header. If the charge of the word following x_1 in $ORB(\mathbf{w})$ has charge r_1-1 then include that word in column x_1 and look at the next word in $ORB(\mathbf{w})$. If this next word has charge r_1-2 then include this word in column x_1 as well. Continue until there are d_1+1 words in column x_1 or until the charge on the next word in $ORB(\mathbf{w})$ weakly increases. This word will be a column header and we will label it x_2. We will now try to fill column x_2 with d_2+1 words in the same manner, adding words until we fill the column or we reach another column header which we will label x_3. Once we have filled column x_2, we will try to finish filling column x_1. If we fill column x_1 before reaching a column header, we will jump to the next column header in $ORB(\mathbf{w})$, label it x_2 and begin filling the x_2 column. We will continue this process of "nested filling" until all columns have been filled.

For example, in the following table we show $ORB(\mathbf{w})$, the corresponding charge on each word in $ORB(\mathbf{w})$, and the column number that each word belongs to for $\mathbf{w} = 215321431$.

Word	Charge	Column number
2 1 5 3 2 1 4 3 1	3	x_3

1 2 1 5 3 2 1 4 3	3	x_1
3 1 2 1 5 3 2 1 4	2	x_3
4 3 1 2 1 5 3 2 1	1	x_3
1 4 3 1 2 1 5 3 2	3	x_2
2 1 4 3 1 2 1 5 3	2	x_2
3 2 1 4 3 1 2 1 5	1	x_2
5 3 2 1 4 3 1 2 1	0	x_3
1 5 3 2 1 4 3 1 2	4	x_3

In this example we have

$$\begin{array}{ll} \text{Column header } x_1 = & 1\ \ 2\ \ 1\ \ 5\ \ 3\ \ 2\ \ 1\ \ 4\ \ 3 \\ d_1 + 1 = 1 & r_1 = 3 \\ \text{Column header } x_2 = & 1\ \ 4\ \ 3\ \ 1\ \ 2\ \ 1\ \ 5\ \ 3\ \ 2 \\ d_2 + 1 = 3 & r_2 = 3 \\ \text{Column header } x_3 = & 1\ \ 5\ \ 3\ \ 2\ \ 1\ \ 4\ \ 3\ \ 1\ \ 2 \\ d_3 + 1 = 5 & r_3 = 4 \end{array}$$

Now we can place the words in a cyclic orbit into a Ferrers diagram according to their placement in each column.

(4) 1 5 3 2 1 4 3 1 2	(3) 1 4 3 1 2 1 5 3 2	(3) 1 2 1 5 3 2 1 4 3
(3) 2 1 5 3 2 1 4 3 1	(2) 2 1 4 3 1 2 1 5 3	
(2) 3 1 2 1 5 3 2 1 4	(1) 3 2 1 4 3 1 2 1 5	
(1) 4 3 1 2 1 5 3 2 1		
(0) 5 3 2 1 4 3 1 2 1		

Charge, given in parentheses, decreases by one going down each column.

5.2 Combinatorial Proof of the Garsia-Procesi Recurrence

Let $RW_{i,\mu}$ denote all the words in W_μ which map to row i under ϕ.

Choose any word $\mathbf{v}$ in $ORB(\mathbf{w})$ and suppose $\phi(\mathbf{v}) = (j, l)$. Define

$$\alpha : ORB(\mathbf{w}) \to ORB_1^+(\mathbf{w})$$

by

$$\alpha(\mathbf{v}) = \mathbf{z} \text{ where } \phi(\mathbf{z}) = (1, l).$$

That is, α "projects" $ORB(\mathbf{w})$ onto $ORB_1^+(\mathbf{w})$ up the column of the Ferrers diagram.

Now define β from words $1\mathbf{u}$ of content (μ_j, λ) where $\mu_j \neq 0$ and λ is a partition by $\beta(1\mathbf{u}) = \mathbf{u}$.

Theorem 7. *There is a bijection* $\psi = \rho_i^{-1} \circ \beta \circ \rho_i \circ \alpha$ *from* $RW_{i,\mu}$ *to* $W_{\mu^{(i)}}$ *such that if* $\mathbf{w} \in RW_{i,\mu}$ *and* $\tilde{\mathbf{w}} = \psi(\mathbf{w}) \in W_{\mu^{(i)}}$*, then*

$$ch(\mathbf{w}) = ch(\tilde{\mathbf{w}}) + r_{i,\mu}.$$

For example, let

$$\mathbf{w} = 431215321$$

so that $\phi(\mathbf{w}) = (4, 1)$ from the example at the end of the previous chapter.

Then

$$\alpha(\mathbf{w}) = 153214312.$$

Since $\mathbf{w}$ appears in row 4 under the bijection ϕ, we know from Lemma 4 that applying ρ_4 to $\alpha(\mathbf{w})$ leaves a one in the leftmost position. From our example,

$$\rho_4(\alpha(\mathbf{w})) = \rho_3(153214312) = 154324322.$$

Then

$$\beta \circ \rho_4 \circ \alpha(\mathbf{w}) = \beta(154324322) = 54324322$$

and

$$\rho_4^{-1} \circ \beta \circ \rho_4 \circ \alpha(\mathbf{w}) = \rho_4^{-1}(54324322) = 53213211.$$

Thus $\psi(\mathbf{w}) = 53213211 = \tilde{\mathbf{w}}$. Since $ch(\mathbf{w}) = 1$ and $ch(\tilde{\mathbf{w}}) = ch(53213211) = ch(43213211) = 0$ and $r_{4,(32211)} = 1$, we can see that $ch(\mathbf{w}) = ch(\tilde{\mathbf{w}}) + r_{4,(32211)}$.

Proof. That ψ is a well-defined bijection follows directly from Corollary 1. That ψ changes charge by the correct amount follows from the next lemma.

Lemma 6. *If* ν *is a* j*-protopartition of* μ *and* $1\mathbf{z} \in W_\nu$*, then* $ch(1\mathbf{z}) = ch(\mathbf{z}) + j - 1$*.*

Proof. The proof of this lemma follows directly from the definition of $\tilde{ch}$ which by Theorem 3 is equal to charge. By definition of $\tilde{ch}$, the leftmost one in the word $\rho_i(\alpha(\mathbf{w}))$ belongs to the last subword in the subword decomposition which contains a one, i.e., in the jth subword. The first 2 reading right to left which is not contained in any earlier subword will also be in the jth subword.. When we remove this 1 under the action of β and compute $\tilde{ch}$, the subwords $\mathbf{w}^1$ through $\mathbf{w}^{j-1}$ will remain the same. When we select the jth subword, we will no longer have any ones, so we will read from right to left and select the first 2 which is not in any of the first $j-1$ subwords. This will be the same 2 which was in the jth subword before applying β. Thus the jth subword will be the same except it will be missing a 1 in the leftmost position. The remaining subwords will also remain the same. Thus the charge on all of the subwords will remain the same after applying β except for the charge on the jth subword. The charge value on each letter in this subword will go down by one, thus $ch(1\mathbf{z}) = ch(\mathbf{z}) + j - 1$. □

The proof of Theorem 7 now follows immediately from Lemma 4 and Lemma 6. □

6 A Simple Description

In this section we give a simple description of the bijection given in Theorem 7. This description bypasses the plactic action required by ρ_j and ρ_j^{-1} and uses instead a charge-like calculation.

For a partition μ, let $C(\mu) = \{i : \mu'_i > \mu'_{i+1}\}$. Thus, $C(\mu)$ is the set of column indices of the outer corners of the Ferrers diagram of μ. In other words, $C(\mu) = \{\mu_i\}$ = the set of part sizes of μ. For example, if $\mu = (4, 4, 3, 3, 3, 1, 1)$, then $C(\mu) = \{1, 3, 4\}$.

Suppose $\mathbf{w} \in W^1_\mu$. For each $i \in C(\mu)$, we now describe a special standard subword of $\mathbf{w}$, which we call $\Delta_i(\mathbf{w})$.

We begin by finding the standard subwords $\mathbf{w}^1$, $\mathbf{w}^2$, …, $\mathbf{w}^{i-1}$ as in the standard charge calculation. We call this set of standard subwords the i-th *charge set*. Now underline the unique μ'_i in $\mathbf{w}$ which is not in the i-th charge set. Next, underline the first $\mu'_i - 1$ which occurs to the right of the underlined μ'_i and is not in the i-th charge set. Continue by underlining the first $\mu'_i - 2$ which occurs to the right of the underlined $\mu'_i - 1$ and is not in the i-th charge set, and so on. If at any stage there is no $l - 1$ to the right of the underlined l, then search for the first $l - 1$ beginning at w_n, reading left to right. Continue until a 1 is underlined. The standard subword of underlined letters is $\Delta_i(\mathbf{w})$.

As an example, let

$$\mathbf{w} = 1\,2\,1\,4\,3\,5\,2\,1\,3\,1\,7\,5\,2\,6\,5\,2\,4\,4\,3.$$

Note that $\mathbf{w}$ has type $(4, 4, 3, 3, 3, 1, 1)$.

Below, we underline the three possible $\Delta_i(\mathbf{w})$, and note with superscripts the i-th charge set:

$$\Delta_1(\mathbf{w}) : \quad 1\,\underline{2}\,\underline{1}\,4\,3\,5\,2\,1\,3\,1\,\underline{7}\,5\,2\,\underline{6}\,\underline{5}\,2\,\underline{4}\,4\,\underline{3},$$

$$\Delta_3(\mathbf{w}) : \quad \underline{1}\,2^2\,1\,4^1\,3^1\,\underline{5}\,2^1\,1^2\,\underline{3}\,1^1\,7^1\,5^2\,\underline{2}\,6^1\,5^1\,2\,\underline{4}\,4^2\,3^2,$$

and

$$\Delta_4(\mathbf{w}) : \quad \underline{1}\,2^2\,1^3\,4^1\,3^1\,5^3\,2^1\,1^2\,3^3\,1^1\,7^1\,5^2\,\underline{2}\,6^1\,5^1\,2^3\,4^3\,4^2\,3^2.$$

We will say a standard subword of $\mathbf{w} \in W^1_\mu$ is *pinned* if the 1 of the standard subword is w_n.

We can now describe a portion of the bijection in Theorem 7. If $\Delta_i(\mathbf{w})$ is pinned, then define $\tau_i(\mathbf{w})$ by removing w_n from $\mathbf{w}$ and subtracting 1 from the other letters in $\Delta_i(\mathbf{w})$.

Theorem 8. *If $\Delta_i(\mathbf{w})$ is pinned, then*

$$\rho^{-1}_{\mu'_i} \circ \beta \circ \rho_{\mu'_i}(\mathbf{w}) = \tau_i(\mathbf{w}).$$

If $\Delta_i(\mathbf{w})$ is not pinned, then $\rho^{-1}_{\mu'_i} \circ \beta \circ \rho_{\mu'_i}(\mathbf{w})$ is not defined.

Proof. Let $\mu_i' = k$ and let the k and $k-1$ below be those in $\Delta_i(\mathbf{w})$.

$$\begin{array}{ccccccc} \cdots & k & \cdots & k-1 & \cdots \\ & & A & & \end{array} \tag{3}$$

In the above diagram and in the following discussion we consider everything to be cyclic, thus the $k-1$ above is cyclically to the right of the region A and k. We now describe what happens to $\Delta_i(\mathbf{w})$ under $\rho_{\mu_i'}^{-1} \circ \beta \circ \rho_{\mu_i'}$.

First, note that when we apply σ_{k-1} to $\mathbf{w}$ we will be changing $k-1$'s to k's, since $\mathbf{w}$ has partition content. If there are $k-1$'s in region A then they must pair with k's which are also in A, otherwise either the k in (3) is in the ith charge set or the $k-1$ in (3) is not the first available $k-1$ to the right of k. By first available $k-1$ we mean it is the first $k-1$ not in the ith charge set which is cyclically to the right of the k in (3).

Second, observe that there could be additional k's in region A which are paired with $k-1$'s to the right of the $k-1$ in (3), but in any case the $k-1$ in the $\Delta_i(\mathbf{w})$ is paired under the plactic swap σ_{k-1} and thus will not change to a k. Continuing to apply σ_j in decreasing succession for $1 \leq j \leq k-2$ we find that the 1 in $\Delta_i(\mathbf{w})$ does not change to a 2 when σ_1 is applied.

Thus we conclude that each of the letters in $\Delta_i(\mathbf{w})$ remains unchanged in ρ_k.

Now under β we remove the leftmost 1 in $\rho_k(\mathbf{w})$ and show that the 2 in $\Delta_i(\mathbf{w})$ changes to a 1 under σ_1 and then the 3 in $\Delta_i(\mathbf{w})$ changes to a 2 under σ_2, etc., until we have applied all of ρ_k^{-1} and each of the letters in $\Delta_i(\mathbf{w})$ have decreased by one, except the 1 which was removed. The letters not in $\Delta_i(\mathbf{w})$ remain unchanged under $\rho_k^{-1} \circ \beta \circ \rho_k(\mathbf{w})$ since at most one i can change to an $i-1$ for each i and we will show that the letters which change are those in $\Delta_i(\mathbf{w})$.

Suppose when we remove the leftmost 1, the 2 which was in $\Delta_i(\mathbf{w})$ remains paired under the plactic swap σ_1. Then this 2 would have been in some jth subword of $\mathbf{w}$ for $j < i$ since we have only μ_i 1's and the 1's and the 2's in the first $(\mu_i - 1)$ subwords will be paired. Then this 2 could not be in $\Delta_i(\mathbf{w})$. Thus this 2 must be unpaired under σ_1 and so will change to a 1. We can make this argument for each of the letters in $\Delta_i(\mathbf{w})$, so each letter will be decreased by 1 as we apply ρ_k^{-1}. This proves the result of our theorem.

□

In the example,

$$\tau_3(\mathbf{w}) = 2\,1\,4\,3\,4\,2\,1\,2\,1\,7\,5\,1\,6\,5\,2\,3\,4\,3$$

and

$$\tau_4(\mathbf{w}) = 2\,1\,4\,3\,5\,2\,1\,3\,1\,7\,5\,1\,6\,5\,2\,4\,4\,3.$$

The reader may verify that these two words are the results of applying $\rho_5^{-1} \circ \beta \circ \rho_5$ and $\rho_2^{-1} \circ \beta \circ \rho_2$, respectively, to $\mathbf{w}$.

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